SCIENTIFIC REPORTS

Statistical Mechanics and Singular SPDEs

04 May 2026–22 May 2026

Organizing Committee

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FLUCTUATIONS IN THE WEAKLY COUPLED 4D ANDERSON HAMILTONIAN

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Classification AMS 2020: 60H15; 35R60.

Keywords: Scaling critical singular SPDEs; Anderson model; BPHZ renormalisation.

We study the effect of time-independent white noise disorder on a Brownian particle in four dimensions. The problem can be reformulated as a scaling-critical singular SPDE. By tuning the noise strength appropriately, we obtain an exact analysis of the perturbative terms arising in the corresponding Green function. This gives a first systematic, though still perturbative, renormalisation analysis for a scaling-critical singular SPDE. This report is based on the joint work [1] with Tommaso Rosati (University of Warwick).

1. Setup and main result

We study the elliptic Anderson model in resolvent form

$$(1 - \Delta)v = \lambda\xi v + \delta_y, \qquad x, y \in \mathbb{T}^4, \tag{1.1}$$

where ξ is space white noise. The Anderson Hamiltonian was introduced as a model for quantum particles in a random potential [2]. In the continuum, the equation with white-noise potential is well understood only in subcritical dimensions $d < 4$ [3, 4]. These constructions fit the general paradigm of singular SPDEs, where one removes local divergences by renormalisation and then solves a subcritical fixed point problem. See for example the theory of regularity structures [5], paracontrolled distributions [6], or the flow approach [7]. Dimension $d = 4$ is the borderline case for the continuum Anderson Hamiltonian. Indeed, under the spatial rescaling $x \mapsto x/\zeta$, the Laplacian scales like ζ^{-2}, while white noise has homogeneity $-d/2$. Thus the two terms Δv and ξv have the same scaling strength exactly when $d = 4$. Hence, the equation is critical: the random potential is no longer a lower-order perturbation of the Laplacian, as it is in $d < 4$.

Without further regularisation (1.1) is ill–posed. We therefore study a mollified resolvent equation

$$(1 - \Delta)v_\varepsilon = \rho_\varepsilon * (\lambda_\varepsilon \xi v_\varepsilon + \delta_y) \ ,$$

where ρ_ε is a smooth radial mollifier on $\mathbb{T}^4$. We tune the coupling constant according to the small–scale cutoff and set

$$\lambda_\varepsilon = \hat{\lambda} \left(\log \frac{1}{\varepsilon} \right)^{-1/2} .$$

Writing G for the Green function of $1 - \Delta$ on $\mathbb{T}^4$, and $G_\varepsilon = \rho_\varepsilon * G$, the Wild expansion is interpreted as a formal series

$$v_\varepsilon(x, y) \text{ “ = ” } G_\varepsilon(x - y) + \sum_{n \geq 1} \lambda_\varepsilon^n \mathcal{I}_{n,\varepsilon}(x, y) \in \mathbb{R}[[\hat{\lambda}]] \,,$$

where each $\mathcal{I}_{n,\varepsilon}$ denotes a sum of iterated stochastic integrals in the n-th inhomogeneous Wiener chaos, generated by n occurrences of the noise ξ. The terms $\mathcal{I}_{n,\varepsilon}$ contain ultraviolet divergences: already for $n = 2$ one sees the "tadpole" divergence

$$\mathbb{E}\mathcal{I}_{2,\varepsilon}(x,y) = G_\varepsilon(0)(G_\varepsilon * G_\varepsilon)(x-y)\,,$$

or for $n = 4$ the "sunset" diagram

$$G_\varepsilon * G_\varepsilon^3 * G_\varepsilon(x-y)\,,$$

which both diverge like ε^{-2}. Higher terms contain products and nested versions of such divergences. These polynomial divergences have to be removed by a renormalisation map $\mathcal{R}_\varepsilon$, analogous to the BPHZ renormalisation theory for subcritical singular SPDEs [8, 9]. Formally, this corresponds to the counterterm equation

$$(1-\Delta)\mathcal{R}_\varepsilon v_\varepsilon = \rho_\varepsilon * (\lambda_\varepsilon(\xi - c_\varepsilon)\mathcal{R}_\varepsilon v_\varepsilon + \delta_y)\,, \qquad c_\varepsilon \sim \lambda_\varepsilon \varepsilon^{-2}. \tag{1.2}$$

The following statement concerns the renormalised terms and the convergence of the Wild expansion as a formal power series, thus providing a candidate solution of (1.2) in a subcritical regime.

Theorem 1.1 (G.–Rosati). *Let $\mathcal{J}$ be the centred Gaussian distribution on $(\mathbb{T}^4)^2$ whose covariance kernel is formally given by*

$$\mathbf{E}\left[\mathcal{J}(x,y)\mathcal{J}(x',y')\right] = \int_{\mathbb{T}^4} G(x-z)G(y-z)G(x'-z)G(y'-z)\,\mathrm{d}z.$$

For every $n \geq 1$ there is a constant $\sigma_n > 0$ such that

$$\left(\lambda_\varepsilon^{n-1}\mathcal{R}_\varepsilon \mathcal{I}_{n,\varepsilon}\right)_{n\geq 1} \Longrightarrow \left(\sigma_n \hat{\lambda}^{n-1}\mathcal{J}^{(n)}\right)_{n\geq 1}\,, \tag{1.3}$$

in distribution in $\mathcal{S}'((\mathbb{T}^4)^2;\mathbb{R})^{\mathbb{N}}$. Here each $\mathcal{J}^{(n)}$ has the same marginal law as $\mathcal{J}$, but the fields $\mathcal{J}^{(n)}$ are not independent. Moreover, if $|\hat{\lambda}| < \sqrt{2}\pi$, then

$$\mathcal{U} = \sum_{n\geq 1} \sigma_n \hat{\lambda}^{n-1}\mathcal{J}^{(n)}$$

converges almost surely as a random distribution and satisfies

$$\mathcal{U} \stackrel{d}{=} \sigma_{\mathrm{eff}}(\hat{\lambda})\mathcal{J}\,, \qquad \sigma_{\mathrm{eff}}^2(\hat{\lambda}) = \frac{2\pi^2}{2\pi^2 - \hat{\lambda}^2}.$$

The theorem shows that below the threshold $|\hat{\lambda}| = \sqrt{2}\pi$, the perturbative fluctuations around the Laplacian Green function are Gaussian and vanish as $\varepsilon \to 0$. This is reflected by the normalisation in (1.3), where we have $\lambda_\varepsilon^{n-1}$ instead of λ_ε^n. More precisely, formally we have

$$\mathcal{R}_\varepsilon v_\varepsilon - G_\varepsilon = \lambda_\varepsilon \mathcal{U} + o(\lambda_\varepsilon)\,.$$

The variance of $\mathcal{U}$ diverges at the critical threshold, which is the perturbative signature of a change of fluctuation behaviour. This also justifies the choice of weakly tuned coupling constant λ_ε, since the above theorem suggests that the variance blowup at the critical threshold may compensate the vanishing fluctuations, leading to fluctuations of the same order as G_ε which are expected to be non–Gaussian. A closely related transition has been observed for the two-dimensional stochastic heat equation driven by space-time white noise [10, 11]. Thanks to the independent resampling of noise over time, the multiplicative stochastic heat equation admits an explicit Itô chaos expansion of the solution, while in the present Anderson model the perturbative expansion contains ultraviolet divergent contractions that must be compensated by a more involved BPHZ renormalisation.

2. Idea of the proof

The proof in [1] turns the stochastic problem into a combinatorial and analytic statement about Feynman diagrams. First, moments of stochastic integrals $\mathcal{I}_{n,\varepsilon}$ are written diagrammatically as sums over pairings. Each pairing gives a deterministic Anderson–Feynman diagram, whose vertices have degree four and whose edges encode Green kernels. BPHZ renormalisation introduces a consistent subtraction scheme at the level of such Feynman diagrams, which removes polynomial ultraviolet divergences as $\varepsilon \to 0$. Logarithmic zero-degree divergences are left unsubtracted and are instead balanced by the weak-coupling scaling.

The key classification of divergences within a Feynman diagram Γ is by the degree of full subdiagrams $\overline{\Gamma} = (\overline{V}, \overline{E}) \subset \Gamma$, which is defined by

$$\deg \overline{\Gamma} = 4(|\overline{V}| - 1) - 2|\overline{E}| \,.$$

Positive Anderson-Feynman diagrams (that is diagrams that only contain full subdiagrams of positive degree) have uniformly bounded integrals and disappear after the weak-coupling scaling. Negative diagrams (that is diagrams that contain at least one full subdiagram of negative degree) contain ultraviolet divergences of order ε^{-2}. Our BPHZ subtraction cancels the dangerous terms, and we show that the corresponding Taylor remainders vanish in the weak coupling limit. This part also indicates that no Laplacian renormalisation survives for the Anderson model.

The remaining diagrams, so-called zero-degree diagrams, are the only possible contributors. Our detailed analysis shows that a zero-degree connected diagram contributes precisely when it is a connected *nested bubble diagram* that contains nested occurrences of the kernel $G_\varepsilon^2(x-y)$ (which we call a primitive bubble). Equivalently, one can successively contract these primitive bubbles within an Anderson-Feynman diagram until only the external legs remain. Each such primitive bubble contributes one logarithm after integration, and the weak-coupling factor balances it by

$$\lambda_\varepsilon^2 \log \tfrac{1}{\varepsilon} \longrightarrow \hat{\lambda}^2.$$

This classification has two consequences. First, a nested bubble diagram necessarily has four external legs, so every limiting cumulant of order larger than two vanishes. Together with tightness and the fourth moment theorem, this gives Gaussianity. Second, the exact contribution of each covariance diagram is given by the number of its bubble extraction sequences, depending on how they are nested within the diagram. Counting and summing these contributions yields

$$\sigma_{\text{eff}}^2(\hat{\lambda}) = \sum_{m \geq 0} \Big(\frac{\hat{\lambda}^2}{2\pi^2}\Big)^m,$$

which is the geometric series corresponding to the formula in the theorem.

The open problem left by the perturbative analysis of the theorem above is to prove that the full renormalised resolvent is approximated by a suitable truncation of the expansion, as was possible in a different critical problem for the Allen–Cahn equation with random initial data [12].

References

[1] S. Gabriel and T. Rosati. Fluctuations in the weakly coupled 4D Anderson Hamiltonian. *arXiv preprint arXiv:2602.22509*, 2026.

[2] P. W. Anderson. Absence of diffusion in certain random lattices. *Phys. Rev.*, 109(5):1492–1505, 1958.

[3] R. Allez and K. Chouk. The continuous Anderson hamiltonian in dimension two. *arXiv preprint arXiv:1511.02718*, 2015.

