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Statistical Mechanics and Singular SPDEs

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University of Warwick

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ANOMALOUS SUPERDIFFUSION OF BROWNIAN MOTION IN RANDOM INCOMPRESSIBLE FLOWS

SCOTT ARMSTRONG

Classification AMS 2020: 60K37, 35R60, 82B28

Keywords: Probability, Mathematical Physics, Analysis of PDEs

We study the long-time behavior of a particle in $\mathbb{R}^d$, $d \geq 2$, subject to molecular diffusion and advection by a random incompressible flow. The velocity field is the divergence of a stationary random stream matrix $\mathbf{k}$ with positive Hurst exponent $\gamma > 0$, so the resulting random environment is multiscale and self-similar. In the perturbative regime $\gamma \ll 1$, we prove quenched power-law superdiffusion: for a typical realization of the environment, the displacement variance at time t grows like $t^{2/(2-\gamma)}$, the scaling predicted by renormalization group heuristics. We also identify the leading prefactor up to a random (quenched) relative error of order $\gamma^{-\frac{1}{2}}|\log\gamma|^{\frac{9}{2}}$. The proof implements a Wilsonian renormalization group scheme at the level of the infinitesimal generator

$$\nabla \cdot (\nu I_d + \mathbf{k})\nabla,$$

based on a self-similar induction across scales. We demonstrate that the coarse-grained generator is well-approximated, at each scale r, by a constant-coefficient Laplacian with effective diffusivity growing like r^γ. This approximation is inherently scale-local: reflecting the multifractal nature of the environment, the relative error does not decay with the scale, but remains of order $\gamma^{\frac{1}{2}}|\log\gamma|^{\frac{7}{2}}$. We also prove anomalous regularization under the quenched law: for almost every realization of the drift, solutions of the associated elliptic equation are Hölder continuous with exponent $1 - C\gamma^{\frac{1}{2}}$ and satisfy estimates which are uniform in the molecular diffusivity ν and the scale.

The talk is based on joint work with Ahmed Bou-Rabee (UPenn) and Tuomo Kuusi (Helsinki) which is available here: https://arxiv.org/abs/2601.22142.

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SORBONNE UNIVERSITÉ, FRANCE
Email address: scottnarmstrong@gmail.com

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AN INTRODUCTION TO THE STOCHASTIC HEAT FLOW

QUENTIN BERGER

Classification AMS 2020: 82B44, 60K35, 82D60

Keywords: Stochastic Heat Flow, Directed Polymers, Fractional Moments, Coarse-Graining, Critical Phenomena, Strong Disorder, Local Extinction, Superdiffusivity

The Stochastic Heat Equation (SHE) with multiplicative white noise is a stochastic PDE which is ill-defined in its critical dimension $d = 2$. In this case, recent results have shown that a subtle normalisation procedure is needed to make sense of it. For instance, the "probabilistic" approach consists of studying the directed polymer model (which can be thought of as a discretised SHE) in a critical window of parameters. This approach was followed by Caravenna, Sun and Zygouras, who established the existence of the so-called critical 2D Stochastic Heat Flow (SHF). Another approach, developed by Tsai, provides an axiomatic characterisation of the SHF.

The report reviews some of the properties of this object, highlighting contributions from a wide range of ideas, spanning stochastic analysis, probability and statistical mechanics. Particular attention is given to recent developments concerning the long-time behaviour, strong-disorder regime and fractional moments of the critical 2D SHF and its connection to directed polymers. These include results establishing sharp estimates for local extinction, identifying the spatial scales governing transitions between vanishing and diverging mass, and demonstrating superdiffusive fluctuations in regimes of strong disorder. Further results provide estimates for the fractional moments of the mass assigned to small regions, revealing their close relationship with the divergence of the second moment. The analysis also highlights the role of change-of-measure techniques and coarse-graining procedures in understanding these phenomena, illustrating how probabilistic methods contribute to the study of critical stochastic PDEs.

Further details of these developments can be found in the following research papers:

- https://arxiv.org/abs/2608.13359
- https://arxiv.org/abs/2508.02478

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UNIVERSITÉ SORBONNE PARIS NORD, FRANCE
Email address: quentin.berger@univ-paris13.fr

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CONSTRUCTION OF THE 2D NON-ABELIAN YANG-MILLS-HIGGS MEASURE VIA STOCHASTIC QUANTISATION

BJOERN BRINGMANN AND SKY CAO

Classification AMS 2020: 60H15, 81T13, 81T08, 35R60

Keywords: Stochastic Quantisation, Yang–Mills–Higgs Measure, Gauge Theory, Stochastic Partial Differential Equations, Renormalisation, Paracontrolled Calculus

We provide the first mathematical construction of the non-Abelian Yang-Mills-Higgs measure on a two-dimensional torus by implementing Parisi and Wu's "stochastic quantisation" procedure. We use a specific gauge covariant (but non-Markovian) approximation to the previously constructed associated Langevin dynamics and develop a covariant paracontrolled calculus for it. This allows us to obtain short-time control over times that are much longer than what the cubic nonlinearity naïvely suggests. When combined with the decay properties of the Yang-Mills-Higgs action under the deterministic gradient flow, as well as a quantitative measure of how the noise pushes the dynamic away from Yang-Mills connections, this is good enough to yield uniform-in-time bounds.

We finally show that there exists a renormalisation under which our gauge covariant approximation converges to the Langevin dynamics (modulo gauge transformation) and that the latter is topologically irreducible and strong Feller. It therefore admits a unique invariant measure, providing the sought-after construction. Our construction is non-perturbative and works in particular for any fixed values of the coupling constants.

Further details of these developments can be found in the following research paper:

- https://arxiv.org/abs/2611.03640v1

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California Institute of Technology, USA
Email address: bjoern@caltech.edu

University of Pennsylvania, USA
Email address: skycao@wharton.upenn.edu

.

A GENERAL PARACONTROLLED ANSATZ FOR SINGULAR SPDES

YVAIN BRUNED

Classification AMS 2020: 60H15, 35R60, 60H30, 35A01

Keywords: Singular Stochastic Partial Differential Equations, Paracontrolled Distributions, Regularity Structures, Renormalisation, BPHZ Renormalisation, Decorated Trees

We provide a general ansatz for the paracontrolled approach introduced by Gubinelli, Imkeller, Perkowski for treating singular SPDEs. The ansatz proposed covers a large class of equations. It is described via decorated trees that encode its coefficients and its stochastic data. The main novelty is the recursive definition of the paracontrolled stochastic iterated integrals that involve a well-chosen combinatorial set of words. The paralinearisation is performed via a lift of iterated paraproducts to modelled distributions. The main idea is to integrate localised iterated paraproducts in order to get a global object with a good regularity. We also provide a fixed point argument and the renormalised equation with this ansatz when one uses the preparation map formalism that encompasses the BPHZ renormalisation. The last main result is the convergence of the renormalised stochastic data via the BPHZ renormalisation using an adaptation of the spectral gap approach. One obtains in the end a general solution theory for parabolic singular SPDEs via the paracontrolled approach.

Further details of these developments can be found in the following research papers:

- https://arxiv.org/abs/2608.09700

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UNIVESITY OF LORRAINE, FRANCE
Email address: yvain.bruned@univ-lorraine.fr

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MARTINGALE DESCRIPTION OF THE TWO-DIMENSIONAL STOCHASTIC HEAT EQUATION

YU-TING CHEN

Classification AMS 2020: 60H05, 60H15, 60H17

Keywords: Stochastic heat equation; continuum directed random polymer; delta-Bose gas

We characterize the sample-path structure of the two-dimensional stochastic heat equation at criticality. The main theorems provide a complete, pathwise description of its semimartingale characteristics and establish explicit, exact formulas for the covariation measure. The proof derives an asymptotic expansion for the covariation measures of regularized approximate solutions by integrating techniques from quantum-solvable operators into stochastic integration theory.

Further details of these developments can be found in the following research paper:

- https://arxiv.org/abs/2504.21791

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UNIVERSITY OF VICTORIA, CANADA
Email address: chenyuting@uvic.ca

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CONTINUUM POLYMER MEASURES CORRESPONDING TO THE CRITICAL 2D STOCHASTIC HEAT FLOW

JEREMY CLARK

Classification AMS 2020: 82B44, 35R60, 82D60, 60H15, 60G57

Keywords: Critical 2D Stochastic Heat Flow, Continuum Directed Polymers, Chapman–Kolmogorov Relation, Second Moments, Conditional Gaussian Multiplicative Chaos, Disorder Strength

We construct continuum directed polymer measures corresponding to the critical 2d stochastic heat flow (2d SHF) introduced by Caravenna, Sun, and Zygouras in their recent article [Inventiones mathematicae 233, 325–460 (2023)]. For this purpose, we prove a Chapman-Kolmogorov relation for the 2d SHF along with a related elementary conditional expectation formula. We explore some basic properties of the continuum polymer measures, with our main focus being on their second moments. In particular, we show that the form of their second moments is consistent with the family of continuum polymer measures, indexed by a disorder strength parameter, having a conditional Gaussian multiplicative chaos distributional interrelationship similar to that previously found in an analogous hierarchical toy model.

This is joint work with Barkat Mian.

Further details of these developments can be found in the following research papers:

- https://arxiv.org/pdf/2409.01510

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UNIVERSITY OF MISSISSIPPI, USA
Email address: jeremy@olemiss.edu

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MAXIMUM OF THE TWO-DIMENSIONAL DIRECTED POLYMER IN THE CRITICAL WINDOW

CLEMENT COSCO

Classification AMS 2020: 60K37, 60G70, 82D60, 60F05

Keywords: Two-Dimensional Directed Polymers, Critical Window, Partition Functions, Extreme Value Theory, Random Environments, Stochastic Heat Flow

Let $Z_N^{\beta_N}(x)$ denote the partition function of the two-dimensional directed polymer started at $x \in \mathbb{Z}^2$, in Gaussian environemnt, in the critical window, with parameter $\beta_N = \sqrt{(1-\vartheta/\log N)/R_N}$. Here $R_N \sim (\log N)/\pi$ is the truncated collision Green function at the origin for the simple random walk on $\mathbb{Z}^2$. Let $\mathfrak{M}_N$ denote the maximum of $\log Z_N(x)$ over $|x|_\infty \leq \sqrt{N}$. We prove that, for any $\vartheta \in \mathbb{R}$, in probability,

$$\mathfrak{M}_N/\sqrt{\log N} \to 2\sqrt{2} \text{ as } N \to \infty.$$

This is joint work with Shuta Nakajima and Ofer Zeitouni.

Further details of these developments can be found in the following research papers:

- https://arxiv.org/pdf/2610.41989

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UNIVERSITÉ PARIS DAUPHINE, PARIS, FRANCE
Email address: cosco@ceremade.dauphine.fr

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INVARIANT MEASURES FOR THE OPEN KPZ EQUATION: AN ANALYTIC PERSPECTIVE

ALEXANDER DUNLAP

Classification AMS 2020: 60H15, 60J25, 60F05, 82C22

Keywords: Kardar–Parisi–Zhang Equation, Invariant Measures, Stochastic Analysis, Boundary Conditions, Regularity Structures, Stochastic Quantisation

The ergodic theory of the open KPZ equation has seen significant progress in recent years, with explicit invariant measures described in a series of works by Corwin–Knizel, Barraquand–Le Doussal, and Bryc–Kuznetsov–Wang–Wesołowski. In this paper, we provide a stochastic analytic proof of the formula for the invariant measures. Our approach starts from the Gaussian invariant measure for the case of homogeneous boundary conditions. We approximate the inhomogeneous problem by a homogeneous one with a singular boundary potential. Using tools including change of measure, time reversal for Markov processes, and Itô's formula, we then reduce the problem to analyzing the KPZ nonlinearity in a thin boundary layer. Finally, using the theory of regularity structures, we establish a central limit theorem for the time-integrated nonlinearity near the boundary, which completes the proof of the invariance. Although it is known that different boundary parameters give rise to distinct physical regimes for the invariant measures, our method is robust and does not rely on any particular choice of boundary parameters.

This is joint work with Yu GU (University of Maryland) and Tommaso Rosati (University of Warwick).

Further details of these developments can be found in the following research papers:

- https://arxiv.org/pdf/2512.03328

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DEPARTMENT OF MATHEMATICS, DUKE UNIVERSITY, DURHAM, NC 27708, USA
Email address: alexander.dunlap@duke.edu

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FLUCTUATIONS IN THE WEAKLY COUPLED 4D ANDERSON HAMILTONIAN

SIMON GABRIEL

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Keywords: Scaling critical singular SPDEs; Anderson model; BPHZ renormalisation.

We study the effect of time-independent white noise disorder on a Brownian particle in four dimensions. The problem can be reformulated as a scaling-critical singular SPDE. By tuning the noise strength appropriately, we obtain an exact analysis of the perturbative terms arising in the corresponding Green function. This gives a first systematic, though still perturbative, renormalisation analysis for a scaling-critical singular SPDE. This report is based on the joint work [1] with Tommaso Rosati (University of Warwick).

1. Setup and main result

We study the elliptic Anderson model in resolvent form

$$(1-\Delta)v = \lambda\xi v + \delta_y, \qquad x, y \in \mathbb{T}^4, \tag{1.1}$$

where ξ is space white noise. The Anderson Hamiltonian was introduced as a model for quantum particles in a random potential [2]. In the continuum, the equation with white-noise potential is well understood only in subcritical dimensions $d < 4$ [3, 4]. These constructions fit the general paradigm of singular SPDEs, where one removes local divergences by renormalisation and then solves a subcritical fixed point problem. See for example the theory of regularity structures [5], paracontrolled distributions [6], or the flow approach [7]. Dimension $d = 4$ is the borderline case for the continuum Anderson Hamiltonian. Indeed, under the spatial rescaling $x \mapsto x/\zeta$, the Laplacian scales like ζ^{-2}, while white noise has homogeneity $-d/2$. Thus the two terms Δv and ξv have the same scaling strength exactly when $d = 4$. Hence, the equation is critical: the random potential is no longer a lower-order perturbation of the Laplacian, as it is in $d < 4$.

Without further regularisation (1.1) is ill–posed. We therefore study a mollified resolvent equation

$$(1-\Delta)v_\varepsilon = \rho_\varepsilon * (\lambda_\varepsilon \xi v_\varepsilon + \delta_y) \ ,$$

where ρ_ε is a smooth radial mollifier on $\mathbb{T}^4$. We tune the coupling constant according to the small–scale cutoff and set

$$\lambda_\varepsilon = \hat{\lambda}\left(\log\frac{1}{\varepsilon}\right)^{-1/2} .$$

Writing G for the Green function of $1-\Delta$ on $\mathbb{T}^4$, and $G_\varepsilon = \rho_\varepsilon * G$, the Wild expansion is interpreted as a formal series

$$v_\varepsilon(x,y) \text{ “} = \text{ ” } G_\varepsilon(x-y) + \sum_{n\geq 1} \lambda_\varepsilon^n \mathcal{I}_{n,\varepsilon}(x,y) \in \mathbb{R}[[\hat{\lambda}]] \, ,$$

where each $\mathcal{I}_{n,\varepsilon}$ denotes a sum of iterated stochastic integrals in the n-th inhomogeneous Wiener chaos, generated by n occurrences of the noise ξ. The terms $\mathcal{I}_{n,\varepsilon}$ contain ultraviolet divergences: already for $n = 2$ one sees the "tadpole" divergence

$$\mathbb{E}\mathcal{I}_{2,\varepsilon}(x, y) = G_\varepsilon(0)(G_\varepsilon * G_\varepsilon)(x - y)\,,$$

or for $n = 4$ the "sunset" diagram

$$G_\varepsilon * G_\varepsilon^3 * G_\varepsilon(x - y)\,,$$

which both diverge like ε^{-2}. Higher terms contain products and nested versions of such divergences. These polynomial divergences have to be removed by a renormalisation map $\mathcal{R}_\varepsilon$, analogous to the BPHZ renormalisation theory for subcritical singular SPDEs [8, 9]. Formally, this corresponds to the counterterm equation

$$(1 - \Delta)\mathcal{R}_\varepsilon v_\varepsilon = \rho_\varepsilon * (\lambda_\varepsilon(\xi - c_\varepsilon)\mathcal{R}_\varepsilon v_\varepsilon + \delta_y)\,, \qquad c_\varepsilon \sim \lambda_\varepsilon \varepsilon^{-2}. \tag{1.2}$$

The following statement concerns the renormalised terms and the convergence of the Wild expansion as a formal power series, thus providing a candidate solution of (1.2) in a subcritical regime.

Theorem 1.1 (G.–Rosati)**.** *Let $\mathcal{J}$ be the centred Gaussian distribution on $(\mathbb{T}^4)^2$ whose covariance kernel is formally given by*

$$\mathbf{E}\left[\mathcal{J}(x, y)\mathcal{J}(x', y')\right] = \int_{\mathbb{T}^4} G(x - z)G(y - z)G(x' - z)G(y' - z)\,\mathrm{d}z.$$

For every $n \geq 1$ there is a constant $\sigma_n > 0$ such that

$$\left(\lambda_\varepsilon^{n-1}\mathcal{R}_\varepsilon\mathcal{I}_{n,\varepsilon}\right)_{n\geq 1} \Longrightarrow \left(\sigma_n\hat{\lambda}^{n-1}\mathcal{J}^{(n)}\right)_{n\geq 1}\,, \tag{1.3}$$

in distribution in $\mathcal{S}'((\mathbb{T}^4)^2; \mathbb{R})^{\mathbb{N}}$. Here each $\mathcal{J}^{(n)}$ has the same marginal law as $\mathcal{J}$, but the fields $\mathcal{J}^{(n)}$ are not independent. Moreover, if $|\hat{\lambda}| < \sqrt{2}\pi$, then

$$\mathcal{U} = \sum_{n\geq 1} \sigma_n\hat{\lambda}^{n-1}\mathcal{J}^{(n)}$$

converges almost surely as a random distribution and satisfies

$$\mathcal{U} \overset{d}{=} \sigma_{\mathrm{eff}}(\hat{\lambda})\mathcal{J}, \qquad \sigma_{\mathrm{eff}}^2(\hat{\lambda}) = \frac{2\pi^2}{2\pi^2 - \hat{\lambda}^2}.$$

The theorem shows that below the threshold $|\hat{\lambda}| = \sqrt{2}\pi$, the perturbative fluctuations around the Laplacian Green function are Gaussian and vanish as $\varepsilon \to 0$. This is reflected by the normalisation in (1.3), where we have $\lambda_\varepsilon^{n-1}$ instead of λ_ε^n. More precisely, formally we have

$$\mathcal{R}_\varepsilon v_\varepsilon - G_\varepsilon = \lambda_\varepsilon\mathcal{U} + o(\lambda_\varepsilon)\,.$$

The variance of $\mathcal{U}$ diverges at the critical threshold, which is the perturbative signature of a change of fluctuation behaviour. This also justifies the choice of weakly tuned coupling constant λ_ε, since the above theorem suggests that the variance blowup at the critical threshold may compensate the vanishing fluctuations, leading to fluctuations of the same order as G_ε which are expected to be non–Gaussian. A closely related transition has been observed for the two-dimensional stochastic heat equation driven by space-time white noise [10, 11]. Thanks to the independent resampling of noise over time, the multiplicative stochastic heat equation admits an explicit Itô chaos expansion

of the solution, while in the present Anderson model the perturbative expansion contains ultraviolet divergent contractions that must be compensated by a more involved BPHZ renormalisation.

2. Idea of the proof

The proof in [1] turns the stochastic problem into a combinatorial and analytic statement about Feynman diagrams. First, moments of stochastic integrals $\mathcal{I}_{n,\varepsilon}$ are written diagrammatically as sums over pairings. Each pairing gives a deterministic Anderson–Feynman diagram, whose vertices have degree four and whose edges encode Green kernels. BPHZ renormalisation introduces a consistent subtraction scheme at the level of such Feynman diagrams, which removes polynomial ultraviolet divergences as $\varepsilon \to 0$. Logarithmic zero-degree divergences are left unsubtracted and are instead balanced by the weak-coupling scaling.

The key classification of divergences within a Feynman diagram Γ is by the degree of full subdiagrams $\overline{\Gamma} = (\overline{V}, \overline{E}) \subset \Gamma$, which is defined by

$$\deg \overline{\Gamma} = 4(|\overline{V}| - 1) - 2|\overline{E}|\,.$$

Positive Anderson-Feynman diagrams (that is diagrams that only contain full subdiagrams of positive degree) have uniformly bounded integrals and disappear after the weak-coupling scaling. Negative diagrams (that is diagrams that contain at least one full subdiagram of negative degree) contain ultraviolet divergences of order ε^{-2}. Our BPHZ subtraction cancels the dangerous terms, and we show that the corresponding Taylor remainders vanish in the weak coupling limit. This part also indicates that no Laplacian renormalisation survives for the Anderson model.

The remaining diagrams, so-called zero-degree diagrams, are the only possible contributors. Our detailed analysis shows that a zero-degree connected diagram contributes precisely when it is a connected *nested bubble diagram* that contains nested occurrences of the kernel $G_\varepsilon^2(x-y)$ (which we call a primitive bubble). Equivalently, one can successively contract these primitive bubbles within an Anderson-Feynman diagram until only the external legs remain. Each such primitive bubble contributes one logarithm after integration, and the weak-coupling factor balances it by

$$\lambda_\varepsilon^2 \log \tfrac{1}{\varepsilon} \longrightarrow \hat{\lambda}^2.$$

This classification has two consequences. First, a nested bubble diagram necessarily has four external legs, so every limiting cumulant of order larger than two vanishes. Together with tightness and the fourth moment theorem, this gives Gaussianity. Second, the exact contribution of each covariance diagram is given by the number of its bubble extraction sequences, depending on how they are nested within the diagram. Counting and summing these contributions yields

$$\sigma_{\text{eff}}^2(\hat{\lambda}) = \sum_{m\geq 0} \Big(\frac{\hat{\lambda}^2}{2\pi^2}\Big)^m,$$

which is the geometric series corresponding to the formula in the theorem.

The open problem left by the perturbative analysis of the theorem above is to prove that the full renormalised resolvent is approximated by a suitable truncation of the

expansion, as was possible in a different critical problem for the Allen–Cahn equation with random initial data [12].

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UNIVERSITY DR, BERKELEY, CA 94720, UNITED STATES

Email address: s.gabriel@berkeley.edu

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FERNIQUE-TYPE BOUNDS FOR BPHZ MODELS AND THEIR APPLICATIONS

MASATO HOSHINO

Classification AMS 2020: 60H15; 60F10.

Keywords: singular SPDEs; regularity structures; transportation cost inequalities; large deviations.

This note is based on joint work with Ismaël Bailleul and Ryoji Takano [1] on concentration estimates for BPHZ models. In the theory of regularity structures, the random input of a singular SPDE is first lifted to a renormalized *model*, and the solution is then obtained by a deterministic continuous map of models. It is therefore natural to ask which concentration properties of the noise survive this nonlinear and renormalized lift. The answer is positive for a broad class of parabolic singular SPDEs: a Talagrand-type transportation cost inequality for the noise yields both Talagrand-type inequalities and Gaussian Fernique-type bounds for the BPHZ model.

We consider equations of the form

$$(\partial_t - \Delta)u = b(u, \nabla u) + \sigma(u, \nabla u)\xi \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^d. \tag{0.1}$$

Let Ω be a separable Banach space of space-time distributions and let $H \subset \Omega$ be a continuously and densely embedded Hilbert subspace, typically $H = H^{s_0}$. We assume that the noise is the identity map on Ω, that its law $\mathbb{P}$ is centered and invariant under space-time translations, and that $\mathbb{P}$ satisfies the Talagrand-type transportation cost inequality

$$a_0\, \mathcal{W}_{c_H}(\mathbb{P}, \mathbb{Q}) \le \mathcal{H}(\mathbb{Q} \mid \mathbb{P}), \tag{0.2}$$

for any Borel probability measures $\mathbb{Q}$ on Ω and for some $a_0 > 0$, where

$$c_H(\omega_1, \omega_2) = \begin{cases} \|h\|_H^2, & \omega_1 - \omega_2 = h \in H, \\ +\infty, & \text{otherwise.} \end{cases}$$

The c_H-transportation cost between $\mathbb{P}$ and $\mathbb{Q}$ is defined as $\mathcal{W}_{c_H}(\mathbb{P}, \mathbb{Q}) = \inf E[c_H(X_1, X_2)]$, for an infimum over the set of all random variables defined on some probability space such that X_1 has law $\mathbb{P}$ and X_2 has law $\mathbb{Q}$. The relative entropy $\mathcal{H}(\mathbb{Q} \mid \mathbb{P})$ of $\mathbb{Q}$ with respect to $\mathbb{P}$ is defined as $\int_\Omega \log(\frac{d\mathbb{Q}}{d\mathbb{P}}) d\mathbb{Q}$ if $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$, otherwise we set it equal to ∞. This assumption includes Gaussian noises with H their Cameron-Martin space and implies a spectral gap inequality in the H-directions.

Let $\mathcal{F}$ be the finite set of "rooted decorated trees" that spans the structure space of the regularity structure associated with the equation (0.1). Each symbol τ carries a regularity $r(\tau)$ and contains $n(\tau)$ occurrences of the noise. (See [3, Section 8] for details.) We work under two standard hypotheses. The first is *local subcriticality*, which says that the set of relevant trees is locally finite in regularity. As the second one, we assume that the phenomenon of *variance blow-up*, in the sense of Hairer [4], does not occur. This

condition ensures that the BPHZ renormalized model considered in [2] is well-defined. A model is a pair (Π, Γ), where

$$\Pi = \{\Pi_x : \mathcal{F} \to \mathcal{S}'\}_{x \in \mathbb{R}^{1+d}}$$

is a family of maps and Γ is the corresponding reexpansion map. The BPHZ model is constructed by first replacing the noise ξ with a smooth approximation ξ_ε, lifting it canonically to a model, subtracting the appropriate counterterms, and then taking the limit. The homogeneous model norm on a compact set $K \subset \mathbb{R}^{1+d}$ is defined by

$$\|M\|_{\mathrm{hom};K} := \max_{\tau \in \mathcal{F}} \sup_{\lambda \in (0,1],\, \varphi \in \mathcal{B}_N} \left(\lambda^{-r(\tau)} \sup_{x \in K} |(\Pi_x \tau)(\varphi_x^\lambda)| \right)^{1/n(\tau)}, \tag{0.3}$$

where $\mathcal{B}_N$ denotes the set of all smooth functions φ supported in the unit ball and satisfying $\|\varphi\|_{C^N} \le 1$. (For simplicity, we ignore the norm for the reexpansion map Γ here.) We also define the metric $\|M; \bar{M}\|_{\mathrm{hom};K}$ by replacing Π_x with $\Pi_x - \bar{\Pi}_x$. The exponent $1/n(\tau)$ records the polynomial degree in the noise and is essential for our main theorem.

Theorem 0.1 ([1, Theorem 2])**.** *Assume the local subcriticality and the no-variance-blow-up condition. Let M^ξ be the BPHZ model and put $m = \max_{\tau \in \mathcal{F}} n(\tau)$. Then, for every compact K,*

$$\|M^{\xi+h}; M^\xi\|_{\mathrm{hom};K} \le C(\xi)\big(\|h\|_H \vee \|h\|_H^{1/m}\big) \tag{0.4}$$

for all $h \in H$ almost surely, with $C(\xi) \in \cap_{p \ge 1} L^p(\mathbb{P})$. Consequently, the law P of M^ξ satisfies Talagrand-type transportation cost inequalities

$$a_1 \, \mathcal{W}_{\|\cdot;\cdot\|_{\mathrm{hom};K}}(P, Q) \le \mathcal{H}(Q \mid P)^{1/2} + \mathcal{H}(Q \mid P)^{1/(2m)}$$

for any Borel probability measures Q on the space of models and for some $a_1 > 0$. In particular, there is $\kappa > 0$ such that

$$\mathbb{E}\big[\exp\{\kappa \|M^\xi\|_{\mathrm{hom};K}^2\}\big] < \infty. \tag{0.5}$$

The result applies directly to solutions of (0.1). If the deterministic solution map is globally well-posed and satisfies an a priori estimate in terms of the homogeneous model norm, then the law of the solution inherits a Talagrand-type inequality and the corresponding exponential integrability. If the estimate is independent of the initial condition, invariant measures inherit the same concentration property. A second application is a large deviation principle in the small-noise regime

$$(\partial_t - \Delta) u^\varepsilon = b(u^\varepsilon, \nabla u^\varepsilon) + \varepsilon \sigma(u^\varepsilon, \nabla u^\varepsilon) \xi.$$

Whenever $\varepsilon\xi$ satisfies an LDP with rate ε^2 and rate function $\frac{1}{2}\|h\|_H^2$, the BPHZ models and, by continuity, the corresponding SPDE solutions satisfy the natural contracted LDP. This extends the Hairer–Weber large deviation framework to the present general BPHZ setting.

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INSTITUTE OF SCIENCE TOKYO, 2-12-1 OOKAYAMA, MEGURO-KU, TOKYO, 152-8551, JAPAN
Email address: hoshino@math.sci.isct.ac.jp

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EQUIVALENCE OF FLUCTUATIONS FOR THE SHE AND KPZ EQUATION IN THE SUBCRITICAL WEAK DISORDER REGIME

SHUTA NAKAJIMA

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Keywords: stochastic heat equation, KPZ equation, directed polymers, weak disorder, fluctuations.

This extended abstract is based on joint work with Stefan Junk (Gakushuin University).

The Kardar–Parisi–Zhang equation is the formal stochastic PDE

$$\partial_t h(t,x) = \frac{1}{2}\Delta h(t,x) + \frac{1}{2}|\nabla h(t,x)|^2 + \beta\xi(t,x),$$

where ξ denotes space-time white noise. Since ξ is a Schwartz distribution, the nonlinear term must be defined by a regularization and renormalization procedure. If the noise is mollified in space at scale ϵ by $\xi^\epsilon = \phi^\epsilon * \xi$, one considers

$$\partial_t h_\epsilon(t,x) = \frac{1}{2}\Delta h_\epsilon(t,x) + \frac{1}{2}|\nabla h_\epsilon(t,x)|^2 + \beta_\epsilon \xi^\epsilon(t,x) - C_\epsilon.$$

The Cole–Hopf transform $u_\epsilon = \exp(h_\epsilon)$ maps this equation to the multiplicative stochastic heat equation

$$\partial_t u_\epsilon(t,x) = \frac{1}{2}\Delta u_\epsilon(t,x) + \beta_\epsilon \xi^\epsilon(t,x) u_\epsilon(t,x) - C_\epsilon u_\epsilon(t,x).$$

With flat initial data, the Feynman–Kac formula identifies u_ϵ with the partition function of a continuum directed polymer [7]. Thus the relation between the SHE and KPZ equation can be studied by comparing a partition function with its logarithm.