[4] C. Labbé. The continuous Anderson Hamiltonian in $d \leq 3$. *J. Funct. Anal.*, 277(9):3187–3235, 2019.

[5] M. Hairer. A theory of regularity structures. *Invent. Math.*, 198(2):269–504, 2014.

[6] M. Gubinelli, P. Imkeller, and N. Perkowski. Paracontrolled distributions and singular PDEs. *Forum Math. Pi*, 3:e6, 75, 2015.

[7] P. Duch. Flow equation approach to singular stochastic PDEs. *Probab. Math. Phys.*, 6(2):327–437, 2025.

[8] A. Chandra and M. Hairer. An analytic BPHZ theorem for regularity structures. *arXiv preprint arXiv:1612.08138*, 2016.

[9] M. Hairer. An analyst's take on the BPHZ theorem. In *Computation and combinatorics in dynamics, stochastics and control*, volume 13 of *Abel Symp.*, pages 429–476. Springer, Cham, 2018.

[10] F. Caravenna, R. Sun, and N. Zygouras. Universality in marginally relevant disordered systems. *Ann. Appl. Probab.*, 27(5):3050–3112, 2017.

[11] F. Caravenna, R. Sun, and N. Zygouras. The critical 2d stochastic heat flow. *Invent. Math.*, 233(1):325–460, 2023.

[12] S. Gabriel, T. Rosati, and N. Zygouras. The Allen–Cahn equation with weakly critical random initial datum. *Probab. Theory Related Fields*, 192(3–4):1373–1446, 2024.

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FERNIQUE-TYPE BOUNDS FOR BPHZ MODELS AND THEIR APPLICATIONS

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Classification AMS 2020: 60H15; 60F10.

Keywords: singular SPDEs; regularity structures; transportation cost inequalities; large deviations.

This note is based on joint work with Ismaël Bailleul and Ryoji Takano [1] on concentration estimates for BPHZ models. In the theory of regularity structures, the random input of a singular SPDE is first lifted to a renormalized *model*, and the solution is then obtained by a deterministic continuous map of models. It is therefore natural to ask which concentration properties of the noise survive this nonlinear and renormalized lift. The answer is positive for a broad class of parabolic singular SPDEs: a Talagrand-type transportation cost inequality for the noise yields both Talagrand-type inequalities and Gaussian Fernique-type bounds for the BPHZ model.

We consider equations of the form

$$(\partial_t - \Delta)u = b(u, \nabla u) + \sigma(u, \nabla u)\xi \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^d. \tag{0.1}$$

Let Ω be a separable Banach space of space-time distributions and let $H \subset \Omega$ be a continuously and densely embedded Hilbert subspace, typically $H = H^{s_0}$. We assume that the noise is the identity map on Ω, that its law $\mathbb{P}$ is centered and invariant under space-time translations, and that $\mathbb{P}$ satisfies the Talagrand-type transportation cost inequality

$$a_0\, \mathcal{W}_{c_H}(\mathbb{P}, \mathbb{Q}) \le \mathcal{H}(\mathbb{Q} \mid \mathbb{P}), \tag{0.2}$$

for any Borel probability measures $\mathbb{Q}$ on Ω and for some $a_0 > 0$, where

$$c_H(\omega_1, \omega_2) = \begin{cases} \|h\|_H^2, & \omega_1 - \omega_2 = h \in H, \\ +\infty, & \text{otherwise.} \end{cases}$$

The c_H-transportation cost between $\mathbb{P}$ and $\mathbb{Q}$ is defined as $\mathcal{W}_{c_H}(\mathbb{P}, \mathbb{Q}) = \inf E[c_H(X_1, X_2)]$, for an infimum over the set of all random variables defined on some probability space such that X_1 has law $\mathbb{P}$ and X_2 has law $\mathbb{Q}$. The relative entropy $\mathcal{H}(\mathbb{Q} \mid \mathbb{P})$ of $\mathbb{Q}$ with respect to $\mathbb{P}$ is defined as $\int_\Omega \log(\frac{d\mathbb{Q}}{d\mathbb{P}})d\mathbb{Q}$ if $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$, otherwise we set it equal to ∞. This assumption includes Gaussian noises with H their Cameron-Martin space and implies a spectral gap inequality in the H-directions.

Let $\mathcal{F}$ be the finite set of "rooted decorated trees" that spans the structure space of the regularity structure associated with the equation (0.1). Each symbol τ carries a regularity $r(\tau)$ and contains $n(\tau)$ occurrences of the noise. (See [3, Section 8] for details.) We work under two standard hypotheses. The first is *local subcriticality*, which says that the set of relevant trees is locally finite in regularity. As the second one, we assume that the phenomenon of *variance blow-up*, in the sense of Hairer [4], does not occur. This condition ensures that the BPHZ renormalized model considered in [2] is well-defined. A model is a pair (Π, Γ), where

$$\Pi = \{\Pi_x : \mathcal{F} \to \mathcal{S}'\}_{x\in\mathbb{R}^{1+d}}$$

is a family of maps and Γ is the corresponding reexpansion map. The BPHZ model is constructed by first replacing the noise ξ with a smooth approximation ξ_ε, lifting it canonically to a model, subtracting the appropriate counterterms, and then taking the limit. The homogeneous model norm on a compact set $K \subset \mathbb{R}^{1+d}$ is defined by

$$\|M\|_{\mathrm{hom};K} := \max_{\tau\in\mathcal{F}} \sup_{\lambda\in(0,1],\, \varphi\in\mathcal{B}_N} \left(\lambda^{-r(\tau)} \sup_{x\in K} |(\Pi_x\tau)(\varphi_x^\lambda)|\right)^{1/n(\tau)}, \tag{0.3}$$

where $\mathcal{B}_N$ denotes the set of all smooth functions φ supported in the unit ball and satisfying $\|\varphi\|_{C^N} \le 1$. (For simplicity, we ignore the norm for the reexpansion map Γ here.) We also define the metric $\|M;\bar{M}\|_{\mathrm{hom};K}$ by replacing Π_x with $\Pi_x - \bar{\Pi}_x$. The exponent $1/n(\tau)$ records the polynomial degree in the noise and is essential for our main theorem.

Theorem 0.1 ([1, Theorem 2]). *Assume the local subcriticality and the no-variance-blow-up condition. Let M^ξ be the BPHZ model and put $m = \max_{\tau\in\mathcal{F}} n(\tau)$. Then, for every compact K,*

$$\|M^{\xi+h}; M^\xi\|_{\mathrm{hom};K} \le C(\xi)\big(\|h\|_H \vee \|h\|_H^{1/m}\big) \tag{0.4}$$

for all $h \in H$ almost surely, with $C(\xi) \in \cap_{p\ge1} L^p(\mathbb{P})$. Consequently, the law P of M^ξ satisfies Talagrand-type transportation cost inequalities

$$a_1\, \mathcal{W}_{\|\cdot;\cdot\|_{\mathrm{hom};K}}(P,Q) \le \mathcal{H}(Q \mid P)^{1/2} + \mathcal{H}(Q \mid P)^{1/(2m)}$$

for any Borel probability measures Q on the space of models and for some $a_1 > 0$. In particular, there is $\kappa > 0$ such that

$$\mathbb{E}\big[\exp\{\kappa\|M^\xi\|_{\mathrm{hom};K}^2\}\big] < \infty. \tag{0.5}$$

The result applies directly to solutions of (0.1). If the deterministic solution map is globally well-posed and satisfies an a priori estimate in terms of the homogeneous model norm, then the law of the solution inherits a Talagrand-type inequality and the corresponding exponential integrability. If the estimate is independent of the initial condition, invariant measures inherit the same concentration property. A second application is a large deviation principle in the small-noise regime

$$(\partial_t - \Delta)u^\varepsilon = b(u^\varepsilon, \nabla u^\varepsilon) + \varepsilon\sigma(u^\varepsilon, \nabla u^\varepsilon)\xi.$$

Whenever $\varepsilon\xi$ satisfies an LDP with rate ε^2 and rate function $\frac{1}{2}\|h\|_H^2$, the BPHZ models and, by continuity, the corresponding SPDE solutions satisfy the natural contracted LDP. This extends the Hairer–Weber large deviation framework to the present general BPHZ setting.

References

[1] I. Bailleul, M. Hoshino and R. Takano. Transportation cost inequalities for singular SPDEs. Preprint, arXiv:2511.19216.

[2] Y. Bruned, M. Hairer and L. Zambotti. Algebraic renormalisation of regularity structures. *Invent. Math.* **215** (2019), 1039–1156.

[3] M. Hairer. A theory of regularity structures. *Invent. Math.* **198** (2014), 269–504.
[4] M. Hairer. Renormalisation in the presence of variance blowup. *Ann. Probab.* **53** (2025), 1958–1985.
[5] M. Hairer and H. Weber. Large deviations for white-noise driven, nonlinear stochastic PDEs in two and three dimensions. *Ann. Fac. Sci. Toulouse Math.* **24** (2015), 55–92.

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EQUIVALENCE OF FLUCTUATIONS FOR THE SHE AND KPZ EQUATION IN THE SUBCRITICAL WEAK DISORDER REGIME

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2020 Mathematics Subject Classification: 60H15, 60K37, 60F05, 82D60.

Keywords: stochastic heat equation, KPZ equation, directed polymers, weak disorder, fluctuations.

This extended abstract is based on joint work with Stefan Junk (Gakushuin University).

The Kardar–Parisi–Zhang equation is the formal stochastic PDE

$$\partial_t h(t,x) = \frac{1}{2}\Delta h(t,x) + \frac{1}{2}|\nabla h(t,x)|^2 + \beta\xi(t,x),$$

where ξ denotes space-time white noise. Since ξ is a Schwartz distribution, the nonlinear term must be defined by a regularization and renormalization procedure. If the noise is mollified in space at scale ϵ by $\xi^\epsilon = \phi^\epsilon * \xi$, one considers

$$\partial_t h_\epsilon(t,x) = \frac{1}{2}\Delta h_\epsilon(t,x) + \frac{1}{2}|\nabla h_\epsilon(t,x)|^2 + \beta_\epsilon\xi^\epsilon(t,x) - C_\epsilon.$$

The Cole–Hopf transform $u_\epsilon = \exp(h_\epsilon)$ maps this equation to the multiplicative stochastic heat equation

$$\partial_t u_\epsilon(t,x) = \frac{1}{2}\Delta u_\epsilon(t,x) + \beta_\epsilon\xi^\epsilon(t,x)u_\epsilon(t,x) - C_\epsilon u_\epsilon(t,x).$$

With flat initial data, the Feynman–Kac formula identifies u_ϵ with the partition function of a continuum directed polymer [7]. Thus the relation between the SHE and KPZ equation can be studied by comparing a partition function with its logarithm.