We first recall the known phase diagram. In $d = 1$, Bertini and Giacomin constructed the Cole–Hopf solution of the KPZ equation as the scaling limit of weakly asymmetric particle systems [1]. In $d = 2$, the mollified KPZ equation has a nontrivial subcritical regime: Caravenna, Sun, and Zygouras proved convergence for $\hat\beta < 1$ and divergence to $-\infty$ for $\hat\beta > 1$ under the logarithmic scaling of the coupling constant [2]. In dimensions $d \geq 3$, the natural weak-noise scaling is $\beta_\epsilon = \hat\beta \epsilon^{(d-2)/2}$ [7, 3]. There are critical parameters $0 < \hat\beta_{L^2} \leq \hat\beta_c$. For $\hat\beta < \hat\beta_c$, the SHE partition functions satisfy a law of large numbers, whereas for $\hat\beta > \hat\beta_c$ they converge to zero [7, 3]. In the L^2 region, $\hat\beta < \hat\beta_{L^2}$, the centered fluctuations of u_ϵ and $h_\epsilon = \log u_\epsilon$ converge to Edwards–Wilkinson type Gaussian limits [4, 3]. The main motivation is the weak disorder region beyond the L^2 region, where non-Gaussian fluctuations are expected.

The main results are joint work with Stefan Junk and are stated for the discrete directed polymer model. Let $(\omega(n,x))_{n\in\mathbb{N},x\in\mathbb{Z}^d}$ be i.i.d. random variables with compact support, and set

$$\lambda(\beta) = \log \mathbb{E}[e^{\beta\omega(n,x)}].$$

For the simple random walk (X_k) on $\mathbb{Z}^d$, define the polymer partition function starting from x by

$$Z_N(x) = \mathrm{E}_x\left[\exp\left(\beta\sum_{k=1}^{N}\omega(k,X_k) - N\lambda(\beta)\right)\right].$$

For brevity, we write $Z_N = Z_N(0)$. The weak disorder regime is the set of β for which $\mathbb{P}(\liminf_N Z_N > 0) = 1$. Let

$$p_*(\beta) = \sup\{p \geq 0 : \sup_N \mathbb{E}[Z_N^p] < \infty\}$$

be the moment exponent. Junk proved that throughout weak disorder one has $p_*(\beta) \geq (d+2)/d$, and for a large class of finite-support environments the strict inequality $p_*(\beta) > (d+2)/d$ is equivalent to $\beta < \beta_c$ [5].

For a compactly supported test function $f : \mathbb{R}^d \to \mathbb{R}$, define the discrete SHE fluctuation

$$\chi_N(f) = N^{-d/2}\sum_{x\in\mathbb{Z}^d} f(x/\sqrt{N})(Z_N(x) - \mathbb{E}Z_N(x)).$$

Put

$$\xi(\beta) = -\frac{d}{2} + \frac{d+2}{2(p_*(\beta)\wedge 2)}.$$

Junk proved that, in weak disorder, for every $\epsilon > 0$ and $f \not\equiv 0$ [5],

$$\mathbb{P}\left(N^{-\xi(\beta)-\epsilon} < |\chi_N(f)| < N^{-\xi(\beta)+\epsilon}\right) \longrightarrow 1.$$

We then focused on the corresponding KPZ fluctuation

$$\kappa_N(f) = N^{-d/2}\sum_{x\in\mathbb{Z}^d} f(x/\sqrt{N})\left(\log Z_N(x) - \mathbb{E}\log Z_N(x)\right).$$

Our main theorem asserts that, in the subcritical weak disorder regime, the two fluctuation fields are asymptotically equivalent up to an error of lower polynomial order than their common magnitude [6]. More precisely, if $p_*(\beta) > (d+2)/d$, then there exists $\delta > 0$ such that

$$\mathbb{P}\left(|\chi_N(f) - \kappa_N(f)| \leq N^{-\xi(\beta)-\delta}\right) \longrightarrow 1.$$

Consequently,

$$\mathbb{P}\left(\frac{|\chi_N(f) - \kappa_N(f)|}{\min\{|\chi_N(f)|, |\kappa_N(f)|\}} \leq N^{-\delta}\right) \longrightarrow 1.$$

In particular, the SHE and KPZ fluctuations have the same polynomial order. This is significant because in the region $\beta_{L^2} < \beta < \beta_c$ the SHE fluctuation is expected to have a stable, non-Gaussian limit, whereas the logarithm has stronger moment and covariance estimates than the partition function itself. The theorem shows that this regularizing effect of the logarithm does not change the leading fluctuation scale.

The common approximation is a martingale argument built from a short backward polymer near the observation time. Let $\ell_k = k^{1/8}$ and

$$E_k(y) = e^{\beta\omega(k,y)-\lambda(\beta)} - 1.$$

If $\overleftarrow{Z}_{[k-\ell_k,k)}((k,y))$ denotes the backward partition function ending at (k,y) over the time interval $[k-\ell_k,k)$ and $p_N(x,y)$ is the simple random walk transition probability, set

$$\rho_N(f) = N^{-d/2} \sum_{x\in\mathbb{Z}^d} f(x/\sqrt{N}) \sum_{k=1}^{N} \sum_{y\in\mathbb{Z}^d} \overleftarrow{Z}_{[k-\ell_k,k)}((k,y)) E_k(y) p_N(x,y).$$

It is proved separately that $\chi_N(f)$ and $\kappa_N(f)$ are both close to $\rho_N(f)$ at the required precision [6].

For $\chi_N(f)$, this follows from expanding $Z_N(x)-1$ according to the first time at which the environment is sampled, followed by a local limit theorem and a coarse-graining estimate for point-to-point polymer partition functions. For $\kappa_N(f)$, one uses a martingale decomposition of $\log Z_N(x) - \mathbb{E}\log Z_N(x)$ along the time coordinate. The increment is

$$\log \frac{Z_k(x)}{Z_{k-1}(x)} = \log\left(1 + \sum_{y\in\mathbb{Z}^d} \mu_k(x,y) E_k(y)\right),$$

where $\mu_k(x,y)$ is the polymer endpoint distribution at time k. The logarithm can be linearized with an error controlled by lower-tail estimates for Z_N. The endpoint distribution is then replaced by

$$\overleftarrow{Z}_{[k-\ell_k,k)}((k,y)) p_k(x,y),$$

using a point-to-point factorization result for directed polymers. The main inputs are moment estimates governed by $p_*(\beta)$ and a stretched Gaussian lower-tail bound of the form

$$\mathbb{P}(Z_N < 1/u) \le C\exp\{-(\log u)^2/C\}, \qquad u \ge 1.$$

These estimates make it possible to compare the nonlinear KPZ fluctuations with the linear SHE fluctuations throughout the subcritical weak disorder region [6].

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DEPARTMENT OF MATHEMATICS, KEIO UNIVERSITY
Email address: njima@keio.jp

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MOMENTS OF CRITICAL 2D STOCHASTIC HEAT FLOW

KYEONGSIK NAM

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Keywords: Critical 2D Stochastic Heat Flow, directed polymers, intermittency, Gaussian free field

The stochastic heat equation with multiplicative space-time white noise is one of the central continuum models in the Kardar–Parisi–Zhang universality class. In spatial dimension two, however, the equation is ill-posed without a delicate renormalization procedure. This dimension is especially subtle: it is neither ultraviolet nor infrared super-renormalizable, and the corresponding polymer models exhibit a weak-to-strong disorder transition at a critical logarithmic scale. A major recent breakthrough of Caravenna, Sun, and Zygouras [2] constructed the scaling limit of the critical $2+1$-dimensional directed polymer partition function. This limiting random measure is called the critical two-dimensional stochastic heat flow, or *critical 2D SHF*. We investigate a fundamental manifestation of its intermittency: the asymptotic growth of high moments and the resulting upper-tail behavior.

Let $\mathscr{Z}^\theta$ denote the critical 2D SHF with parameter $\theta \in \mathbb{R}$. For a nonnegative smooth test function φ on $\mathbb{R}^2$, define the scalar observable

$$X_\varphi := \mathscr{Z}_1^\theta(\varphi) = \int_{\mathbb{R}^2} \varphi(x)\, \mathscr{Z}_1^\theta(dx, 1),$$

where the second endpoint of the flow is integrated over the whole plane. This normalization is natural from the polymer point of view, since it corresponds to testing the initial endpoint by φ while keeping the terminal endpoint free. Earlier work established that the critical SHF is singular with respect to Lebesgue measure and is not a Gaussian multiplicative chaos [3, 4]. Nevertheless, it is expected to display rich intermittent behavior reminiscent of other random measures arising in critical field theories and directed polymer models.

The main problem addressed in this paper is the high-moment growth of X_φ. A prediction going back to Rajeev [7] suggested that the h-th moment should grow double exponentially in h, namely on the order of $\exp\{\exp(ch)\}$ for some positive constant c. Prior rigorous bounds were far from this prediction. The best available upper bound, due to Gu, Quastel, and Tsai [5], was of the form

$$\mathbb{E}[X_\varphi^h] \le \exp\{\exp(Ch^2)\},$$

whereas the known lower bound, obtainable for instance from the Gaussian correlation inequality, was only of order $\exp(ch^2)$ [3].

The first main result gives an exponential improvement of the lower bound and matches the predicted double-exponential scale from below. More precisely, for every fixed $\theta \in \mathbb{R}$ and every smooth nonnegative test function φ with $\varphi(0) > 0$, there exists an

absolute constant $c_0 > 0$ such that, for all sufficiently large integers h,

$$\mathbb{E}[X_\varphi^h] \geq \exp\{\exp(c_0 h)\}.$$

This establishes the predicted order of magnitude as a genuine feature of the critical SHF and provides a new quantitative signature of its intermittency.

A key novelty of the proof is a new connection between the moment formula for the critical SHF and the Gaussian free field on certain Feynman diagrams. The moments of the SHF admit an expansion over collision patterns of multiple random walks, as developed in the construction and moment analysis of the critical 2D SHF [1, 2, 5]. Each diagram contains time variables encoding the beginning and end of effective pairwise collision intervals, and spatial variables encoding the corresponding collision locations. After fixing a collision pattern and the associated time variables, the spatial integral can be interpreted as the partition function of a Gaussian free field on a weighted graph naturally associated with the diagram. The conductances of this graph are determined by inverse time gaps between successive collision events.

This observation brings into the problem the algebraic structure of the Gaussian free field. In particular, the Gaussian integral over spatial variables is expressed through the determinant of a weighted graph Laplacian. By Kirchhoff's Matrix–Tree theorem, this determinant is equal to the weighted sum of spanning trees of the corresponding graph. Thus the estimation of SHF moments is reduced to a combinatorial problem of controlling weighted spanning trees on Feynman diagrams. The dominant contribution comes from diagrams with exponentially many collision events in the moment order h, leading precisely to the $\exp\{\exp(ch)\}$ lower bound.

The second main consequence concerns upper-tail asymptotics. Combining the new moment lower bound with the best known moment upper bound, it is obtained that for compactly supported smooth nonnegative φ with $\varphi(0) > 0$, the tail of X_φ satisfies, for large z,

$$\exp\left\{-(\log z)^{(\log\log z)^{1+o(1)}}\right\} \leq \mathbb{P}(X_\varphi > z) \leq \exp\left\{-\Omega(1)\,\log z\,\sqrt{\log\log z}\right\}.$$

Thus the upper tail is barely super-polynomial: it decays faster than any fixed power of z, but much more slowly than a stretched exponential in z. This behavior is markedly different from higher-dimensional weak disorder directed polymers, where limiting partition functions may have power-law tails [6]. The result therefore identifies a distinctive feature of criticality in two dimensions.

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DEPARTMENT OF MATHEMATICAL SCIENCES, KAIST, SOUTH KOREA
Email address: ksnam@kaist.ac.kr

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GUE FLUCTUATIONS NEAR THE AXIS IN ONE-SIDED BALLISTIC DEPOSITION

ALEJANDRO RAMIREZ

Classification AMS 2020: 60K35, 82C41

Keywords: Ballistic Deposition, Last Passage Percolation, GUE Tracy–Widom Distribution, Height Fluctuations, Geodesics, Transversal Fluctuations, KPZ Universality

We introduce a variation of the classic ballistic deposition model in which vertically falling blocks can only stick to the top or the upper right corner of growing columns. We establish that fluctuations of the height function at points near the t-axis are given by the GUE ensemble and its corresponding Tracy–Widom limiting distribution. The proof is based on a graphical construction of the process in terms of a directed Last Passage Percolation model. Using this graphical construction, we define the notion of geodesics for the height function and show that the wandering exponent governing the transversal fluctuations of these geodesics is $2/3$.

This is joint work with Pablo Groisman, Santiago Saglietti and Sebastián Zaninovich.

Further details of these developments can be found in the following research papers:

- https://arxiv.org/abs/2504.15837v2

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NYU SHANGHAI, CHINA
Email address: ar23@nyu.edu

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GLOBAL WELL-POSEDNESS FOR $\exp(\Phi)$

TOMMASO ROSATI

Classification AMS 2020: Primary 60H15; Secondary 60G57, 60L40, 81T08.

Keywords: parabolic Liouville equation; exponential ($\exp(\Phi)_2$) model; Gaussian multiplicative chaos; global well-posedness; stochastic quantization; Da Prato–Debussche decomposition.

This talk reports on joint work with G. Cannizzaro and T. Galanis on the global-in-time solution theory of the massless parabolic Liouville (or $\exp(\Phi)_2$) equation on the two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$,

$$\partial_t u = \tfrac{1}{4\pi}\Delta u + \alpha - e^{\gamma u} + \xi, \tag{0.1}$$

where ξ is space–time white noise, $\gamma > 0$ a coupling constant, and $\alpha > 0$ a constant drift. Equation (0.1) is the natural Langevin dynamics of the Liouville-type measure formally proportional to $\exp\{-\int_{\mathbb{T}^2} |\nabla\phi|^2 + \alpha \int_{\mathbb{T}^2} \phi - \int_{\mathbb{T}^2} :e^{\gamma\phi}:\}$, whose exponential interaction is built from Gaussian multiplicative chaos (GMC). Because the field is log-correlated, $e^{\gamma u}$ is almost surely infinite, and (0.1) only makes sense after multiplicative Wick renormalisation, the renormalised nonlinearity being a dynamical GMC.

Following Da Prato–Debussche, we decompose $u = \Psi + v$, where Ψ solves the linear equation $\partial_t \Psi = \frac{1}{4\pi}\Delta\Psi + \xi$ and is a log-correlated Gaussian field, and $\eta := :e^{\gamma\Psi}:$ is the associated dynamical GMC, a positive distribution of negative Besov regularity. The remainder v then solves a random PDE whose nonlinearity is $\eta\, e^{\gamma v}$.

A central difficulty of the massless model is the zeroth Fourier mode: absent a mass term, it carries no restoring force and is driven by a spatially constant Brownian motion W_t and the drift α. We handle it through a low-mode decoupling, obtaining an effective dynamics for the zeroth mode in which we see nonlinear damping: since $\eta \geq 0$ and the prefactor $e^{\gamma(\alpha t + W_t)} > 0$, the nonlinearity in the equation for v is sign-definite and dissipative.

Main result (informal). Fix γ in the subcritical (Da Prato–Debussche) regime. There exists $\alpha_0 = \alpha_0(\gamma) > 0$ such that, for every drift $0 < \alpha < \alpha_0$ and every initial datum $u_0 \in L^\infty(\mathbb{T}^2)$, the renormalised equation (0.1) has a unique global-in-time solution $u = \Psi + v$. The remainder v is tight in time, and the dynamics admits an invariant probability measure.

Related work. The model was introduced by Garban [1], who solved the classical Da Prato–Debussche phase $\gamma \in [0, 2\sqrt{2} - \sqrt{6})$ and, exploiting positivity of the nonlinearity, obtained a weaker notion of solution up to $\gamma_{\mathrm{pos}} = 2\sqrt{2} - 2$. Oh–Robert–Wang [4] proved local well-posedness for $\gamma^2 < 8\pi/(3 + 2\sqrt{2})$ and, in the defocusing case, global well-posedness for $\gamma^2 < 4\pi$ together with invariance of the Gibbs measure, again using sign-definiteness and positivity of GMC; Oh–Robert–Tzvetkov–Wang [3] treated the stochastic

quantization of Liouville conformal field theory on closed surfaces. Hoshino–Kawabi–Kusuoka [2] constructed the $\exp(\phi)_2$ dynamics in the full L^1-regime by Dirichlet-form methods. All of these works consider the stochastic quantisation problem only in the massive case; our contribution is to prove that there exists an invariant measure for the equation in the massless case, under a small-drift assumption.

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MATHEMATICS INSTITUTE, UNIVERSITY OF WARWICK, COVENTRY CV4 7AL, UNITED KINGDOM
Email address: t.rosati@warwick.ac.uk

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SCALING LIMIT OF THE 3D ABELIAN YANG–MILLS LANGEVIN DYNAMICS

HAO SHEN

Classification AMS 2020: 60H15, 81T13, 82B20, 60F17.

Keywords: abelian Yang–Mills theory, lattice gauge theory, Langevin dynamics, stochastic quantization, scaling limit, singular SPDE.

This talk is based on our ongoing work joint with Ilya Chevyrev and Yahui Qu. It concerned the continuum limit of the Langevin dynamics associated with three-dimensional abelian lattice gauge theory. In the continuum, for structure group $U(1)$ on the torus, a connection is a real-valued one-form A, its curvature is the linear two-form $F_A = dA$, and the Euclidean Yang–Mills action is

$$S(A) = \frac{1}{2}\int_{\mathbb{T}^3} |dA|^2 \, dx.$$

Gauge transformations $A \mapsto A + d\phi$ leave the curvature invariant; after gauge fixing the formal Yang–Mills measure is Gaussian. This linearity is the main distinction from non-abelian Yang–Mills theory, where $F_A = dA + [A, A]$ and the action is no longer quadratic.

On the lattice, a $U(1)$ gauge field assigns to every oriented edge e a variable $U(e) = e^{i\theta(e)}$, with $\theta(e) \in (-\pi, \pi]$ and $U(-e) = U(e)^{-1}$. For a plaquette $p = (e^{(1)}, \dots, e^{(4)})$, the gauge-invariant curvature variable is $U(\partial p) = \exp(i\sum_j \theta(e^{(j)}))$. We consider an even, smooth, 2π-periodic single-plaquette potential h, so that the finite-volume Gibbs measure is

$$\mu_N(d\theta) = Z^{-1} \exp\Big\{ -\beta \sum_p h(\theta(\partial p))\Big\} \prod_e d\theta(e).$$

This includes the Wilson, Manton, and Villain actions. Under the usual scaling $\beta \mapsto \varepsilon^{d-4}\beta$ and $\theta \mapsto \varepsilon\theta$, the action in dimension $d = 3$ is formally close to the quadratic continuum action.

Let d_ε be the discrete exterior derivative and d_ε^* its adjoint. With the Hodge Laplacian $\Delta_\varepsilon = -d_\varepsilon^* d_\varepsilon - d_\varepsilon d_\varepsilon^*$, the DeTurck-gauge lattice Langevin equation studied in the talk is

$$\partial_t \theta_\varepsilon = -\varepsilon^{-2} d_\varepsilon^* h'(\varepsilon^2 d_\varepsilon \theta_\varepsilon) - d_\varepsilon d_\varepsilon^* \theta_\varepsilon + \xi_\varepsilon.$$

Here ξ_ε is lattice space-time white noise, and the exact DeTurck term does not change the gauge orbit. Normalizing $h''(0) = 1$ and writing

$$h'(z) = z + a_3 z^3 + R(z), \qquad R(z) = O(z^5),$$

one obtains

$$\partial_t \theta_\varepsilon = \Delta_\varepsilon \theta_\varepsilon + a_3 \varepsilon^4 d_\varepsilon^* (d_\varepsilon \theta_\varepsilon)^3 + \varepsilon^{-2} d_\varepsilon^* R(\varepsilon^2 d_\varepsilon \theta_\varepsilon) + \xi_\varepsilon.$$

For example, $a_3 = -1/6$ for the Wilson action. Although the expected limit is linear, these nonlinear error terms are singular uniformly in the lattice spacing.

Main result. Let $d = 3$ and let h be as above. If the interpolated initial data converge in a negative Hölder topology to A_0, then, for every finite T and every $\kappa > 0$, the interpolated solutions converge in law in $C([0,T], \mathcal{C}^{-1/2-\kappa})$ to the solution of

$$\partial_t A = \Delta A + \xi, \qquad A(0) = A_0.$$

Consequently the gauge-fixed continuum limit is Gaussian; in equilibrium its curvature is spatial white noise.

The proof uses a lattice Da Prato–Debussche decomposition $\theta_\varepsilon = \Psi_\varepsilon + v_\varepsilon$, where Ψ_ε solves the discrete stochastic heat equation and v_ε solves the remainder equation. In dimension three, Ψ_ε has spatial regularity $\mathcal{C}^{-1/2-}$, and the cubic term has the same power-counting singularity as the dynamical Φ^4_3 nonlinearity. The crucial simplification is that the powers of ε in the expanded equation can be traded for regularity in discrete Hölder spaces $\mathcal{C}^\alpha_\varepsilon$, which measure scales between ε and 1. Typical deterministic bounds are

$$\|\varepsilon^{3/2+\kappa} d_\varepsilon \Psi_\varepsilon\|_{L^\infty_\varepsilon} \lesssim \|\Psi_\varepsilon\|_{\mathcal{C}^{-1/2-\kappa}_\varepsilon}, \qquad \|\varepsilon^{1-\alpha} d_\varepsilon v_\varepsilon\|_{L^\infty_\varepsilon} \lesssim \|v_\varepsilon\|_{\mathcal{C}^\alpha_\varepsilon}.$$

These estimates allow the nonlinearities to be controlled by elementary L^∞_ε product bounds rather than by a full regularity-structure expansion. The probabilistic estimates are also favorable: uniformly on compact time intervals the stochastic heat solution has the required discrete Hölder bounds, and no Wick renormalization is needed since

$$\varepsilon^3 \mathbf{E}\big[(d_\varepsilon \Psi_\varepsilon(p))^2\big] = O(1).$$

A stopping-time argument then shows $v_\varepsilon \to 0$ in positive discrete Hölder norms, reducing the limit to convergence of the linear discrete stochastic heat equation.

The result fits with Gross's convergence theorem for $U(1)_3$ lattice gauge theory [1] and Driver's four-dimensional abelian convergence theorem at small coupling [2]. It is also related in spirit to weak universality results for singular SPDEs, such as [5]. In comparison with the two-dimensional non-abelian Yang–Mills Langevin theory and its lattice universality results [3, 4], the present three-dimensional abelian problem has a Gaussian limit but still contains singular lattice nonlinearities. Natural future directions include four-dimensional abelian Langevin dynamics and three-dimensional non-abelian lattice models, where genuine Yang–Mills nonlinearities should interact with singular error terms.

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University of Wisconsin–Madison, USA

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DIMERS WITH LAYERED DISORDER

FABIO TONINELLI

Classification AMS 2020: 60K35, 82B20, 82B44, 60G60

Keywords: Dimer Models, Layered Disorder, Random Environments, Statistical Mechanics, Random Tilings, Height Functions

We study the dimer model on the square grid, with quenched random edge weights. Randomness is chosen to have a layered structure, similar to that of the celebrated McCoy-Wu disordered Ising model. Disorder has a highly non-trivial effect and it produces an essential singularity of the free energy, with $e^{-\sqrt{\text{distance}}}$ decay of dimer-dimer correlations, at a point of the "liquid" (or "massless") phase where the homogeneous dimer model has instead a real analytic free energy and correlations decaying like $1/(\text{distance})^2$. Moreover, at a point where the homogeneous model has a transition between a massive (gaseous) and massless (liquid) phase, the critical exponent $3/2$ (Pokrovsky-Talapov law), characteristic of the transition between the two regimes, is modified by disorder into an exponent that ranges continuously between $3/2$ and infinity.

This is joint work with Quentin Moulard.

Further details of these developments can be found in the following research papers:

- https://arxiv.org/abs/2507.11964

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TU WIEN, AUSTRIA
Email address: fabio.toninelli@tuwien.ac.at

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STRONG-DISORDER ASYMPTOTICS FOR 2D DIRECTED POLYMERS AND STOCHASTIC HEAT FLOW

NICOLA TURCHI

Classification AMS 2020: Primary 82B44; Secondary 60K35, 82D60.

Keywords: Change of Measure, Coarse-Graining, Directed Polymer in Random Environment, Disordered Systems, Size Bias, Stochastic Heat Equation, Stochastic Heat Flow.

We consider the critical two-dimensional Stochastic Heat Flow and its discrete approximation by directed polymers in random environment. The aim is to describe their behavior in the strong-disorder and large-time regimes, with particular emphasis on quantitative local extinction and on the spatial scale at which the mass becomes visible. The results presented here are joint work with Quentin Berger and Francesco Caravenna [1].

Let $S = (S_n)_{n\geq 0}$ be the simple symmetric random walk on $\mathbb{Z}^2$, and let $\omega = (\omega(n,x))_{n\in\mathbb{N},\, x\in\mathbb{Z}^2}$ be a family of independent and identically distributed random variables, independent of S, such that

$$\mathbb{E}[\omega] = 0, \qquad \mathbb{E}[\omega^2] = 1, \qquad \lambda(\beta) := \log \mathbb{E}[\mathrm{e}^{\beta\omega}] < \infty \quad \text{for } -2 < \beta < 2.$$

For $N \in \mathbb{N}$, and $x \in \mathbb{Z}^2$, the point-to-plane partition function is

$$Z_N^{\beta,\omega}(x) := \mathbf{E}_x\left[\exp\left(\sum_{n=1}^{N}\big(\beta\omega(n,S_n) - \lambda(\beta)\big)\right)\right].$$

For a probability mass function f on $\mathbb{Z}^2$, set

$$Z_N^{\beta,\omega}(f) := \sum_{x\in\mathbb{Z}^2} f(x) Z_N^{\beta,\omega}(x), \qquad \mathbb{E}[Z_N^{\beta,\omega}(f)] = 1.$$

The relevant parameter measuring the strength of the disorder is determined by the relation between β and N. Define

$$\sigma^2(\beta) := \mathbb{V}\mathrm{ar}\big(\mathrm{e}^{\beta\omega-\lambda(\beta)}\big) = \mathrm{e}^{\lambda(2\beta)-2\lambda(\beta)} - 1$$

and

$$R_N := \sum_{n=1}^{N} \mathbf{P}(S_{2n} = 0) = \frac{1}{\pi}\big(\log N + \alpha_N\big), \qquad \alpha_N \longrightarrow \alpha := 4\log 2 + \gamma - \pi.$$

The critical window introduced in [2] is characterized by

$$\sigma^2(\beta_N) = \frac{1}{R_N - \frac{\vartheta + o(1)}{\pi}}, \qquad \vartheta \in \mathbb{R}.$$

For arbitrary N and $\beta \in (0,1)$, this suggests defining the effective disorder parameter

$$\vartheta(N,\beta) := \pi R_N - \frac{\pi}{\sigma^2(\beta)}. \tag{0.1}$$

Thus the critical window corresponds to $\vartheta(N,\beta_N) \to \vartheta \in \mathbb{R}$, whereas the supercritical regime is given by $\vartheta(N,\beta_N) \to +\infty$. Moreover,

$$\mathrm{e}^{\vartheta(N,\beta)} = \mathrm{e}^{\alpha_N} N \exp\Big(-\frac{\pi}{\sigma^2(\beta)}\Big). \tag{0.2}$$

The critical 2D Stochastic Heat Flow $\mathscr{Z}^{\vartheta}$ was constructed in [2] as the universal scaling limit of diffusively rescaled directed-polymer partition functions in this critical window. Denoting its one-time marginal by $\mathscr{Z}_t^{\vartheta}(\mathrm{d}x)$, for a probability density φ on $\mathbb{R}^2$ we write

$$\mathscr{Z}_t^{\vartheta}(\varphi) := \int_{\mathbb{R}^2} \varphi(x)\mathscr{Z}_t^{\vartheta}(\mathrm{d}x), \qquad \mathbb{E}[\mathscr{Z}_t^{\vartheta}(\varphi)] = 1.$$

Although the mean remains equal to one, this random variable typically vanishes in the strong-disorder or large-time limits; the mean is sustained by increasingly rare and high peaks. Local extinction for fixed disorder and large time was obtained in [3], while strong-disorder local extinction follows from the conditional Gaussian multiplicative chaos structure established in [4]. Our purpose is to identify the extinction rate and to obtain bounds uniform throughout the supercritical regime.

For $r > 0$, let $\mathcal{M}_1^{\mathrm{disc}}(r)$ denote the set of probability mass functions on $\mathbb{Z}^2$ supported in the ball of radius r. There exist constants $c_0, c_1, c_2 \in (0,\infty)$ such that, uniformly over $N \in \mathbb{N}$ and $\beta \in (0,1)$,

$$\frac{1}{c_1}\exp\big(-c_1\mathrm{e}^{\vartheta(N,\beta)}\big) \le \sup_{f\in\mathcal{M}_1^{\mathrm{disc}}(\mathrm{e}^{c_0\mathrm{e}^{\vartheta(N,\beta)}}\sqrt{N})} \mathbb{E}\big[Z_N^{\beta,\omega}(f)\wedge 1\big] \le \frac{1}{c_2}\exp\big(-c_2\mathrm{e}^{\vartheta(N,\beta)}\big). \tag{0.3}$$

The same conclusion holds for fractional moments $\mathbb{E}[Z_N^{\beta,\omega}(f)^\gamma]$, $\gamma \in (0,1)$, with constants depending on γ. In particular, whenever $\vartheta(N,\beta_N) \to +\infty$, one has $Z_N^{\beta_N,\omega}(f_N) \to 0$ in probability for every sequence of initial conditions whose support is contained in a ball of radius $\mathrm{e}^{c_0\mathrm{e}^{\vartheta(N,\beta_N)}}\sqrt{N}$. The estimate is sharp at the level of the exponent: the decay is exponential in $\mathrm{e}^{\vartheta(N,\beta)}$.