We first recall the known phase diagram. In $d = 1$, Bertini and Giacomin constructed the Cole–Hopf solution of the KPZ equation as the scaling limit of weakly asymmetric particle systems [1]. In $d = 2$, the mollified KPZ equation has a nontrivial subcritical regime: Caravenna, Sun, and Zygouras proved convergence for $\hat\beta < 1$ and divergence to $-\infty$ for $\hat\beta > 1$ under the logarithmic scaling of the coupling constant [2]. In dimensions $d \geq 3$, the natural weak-noise scaling is $\beta_\epsilon = \hat\beta\epsilon^{(d-2)/2}$ [7, 3]. There are critical parameters $0 < \hat\beta_{L^2} \leq \hat\beta_c$. For $\hat\beta < \hat\beta_c$, the SHE partition functions satisfy a law of large numbers, whereas for $\hat\beta > \hat\beta_c$ they converge to zero [7, 3]. In the L^2 region, $\hat\beta < \hat\beta_{L^2}$, the centered fluctuations of u_ϵ and $h_\epsilon = \log u_\epsilon$ converge to Edwards–Wilkinson type Gaussian limits [4, 3]. The main motivation is the weak disorder region beyond the L^2 region, where non-Gaussian fluctuations are expected.

The main results are joint work with Stefan Junk and are stated for the discrete directed polymer model. Let $(\omega(n,x))_{n\in\mathbb{N},x\in\mathbb{Z}^d}$ be i.i.d. random variables with compact support, and set

$$\lambda(\beta) = \log \mathbb{E}[e^{\beta\omega(n,x)}].$$

For the simple random walk (X_k) on $\mathbb{Z}^d$, define the polymer partition function starting from x by

$$Z_N(x) = \mathrm{E}_x\left[\exp\left(\beta\sum_{k=1}^{N}\omega(k, X_k) - N\lambda(\beta)\right)\right].$$

For brevity, we write $Z_N = Z_N(0)$. The weak disorder regime is the set of β for which $\mathbb{P}(\liminf_N Z_N > 0) = 1$. Let

$$p_*(\beta) = \sup\{p \geq 0 : \sup_N \mathbb{E}[Z_N^p] < \infty\}$$

be the moment exponent. Junk proved that throughout weak disorder one has $p_*(\beta) \geq (d+2)/d$, and for a large class of finite-support environments the strict inequality $p_*(\beta) > (d+2)/d$ is equivalent to $\beta < \beta_c$ [5].

For a compactly supported test function $f : \mathbb{R}^d \to \mathbb{R}$, define the discrete SHE fluctuation

$$\chi_N(f) = N^{-d/2}\sum_{x\in\mathbb{Z}^d} f(x/\sqrt{N})(Z_N(x) - \mathbb{E}Z_N(x)).$$

Put

$$\xi(\beta) = -\frac{d}{2} + \frac{d+2}{2(p_*(\beta)\wedge 2)}.$$

Junk proved that, in weak disorder, for every $\epsilon > 0$ and $f \not\equiv 0$ [5],

$$\mathbb{P}\left(N^{-\xi(\beta)-\epsilon} < |\chi_N(f)| < N^{-\xi(\beta)+\epsilon}\right) \longrightarrow 1.$$

We then focused on the corresponding KPZ fluctuation

$$\kappa_N(f) = N^{-d/2}\sum_{x\in\mathbb{Z}^d} f(x/\sqrt{N})\left(\log Z_N(x) - \mathbb{E}\log Z_N(x)\right).$$

Our main theorem asserts that, in the subcritical weak disorder regime, the two fluctuation fields are asymptotically equivalent up to an error of lower polynomial order than their common magnitude [6]. More precisely, if $p_*(\beta) > (d+2)/d$, then there exists $\delta > 0$ such that

$$\mathbb{P}\left(|\chi_N(f) - \kappa_N(f)| \leq N^{-\xi(\beta)-\delta}\right) \longrightarrow 1.$$

Consequently,

$$\mathbb{P}\left(\frac{|\chi_N(f) - \kappa_N(f)|}{\min\{|\chi_N(f)|, |\kappa_N(f)|\}} \leq N^{-\delta}\right) \longrightarrow 1.$$

In particular, the SHE and KPZ fluctuations have the same polynomial order. This is significant because in the region $\beta_{L^2} < \beta < \beta_c$ the SHE fluctuation is expected to have a stable, non-Gaussian limit, whereas the logarithm has stronger moment and covariance estimates than the partition function itself. The theorem shows that this regularizing effect of the logarithm does not change the leading fluctuation scale.

The common approximation is a martingale argument built from a short backward polymer near the observation time. Let $\ell_k = k^{1/8}$ and

$$E_k(y) = e^{\beta\omega(k,y) - \lambda(\beta)} - 1.$$

If $\overleftarrow{Z}_{[k-\ell_k,k)}((k,y))$ denotes the backward partition function ending at (k,y) over the time interval $[k-\ell_k,k)$ and $p_N(x,y)$ is the simple random walk transition probability, set

$$\rho_N(f) = N^{-d/2} \sum_{x\in\mathbb{Z}^d} f(x/\sqrt{N}) \sum_{k=1}^{N} \sum_{y\in\mathbb{Z}^d} \overleftarrow{Z}_{[k-\ell_k,k)}((k,y)) E_k(y) p_N(x,y).$$

It is proved separately that $\chi_N(f)$ and $\kappa_N(f)$ are both close to $\rho_N(f)$ at the required precision [6].

For $\chi_N(f)$, this follows from expanding $Z_N(x)-1$ according to the first time at which the environment is sampled, followed by a local limit theorem and a coarse-graining estimate for point-to-point polymer partition functions. For $\kappa_N(f)$, one uses a martingale decomposition of $\log Z_N(x) - \mathbb{E}\log Z_N(x)$ along the time coordinate. The increment is

$$\log \frac{Z_k(x)}{Z_{k-1}(x)} = \log\left(1 + \sum_{y\in\mathbb{Z}^d} \mu_k(x,y) E_k(y)\right),$$

where $\mu_k(x,y)$ is the polymer endpoint distribution at time k. The logarithm can be linearized with an error controlled by lower-tail estimates for Z_N. The endpoint distribution is then replaced by

$$\overleftarrow{Z}_{[k-\ell_k,k)}((k,y)) p_k(x,y),$$

using a point-to-point factorization result for directed polymers. The main inputs are moment estimates governed by $p_*(\beta)$ and a stretched Gaussian lower-tail bound of the form

$$\mathbb{P}(Z_N < 1/u) \le C\exp\{-(\log u)^2/C\}, \qquad u \ge 1.$$

These estimates make it possible to compare the nonlinear KPZ fluctuations with the linear SHE fluctuations throughout the subcritical weak disorder region [6].

References

[1] L. Bertini and G. Giacomin. Stochastic Burgers and KPZ equations from particle systems. *Comm. Math. Phys.*, 183:571–607, 1997.

[2] F. Caravenna, R. Sun, and N. Zygouras. The two-dimensional KPZ equation in the entire subcritical regime. *Ann. Probab.*, 48:1086–1127, 2020.

[3] C. Cosco, S. Nakajima, and M. Nakashima. Law of large numbers and fluctuations in the sub-critical and L^2 regions for SHE and KPZ equation in dimension $d \ge 3$. *Stochastic Process. Appl.*, 2022.

[4] Y. Gu, L. Ryzhik, and O. Zeitouni. The Edwards–Wilkinson limit of the random heat equation in dimensions three and higher. *Comm. Math. Phys.*, 2018.

[5] S. Junk. Fluctuations of partition functions of directed polymers in weak disorder beyond the L^2-phase. *Ann. Probab.*, 2025.

[6] S. Junk and S. Nakajima. Equivalence of fluctuations of discretized SHE and KPZ equations. *Probab. Math. Phys.*, 2025.

[7] D. Mukherjee, A. Shamov, and O. Zeitouni. Weak and strong disorder for the stochastic heat equation and continuous directed polymers in $d \ge 3$. *Electron. Commun. Probab.*, 2016.

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MOMENTS OF CRITICAL 2D STOCHASTIC HEAT FLOW

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Classification AMS 2020: Primary 60H17; Secondary 60H15, 60K37, 82B44

Keywords: Critical 2D Stochastic Heat Flow, directed polymers, intermittency, Gaussian free field

The stochastic heat equation with multiplicative space-time white noise is one of the central continuum models in the Kardar–Parisi–Zhang universality class. In spatial dimension two, however, the equation is ill-posed without a delicate renormalization procedure. This dimension is especially subtle: it is neither ultraviolet nor infrared super-renormalizable, and the corresponding polymer models exhibit a weak-to-strong disorder transition at a critical logarithmic scale. A major recent breakthrough of Caravenna, Sun, and Zygouras [2] constructed the scaling limit of the critical $2+1$-dimensional directed polymer partition function. This limiting random measure is called the critical two-dimensional stochastic heat flow, or *critical 2D SHF*. We investigate a fundamental manifestation of its intermittency: the asymptotic growth of high moments and the resulting upper-tail behavior.

Let $\mathscr{Z}^\theta$ denote the critical 2D SHF with parameter $\theta \in \mathbb{R}$. For a nonnegative smooth test function φ on $\mathbb{R}^2$, define the scalar observable

$$X_\varphi := \mathscr{Z}_1^\theta(\varphi) = \int_{\mathbb{R}^2} \varphi(x)\, \mathscr{Z}_1^\theta(dx, 1),$$

where the second endpoint of the flow is integrated over the whole plane. This normalization is natural from the polymer point of view, since it corresponds to testing the initial endpoint by φ while keeping the terminal endpoint free. Earlier work established that the critical SHF is singular with respect to Lebesgue measure and is not a Gaussian multiplicative chaos [3, 4]. Nevertheless, it is expected to display rich intermittent behavior reminiscent of other random measures arising in critical field theories and directed polymer models.

The main problem addressed in this paper is the high-moment growth of X_φ. A prediction going back to Rajeev [7] suggested that the h-th moment should grow double exponentially in h, namely on the order of $\exp\{\exp(ch)\}$ for some positive constant c. Prior rigorous bounds were far from this prediction. The best available upper bound, due to Gu, Quastel, and Tsai [5], was of the form

$$\mathbb{E}[X_\varphi^h] \leq \exp\{\exp(Ch^2)\},$$

whereas the known lower bound, obtainable for instance from the Gaussian correlation inequality, was only of order $\exp(ch^2)$ [3].

The first main result gives an exponential improvement of the lower bound and matches the predicted double-exponential scale from below. More precisely, for every fixed $\theta \in \mathbb{R}$ and every smooth nonnegative test function φ with $\varphi(0) > 0$, there exists an absolute constant $c_0 > 0$ such that, for all sufficiently large integers h,

$$\mathbb{E}[X_\varphi^h] \geq \exp\{\exp(c_0 h)\}.$$

This establishes the predicted order of magnitude as a genuine feature of the critical SHF and provides a new quantitative signature of its intermittency.

A key novelty of the proof is a new connection between the moment formula for the critical SHF and the Gaussian free field on certain Feynman diagrams. The moments of the SHF admit an expansion over collision patterns of multiple random walks, as developed in the construction and moment analysis of the critical 2D SHF [1, 2, 5]. Each diagram contains time variables encoding the beginning and end of effective pairwise collision intervals, and spatial variables encoding the corresponding collision locations. After fixing a collision pattern and the associated time variables, the spatial integral can be interpreted as the partition function of a Gaussian free field on a weighted graph naturally associated with the diagram. The conductances of this graph are determined by inverse time gaps between successive collision events.

This observation brings into the problem the algebraic structure of the Gaussian free field. In particular, the Gaussian integral over spatial variables is expressed through the determinant of a weighted graph Laplacian. By Kirchhoff's Matrix–Tree theorem, this determinant is equal to the weighted sum of spanning trees of the corresponding graph. Thus the estimation of SHF moments is reduced to a combinatorial problem of controlling weighted spanning trees on Feynman diagrams. The dominant contribution comes from diagrams with exponentially many collision events in the moment order h, leading precisely to the $\exp\{\exp(ch)\}$ lower bound.

The second main consequence concerns upper-tail asymptotics. Combining the new moment lower bound with the best known moment upper bound, it is obtained that for compactly supported smooth nonnegative φ with $\varphi(0) > 0$, the tail of X_φ satisfies, for large z,

$$\exp\left\{-(\log z)^{(\log\log z)^{1+o(1)}}\right\} \leq \mathbb{P}(X_\varphi > z) \leq \exp\left\{-\Omega(1)\,\log z\,\sqrt{\log\log z}\right\}.$$

Thus the upper tail is barely super-polynomial: it decays faster than any fixed power of z, but much more slowly than a stretched exponential in z. This behavior is markedly different from higher-dimensional weak disorder directed polymers, where limiting partition functions may have power-law tails [6]. The result therefore identifies a distinctive feature of criticality in two dimensions.

References

[1] Francesco Caravenna, Rongfeng Sun, and Nikos Zygouras. On the moments of the $(2+1)$-dimensional directed polymer and stochastic heat equation in the critical window. *Communications in Mathematical Physics*, 372(2):385–440, 2019.

[2] Francesco Caravenna, Rongfeng Sun, and Nikos Zygouras. The critical 2D stochastic heat flow. *Inventiones Mathematicae*, 233(1):325–460, 2023.

[3] Francesco Caravenna, Rongfeng Sun, and Nikos Zygouras. The critical 2D stochastic heat flow is not a Gaussian multiplicative chaos. *Annals of Probability*, 51(6):2265–2300, 2023.

[4] Francesco Caravenna, Rongfeng Sun, and Nikos Zygouras. Singularity and regularity of the critical 2D stochastic heat flow. *arXiv preprint arXiv:2504.06128*, 2025.

[5] Yu Gu, Jeremy Quastel, and Li-Cheng Tsai. Moments of the 2D SHE at criticality. *Probability and Mathematical Physics*, 2(1):179–219, 2021.

[6] Stefan Junk and Hubert Lacoin. The Tail Distribution of the Partition Function for Directed Polymers in the Weak Disorder Phase. *Communications in Mathematical Physics*, 406(48), 2025.
[7] S. G. Rajeev. A condensation of interacting bosons in two dimensional space. *arXiv preprint hep-th/9905120*, 1999.

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GLOBAL WELL-POSEDNESS FOR $\exp(\Phi)$

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Classification AMS 2020: Primary 60H15; Secondary 60G57, 60L40, 81T08.

Keywords: parabolic Liouville equation; exponential $(\exp(\Phi)_2)$ model; Gaussian multiplicative chaos; global well-posedness; stochastic quantization; Da Prato–Debussche decomposition.

This talk reports on joint work with G. Cannizzaro and T. Galanis on the global-in-time solution theory of the massless parabolic Liouville (or $\exp(\Phi)_2$) equation on the two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$,

$$\partial_t u = \tfrac{1}{4\pi}\Delta u + \alpha - e^{\gamma u} + \xi, \tag{0.1}$$

where ξ is space–time white noise, $\gamma > 0$ a coupling constant, and $\alpha > 0$ a constant drift. Equation (0.1) is the natural Langevin dynamics of the Liouville-type measure formally proportional to $\exp\{-\int_{\mathbb{T}^2} |\nabla\phi|^2 + \alpha \int_{\mathbb{T}^2} \phi - \int_{\mathbb{T}^2} {:}e^{\gamma\phi}{:}\}$, whose exponential interaction is built from Gaussian multiplicative chaos (GMC). Because the field is log-correlated, $e^{\gamma u}$ is almost surely infinite, and (0.1) only makes sense after multiplicative Wick renormalisation, the renormalised nonlinearity being a dynamical GMC.

Following Da Prato–Debussche, we decompose $u = \Psi + v$, where Ψ solves the linear equation $\partial_t\Psi = \frac{1}{4\pi}\Delta\Psi + \xi$ and is a log-correlated Gaussian field, and $\eta := {:}e^{\gamma\Psi}{:}$ is the associated dynamical GMC, a positive distribution of negative Besov regularity. The remainder v then solves a random PDE whose nonlinearity is $\eta\, e^{\gamma v}$.

A central difficulty of the massless model is the zeroth Fourier mode: absent a mass term, it carries no restoring force and is driven by a spatially constant Brownian motion W_t and the drift α. We handle it through a low-mode decoupling, obtaining an effective dynamics for the zeroth mode in which we see nonlinear damping: since $\eta \geq 0$ and the prefactor $e^{\gamma(\alpha t + W_t)} > 0$, the nonlinearity in the equation for v is sign-definite and dissipative.

Main result (informal). Fix γ in the subcritical (Da Prato–Debussche) regime. There exists $\alpha_0 = \alpha_0(\gamma) > 0$ such that, for every drift $0 < \alpha < \alpha_0$ and every initial datum $u_0 \in L^\infty(\mathbb{T}^2)$, the renormalised equation (0.1) has a unique global-in-time solution $u = \Psi + v$. The remainder v is tight in time, and the dynamics admits an invariant probability measure.

Related work. The model was introduced by Garban [1], who solved the classical Da Prato–Debussche phase $\gamma \in [0, 2\sqrt{2} - \sqrt{6})$ and, exploiting positivity of the nonlinearity, obtained a weaker notion of solution up to $\gamma_{\text{pos}} = 2\sqrt{2} - 2$. Oh–Robert–Wang [4] proved local well-posedness for $\gamma^2 < 8\pi/(3 + 2\sqrt{2})$ and, in the defocusing case, global well-posedness for $\gamma^2 < 4\pi$ together with invariance of the Gibbs measure, again using sign-definiteness and positivity of GMC; Oh–Robert–Tzvetkov–Wang [3] treated the stochastic quantization of Liouville conformal field theory on closed surfaces. Hoshino–Kawabi–Kusuoka [2] constructed the $\exp(\phi)_2$ dynamics in the full L^1-regime by Dirichlet-form methods. All of these works consider the stochastic quantisation problem only in the massive case; our contribution is to prove that there exists an invariant measure for the equation in the massless case, under a small-drift assumption.

References

[1] C. Garban. Dynamical Liouville. arXiv:1805.04507 [math], 2019.

[2] M. Hoshino, H. Kawabi, and S. Kusuoka. Stochastic quantization associated with the $\exp(\phi)_2$-quantum field model driven by space-time white noise on the torus in the full L^1-regime. *Probability Theory and Related Fields*, 185(1–2):391–447, 2023.

[3] T. Oh, T. Robert, N. Tzvetkov, and Y. Wang. Stochastic quantization of Liouville conformal field theory. arXiv:2004.04194 [math], 2020.

[4] T. Oh, T. Robert, and Y. Wang. On the parabolic and hyperbolic Liouville equations. *Communications in Mathematical Physics*, 387(3):1281–1351, 2021.

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SCALING LIMIT OF THE 3D ABELIAN YANG–MILLS LANGEVIN DYNAMICS

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Classification AMS 2020: 60H15, 81T13, 82B20, 60F17.

Keywords: abelian Yang–Mills theory, lattice gauge theory, Langevin dynamics, stochastic quantization, scaling limit, singular SPDE.

This talk is based on our ongoing work joint with Ilya Chevyrev and Yahui Qu. It concerned the continuum limit of the Langevin dynamics associated with three-dimensional abelian lattice gauge theory. In the continuum, for structure group $U(1)$ on the torus, a connection is a real-valued one-form A, its curvature is the linear two-form $F_A = dA$, and the Euclidean Yang–Mills action is

$$S(A) = \frac{1}{2}\int_{\mathbb{T}^3} |dA|^2 \, dx.$$

Gauge transformations $A \mapsto A + d\phi$ leave the curvature invariant; after gauge fixing the formal Yang–Mills measure is Gaussian. This linearity is the main distinction from non-abelian Yang–Mills theory, where $F_A = dA + [A, A]$ and the action is no longer quadratic.

On the lattice, a $U(1)$ gauge field assigns to every oriented edge e a variable $U(e) = e^{i\theta(e)}$, with $\theta(e) \in (-\pi, \pi]$ and $U(-e) = U(e)^{-1}$. For a plaquette $p = (e^{(1)}, \dots, e^{(4)})$, the gauge-invariant curvature variable is $U(\partial p) = \exp(i \sum_j \theta(e^{(j)}))$. We consider an even, smooth, 2π-periodic single-plaquette potential h, so that the finite-volume Gibbs measure is

$$\mu_N(d\theta) = Z^{-1} \exp\Big\{ -\beta \sum_p h(\theta(\partial p)) \Big\} \prod_e d\theta(e).$$

This includes the Wilson, Manton, and Villain actions. Under the usual scaling $\beta \mapsto \varepsilon^{d-4}\beta$ and $\theta \mapsto \varepsilon\theta$, the action in dimension $d = 3$ is formally close to the quadratic continuum action.