The truncated mean in (0.3) is closely related to the total variation distance between the original disorder law and its size-biased version. It also gives corresponding upper-tail estimates. For every $\varepsilon \in (0,1)$,

$$C_{1,\varepsilon}\exp\big(-c_1\mathrm{e}^{\vartheta(N,\beta)}\big) \le \sup_f \mathbb{P}\big(Z_N^{\beta,\omega}(f) \ge \varepsilon\big) \le C_{2,\varepsilon}\exp\big(-c_2\mathrm{e}^{\vartheta(N,\beta)}\big),$$

where the supremum is taken over the same class of initial conditions. For the lower bound it is enough to take f uniform on the ball of radius $\sqrt{N}$.

A consequence concerns the quenched free energy

$$\mathtt{F}(\beta) := \lim_{N\to\infty}\frac{1}{N}\log Z_N^{\beta,\omega}(0) = \lim_{N\to\infty}\frac{1}{N}\mathbb{E}[\log Z_N^{\beta,\omega}(0)] \in (-\infty,0].$$

Earlier work by Lacoin [5] and Berger–Lacoin [6] identified its leading exponential behavior. The estimate (0.3), together with a complementary lower bound based on superadditivity and concentration, gives the more precise bounds

$$-\frac{c'}{\sigma^2(\beta)^4}\exp\Big(-\frac{\pi}{\sigma^2(\beta)}\Big) \le \mathtt{F}(\beta) \le -c\exp\Big(-\frac{\pi}{\sigma^2(\beta)}\Big), \qquad \beta \in (0,1). \tag{0.4}$$

Thus the exact exponential scale is expressed in terms of $\sigma^2(\beta)$ rather than merely β^2, and the error term is brought outside the exponential.

We next pass to the Stochastic Heat Flow. Let $\mathcal{M}_1(r)$ be the set of probability densities on $\mathbb{R}^2$ supported in the ball of radius r. By convergence of directed polymers to the SHF and by its scaling covariance, (0.3) yields universal constants $c_0, c_1, c_2 > 0$ such that, for all $t > 0$ and $\vartheta \in \mathbb{R}$,

$$\frac{1}{c_1}\mathrm{e}^{-c_1 t\mathrm{e}^{\vartheta}} \le \sup_{\varphi\in\mathcal{M}_1(\mathrm{e}^{c_0 t\mathrm{e}^{\vartheta}}\sqrt{t})} \mathbb{E}\big[\mathscr{Z}_t^{\vartheta}(\varphi)\wedge 1\big] \le \frac{1}{c_2}\mathrm{e}^{-c_2 t\mathrm{e}^{\vartheta}}. \tag{0.5}$$

The same bounds hold for fractional moments, and corresponding two-sided estimates hold for the probability that $\mathscr{Z}_t^{\vartheta}(\varphi)$ exceeds a fixed positive level. Thus the effective parameter controlling local extinction is $t\mathrm{e}^{\vartheta}$: the decay is exponential in time and doubly exponential in the disorder strength. The initial condition may spread over a radius exponentially larger than the diffusive scale $\sqrt{t}$.

The polymer result also gives information on a space-time discretization of the two-dimensional stochastic heat equation

$$\partial_t u = \frac{1}{2}\Delta u + \beta\xi u, \qquad u(0,x) = 1.$$

The discretized solution u_1^{β} coincides, after time reversal of the environment, with a directed-polymer partition function. Its diffusively rescaled version is $u_N^{\beta}(t,x) := u_1^{\beta}(Nt, \sqrt{N}x)$. In every supercritical regime $\vartheta(N,\beta_N) \to +\infty$, spatial averages of $u_N^{\beta_N}$ converge to zero, so the diffusive scale does not capture the fluctuations. For $c > 0$, introduce

$$\widehat{u}_N^{\beta,c}(t,x) := u_1^{\beta}\big(Nt, \rho_{Nt}^{\beta,c}\sqrt{Nt}\,x\big), \qquad \rho_{Nt}^{\beta,c} := \exp\big(c\mathrm{e}^{\vartheta(\lfloor Nt\rfloor,\beta)}\big).$$

There exist constants $0 < c' < c'' < \infty$ such that, if $\vartheta(N,\beta_N) \to +\infty$, then for every $t > 0$ and every fixed probability density φ,

$$\int_{\mathbb{R}^2} \varphi(x)\widehat{u}_N^{\beta_N,c}(t,x)\mathrm{d}x \xrightarrow[N\to\infty]{\mathrm{d}} \begin{cases} 0, & c < c', \\ 1, & c > c''. \end{cases} \tag{0.6}$$

This includes the case $\beta_N \equiv \beta \in (0,1)$. By (0.2), the spatial factor is then of order

$$\rho_{Nt}^{\beta,c} = \exp\Big(c\mathrm{e}^{\alpha}Nt\exp\Big(-\frac{\pi}{\sigma^2(\beta)}\Big)(1+o(1))\Big),$$

which is genuinely exponential in N. The first line of (0.6) proves that, up to the constant in the exponent, this scale cannot be reduced.

We finally describe the proof of the upper bound in (0.3). The lower bound follows from a variance estimate and the Paley–Zygmund inequality. The upper bound is obtained in three steps. First, for fixed large ϑ in the critical window, one proves

$$\limsup_{N\to\infty} \sup_{f\in\mathcal{M}_1^{\mathrm{disc}}(\sqrt{N})} \mathbb{E}[Z_N^{\beta_N,\omega}(f)\wedge 1] \le \frac{C}{\vartheta}.$$

The key device is size biasing. If $Z \ge 0$, $\mathbb{E}[Z] = 1$, and $\mathrm{d}\widetilde{\mathbb{P}} = Z\mathrm{d}\mathbb{P}$, then for any event A and any $M > 0$,

$$\mathbb{E}[Z\wedge M] \le M\mathbb{P}(A) + \widetilde{\mathbb{P}}(A^c).$$

The event is defined through a tractable proxy X for the partition function. Starting from the polynomial chaos expansion

$$Z_N^{\beta,\omega}(f) = \sum_B q^{(f)}(B) \prod_{(n,x)\in B} \xi_{n,x}, \qquad \xi_{n,x} := \mathrm{e}^{\beta\omega(n,x)-\lambda(\beta)} - 1,$$

we retain chaos terms whose time support has width at most an intermediate scale $\widetilde{N}$, whose starting time stays away from the endpoints, and whose order is at most $\log N$. This is the first-order term in a coarse-grained chaos expansion. Taking $A = \{X \geq \frac{1}{2}\widetilde{\mathbb{E}}[X]\}$, Chebyshev's inequality reduces the argument to estimates on $\mathbb{V}\mathrm{ar}(X)$, $\widetilde{\mathbb{E}}[X]$ and $\widetilde{\mathbb{V}\mathrm{ar}}(X)$. The first two are essentially second-moment calculations; the last requires a more delicate analysis of pairwise and triple intersections.

Second, a finite-volume criterion converts this polynomial estimate into exponential decay. If, at some scale L,

$$\sup_{f\in\mathcal{M}_1^{\mathrm{disc}}(\sqrt{L})} \mathbb{E}[Z_L^{\beta,\omega}(f)^{1/2}] \leq c_*,$$

for a sufficiently small universal constant c_*, then block coarse graining gives

$$\sup_{f\in\mathcal{M}_1^{\mathrm{disc}}(\sqrt{L})} \mathbb{E}[Z_N^{\beta,\omega}(f)^{1/2}] \leq 3\mathrm{e}^{-N/L}, \qquad N \geq L.$$

Finally, a change-of-scale estimate allows the support of the initial condition to be enlarged. For $1 \leq A \leq B$ and $\gamma \in (0,1)$,

$$\sup_{f\in\mathcal{M}_1^{\mathrm{disc}}(\sqrt{B})} \mathbb{E}[Z_N^{\beta,\omega}(f)^{\gamma}] \leq \Big(4\frac{B}{A}\Big)^{1-\gamma} \sup_{g\in\mathcal{M}_1^{\mathrm{disc}}(\sqrt{A})} \mathbb{E}\big[Z_N^{\beta,\omega}(g)^{\gamma}\Big].$$

At the relevant block scale one has $N/L \asymp \mathrm{e}^{\vartheta(N,\beta)}$. The exponential gain $\mathrm{e}^{-N/L}$ absorbs the entropy cost of covering balls whose radius is of order $\exp(c\mathrm{e}^{\vartheta(N,\beta)})\sqrt{N}$, leading to (0.3).

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NICOLA TURCHI, DIPARTIMENTO DI MATEMATICA E APPLICAZIONI, UNIVERSITÀ DEGLI STUDI DI MILANO-BICOCCA, VIA COZZI 55, 20125 MILANO, ITALY

Email address: nicola.turchi@unimib.it

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MOMENTS OF A LONG-RANGE CORRELATED 2D STOCHASTIC HEAT EQUATION AT CRITICALITY

TE-CHUN WANG

Classification AMS 2020: 60K37, 60H15, 82D60.

Keywords: Kardar-Parisi-Zhang equation; stochastic heat equation; continuum directed polymer; critical 2d stochastic heat flow; long-range correlation.

We consider the following formal stochastic heat equation (SHE):

$$\partial_t \mathbf{X}^\beta(x,t) = \frac{\Delta_x}{2}\mathbf{X}^\beta(x,t) + \mathbf{X}^\beta(x,t)\cdot \beta\xi(x,t), \quad (x,t)\in\mathbb{R}^2\times\mathbb{R}_+. \tag{0.1}$$

The driving noise is a centered Gaussian field with correlation function

$$\mathbb{E}[\xi(x,t)\xi(y,s)] = \delta(t-s)\times|x-y|^{-2}. \tag{0.2}$$

The coupling constant β describes the strength of the noise. From a physical perspective, the stochastic heat equation has a rich interpretation. On the one hand, it arises in the study of the KPZ equation through the Cole–Hopf transformation [1]. On the other hand, the solution to the SHE can also be viewed as the partition function of the continuous directed polymer. The polymer model is a fundamental model in statistical physics that describes the fluctuations of a polymer chain affected by disorder [2, 3].

From a mathematical perspective, beyond the SHE driven by space-time white noise, it is natural to ask how spatial correlations affect the behavior of the SHE. This motivates the study of SHEs driven by Riesz-type correlated noise, such as (0.1).

Because of the exponent 2 in the spatial correlation, a formal scaling argument shows that two is a critical dimension for this SPDE. Due to the singularity of the noise, (0.1) is not well-defined in the sense of the classical Itô theory. To circumvent this problem, a common approach is to consider a regularized version of (0.1) and analyze its scaling limit as the regularization vanishes.

For the same correlation as in (0.2), and in dimensions greater than or equal to three, A. Dunlap, M. Hairer, and X.-M. Li [4] proved the scaling limit of a nonlinear version of (0.1). In addition, C. Mueller and R. Tribe [5] proved a more critical scaling limit for (0.1). Based on the latter result, S. Kuzgun and R. Tao [6] studied a time-dependent average. Versions with different exponents have also been studied in [7, 8, 9].

In the polymer context, C. Rovira, S. Tindel [10], and H. Lacoin [11] proved several results on how long-range spatial correlations affect the polymer model. In particular, our work is mainly motivated by the work of C. Cosco, F. Cottini, and A. Donadini [12]. They showed that a discrete directed polymer with long-range spatial correlations behaving at large scales like $(\log|x|)^a/|x|^2$, for $a>-1$, exhibits a phase transition at a certain critical point. This result naturally raises the question of the scaling limit of the partition function exactly at criticality. Our result provides an answer to this question in the continuous setting.

More precisely, we consider the following approximation scheme for (0.1):

$$\partial_t \mathbf{X}_\varepsilon^{\lambda,a}(x,t) = \frac{\Delta_x}{2}\mathbf{X}_\varepsilon^{\lambda,a}(x,t) + \mathbf{X}_\varepsilon^{\lambda,a}(x,t)\cdot \beta_\varepsilon^{\lambda,a}\xi_\varepsilon^a(x,t), \quad \mathbf{X}_\varepsilon^{\lambda,a}(x,0) = X_0(x),$$

where the regularized driving noise is a centered Gaussian field with correlation function

$$\mathbb{E}[\xi_\varepsilon^a(x,t)\xi_\varepsilon^a(y,s)] = \delta(t-s)\cdot R_\varepsilon^a(x-y), \quad R_\varepsilon^a(x) := \varepsilon^{-2}R^a(x/\varepsilon).$$

The primary kernel we consider is $1/(1+|x|^2)$, since in this case the rescaled kernel converges to $1/|x|^2$ as $\varepsilon \to 0$. More generally, our model also includes kernels that behave at large scales like

$$R^a(x) \overset{x\to\infty}{\sim} \frac{(\log|x|)^a}{|x|^2}, \quad a > -1.$$

We also assume that $R^a \in C_b(\mathbb{R}^2;\mathbb{R}_{\geq 0})$ is radially symmetric, and we impose certain decay conditions on R^a. One important feature of this model is that, compared with the two-dimensional stochastic heat equation driven by regularized space-time white noise, the correlation kernel here is not integrable. Hence, one of the key points of our study is to understand how this singularity, together with the logarithmic perturbation, affects the limit of the regularized solution.

To obtain a nontrivial limit, we tune down the strength of the deriving noise by the following scaling:

$$(\beta_\varepsilon^{\lambda,a})^2 := \underbrace{\frac{(\beta_c^a)^2}{(\log\varepsilon^{-1})^{a+2}}}_{\text{rescaled coupling constant}} \times \underbrace{\left(1 + \text{“correction term”} + \frac{\lambda}{\log\varepsilon^{-1}}\right)}_{\text{perturbation of the coupling constant}}. \tag{0.3}$$

Here the coupling constant is set to its critical value, given by

$$(\beta_c^a)^2 := \left(\frac{\mathbf{j}_{-\nu_a,0}}{2\nu_a}\right)^2, \quad \nu_a := \frac{1}{a+2},$$

where $\mathbf{j}_{-\nu,0}$ is the first zero of the Bessel function of the first kind:

$$\mathcal{J}_{-\nu}(z) := \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m-\nu+1)}\left(\frac{z}{2}\right)^{2m-\nu}.$$

Under the critical scaling (0.3), we establish the following universal limit.