Let d_ε be the discrete exterior derivative and d_ε^* its adjoint. With the Hodge Laplacian $\Delta_\varepsilon = -d_\varepsilon^* d_\varepsilon - d_\varepsilon d_\varepsilon^*$, the DeTurck-gauge lattice Langevin equation studied in the talk is

$$\partial_t \theta_\varepsilon = -\varepsilon^{-2} d_\varepsilon^* h'(\varepsilon^2 d_\varepsilon \theta_\varepsilon) - d_\varepsilon d_\varepsilon^* \theta_\varepsilon + \xi_\varepsilon.$$

Here ξ_ε is lattice space-time white noise, and the exact DeTurck term does not change the gauge orbit. Normalizing $h''(0) = 1$ and writing

$$h'(z) = z + a_3 z^3 + R(z), \qquad R(z) = O(z^5),$$

one obtains

$$\partial_t \theta_\varepsilon = \Delta_\varepsilon \theta_\varepsilon + a_3 \varepsilon^4 d_\varepsilon^* (d_\varepsilon \theta_\varepsilon)^3 + \varepsilon^{-2} d_\varepsilon^* R(\varepsilon^2 d_\varepsilon \theta_\varepsilon) + \xi_\varepsilon.$$

For example, $a_3 = -1/6$ for the Wilson action. Although the expected limit is linear, these nonlinear error terms are singular uniformly in the lattice spacing.

Main result. Let $d = 3$ and let h be as above. If the interpolated initial data converge in a negative Hölder topology to A_0, then, for every finite T and every $\kappa > 0$, the interpolated solutions converge in law in $C([0, T], \mathcal{C}^{-1/2-\kappa})$ to the solution of

$$\partial_t A = \Delta A + \xi, \qquad A(0) = A_0.$$

Consequently the gauge-fixed continuum limit is Gaussian; in equilibrium its curvature is spatial white noise.

The proof uses a lattice Da Prato–Debussche decomposition $\theta_\varepsilon = \Psi_\varepsilon + v_\varepsilon$, where Ψ_ε solves the discrete stochastic heat equation and v_ε solves the remainder equation. In dimension three, Ψ_ε has spatial regularity $\mathcal{C}^{-1/2-}$, and the cubic term has the same power-counting singularity as the dynamical Φ^4_3 nonlinearity. The crucial simplification is that the powers of ε in the expanded equation can be traded for regularity in discrete Hölder spaces $\mathcal{C}^\alpha_\varepsilon$, which measure scales between ε and 1. Typical deterministic bounds are

$$\|\varepsilon^{3/2+\kappa} d_\varepsilon \Psi_\varepsilon\|_{L^\infty_\varepsilon} \lesssim \|\Psi_\varepsilon\|_{\mathcal{C}^{-1/2-\kappa}_\varepsilon}, \qquad \|\varepsilon^{1-\alpha} d_\varepsilon v_\varepsilon\|_{L^\infty_\varepsilon} \lesssim \|v_\varepsilon\|_{\mathcal{C}^\alpha_\varepsilon}.$$

These estimates allow the nonlinearities to be controlled by elementary L^∞_ε product bounds rather than by a full regularity-structure expansion. The probabilistic estimates are also favorable: uniformly on compact time intervals the stochastic heat solution has the required discrete Hölder bounds, and no Wick renormalization is needed since

$$\varepsilon^3 \mathbf{E}\big[(d_\varepsilon \Psi_\varepsilon(p))^2\big] = O(1).$$

A stopping-time argument then shows $v_\varepsilon \to 0$ in positive discrete Hölder norms, reducing the limit to convergence of the linear discrete stochastic heat equation.

The result fits with Gross's convergence theorem for $U(1)_3$ lattice gauge theory [1] and Driver's four-dimensional abelian convergence theorem at small coupling [2]. It is also related in spirit to weak universality results for singular SPDEs, such as [5]. In comparison with the two-dimensional non-abelian Yang–Mills Langevin theory and its lattice universality results [3, 4], the present three-dimensional abelian problem has a Gaussian limit but still contains singular lattice nonlinearities. Natural future directions include four-dimensional abelian Langevin dynamics and three-dimensional non-abelian lattice models, where genuine Yang–Mills nonlinearities should interact with singular error terms.

References

[1] L. Gross. Convergence of $U(1)_3$ lattice gauge theory to its continuum limit. *Communications in Mathematical Physics*, 92(2):137–162, 1983.

[2] B. K. Driver. Convergence of the $U(1)_4$ lattice gauge theory to its continuum limit. *Communications in Mathematical Physics*, 110(3):479–501, 1987.

[3] A. Chandra, I. Chevyrev, M. Hairer, and H. Shen. Langevin dynamic for the 2D Yang–Mills measure. *Publications Mathématiques de l'IHES*, 136:1–147, 2022.

[4] I. Chevyrev and H. Shen. Invariant measure and universality of the 2D Yang–Mills Langevin dynamic. *Communications on Pure and Applied Mathematics*, 2026. DOI: 10.1002/cpa.70043.

[5] M. Hairer and J. Quastel. A class of growth models rescaling to KPZ. *Forum of Mathematics, Pi*, 6:e3, 2018.

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STRONG-DISORDER ASYMPTOTICS FOR 2D DIRECTED POLYMERS AND STOCHASTIC HEAT FLOW

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Classification AMS 2020: Primary 82B44; Secondary 60K35, 82D60.

Keywords: Change of Measure, Coarse-Graining, Directed Polymer in Random Environment, Disordered Systems, Size Bias, Stochastic Heat Equation, Stochastic Heat Flow.

We consider the critical two-dimensional Stochastic Heat Flow and its discrete approximation by directed polymers in random environment. The aim is to describe their behavior in the strong-disorder and large-time regimes, with particular emphasis on quantitative local extinction and on the spatial scale at which the mass becomes visible. The results presented here are joint work with Quentin Berger and Francesco Caravenna [1].

Let $S = (S_n)_{n\geq 0}$ be the simple symmetric random walk on $\mathbb{Z}^2$, and let $\omega = (\omega(n,x))_{n\in\mathbb{N},\, x\in\mathbb{Z}^2}$ be a family of independent and identically distributed random variables, independent of S, such that

$$\mathbb{E}[\omega] = 0, \qquad \mathbb{E}[\omega^2] = 1, \qquad \lambda(\beta) := \log \mathbb{E}[\mathrm{e}^{\beta\omega}] < \infty \quad \text{for } -2 < \beta < 2.$$

For $N \in \mathbb{N}$, and $x \in \mathbb{Z}^2$, the point-to-plane partition function is

$$Z_N^{\beta,\omega}(x) := \mathbf{E}_x\left[\exp\left(\sum_{n=1}^{N}\big(\beta\omega(n,S_n) - \lambda(\beta)\big)\right)\right].$$

For a probability mass function f on $\mathbb{Z}^2$, set

$$Z_N^{\beta,\omega}(f) := \sum_{x\in\mathbb{Z}^2} f(x) Z_N^{\beta,\omega}(x), \qquad \mathbb{E}[Z_N^{\beta,\omega}(f)] = 1.$$

The relevant parameter measuring the strength of the disorder is determined by the relation between β and N. Define

$$\sigma^2(\beta) := \mathbb{V}\mathrm{ar}\big(\mathrm{e}^{\beta\omega - \lambda(\beta)}\big) = \mathrm{e}^{\lambda(2\beta) - 2\lambda(\beta)} - 1$$

and

$$R_N := \sum_{n=1}^{N} \mathbf{P}(S_{2n} = 0) = \frac{1}{\pi}\big(\log N + \alpha_N\big), \qquad \alpha_N \longrightarrow \alpha := 4\log 2 + \gamma - \pi.$$

The critical window introduced in [2] is characterized by

$$\sigma^2(\beta_N) = \frac{1}{R_N - \frac{\vartheta + o(1)}{\pi}}, \qquad \vartheta \in \mathbb{R}.$$

For arbitrary N and $\beta \in (0,1)$, this suggests defining the effective disorder parameter

$$\vartheta(N,\beta) := \pi R_N - \frac{\pi}{\sigma^2(\beta)}. \tag{0.1}$$

Thus the critical window corresponds to $\vartheta(N,\beta_N) \to \vartheta \in \mathbb{R}$, whereas the supercritical regime is given by $\vartheta(N,\beta_N) \to +\infty$. Moreover,

$$\mathrm{e}^{\vartheta(N,\beta)} = \mathrm{e}^{\alpha_N} N \exp\Big(-\frac{\pi}{\sigma^2(\beta)}\Big). \tag{0.2}$$

The critical 2D Stochastic Heat Flow $\mathscr{Z}^\vartheta$ was constructed in [2] as the universal scaling limit of diffusively rescaled directed-polymer partition functions in this critical window. Denoting its one-time marginal by $\mathscr{Z}_t^\vartheta(\mathrm{d}x)$, for a probability density φ on $\mathbb{R}^2$ we write

$$\mathscr{Z}_t^\vartheta(\varphi) := \int_{\mathbb{R}^2} \varphi(x)\mathscr{Z}_t^\vartheta(\mathrm{d}x), \qquad \mathbb{E}[\mathscr{Z}_t^\vartheta(\varphi)] = 1.$$

Although the mean remains equal to one, this random variable typically vanishes in the strong-disorder or large-time limits; the mean is sustained by increasingly rare and high peaks. Local extinction for fixed disorder and large time was obtained in [3], while strong-disorder local extinction follows from the conditional Gaussian multiplicative chaos structure established in [4]. Our purpose is to identify the extinction rate and to obtain bounds uniform throughout the supercritical regime.

For $r > 0$, let $\mathcal{M}_1^{\mathrm{disc}}(r)$ denote the set of probability mass functions on $\mathbb{Z}^2$ supported in the ball of radius r. There exist constants $c_0, c_1, c_2 \in (0,\infty)$ such that, uniformly over $N \in \mathbb{N}$ and $\beta \in (0,1)$,

$$\frac{1}{c_1}\exp\big(-c_1\mathrm{e}^{\vartheta(N,\beta)}\big) \le \sup_{f\in\mathcal{M}_1^{\mathrm{disc}}(\mathrm{e}^{c_0\mathrm{e}^{\vartheta(N,\beta)}}\sqrt{N})} \mathbb{E}\big[Z_N^{\beta,\omega}(f)\wedge 1\big] \le \frac{1}{c_2}\exp\big(-c_2\mathrm{e}^{\vartheta(N,\beta)}\big). \tag{0.3}$$

The same conclusion holds for fractional moments $\mathbb{E}[Z_N^{\beta,\omega}(f)^\gamma]$, $\gamma \in (0,1)$, with constants depending on γ. In particular, whenever $\vartheta(N,\beta_N) \to +\infty$, one has $Z_N^{\beta_N,\omega}(f_N) \to 0$ in probability for every sequence of initial conditions whose support is contained in a ball of radius $\mathrm{e}^{c_0\mathrm{e}^{\vartheta(N,\beta_N)}}\sqrt{N}$. The estimate is sharp at the level of the exponent: the decay is exponential in $\mathrm{e}^{\vartheta(N,\beta)}$.