Theorem 0.1. *Assume that $a > -1$, $\lambda \in \mathbb{R}$, and that R^a satisfies suitable decay conditions. Then, for each $N \geq 2$, $f \in C_c(\mathbb{R}^2)$, and $X_0 \in C_b(\mathbb{R}^2)$,*

$$\mathbb{E}[\mathbf{X}_\varepsilon^{\lambda,a}(f,t)^N] \xrightarrow{\varepsilon\to 0} \mathbb{E}[\mathscr{Z}_{0,t}^{\lambda,a}(X_0,f)^N]. \tag{0.4}$$

Here $\mathscr{Z}_{s,t}^{\lambda,a}(dx,dy)$ is the critical two-dimensional stochastic heat flow (SHF), where the function describing the two-body motion is given by

$$\mathfrak{s}^{\vartheta_a}(t) := (2\pi)\int_0^\infty \frac{(\vartheta_a)^u t^{u-1}}{\Gamma(u)}du, \quad \log\vartheta_a := \frac{2}{a+2}\lambda + \log 2 - 2\gamma_{\text{EM}}. \tag{0.5}$$

The critical 2d SHF arises both as a normalized limit of the two-dimensional SHE driven by space-time white noise [13, 14] and as a scaling limit of the two-dimensional

directed polymer model [14], both under critical scaling. The correction term in (0.3) is given by

"correction term"

$$:= \begin{cases} 0, & \text{if } a > 0, \\ -\dfrac{1}{\log \varepsilon^{-1}} \left(\dfrac{1}{2} \displaystyle\int_{-\infty}^{0} \mathfrak{u}^a(e^{\frac{z}{2}})^2 dz + \int_0^{\infty} \mathfrak{u}_0^a(e^{\frac{z}{2}})^2 dz + \int_0^{\infty} \mathfrak{u}_0^a(e^{\frac{z}{2}}) dz \right), & \text{if } a = 0, \\ \dfrac{\|\mathbf{K}_0^a\|_{\mathrm{op}}}{\|\mathcal{K}_\varepsilon^a\|_{\mathrm{op}}} - 1, & \text{if } a \in (-1, 0). \end{cases}$$

Here $\mathfrak{u}^a(r)$ and $\mathfrak{u}_0^a(r)$ are related to the large-scale behavior of R^a and are given by

$$\mathfrak{u}^a(r) := \big(2r^2 R^a(\sqrt{2}r)\big)^{\frac{1}{2}}, \quad \frac{1}{(\log r)^{\frac{a}{2}}} \mathfrak{u}^a(r) = 1 - \left(1 - \left(\frac{2r^2}{(\log r)^a} R^a(\sqrt{2}r)\right)^{\frac{1}{2}}\right) =: 1 + \mathfrak{u}_0^a(r).$$

In addition, $\mathbf{K}_0^a$ is a bounded operator on $L^2((0,1))$ with kernel

$$\mathbf{K}_0^a(z', z) := (z')^{\frac{a}{2}} (1 - z' \vee z) z^{\frac{a}{2}},$$

which arises as the scaling limit of a Birman–Schwinger kernel. Moreover, $\mathcal{K}_\varepsilon^a$ is a small perturbation of $\mathbf{K}_0^a$. It can be shown that, for every sufficiently small $\varepsilon > 0$,

$$\frac{C}{(\log \varepsilon^{-1})^{1+a}} \le \left| \frac{\|\mathbf{K}_0^a\|_{\mathrm{op}}}{\|\mathcal{K}_\varepsilon^a\|_{\mathrm{op}}} - 1 \right| \le \frac{C'}{(\log \varepsilon^{-1})^{1+a}}, \quad 0 < C < C'.$$

We remark that the main purpose of the correction term is to put the function $\mathfrak{s}^{\vartheta_a}$ in the form (0.5). In particular, when $a \in (-1, 0)$, the correction term is necessary to obtain the non-Gaussian limit (0.4): if the correction term is set to zero, then the limit of the higher moments is given only by the Gaussian contribution. This phenomenon does not appear for the two-dimensional SHE driven by space-time white noise [15, 16, 17], but a similar phenomenon appears in the mean-zero case [18].

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INSTITUTE OF MATHEMATICS, ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE, SWITZERLAND.
Email address: te-chun.wang@epfl.ch

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SCALING LIMIT OF A WEAKLY ASYMMETRIC SIMPLE EXCLUSION PROCESS IN THE FRAMEWORK OF REGULARITY STRUCTURES

HENDRIK WEBER

Classification AMS 2020: 82C22, 60K35, 60H17, 37E20

Keywords: Weakly Asymmetric Simple Exclusion Process, KPZ Equation, Regularity Structures, Scaling Limits, Renormalisation, Interacting Particle Systems, Discrete Stochastic Analysis

We prove that a parabolically rescaled and suitably renormalised height function of a weakly asymmetric simple exclusion process on a circle converges to the Cole-Hopf solution of the KPZ equation. This is an analogue of the celebrated result by Bertini and Giacomin from 1997 for the exclusion process on a circle with any particle density. The main goal of this article is to analyse the interacting particle system using the framework of regularity structures without applying the Gaertner transform, a discrete version of the Cole-Hopf transform which linearises the KPZ equation. Our analysis relies on discretisation framework for regularity structures developed by Erhard and Hairer as well as estimates for iterated integrals with respect to cadlag martingales derived by Grazieschi, Matetski and Weber. The main technical challenge addressed in this work is the renormalisation procedure which requires a subtle analysis of regularity preserving discrete convolution operators.

This is joint work with Ruojun Huang and Konstantin Matetski.

Further details of these developments can be found in the following research papers:

- https://arxiv.org/abs/2505.00621

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UNIVERSITY OF MÜNSTER, GERMANY
Email address: hendrik.weber@uni-muenster.de

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FROM SKEW TO REFLECTED SPDES

LORENZO ZAMBOTTI

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Keywords: Stochastic partial differential equations, Singular drift, Reflection, Bessel process

In this article, we prove that the solution of the skew stochastic heat equation converges to the solution of the stochastic heat equation with reflection in the limit of infinite skewness. The result holds at equilibrium. This provides an SPDE analog of the link between skew and reflected Brownian motion. One of the intermediate results is the convergence in law of a penalised Brownian bridge to a 3-Bessel bridge.

This is joint work with Cyril Labbé and Thomas Le Guerch.

Further details of these developments can be found in the following research papers:

- https://arxiv.org/abs/2609.13014v1

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SORBONNE UNIVERSITÉ, FRANCE
Email address: zambotti@lpsm.paris

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DERIVATION OF THE FULL FOCUSING Φ_1^6 MEASURE FROM MANY-BODY QUANTUM GIBBS STATES

RONGCHAN ZHU

Classification AMS 2020: 81V70, 35Q40, 35Q55.

Keywords: many-body quantum Gibbs states; focusing Φ_1^6 measure; nonlinear Gibbs measures; variational method; normalizability threshold.

This extended abstract is based on joint work with Lin Lü and Phan Thành Nam [5]. The purpose of the talk was to explain how the full one-dimensional focusing Φ_1^6 measure, including the sharp critical mass regime, can be obtained as a high-temperature and mean-field limit of many-body bosonic quantum Gibbs states. A point emphasized here is the quantitative subcritical feature: below the critical mass, the singular interaction scale $\varepsilon = \varepsilon(\tau)$ may depend polynomially on the temperature parameter, and the proof uses the sharp subcritical integrability input from [7].

Let $\mathbb{T} = [0, 1]$. The classical object is the focusing Gibbs measure

$$\mathrm{d}\mu_K(u) = \frac{1}{\mathcal{Z}_K} \exp\left\{\frac{1}{6}\int_{\mathbb{T}} |u(x)|^6 \,\mathrm{d}x\right\} \mathbf{1}_{\{\|u\|_{L^2(\mathbb{T})} \le K\}} \,\mathrm{d}\mu_0(u),$$

where μ_0 is the free complex Gaussian field with covariance h^{-1}, $h = \frac{1}{2}(1 - \Delta)$. Since μ_0 is supported on $H^{1/2-}(\mathbb{T})$, no Wick renormalization is needed for the sixth order term in dimension one. This measure is formally invariant under the focusing quintic nonlinear Schrödinger equation

$$i\partial_t u + \partial_x^2 u + |u|^4 u = 0.$$

The focusing sign makes the Hamiltonian unbounded from below, and the L^2 cutoff is essential. The sharp threshold is

$$K_c = \|Q\|_{L^2(\mathbb{R})}, \qquad -Q'' + Q - Q^5 = 0 \quad \text{on } \mathbb{R},$$

where Q is the optimizer of the sharp Gagliardo–Nirenberg–Sobolev inequality. The classical normalizability threshold is sharp: one has a finite partition function for $K \le K_c$ and divergence for $K > K_c$ [4, 7].

We consider the bosonic Fock space $\mathcal{F} = \bigoplus_{n\ge 0} L^2_{\mathrm{sym}}(\mathbb{T}^n)$, the number operator N, and the free Hamiltonian $H_0 = \mathrm{d}\Gamma(h)$. Let $w \ge 0$ be an even compactly supported function with $\int w = 1$, and set $w_\varepsilon(x) = \varepsilon^{-1} w(x/\varepsilon)$, periodized on $\mathbb{T}$. The attractive three-body interaction is

$$W_\varepsilon = \frac{1}{3!}\int_{\mathbb{T}^3} a^*(x)a^*(y)a^*(z)w_\varepsilon(x-y)w_\varepsilon(x-z)a(x)a(y)a(z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

The natural high-temperature scaling is obtained by taking the interaction strength $\lambda = \tau^{-2}$, because under the free Gibbs state the typical particle number is of order τ, so that

the one-body and three-body contributions are both of order one in the Gibbs exponent. With a smooth cutoff $f_\eta \to \mathbf{1}_{[0,K^2)}$, the grand-canonical Gibbs state is

$$\Gamma_\tau^{f_\eta} = \frac{1}{Z_\tau^{f_\eta}} \exp\left\{-\frac{1}{\tau}H_0 + \frac{1}{\tau^3}W_\varepsilon\right\} f_\eta(N/\tau).$$

Here N/τ is the quantum analogue of the classical mass $\|u\|_{L^2}^2$.

The main result states that the quantum Gibbs state converges to the focusing Φ_1^6 measure throughout the full normalizable range. More precisely, if $0 < K \le K_c$, then, along an admissible regime with $\varepsilon \to 0$, $\eta \to 0$, and $\tau \to \infty$, one has

$$\frac{Z_\tau^{f_\eta}}{Z_{\tau,0}} \longrightarrow \int \exp\left\{\frac{1}{6}\int_{\mathbb{T}} |u|^6 \,\mathrm{d}x\right\} \mathbf{1}_{\{\|u\|_{L^2} \le K\}} \,\mathrm{d}\mu_0(u).$$

At the critical threshold $K = K_c$, the interaction range may shrink logarithmically, for instance $\varepsilon \gtrsim (\log\tau)^{-1/2}$. A central subcritical refinement is that, when $K < K_c$, the dependence of the interaction scale on the temperature becomes polynomial: one may take, for example, $\varepsilon \gtrsim \tau^{-1/96}$. The proof of this improvement uses the subcritical exponential-integrability input from [7] to replace rough pointwise bounds of size $e^{O(\varepsilon^{-2})}$ by uniform moment bounds; this is precisely where the mass gap $K < K_c$ enters. Furthermore, for every fixed $k \ge 1$,

$$\frac{k!}{\tau^k}\big(\Gamma_\tau^{f_\eta}\big)^{(k)} \longrightarrow \int |u^{\otimes k}\rangle\langle u^{\otimes k}| \,\mathrm{d}\mu_K(u) \quad \text{in } \mathfrak{S}^1\big(L^2_{\mathrm{sym}}(\mathbb{T}^k)\big).$$

Thus the limiting classical measure is recovered not only at the level of free energy, but also through all reduced density matrices.

The complementary result is a sharp quantum phase transition. If $K > K_c$ and $g \in C_c^\infty([0,\infty);\mathbb{R}_+)$ satisfies $g \equiv 1$ on $[0,K^2]$, then along the corresponding logarithmic admissible regime

$$\frac{\mathrm{Tr}_{\mathcal{F}}\left[\exp\left\{-\frac{1}{\tau}H_0 + \frac{1}{\tau^3}W_\varepsilon\right\} g(N/\tau)\right]}{\mathrm{Tr}_{\mathcal{F}}\left[\exp\left\{-\frac{1}{\tau}H_0\right\}\right]} \longrightarrow \infty.$$

Consequently the many-body model detects exactly the same threshold K_c as the classical focusing theory.

I briefly describe the proof. The first step is a quantum-to-classical limit at a fixed interaction scale $\varepsilon > 0$. The limiting classical measure is a Hartree Gibbs measure with nonlocal potential

$$W_\varepsilon(u) = \frac{1}{3!}\int_{\mathbb{T}} (w_\varepsilon * |u|^2)^2 |u(x)|^2 \,\mathrm{d}x, \qquad \mathrm{d}\mu^{\varepsilon,g}(u) \propto e^{W_\varepsilon(u)} g(\|u\|_{L^2}^2) \,\mathrm{d}\mu_0(u).$$

This passage is proved by a variational comparison between the quantum Gibbs principle and the corresponding classical entropy minimization problem. After localizing to low modes $P = \mathbf{1}(h \le \Lambda)$, the low-frequency part of a quantum state is represented by its Husimi measure. Berezin–Lieb inequalities provide the entropy lower bound, while quantitative de Finetti estimates identify the limiting moments.

The focusing particle-number cutoff creates the main difficulty: it couples low and high modes and is incompatible with the factorized trial states used in defocusing problems. A key new ingredient is a non-factorized trial state that keeps the full cutoff $f_\eta(N/\tau)$ while localizing only the interaction. This preserves the global mass constraint and controls the localization and relative-entropy errors. For the lower bound one applies entropy

duality only to a mass-truncated observable and proves a tail estimate for the interacting lower symbol. Finally, W_ε is sent to the local potential $W(u) = \frac{1}{6}\|u\|_{L^6}^6$ by a mollifier approximation. At the endpoint K_c, the optimal normalizability theorem [7] supplies the required integrable majorant. Strictly below K_c, the projected subcritical moment estimate adapted from [7, Sec. 4] gives uniform L^p control of the focusing weight. This removes the exponential loss in ε^{-2} and is the reason the singular interaction scale can be chosen polynomially in τ, for example $\varepsilon \gtrsim \tau^{-1/96}$, rather than only logarithmically.

This work builds on the construction and sharp normalizability theory of focusing Gibbs measures [4, 1, 7], on many-body derivations of nonlinear Gibbs measures [3, 2, 6], and on focusing grand-canonical limits [8]. The novelty here is the combination of the full optimal mass range, the singular limit $w_\varepsilon \to \delta_0$, the polynomial subcritical interaction scale, and the matching supercritical divergence at the quantum level.

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SCHOOL OF MATHEMATICS AND STATISTICS, BEIJING INSTITUTE OF TECHNOLOGY, BEIJING 100081, CHINA

Email address: zhurongchan@126.com

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FROM MANY-BODY QUANTUM GIBBS STATES TO SINGULAR GIBBS MEASURES

XIANGCHAN ZHU

Classification AMS 2020: 81V70, 82B10, 35Q40, 60H15.

Keywords: many-body quantum Gibbs states; nonlinear Gibbs measures; Bose gases; Φ_3^4 measure; fractional Bessel interaction; stochastic quantization.

This extended abstract is based on joint works with Phan Thành Nam and Rongchan Zhu [9, 10]. The talk discussed a program which derives singular classical Gibbs measures from grand-canonical many-body quantum Gibbs states in the high-temperature, or semiclassical, limit. The basic philosophy is that, after the rescaling $b_x = \sqrt{\lambda}\, a_x$ of annihilation operators, the canonical commutation relation becomes

$$[b_x, b_y^*] = \lambda\delta(x-y) \longrightarrow 0,$$

so the quantum field should converge to a classical random field. The difficulty is that the limiting field theories are singular and require renormalization, while the quantum model is formulated through non-commuting operators and nonlocal interactions.

The first result concerns the derivation of the complex Φ_3^4 measure on $\mathbb{T}^3$. Formally this measure is

$$\mathrm{d}\nu(\Phi) = \frac{1}{Z}\exp\left\{-\int_{\mathbb{T}^3}\left(|\nabla\Phi|^2 + m_0|\Phi|^2 + \frac{1}{2}|\Phi|^4\right)\,\mathrm{d}x\right\} D\Phi,$$

which has to be understood after renormalization since the Gaussian free field is a distribution of regularity $C^{-1/2-}$. On the quantum side one considers, on bosonic Fock space, an interacting Bose gas with a short-range two-body potential $v^\varepsilon = \varepsilon^{-3}v(\cdot/\varepsilon)$ and Gibbs state $\Gamma_\lambda = Z_\lambda^{-1}e^{-H_\lambda}$. The chemical potential is chosen as

$$\vartheta = \frac{\zeta(3/2)}{(4\pi)^{3/2}}\lambda^{-1/2} + C_0 + a_\varepsilon - 6b_\varepsilon + 2C_1 + 2C_2 - m_0,$$

where the ideal-gas term places the system just above the critical density, while $a_\varepsilon \sim \varepsilon^{-1}$, $b_\varepsilon \sim |\log\varepsilon|$, and the finite constants C_1, C_2 encode the mass renormalization needed in dimension three.