The truncated mean in (0.3) is closely related to the total variation distance between the original disorder law and its size-biased version. It also gives corresponding upper-tail estimates. For every $\varepsilon \in (0,1)$,

$$C_{1,\varepsilon}\exp\big(-c_1\mathrm{e}^{\vartheta(N,\beta)}\big) \le \sup_f \mathbb{P}\big(Z_N^{\beta,\omega}(f) \ge \varepsilon\big) \le C_{2,\varepsilon}\exp\big(-c_2\mathrm{e}^{\vartheta(N,\beta)}\big),$$

where the supremum is taken over the same class of initial conditions. For the lower bound it is enough to take f uniform on the ball of radius $\sqrt{N}$.

A consequence concerns the quenched free energy

$$\mathtt{F}(\beta) := \lim_{N\to\infty}\frac{1}{N}\log Z_N^{\beta,\omega}(0) = \lim_{N\to\infty}\frac{1}{N}\mathbb{E}[\log Z_N^{\beta,\omega}(0)] \in (-\infty,0].$$

Earlier work by Lacoin [5] and Berger–Lacoin [6] identified its leading exponential behavior. The estimate (0.3), together with a complementary lower bound based on superadditivity and concentration, gives the more precise bounds

$$-\frac{c'}{\sigma^2(\beta)^4}\exp\Big(-\frac{\pi}{\sigma^2(\beta)}\Big) \le \mathtt{F}(\beta) \le -c\exp\Big(-\frac{\pi}{\sigma^2(\beta)}\Big), \qquad \beta\in(0,1). \tag{0.4}$$

Thus the exact exponential scale is expressed in terms of $\sigma^2(\beta)$ rather than merely β^2, and the error term is brought outside the exponential.

We next pass to the Stochastic Heat Flow. Let $\mathcal{M}_1(r)$ be the set of probability densities on $\mathbb{R}^2$ supported in the ball of radius r. By convergence of directed polymers to the SHF and by its scaling covariance, (0.3) yields universal constants $c_0, c_1, c_2 > 0$ such that, for all $t > 0$ and $\vartheta \in \mathbb{R}$,

$$\frac{1}{c_1}\mathrm{e}^{-c_1 t\mathrm{e}^{\vartheta}} \le \sup_{\varphi\in\mathcal{M}_1(\mathrm{e}^{c_0 t\mathrm{e}^{\vartheta}}\sqrt{t})} \mathbb{E}\big[\mathscr{Z}_t^{\vartheta}(\varphi)\wedge 1\big] \le \frac{1}{c_2}\mathrm{e}^{-c_2 t\mathrm{e}^{\vartheta}}. \tag{0.5}$$

The same bounds hold for fractional moments, and corresponding two-sided estimates hold for the probability that $\mathscr{Z}_t^{\vartheta}(\varphi)$ exceeds a fixed positive level. Thus the effective parameter controlling local extinction is $t\mathrm{e}^{\vartheta}$: the decay is exponential in time and doubly exponential in the disorder strength. The initial condition may spread over a radius exponentially larger than the diffusive scale $\sqrt{t}$.

The polymer result also gives information on a space-time discretization of the two-dimensional stochastic heat equation

$$\partial_t u = \frac{1}{2}\Delta u + \beta\xi u, \qquad u(0,x) = 1.$$

The discretized solution u_1^{β} coincides, after time reversal of the environment, with a directed-polymer partition function. Its diffusively rescaled version is $u_N^{\beta}(t,x) := u_1^{\beta}(Nt, \sqrt{N}x)$. In every supercritical regime $\vartheta(N,\beta_N) \to +\infty$, spatial averages of $u_N^{\beta_N}$ converge to zero, so the diffusive scale does not capture the fluctuations. For $c > 0$, introduce

$$\widehat{u}_N^{\beta,c}(t,x) := u_1^{\beta}\big(Nt, \rho_{Nt}^{\beta,c}\sqrt{Nt}\,x\big), \qquad \rho_{Nt}^{\beta,c} := \exp\big(c\mathrm{e}^{\vartheta(\lfloor Nt\rfloor,\beta)}\big).$$

There exist constants $0 < c' < c'' < \infty$ such that, if $\vartheta(N,\beta_N) \to +\infty$, then for every $t > 0$ and every fixed probability density φ,

$$\int_{\mathbb{R}^2} \varphi(x)\widehat{u}_N^{\beta_N,c}(t,x)\mathrm{d}x \xrightarrow[N\to\infty]{\mathrm{d}} \begin{cases} 0, & c < c', \\ 1, & c > c''. \end{cases} \tag{0.6}$$

This includes the case $\beta_N \equiv \beta \in (0,1)$. By (0.2), the spatial factor is then of order

$$\rho_{Nt}^{\beta,c} = \exp\Big(c\mathrm{e}^{\alpha}Nt\exp\Big(-\frac{\pi}{\sigma^2(\beta)}\Big)(1+o(1))\Big),$$

which is genuinely exponential in N. The first line of (0.6) proves that, up to the constant in the exponent, this scale cannot be reduced.

We finally describe the proof of the upper bound in (0.3). The lower bound follows from a variance estimate and the Paley–Zygmund inequality. The upper bound is obtained in three steps. First, for fixed large ϑ in the critical window, one proves

$$\limsup_{N\to\infty} \sup_{f\in\mathcal{M}_1^{\mathrm{disc}}(\sqrt{N})} \mathbb{E}[Z_N^{\beta_N,\omega}(f)\wedge 1] \le \frac{C}{\vartheta}.$$

The key device is size biasing. If $Z \ge 0$, $\mathbb{E}[Z] = 1$, and $\mathrm{d}\widetilde{\mathbb{P}} = Z\mathrm{d}\mathbb{P}$, then for any event A and any $M > 0$,

$$\mathbb{E}[Z\wedge M] \le M\mathbb{P}(A) + \widetilde{\mathbb{P}}(A^c).$$

The event is defined through a tractable proxy X for the partition function. Starting from the polynomial chaos expansion

$$Z_N^{\beta,\omega}(f) = \sum_B q^{(f)}(B) \prod_{(n,x)\in B} \xi_{n,x}, \qquad \xi_{n,x} := \mathrm{e}^{\beta\omega(n,x)-\lambda(\beta)} - 1,$$

we retain chaos terms whose time support has width at most an intermediate scale $\widetilde{N}$, whose starting time stays away from the endpoints, and whose order is at most $\log N$. This is the first-order term in a coarse-grained chaos expansion. Taking $A = \{X \geq \frac{1}{2}\widetilde{\mathbb{E}}[X]\}$, Chebyshev's inequality reduces the argument to estimates on $\mathbb{V}\mathrm{ar}(X)$, $\widetilde{\mathbb{E}}[X]$ and $\widetilde{\mathbb{V}\mathrm{ar}}(X)$. The first two are essentially second-moment calculations; the last requires a more delicate analysis of pairwise and triple intersections.

Second, a finite-volume criterion converts this polynomial estimate into exponential decay. If, at some scale L,

$$\sup_{f\in\mathcal{M}_1^{\mathrm{disc}}(\sqrt{L})} \mathbb{E}[Z_L^{\beta,\omega}(f)^{1/2}] \leq c_*,$$

for a sufficiently small universal constant c_*, then block coarse graining gives

$$\sup_{f\in\mathcal{M}_1^{\mathrm{disc}}(\sqrt{L})} \mathbb{E}[Z_N^{\beta,\omega}(f)^{1/2}] \leq 3\mathrm{e}^{-N/L}, \qquad N \geq L.$$

Finally, a change-of-scale estimate allows the support of the initial condition to be enlarged. For $1 \leq A \leq B$ and $\gamma \in (0,1)$,

$$\sup_{f\in\mathcal{M}_1^{\mathrm{disc}}(\sqrt{B})} \mathbb{E}[Z_N^{\beta,\omega}(f)^{\gamma}] \leq \Big(4\frac{B}{A}\Big)^{1-\gamma} \sup_{g\in\mathcal{M}_1^{\mathrm{disc}}(\sqrt{A})} \mathbb{E}\big[Z_N^{\beta,\omega}(g)^{\gamma}\big].$$

At the relevant block scale one has $N/L \asymp \mathrm{e}^{\vartheta(N,\beta)}$. The exponential gain $\mathrm{e}^{-N/L}$ absorbs the entropy cost of covering balls whose radius is of order $\exp(c\mathrm{e}^{\vartheta(N,\beta)})\sqrt{N}$, leading to (0.3).

References

[1] Q. Berger, F. Caravenna and N. Turchi. Strong disorder for stochastic heat flow and 2D directed polymers. *Preprint arXiv:2508.02478*, 2025+.

[2] F. Caravenna, R. Sun and N. Zygouras. The critical 2D Stochastic Heat Flow. *Invent. Math.*, 233:325–460, 2023.

[3] F. Caravenna, R. Sun and N. Zygouras. Singularity and regularity of the critical 2D Stochastic Heat Flow. *Preprint arXiv:2504.06128*, 2025+.

[4] J. Clark and L.-C. Tsai. Conditional GMC within the Stochastic Heat Flow. *Preprint arXiv:2507.16056*, 2025+.

[5] H. Lacoin. New bounds for the free energy of directed polymers in dimension $1+1$ and $1+2$. *Comm. Math. Phys.*, 294:471–503, 2010.

[6] Q. Berger and H. Lacoin. The high-temperature behavior for the directed polymer in dimension $1+2$. *Ann. Inst. Henri Poincaré Probab. Stat.*, 53:430–450, 2017.

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MOMENTS OF A LONG-RANGE CORRELATED 2D STOCHASTIC HEAT EQUATION AT CRITICALITY

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Classification AMS 2020: 60K37, 60H15, 82D60.

Keywords: Kardar-Parisi-Zhang equation; stochastic heat equation; continuum directed polymer; critical 2d stochastic heat flow; long-range correlation.

We consider the following formal stochastic heat equation (SHE):

$$\partial_t \mathbf{X}^\beta(x,t) = \frac{\Delta_x}{2}\mathbf{X}^\beta(x,t) + \mathbf{X}^\beta(x,t)\cdot\beta\xi(x,t), \quad (x,t)\in\mathbb{R}^2\times\mathbb{R}_+. \tag{0.1}$$

The driving noise is a centered Gaussian field with correlation function

$$\mathbb{E}[\xi(x,t)\xi(y,s)] = \delta(t-s)\times|x-y|^{-2}. \tag{0.2}$$

The coupling constant β describes the strength of the noise. From a physical perspective, the stochastic heat equation has a rich interpretation. On the one hand, it arises in the study of the KPZ equation through the Cole–Hopf transformation [1]. On the other hand, the solution to the SHE can also be viewed as the partition function of the continuous directed polymer. The polymer model is a fundamental model in statistical physics that describes the fluctuations of a polymer chain affected by disorder [2, 3].