The main theorem of [9] states that, if $\lambda, \varepsilon \to 0$ and $\lambda^\eta \le \varepsilon$ for a sufficiently small $\eta > 0$, then the quantum field observables converge to those of the Φ_3^4 measure. More precisely, for every continuous $f : [0,\infty) \to \mathbb{R}$ satisfying

$$|f(x) - f(y)| \le C|x-y|(1 + x^5 + y^5)$$

and every fixed test function φ with finitely supported Fourier transform,

$$\mathrm{Tr}\left[f(\lambda a^*(\varphi)a(\varphi))\Gamma_\lambda\right] \longrightarrow \int f(|\langle\varphi, u\rangle|^2)\,\mathrm{d}\nu(u).$$

Consequently, for $1 \le n \le 6$,

$$n!\lambda^n\langle\varphi^{\otimes n}, \Gamma_\lambda^{(n)}\varphi^{\otimes n}\rangle \longrightarrow \int |\langle\varphi, u\rangle|^{2n}\,\mathrm{d}\nu(u).$$

Under the stronger logarithmic assumption $|\log\lambda|^{-\eta} \le \varepsilon \to 0$, the same convergence holds for all $n \ge 1$. This gives a rigorous bridge from a regular quantum Bose gas to the renormalized three-dimensional Euclidean field.

A central intermediate object is the Hartree approximation

$$\mathrm{d}\nu^{\varepsilon}(\Psi) = Z_{\varepsilon}^{-1} e^{-D_{\varepsilon}(\Psi)}\, \mathrm{d}\mu_0(\Psi),$$

where μ_0 is the Gaussian free field and

$$D_{\varepsilon}(\Psi) = \frac{1}{2}\int_{\mathbb{T}^3\times\mathbb{T}^3} :|\Psi(x)|^2: v^{\varepsilon}(x-y) :|\Psi(y)|^2:\ \mathrm{d}x\,\mathrm{d}y - (a_{\varepsilon} - 6b_{\varepsilon} - m + 1)\int_{\mathbb{T}^3} :|\Psi|^2:\ \mathrm{d}x.$$

The proof has two layers. First, stochastic quantization and paracontrolled calculus show that ν^{ε} converges to the Φ^4_3 measure; new resonant terms generate the finite constants C_1 and C_2. Second, a variational comparison between quantum relative entropy and classical entropy derives the Hartree measure from Γ_λ. Low frequencies are handled by Husimi measures and quantitative de Finetti theorems; high frequencies are controlled by correlation inequalities and moment estimates. This also improves earlier two-dimensional shrinking-range results, since the same strategy yields all density matrices in $d = 2$ under the polynomial-type constraint $\lambda^{\eta} \le \varepsilon$.

The second result, proved in [10], treats a different singular regime: a fixed fractional Bessel interaction in two dimensions. On $\mathbb{T}^2$, let

$$\widehat{v}_{\beta}(k) = \langle k\rangle^{-\beta} = (1+|k|^2)^{-\beta/2}, \qquad \frac{3}{2} < \beta \le 2.$$

The case $\beta = 2$ is the Yukawa kernel, with the same short-distance singularity as the Coulomb potential in two dimensions. Unlike the local δ-limit, this is already singular at the quantum level: throughout the range $\frac{3}{2} < \beta \le 2$ one has $\sum_{k\in\mathbb{Z}^2} \widehat{v}_{\beta}(k) = \infty$. Therefore the usual density-square rewriting produces an infinite self-energy and cannot be used directly.

The quantum Hamiltonian must instead be renormalized at the operator level. If $N_0 = \mathrm{Tr}(N\Gamma_0)$ is the free expected particle number, the renormalized interaction is

$$W^{\mathrm{re}} = \frac{\lambda^2}{2(2\pi)^2}\sum_{k\neq 0}\widehat{v}_{\beta}(k)\sum_{p,q\in\mathbb{Z}^2} a^*_{p+k}a^*_{q-k}a_q a_p + \frac{\lambda^2}{2(2\pi)^2}(N-N_0)^2,$$

and the Gibbs state is $\Gamma^{\mathrm{re}}_{\lambda} = (Z^{\mathrm{re}}_{\lambda})^{-1} e^{-\lambda\,\mathrm{d}\Gamma(h) - W^{\mathrm{re}}}$, with $h = -\Delta + 1$. The limiting classical Gibbs measure is

$$\mathrm{d}\nu(u) = z_{\mathrm{cl}}^{-1} e^{-D(u)}\,\mathrm{d}\mu_0(u), \qquad D(u) = \frac{1}{2}\int_{\mathbb{T}^2\times\mathbb{T}^2} v_{\beta}(x-y) :|u(x)|^2::|u(y)|^2:\ \mathrm{d}x\,\mathrm{d}y.$$

The main theorem is the convergence, as $\lambda \downarrow 0$, of both the relative free energy and the reduced density matrices:

$$-\log\frac{Z^{\mathrm{re}}_{\lambda}}{Z_0} \longrightarrow -\log z_{\mathrm{cl}},$$

and, for every fixed $k \ge 1$,

$$k!\lambda^k\Gamma^{(k)}_{\lambda} \longrightarrow \int |u^{\otimes k}\rangle\langle u^{\otimes k}|\ \mathrm{d}\nu(u) \quad \text{in } \mathfrak{S}^2(L^2_{\mathrm{sym}}(\mathbb{T}^2)^{\otimes k}).$$

Thus the full classical fractional-Bessel Gibbs measure is obtained from a renormalized Bose gas with the same singular fixed interaction.

The proof remains variational but requires new high-frequency analysis. After localizing to low frequencies $P = \mathbf{1}_{\{h \leq \Lambda^2\}}$, the projected quantum interaction is compared with a finite-dimensional Hartree functional. The error is decomposed into a zero-mode part and a nonzero-mode part; the latter is further split into an intermediate shell and a genuine far tail. The zero-mode contribution is controlled by a second-order correlation inequality and a double-commutator estimate. The shell contribution is rewritten through polarized localized fluctuation observables and estimated mode by mode. The far tail is controlled by combining positivity of the full Bessel operator with small high-frequency particle number. These estimates are precisely what makes it possible to treat non-summable kernels and to recover the Hilbert–Schmidt convergence of all fixed reduced density matrices.

Together, [9] and [10] show two complementary ways in which singular Gibbs measures emerge from many-body quantum mechanics. In the Φ^4_3 problem the singularity comes from the shrinking interaction range and the local field-theory limit. In the fractional-Bessel problem the interaction is fixed but already ultraviolet singular, so the main challenge is the divergent quantum self-energy and the resulting high-frequency remainders. Both results strengthen the bridge between critical Bose gases, constructive quantum field theory, and singular stochastic PDEs, and suggest further questions on Bose–Einstein condensation, Gross–Pitaevskii scaling, Coulomb interactions, and dynamic correlation functions at positive temperature.

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ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE, CHINESE ACADEMY OF SCIENCES, BEIJING 100190, CHINA

Email address: zhuxiangchan@126.com

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WICK RENORMALIZED PARABOLIC STOCHASTIC QUANTIZATION EQUATIONS ON ROUGH METRIC MEASURE SPACES

HONGYI CHEN

Classification AMS 2020: 60H17, 28A80, 31C25, 46E35, 81T08

Keywords: Stochastic Quantisation, Wick Renormalisation, Singular Stochastic Partial Differential Equations, Rough Metric Measure Spaces, Sub-Gaussian Heat Kernels, Fractal Geometry, Invariant Measures

On metric measure spaces with sub-Gaussian heat kernel behavior in small time, we obtain sufficient conditions for the well-posedness of Wick renormalized stochastic quantization equations with polynomial interaction. Given the power of the nonlinearity, the local solution condition depends on the Hausdorff dimension d_h, the walk dimension d_w, and the maximal spatial Hölder regularity Θ of the heat kernel. A slightly more restrictive condition based on the same parameters is required for a global solution. For all global solutions, we construct an invariant measure for the Markov process defined by the solution. Our results apply to many rough spaces such as Barlow–Kigami type fractals as well as their Cartesian products and open up the possibility of making rigorous various structures in quantum field theory and statistical mechanics in non-integer dimensions. In the process, we build entirely from the short-time heat semigroup the necessary analytic framework that accommodates the issues which come with allowing rough local geometry.
This is joint work with Yifan (Johnny)Yang (Arizona State University, Tempe, USA).

Further details of these developments can be found in the following research papers:

- https://arxiv.org/abs/2605.05442

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AARHUS UNIVERSITY, DENMARK
Email address: hchen77@math.au.dk

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NOISE SENSITIVITY AND DIRECTED POLYMERS

ANNA DONADINI

Classification AMS 2020: Primary: 05D40; Secondary: 82B44, 60H15.

Keywords: Directed Polymer in Random Environment, Disordered System, Influence, Noise Sensitivity, Stochastic Heat Equation, Stochastic Heat Flow.

Given a function $f(\omega)$ that depends on independent random variables $\omega = (\omega_1, \omega_2, \dots)$, the concept of noise sensitivity describes the phenomenon where a small perturbation of the variables ω_i significantly alters the function's output. This phenomenon is particularly relevant in the study of Boolean functions [16], where small changes in input bits can make the output unpredictable, and in statistical physics [12], where it describes the sensitivity of systems to random noise, especially in the proximity of a phase transition.

More precisely, we say that a sequence of functions $(f_N)_{N\in\mathbb{N}}$ is noise sensitive if for every $\varepsilon > 0$

$$\lim_{N\to\infty} \mathbb{C}ov(f_N(\omega), f_N(\omega^\varepsilon)) = 0\,, \tag{0.1}$$

where $\omega^\varepsilon = (\omega_1^\varepsilon, \omega_2^\varepsilon, \dots)$ is the modified family where $\omega_i^\varepsilon = \omega_i$ with probability $1-\varepsilon$, while ω_i^ε is independently resampled with probability ε.

The literature on noise sensitivity has largely focused on binary variables ω_i taking two values, say x_+ and x_-:

$$\mathbb{P}(\omega_i = x_+) = p \quad \text{and} \quad \mathbb{P}(\omega_i = x_-) = 1-p \quad \text{for some } p \in (0,1)\,. \tag{0.2}$$

In [2], we extend the classical BKS criterion for noise sensitivity [1] to a general setting, allowing for a wide class of non-Boolean functions $f(\omega) \in \mathbb{R}$ of general random variables ω_i. More specifically, we consider independent random variables $\omega = (\omega_i)_{i\in\mathbb{T}}$, labeled by a finite or countable set $\mathbb{T}$, defined on a probability $(\Omega, \mathcal{A}, \mathbb{P})$ and taking values in measurable spaces $(E_i, \mathcal{E}_i)$:

$$\omega_i \colon \Omega \to E_i \qquad \text{with law} \qquad \mu_i(\cdot) = \mathbb{P}(\omega_i \in \cdot)\,.$$

We then require that the independent random variables $(\omega_i)_{i\in\mathbb{T}}$ and the function $f(\omega) \in L^2$ satisfy the following conditions:

- For any $i \in \mathbb{T}$, there is a closed and separable vector space $\mathcal{V}_i \subseteq L^2(E_i, \mu_i)$ containing the constants, i.e. $1 \in \mathcal{V}_i$, such that $\omega_i \mapsto f(\omega) \in \mathcal{V}_i$ a.s.
- There exist an exponent $q \in (2, \infty)$ and a constant $M_q < \infty$ such that, for any $i \in \mathbb{T}$,

$$\|g(\omega_i)\|_q \le M_q \, \|g(\omega_i)\|_2 \qquad \forall g \in \mathcal{V}_i \quad \text{with} \quad \mathbb{E}[g(\omega_i)] = 0\,,$$

  where $\|\cdot\|_q = \mathbb{E}[|\cdot|^q]^{1/q}$ denotes the usual L^q norm.

These assumptions are quite general. Useful examples for which these conditions hold include all functions $f \in L^2$, when $(\omega_i)_{i\in\mathbb{T}}$ have finite support, and any polynomial chaos $f \in L^2$, when $(\omega_i)_{i\in\mathbb{T}}$ are real-valued satisfying a suitable moment condition.

In this setting, we are able to prove a quantitative version (in the style of [13]) of the classical BKS criterion based on influences (see [1]). First, we extend the classical notion of influence to general functions $f(\omega)$ of general random variables $(\omega_i)_{i\in\mathbb{T}}$. For binary ω's, one can define the influence of f with respect to the k-th variable as

$$I_k(f) = \mathbb{E}[|(f(\omega^k_+) - f(\omega^k_-)|]\,,$$

where $\omega^k_\pm$ denotes the configuration $\omega = (\omega_1, \omega_2, \dots)$ with ω_k fixed to $x_\pm$. For independent variables $(\omega_i)_{i\in\mathbb{T}}$ satisying our assumptions, we define the influence of ω_k on f as

$$Inf_k[f] := \mathbb{E}\big[|f - \mathbb{E}_k[f]|\big] \qquad \text{where we set} \quad \mathbb{E}_k[\cdot] := \mathbb{E}[\cdot|\sigma((\omega_j)_{j\neq k})]\,,$$

which is a quantification of "how much f depends on ω_k".

With this definition, we can extend the Keller-Kindler bound [13] to our general setting, in the form of Van Handel (see [17]). In particular, we prove that for any $\varepsilon \in (0,1)$, there is an exponent $\gamma_{\epsilon,q} > 0$ such that

$$\frac{\mathbb{C}ov\big[f(\omega^\varepsilon), f(\omega)\big]}{\mathbb{V}ar[f]} \leq 4\left(\frac{\mathcal{W}[f]}{\mathbb{V}ar[f]}\right)^{\gamma_{\varepsilon,q}}, \tag{0.3}$$

where $\mathcal{W}[f] := \sum_k Inf_k[f]^2$ is the sum of squared influences. The exponent $\gamma_{\epsilon,q}$ depends on a hypercontractivity constant η_q and satisfies $\gamma_{\varepsilon,q} \sim \alpha_q\,\varepsilon$ as $\varepsilon \downarrow 0$ for an explicit $\alpha_q \in (0, \frac{1}{2})$.

When specialised to the binary setting, our result yields asymptotically optimal rates.

Finally, we can give a uniform lower bound on the exponent $\gamma_{\epsilon,q}$ and apply (0.3) to a sequence of functions $(f_N(\omega))_{N\in\mathbb{N}}$ such that $\limsup_{N\to\infty} \mathbb{V}ar[f_N] < \infty$. This yields the announced generalisation of the BKS criterion for noise sensitivity, i.e.

$$\lim_{N\to\infty} \mathcal{W}[f_N] = 0 \qquad \Longrightarrow \qquad \forall \varepsilon > 0\colon \quad \lim_{N\to\infty} \mathbb{C}ov\big[f_N(\omega^\varepsilon), f_N(\omega)\big] = 0\,.$$

The concept of noise sensitivity is usually formulated in terms of vanishing correlations which — for Boolean function $f(\omega) \in \{0,1\}$ — corresponds to asymptotic independence. For general real functions $f(\omega) \in \mathbb{R}$, however, it is natural to consider an enhanced form of noise sensitivity, where $f(\omega)$ is composed with suitable test functions to ensure asymptotic independence, rather than mere decorrelation. We show that the BKS criterion implies enhanced noise sensitivity, at least when the random variables ω_i take finitely many values.