From a mathematical perspective, beyond the SHE driven by space-time white noise, it is natural to ask how spatial correlations affect the behavior of the SHE. This motivates the study of SHEs driven by Riesz-type correlated noise, such as (0.1).

Because of the exponent 2 in the spatial correlation, a formal scaling argument shows that two is a critical dimension for this SPDE. Due to the singularity of the noise, (0.1) is not well-defined in the sense of the classical Itô theory. To circumvent this problem, a common approach is to consider a regularized version of (0.1) and analyze its scaling limit as the regularization vanishes.

For the same correlation as in (0.2), and in dimensions greater than or equal to three, A. Dunlap, M. Hairer, and X.-M. Li [4] proved the scaling limit of a nonlinear version of (0.1). In addition, C. Mueller and R. Tribe [5] proved a more critical scaling limit for (0.1). Based on the latter result, S. Kuzgun and R. Tao [6] studied a time-dependent average. Versions with different exponents have also been studied in [7, 8, 9].

In the polymer context, C. Rovira, S. Tindel [10], and H. Lacoin [11] proved several results on how long-range spatial correlations affect the polymer model. In particular, our work is mainly motivated by the work of C. Cosco, F. Cottini, and A. Donadini [12]. They showed that a discrete directed polymer with long-range spatial correlations behaving at large scales like $(\log|x|)^a/|x|^2$, for $a>-1$, exhibits a phase transition at a certain critical point. This result naturally raises the question of the scaling limit of the partition function exactly at criticality. Our result provides an answer to this question in the continuous setting.

More precisely, we consider the following approximation scheme for (0.1):

$$\partial_t \mathbf{X}^{\lambda,a}_\varepsilon(x,t) = \frac{\Delta_x}{2}\mathbf{X}^{\lambda,a}_\varepsilon(x,t) + \mathbf{X}^{\lambda,a}_\varepsilon(x,t)\cdot \beta^{\lambda,a}_\varepsilon \xi^a_\varepsilon(x,t), \quad \mathbf{X}^{\lambda,a}_\varepsilon(x,0) = X_0(x),$$

where the regularized driving noise is a centered Gaussian field with correlation function

$$\mathbb{E}[\xi^a_\varepsilon(x,t)\xi^a_\varepsilon(y,s)] = \delta(t-s)\cdot R^a_\varepsilon(x-y), \quad R^a_\varepsilon(x) := \varepsilon^{-2}R^a(x/\varepsilon).$$

The primary kernel we consider is $1/(1+|x|^2)$, since in this case the rescaled kernel converges to $1/|x|^2$ as $\varepsilon \to 0$. More generally, our model also includes kernels that behave at large scales like

$$R^a(x) \overset{x\to\infty}{\sim} \frac{(\log|x|)^a}{|x|^2}, \quad a > -1.$$

We also assume that $R^a \in C_b(\mathbb{R}^2;\mathbb{R}_{\geq 0})$ is radially symmetric, and we impose certain decay conditions on R^a. One important feature of this model is that, compared with the two-dimensional stochastic heat equation driven by regularized space-time white noise, the correlation kernel here is not integrable. Hence, one of the key points of our study is to understand how this singularity, together with the logarithmic perturbation, affects the limit of the regularized solution.

To obtain a nontrivial limit, we tune down the strength of the deriving noise by the following scaling:

$$(\beta^{\lambda,a}_\varepsilon)^2 := \underbrace{\frac{(\beta^a_c)^2}{(\log\varepsilon^{-1})^{a+2}}}_{\text{rescaled coupling constant}} \times \underbrace{\left(1 + \text{“correction term”} + \frac{\lambda}{\log\varepsilon^{-1}}\right)}_{\text{perturbation of the coupling constant}}. \tag{0.3}$$

Here the coupling constant is set to its critical value, given by

$$(\beta^a_c)^2 := \left(\frac{\mathbf{j}_{-\nu_a,0}}{2\nu_a}\right)^2, \quad \nu_a := \frac{1}{a+2},$$

where $\mathbf{j}_{-\nu,0}$ is the first zero of the Bessel function of the first kind:

$$\mathcal{J}_{-\nu}(z) := \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m-\nu+1)}\left(\frac{z}{2}\right)^{2m-\nu}.$$

Under the critical scaling (0.3), we establish the following universal limit.

Theorem 0.1. *Assume that $a > -1$, $\lambda \in \mathbb{R}$, and that R^a satisfies suitable decay conditions. Then, for each $N \geq 2$, $f \in C_c(\mathbb{R}^2)$, and $X_0 \in C_b(\mathbb{R}^2)$,*

$$\mathbb{E}[\mathbf{X}^{\lambda,a}_\varepsilon(f,t)^N] \overset{\varepsilon\to 0}{\longrightarrow} \mathbb{E}[\mathscr{Z}^{\lambda,a}_{0,t}(X_0,f)^N]. \tag{0.4}$$

Here $\mathscr{Z}^{\lambda,a}_{s,t}(dx,dy)$ is the critical two-dimensional stochastic heat flow (SHF), where the function describing the two-body motion is given by

$$\mathfrak{s}^{\vartheta_a}(t) := (2\pi)\int_0^\infty \frac{(\vartheta_a)^u t^{u-1}}{\Gamma(u)}du, \quad \log\vartheta_a := \frac{2}{a+2}\lambda + \log 2 - 2\gamma_{\mathrm{EM}}. \tag{0.5}$$

The critical 2d SHF arises both as a normalized limit of the two-dimensional SHE driven by space-time white noise [13, 14] and as a scaling limit of the two-dimensional directed polymer model [14], both under critical scaling. The correction term in (0.3) is given by

$$\text{“correction term”} := \begin{cases} 0, & \text{if } a > 0, \\ -\frac{1}{\log \varepsilon^{-1}} \Big(\frac{1}{2} \int_{-\infty}^{0} \mathfrak{u}^a (e^{\frac{z}{2}})^2 dz + \int_0^\infty \mathfrak{u}_0^a (e^{\frac{z}{2}})^2 dz + \int_0^\infty \mathfrak{u}_0^a (e^{\frac{z}{2}}) dz \Big), & \text{if } a = 0, \\ \frac{\|\mathbf{K}_0^a\|_{\text{op}}}{\|\mathcal{K}_\varepsilon^a\|_{\text{op}}} - 1, & \text{if } a \in (-1, 0). \end{cases}$$

Here $\mathfrak{u}^a(r)$ and $\mathfrak{u}_0^a(r)$ are related to the large-scale behavior of R^a and are given by

$$\mathfrak{u}^a(r) := \big(2r^2 R^a(\sqrt{2}r)\big)^{\frac{1}{2}}, \quad \frac{1}{(\log r)^{\frac{a}{2}}} \mathfrak{u}^a(r) = 1 - \left(1 - \left(\frac{2r^2}{(\log r)^a} R^a(\sqrt{2}r)\right)^{\frac{1}{2}}\right) =: 1 + \mathfrak{u}_0^a(r).$$

In addition, $\mathbf{K}_0^a$ is a bounded operator on $L^2((0,1))$ with kernel

$$\mathbf{K}_0^a(z', z) := (z')^{\frac{a}{2}} (1 - z' \vee z) z^{\frac{a}{2}},$$

which arises as the scaling limit of a Birman–Schwinger kernel. Moreover, $\mathcal{K}_\varepsilon^a$ is a small perturbation of $\mathbf{K}_0^a$. It can be shown that, for every sufficiently small $\varepsilon > 0$,

$$\frac{C}{(\log \varepsilon^{-1})^{1+a}} \le \left| \frac{\|\mathbf{K}_0^a\|_{\text{op}}}{\|\mathcal{K}_\varepsilon^a\|_{\text{op}}} - 1 \right| \le \frac{C'}{(\log \varepsilon^{-1})^{1+a}}, \quad 0 < C < C'.$$

We remark that the main purpose of the correction term is to put the function $\mathfrak{s}^{\vartheta_a}$ in the form (0.5). In particular, when $a \in (-1, 0)$, the correction term is necessary to obtain the non-Gaussian limit (0.4): if the correction term is set to zero, then the limit of the higher moments is given only by the Gaussian contribution. This phenomenon does not appear for the two-dimensional SHE driven by space-time white noise [15, 16, 17], but a similar phenomenon appears in the mean-zero case [18].

References

[1] L. Bertini and G. Giacomin. Stochastic Burgers and KPZ equations from particle systems. *Communications in Mathematical Physics*, 183(3), 571–607, 1997.

[2] F. Comets, T. Shiga, and N. Yoshida. Probabilistic analysis of directed polymers in a random environment: a review. *Stochastic analysis on large scale interacting systems*, 39, 115–142, 2004.

[3] N. Zygouras. Directed polymers in a random environment: A review of the phase transitions. *Stochastic Processes and their Applications*, 177, 104431, 2024.

[4] A. Dunlap, M. Hairer, and X.-M. Li. A critical stochastic heat equation with long-range noise. *arXiv*, 2025.

[5] C. Mueller and R. Tribe. A Singular Parabolic Anderson Model. *Electronic Journal of Probability*, 9, 98 – 144, 2004.

[6] S. Kuzgun and R. Tao. Time-dependent averages of a critical long-range stochastic heat equation. *arXiv*, 2024.

[7] R. Dalang. Extending the Martingale Measure Stochastic Integral With Applications to Spatially Homogeneous S.P.D.E.'s. *Electronic Journal of Probability*, 4, 1-29, 1999.

[8] L. Gerolla, M. Hairer, and X.-M. Li. Fluctuations of stochastic PDEs with long-range correlations. *The Annals of Applied Probability*, 35(2), 1198-1232, 2025.

[9] L. Gerolla, M. Hairer, and X.-M. Li. Scaling limit of the KPZ equation with non-integrable spatial correlations. *arXiv*, 2024.