Finally, we apply our results on noise sensitivity to the $2d$ directed polymer in random environment. We first establish an instance of enhanced noise sensitivity for the partition functions, in the critical regime where they converge to the Critical 2D Stochastic Heat Flow (SHF) [5]. Next, we prove the independence of the SHF from the white noise originating from the environment, which indicates that the SHF is not the solution of a stochastic PDE driven by white noise.

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PIAZZA DEL CALENDARIO 3, 20126, MILAN (IT)
Email address: `a.donadini@campus.unimib.it`

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LAST PASSAGE PERCOLATION IN HIERARCHICAL ENVIRONMENTS

VICTOR GINSBURG

Classification AMS 2020: 60K35, 60K37, 60G60, 82B44

Keywords: Last Passage Percolation, Hierarchical Random Environments, Heavy-Tailed Distributions, Branching Random Walk, Log-Correlated Gaussian Fields, KPZ Universality, Multiscale Analysis

Last passage percolation (LPP) is a model of a directed metric and a zero-temperature polymer where the main observable is a directed path evolving in a random environment accruing as energy the sum of the random weights along itself. When the environment has light tails and a fast decay of correlation, the fluctuations of LPP are predicted to be explained by the Kardar-Parisi-Zhang (KPZ) universality theory. However, the KPZ theory is not expected to apply for many natural environments, particularly ”critical” ones exhibiting a hierarchical structure often leading to logarithmic correlations.

In this article, we initiate a novel study of LPP in such hierarchical environments by investigating two particularly interesting examples. The first is an i.i.d. environment but with a power-law distribution with an inverse quadratic tail decay which is conjectured to be the critical point for the validity of the KPZ scaling relation. The second is the Branching Random Walk which is a hierarchical approximation of the two-dimensional Gaussian Free Field. The second example may be viewed as a high-temperature (weak coupling) directed version of Liouville Quantum Gravity, which is a model of random geometry driven by the exponential of a logarithmically correlated field. Due to the underlying fractal structure, LPP in such environments is expected to exhibit logarithmic correction terms with novel critical exponents. While discussions about such critical models appear in the physics literature, precise predictions about exponents seem to be missing. Developing a framework based on multi-scale analysis, we obtain bounds on such exponents and prove almost optimal concentration results in all dimensions for both models. As a byproduct of our analysis we answer a long-standing question of Martin on necessary and sufficient conditions for the linear growth of the LPP energy in i.i.d. environments.
This is joint work with Shirshendu Ganguly and Kyeongsik Nam.

Further details of these developments can be found in the following research papers:

- https://arxiv.org/abs/2411.08018

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University of California, Berkeley, USA
Email address: victor@math.berkeley.edu

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A STATISTICAL ANALOGUE OF SOLITON RESOLUTION FOR THE FOCUSING SCHRÖDINGER EQUATION

FABIAN HÖFER

Classification AMS 2020: 60H30, 35Q55, 81T08

Keywords: Focusing Gibbs measures, nonlinear Schrödinger equation, phase transition, soliton resolution, Boué-Dupuis formula.

1. THE PHASE TRANSITION

We study the infinite-volume limit of focusing Gibbs measures associated with the one-dimensional nonlinear Schrödinger equation. Let $L\mathbb{T} = \mathbb{R}/(L\mathbb{Z}) \cong [-L/2, L/2]$, let $2 < p < 6$, and set

$$M(u) = \int_{L\mathbb{T}} |u|^2 \, \mathrm{d}x.$$

For $\alpha > 0$, let μ_L^α be the complex Gaussian measure on $L\mathbb{T}$ with inverse covariance $\alpha - \Delta$. Given $K > \alpha^{-1/2}$, $\sigma > 0$, and $\gamma > 0$, we consider

$$\mathrm{d}\rho_L(u) = \frac{1}{Z_L} \exp\left(\frac{\sigma}{pL^\gamma} \int_{L\mathbb{T}} |u|^p \, \mathrm{d}x\right) \mathbf{1}_{\{M(u) \le KL\}} \mathrm{d}\mu_L^\alpha(u). \tag{1.1}$$

The mass cutoff is needed to make the measure normalisable, while the assumption $K > \alpha^{-1/2}$ means that the cutoff lies above the typical mass of the Gaussian field. For $\beta > 0$, we write μ_{OU}^β for the Gaussian measure on $\mathbb{R}$ with inverse covariance $\beta - \Delta$.

Lebowitz, Rose, and Speer [4] introduced these measures with mass cutoff for $L = 1$ and conjectured a phase transition in the infinite-volume limit. Subsequently, Rider [5] studied the infinite volume limit when $\gamma = 0$ and $p = 4$ and showed that the measure concentrates around a single soliton which is uniformly centred. In the limit, the soliton escapes to infinity and the measures converge locally weakly to the trivial measure $\delta_{u=0}$. More recently, Tolomeo and Weber [6] considered the coefficient $L^{-\gamma}$ in front of the nonlinearity and identified

$$\gamma_0(p) = \frac{p}{2} - 1$$

as the critical scaling exponent. If $\gamma < \gamma_0(p)$, the measure concentrates around a single soliton carrying essentially all the available mass KL and the local limit is δ_0. If $\gamma > \gamma_0(p)$, there is no soliton visible and the Gibbs measures converge locally to μ_{OU}^α. At $\gamma = \gamma_0(p)$, they proved convergence to μ_{OU}^α for sufficiently small σ and showed that no limit point is equal to μ_{OU}^α when σ is sufficiently large. In our joint work [3], we consider the critical scaling $\gamma = \gamma_0(p)$, determine a critical value σ_0 at which the phase transition occurs, and describe the limiting measure on both sides of it.

This report is based on joint work with Justin Forlano and Leonardo Tolomeo.

Define

$$A_0 \coloneqq \sup_{\substack{u\in H^1(\mathbb{R})\\ \|u\|_{L^2(\mathbb{R})}=1}} \left\{ \frac{1}{p}\int_{\mathbb{R}} |u|^p \,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}} |\partial_x u|^2 \,\mathrm{d}x \right\}$$

and, for $0 < x \leq K$,

$$\mathscr{F}_\sigma(x) \coloneqq A_0 \sigma^{\frac{4}{6-p}} (K-x)^{\frac{p+2}{6-p}} - \frac{\alpha}{2}(K-x) - \frac{(x\sqrt{\alpha}-1)^2}{2x}. \tag{1.2}$$

Set

$$\sigma_0 \coloneqq \inf\left\{ \sigma > 0 : \sup_{x\in(0,K]} \mathscr{F}_\sigma(x) > 0 \right\}.$$

This threshold has a closed form. Namely, with $c_p \coloneqq \frac{p+2}{6-p} > 1$ and

$$\xi \coloneqq \frac{2}{1 + c_p + \sqrt{(1+c_p)^2 + 4c_p K\sqrt{\alpha}(K\sqrt{\alpha}-2)}},$$

one has

$$\sigma_0 = \left(\frac{1}{2A_0 c_p}\right)^{\frac{6-p}{4}} \frac{1}{K^2 \xi^{\frac{6-p}{2}} (1-\xi)^{\frac{p-2}{2}}}.$$

Theorem 1.1. *Let $2 < p < 6$, $\alpha > 0$, $\sigma > 0$, and $K > \alpha^{-1/2}$, and assume that $\gamma = \gamma_0(p) = \frac{p}{2} - 1$. Then the following hold.*

(i) **Weak nonlinearity:** *If $\sigma < \sigma_0$, then*

$$\rho_L \to \mu^\alpha_{\mathrm{OU}},$$

weakly as $L \to \infty$.

(ii) **Strong nonlinearity:** *If $\sigma > \sigma_0$, then $\mathscr{F}_\sigma$ has a unique maximiser $\bar{x} \in (0, \alpha^{-\frac{1}{2}})$. With $\widetilde{\alpha} \coloneqq \bar{x}^{-2} > \alpha$ we have*

$$\rho_L \to \mu^{\widetilde{\alpha}}_{\mathrm{OU}},$$

weakly as $L \to \infty$. Furthermore, for fixed $K > 1/\sqrt{\alpha}$ and $2 < p < 6$, we have $\widetilde{\alpha}(K,p,\sigma) \to \infty$ as $\sigma \to \infty$.

Moreover, for every $\sigma > 0$, the partition function satisfies

$$\lim_{L\to\infty} \frac{1}{L} \log Z_L = \max\left(0, \sup_{x\in(0,K]} \mathscr{F}_\sigma(x)\right) = \begin{cases} 0, & \sigma \leq \sigma_0, \\ \mathscr{F}_\sigma(\bar{x}), & \sigma > \sigma_0. \end{cases} \tag{1.3}$$

The parameter x can be interpreted as the mass density of the Gaussian background, so that $L(K-x)$ is the mass available for a localised soliton.

The subcritical regime $\sigma < \sigma_0$ is similar to the subcritical case $\gamma > \gamma_0$ from [6]. Here, the nonlinear effects are weak and it is energetically inconvenient for a soliton to form. Thus, the measure looks more and more like the background Gaussian, and we obtain the limiting measure $\mu^\alpha_{\mathrm{OU}} = \lim_{L\to\infty} \mu^\alpha_L$. In the supercritical regime $\sigma > \sigma_0$, the Gibbs measures concentrate around a single soliton Q_L over a Gaussian background $\mu^{\widetilde{\alpha}}_L$. This soliton is again uniformly centred on the torus $L\mathbb{T}$ and thus escapes to infinity, which gives the local limit $\mu^{\widetilde{\alpha}}_{\mathrm{OU}}$. More precisely, we have the following.

Statistical analogue of soliton resolution. In the *supercritical* regime $\sigma > \sigma_0$, the picture changes dramatically. Here, $\sup_{x\in(0,K]} \mathscr{F}_\sigma(x) > 0$ and it is energetically convenient for a soliton to form. More precisely, there exist couplings Γ_L of ρ_L and $\mu_L^{\widetilde{\alpha}}$ such that, for every $\delta > 0$,

$$\Gamma_L\left(\left\{(u,X) : \inf_{x_0\in L\mathbb{T},\theta\in[0,2\pi)} \|u - X - e^{i\theta}Q_L(\cdot - x_0)\|^2_{H^1(L\mathbb{T})} > \delta L\right\}\right) \to 0, \tag{1.4}$$

as $L \to \infty$. Here Q_L denotes a sequence of solitons with height of order $\sqrt{L}$ and width of order one. At leading order, the soliton carries mass $L(K - \bar{x})$, whereas the Gaussian background carries mass $\frac{L}{\sqrt{\widetilde{\alpha}}} = L\bar{x}$. Thus, unlike in the concentrating regimes previously studied by [5] and [6], not all of the available mass KL is carried by the soliton. The Gaussian background plays the role of the radiation term, since $\mu_L^{\widetilde{\alpha}}$ is invariant under the linear Schrödinger flow on $L\mathbb{T}$, and its infinite-volume limit $\mu_{\mathrm{OU}}^{\widetilde{\alpha}}$ is invariant under the linear Schrödinger flow on $\mathbb{R}$. We call this phenomenon *soliton resolution* of the Gibbs measure ρ_L. Since the soliton is uniformly centred on the torus $L\mathbb{T}$, it escapes to infinity in the local limit as $L \to \infty$. Hence the local limit only sees the background radiation $\mu_{\mathrm{OU}}^{\widetilde{\alpha}}$, with renormalised mass parameter $\widetilde{\alpha} > \alpha$. The fact that $\widetilde{\alpha} = \widetilde{\alpha}(K, p, \sigma)$ is independent of the original Gaussian mass parameter α suggests that an analogous result may hold for the corresponding canonical ensemble at the critical scaling.

2. Proof strategy

The main tool in the proof is the Boué-Dupuis variational formula [2], popularised by Barashkov and Gubinelli [1] in the context of constructive quantum field theory. It rewrites $\log Z_L$ as a variational problem over progressively measurable drifts. Moreover, almost optimisers of this variational problem give a quantitative description of the Gibbs measure. We decompose an almost optimiser into a localised component and a non-localised component. The localised component is controlled by a deterministic variational problem whose non-zero optimisers are solitons. Together with a large-deviation estimate for the mass of the Gaussian field, this reduces the leading-order analysis to the maximisation of $\mathscr{F}_\sigma$. A further analysis of the almost optimisers yields the coupling statement, while translation invariance gives the local limits.

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Mathematics Münster, University of Münster, Orléans-Ring 10, 48149 Münster, Germany
Email address: fabian.hoefer@uni-muenster.de

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HAIRER-QUASTEL UNIVERSALITY FOR KPZ – POLYNOMIAL SMOOTHING MECHANISMS, GENERAL NONLINEARITIES AND POISSON NOISE

FANHAO KONG

Classification AMS 2020: 60H15, 60K35, 60F17, 82C22

Keywords: KPZ Equation, Hairer–Quastel Universality, Weak Universality, Regularity Structures, Polynomial Smoothing Mechanisms, Non-Gaussian Poisson Noise, Spectral Gap Inequality, Malliavin Calculus

We consider a class of weakly asymmetric continuous microscopic growth models with polynomial smoothing mechanisms, general nonlinearities and Poisson-type noise. We show that they converge to the KPZ equation after proper rescaling and re-centering, where the coupling constant depends nontrivially on all details of the smoothing and growth mechanisms in the microscopic model. This confirms earlier predictions concerning KPZ universality and provides a first example of Hairer–Quastel type with both a generic nonlinearity (non-polynomial) and non-Gaussian noise.

The proof builds on the general discretisation framework of regularity structures and employs the spectral gap inequality to control stochastic objects, together with a new observation on the specific structure of the (discrete) Malliavin derivatives in this setting. This structure enables us to reduce the control of mixed L^p space-time norms (for arbitrarily large p) to that of certain L^2 space-time norms.

This is joint work with Haiyi Wang and Weijun Xu.
Further details of these developments can be found in the following research papers:

- https://arxiv.org/abs/2403.06191

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PEKING UNIVERSITY, CHINA
Email address: fanhaokong@pku.edu.cn

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AN FBSDE CONSTRUCTION OF THE SINE-GORDON EQFT FOR $\beta^2 < \frac{6}{7}\,8\pi$ AND PERTURBATIVE RENORMALIZATION IN THE FULL SUBCRITICAL REGIME

SARAH-JEAN MEYER

Classification AMS 2020: 81S20, 60H30

Keywords: Sine-Gordon Model, Euclidean Quantum Field Theory, Forward-Backward Stochastic Differential Equations, Stochastic Control, Renormalisation Group, Multipole Mayer Expansion, Subcritical Regime

We construct the sine-Gordon measure and its renormalized effective potential in finite volume up to the seventh threshold. The construction relies on a weak stochastic control problem, its associated weak FBSDE, and an analysis of the renormalization flow equation using a multipole Mayer expansion. The analysis is fully inductive, and we include a complete order-by-order analysis in the subcritical regime $\beta^2 < 8\pi$.

Further details of these developments can be found in the following research paper:

- https://arxiv.org/abs/2607.20632

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UNIVERSITY OF OXFORD, UK
Email address: sarah-jean.meyer@maths.ox.ac.uk