[10] C. Rovira and S. Tindel. On the Brownian-directed polymer in a Gaussian random environment. *Journal of Functional Analysis*, 222(1), 178-201, 2005.
[11] H. Lacoin. Influence of spatial correlation for directed polymers. *The Annals of Probability*, 39(1), 139-175, 2011.
[12] C. Cosco, F. Cottini, and A. Donadini. A central limit theorem for two-dimensional directed polymers with critical spatial correlation. *arXiv*, 2025.
[13] L.-C. Tsai. Stochastic heat flow by moments. *arXiv*, 2024.
[14] F. Caravenna, R. Sun, and N. Zygouras. The critical $2d$ Stochastic Heat Flow. *Inventiones mathematicae*, 233, 325–460, 2023.
[15] F. Caravenna, R. Sun, and N. Zygouras. On the moments of the (2+ 1)-dimensional directed polymer and stochastic heat equation in the critical window. *Communications in Mathematical Physics*, 372(2), 385–440, 2019.
[16] Y. Gu, J. Quastel, and L.-C. Tasi. Moments of the $2D$ SHE at criticality. *Probability and Mathematical Physics*, 2(1), 179 – 219, 2021.
[17] Y.-T. Chen. Delta-Bose gas from the viewpoint of the two-dimensional stochastic heat equation. *The Annals of Probability*, 52(1), 127 – 187, 2024.
[18] Y.-T. Chen. On point interactions in two dimensions via additive functionals. *arXiv*, 2025.

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DERIVATION OF THE FULL FOCUSING Φ^6_1 MEASURE FROM MANY-BODY QUANTUM GIBBS STATES

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Classification AMS 2020: 81V70, 35Q40, 35Q55.

Keywords: many-body quantum Gibbs states; focusing Φ^6_1 measure; nonlinear Gibbs measures; variational method; normalizability threshold.

This extended abstract is based on joint work with Lin Lü and Phan Thành Nam [5]. The purpose of the talk was to explain how the full one-dimensional focusing Φ^6_1 measure, including the sharp critical mass regime, can be obtained as a high-temperature and mean-field limit of many-body bosonic quantum Gibbs states. A point emphasized here is the quantitative subcritical feature: below the critical mass, the singular interaction scale $\varepsilon = \varepsilon(\tau)$ may depend polynomially on the temperature parameter, and the proof uses the sharp subcritical integrability input from [7].

Let $\mathbb{T} = [0, 1]$. The classical object is the focusing Gibbs measure

$$\mathrm{d}\mu_K(u) = \frac{1}{\mathcal{Z}_K} \exp\left\{\frac{1}{6}\int_{\mathbb{T}} |u(x)|^6 \,\mathrm{d}x\right\} \mathbf{1}_{\{\|u\|_{L^2(\mathbb{T})}\leq K\}} \,\mathrm{d}\mu_0(u),$$

where μ_0 is the free complex Gaussian field with covariance h^{-1}, $h = \frac{1}{2}(1-\Delta)$. Since μ_0 is supported on $H^{1/2-}(\mathbb{T})$, no Wick renormalization is needed for the sixth order term in dimension one. This measure is formally invariant under the focusing quintic nonlinear Schrödinger equation

$$i\partial_t u + \partial_x^2 u + |u|^4 u = 0.$$

The focusing sign makes the Hamiltonian unbounded from below, and the L^2 cutoff is essential. The sharp threshold is

$$K_c = \|Q\|_{L^2(\mathbb{R})}, \qquad -Q'' + Q - Q^5 = 0 \quad \text{on } \mathbb{R},$$

where Q is the optimizer of the sharp Gagliardo–Nirenberg–Sobolev inequality. The classical normalizability threshold is sharp: one has a finite partition function for $K \leq K_c$ and divergence for $K > K_c$ [4, 7].

We consider the bosonic Fock space $\mathcal{F} = \bigoplus_{n\geq 0} L^2_{\mathrm{sym}}(\mathbb{T}^n)$, the number operator N, and the free Hamiltonian $H_0 = \mathrm{d}\Gamma(h)$. Let $w \geq 0$ be an even compactly supported function with $\int w = 1$, and set $w_\varepsilon(x) = \varepsilon^{-1} w(x/\varepsilon)$, periodized on $\mathbb{T}$. The attractive three-body interaction is

$$W_\varepsilon = \frac{1}{3!}\int_{\mathbb{T}^3} a^*(x)a^*(y)a^*(z) w_\varepsilon(x-y) w_\varepsilon(x-z) a(x)a(y)a(z) \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

The natural high-temperature scaling is obtained by taking the interaction strength $\lambda = \tau^{-2}$, because under the free Gibbs state the typical particle number is of order τ, so that the one-body and three-body contributions are both of order one in the Gibbs exponent. With a smooth cutoff $f_\eta \to \mathbf{1}_{[0,K^2)}$, the grand-canonical Gibbs state is

$$\Gamma_\tau^{f_\eta} = \frac{1}{Z_\tau^{f_\eta}} \exp\left\{ -\frac{1}{\tau} H_0 + \frac{1}{\tau^3} W_\varepsilon \right\} f_\eta(N/\tau).$$

Here N/τ is the quantum analogue of the classical mass $\|u\|_{L^2}^2$.

The main result states that the quantum Gibbs state converges to the focusing Φ_1^6 measure throughout the full normalizable range. More precisely, if $0 < K \le K_c$, then, along an admissible regime with $\varepsilon \to 0$, $\eta \to 0$, and $\tau \to \infty$, one has

$$\frac{Z_\tau^{f_\eta}}{Z_{\tau,0}} \longrightarrow \int \exp\left\{ \frac{1}{6} \int_{\mathbb{T}} |u|^6 \, \mathrm{d}x \right\} \mathbf{1}_{\{\|u\|_{L^2} \le K\}} \, \mathrm{d}\mu_0(u).$$

At the critical threshold $K = K_c$, the interaction range may shrink logarithmically, for instance $\varepsilon \gtrsim (\log \tau)^{-1/2}$. A central subcritical refinement is that, when $K < K_c$, the dependence of the interaction scale on the temperature becomes polynomial: one may take, for example, $\varepsilon \gtrsim \tau^{-1/96}$. The proof of this improvement uses the subcritical exponential-integrability input from [7] to replace rough pointwise bounds of size $e^{O(\varepsilon^{-2})}$ by uniform moment bounds; this is precisely where the mass gap $K < K_c$ enters. Furthermore, for every fixed $k \ge 1$,

$$\frac{k!}{\tau^k} \left(\Gamma_\tau^{f_\eta}\right)^{(k)} \longrightarrow \int |u^{\otimes k}\rangle\langle u^{\otimes k}| \, \mathrm{d}\mu_K(u) \quad \text{in } \mathfrak{S}^1\left(L^2_{\mathrm{sym}}(\mathbb{T}^k)\right).$$

Thus the limiting classical measure is recovered not only at the level of free energy, but also through all reduced density matrices.

The complementary result is a sharp quantum phase transition. If $K > K_c$ and $g \in C_c^\infty([0,\infty);\mathbb{R}_+)$ satisfies $g \equiv 1$ on $[0, K^2]$, then along the corresponding logarithmic admissible regime

$$\frac{\mathrm{Tr}_{\mathcal{F}}\left[\exp\left\{-\frac{1}{\tau}H_0 + \frac{1}{\tau^3}W_\varepsilon\right\} g(N/\tau)\right]}{\mathrm{Tr}_{\mathcal{F}}\left[\exp\left\{-\frac{1}{\tau}H_0\right\}\right]} \longrightarrow \infty.$$

Consequently the many-body model detects exactly the same threshold K_c as the classical focusing theory.

I briefly describe the proof. The first step is a quantum-to-classical limit at a fixed interaction scale $\varepsilon > 0$. The limiting classical measure is a Hartree Gibbs measure with nonlocal potential

$$W_\varepsilon(u) = \frac{1}{3!} \int_{\mathbb{T}} (w_\varepsilon * |u|^2)^2 |u(x)|^2 \, \mathrm{d}x, \qquad \mathrm{d}\mu^{\varepsilon,g}(u) \propto e^{W_\varepsilon(u)} g(\|u\|_{L^2}^2) \, \mathrm{d}\mu_0(u).$$

This passage is proved by a variational comparison between the quantum Gibbs principle and the corresponding classical entropy minimization problem. After localizing to low modes $P = \mathbf{1}(h \le \Lambda)$, the low-frequency part of a quantum state is represented by its Husimi measure. Berezin–Lieb inequalities provide the entropy lower bound, while quantitative de Finetti estimates identify the limiting moments.

The focusing particle-number cutoff creates the main difficulty: it couples low and high modes and is incompatible with the factorized trial states used in defocusing problems. A key new ingredient is a non-factorized trial state that keeps the full cutoff $f_\eta(N/\tau)$ while localizing only the interaction. This preserves the global mass constraint and controls the localization and relative-entropy errors. For the lower bound one applies entropy duality only to a mass-truncated observable and proves a tail estimate for the interacting lower symbol. Finally, W_ε is sent to the local potential $W(u) = \frac{1}{6}\|u\|_{L^6}^6$ by a mollifier approximation. At the endpoint K_c, the optimal normalizability theorem [7] supplies the required integrable majorant. Strictly below K_c, the projected subcritical moment estimate adapted from [7, Sec. 4] gives uniform L^p control of the focusing weight. This removes the exponential loss in ε^{-2} and is the reason the singular interaction scale can be chosen polynomially in τ, for example $\varepsilon \gtrsim \tau^{-1/96}$, rather than only logarithmically.

This work builds on the construction and sharp normalizability theory of focusing Gibbs measures [4, 1, 7], on many-body derivations of nonlinear Gibbs measures [3, 2, 6], and on focusing grand-canonical limits [8]. The novelty here is the combination of the full optimal mass range, the singular limit $w_\varepsilon \to \delta_0$, the polynomial subcritical interaction scale, and the matching supercritical divergence at the quantum level.

References

[1] J. Bourgain. Periodic nonlinear Schrödinger equation and invariant measures. *Comm. Math. Phys.* 166, 1–26, 1994.

[2] J. Fröhlich, A. Knowles, B. Schlein and V. Sohinger. Gibbs measures of nonlinear Schrödinger equations as limits of many-body quantum states in dimensions $d \leq 3$. *Comm. Math. Phys.* 356, 883–980, 2017.

[3] M. Lewin, P. T. Nam and N. Rougerie. Derivation of nonlinear Gibbs measures from many-body quantum mechanics. *J. École Polytech. Math.* 2, 65–115, 2015.

[4] J. Lebowitz, H. Rose and E. Speer. Statistical mechanics of the nonlinear Schrödinger equation. *J. Statist. Phys.* 50, 657–687, 1988.

[5] L. Lü, P. T. Nam and R. Zhu. Derivation of the focusing Φ_1^6 measure in the optimal mass regime from many-body quantum Gibbs states. Preprint, 2026.

[6] P. T. Nam, R. Zhu and X. Zhu. Φ_3^4 theory from many-body quantum Gibbs states. Preprint arXiv:2502.04884, 2025.

[7] T. Oh, P. Sosoe and L. Tolomeo. Optimal integrability threshold for Gibbs measures associated with focusing NLS on the torus. *Invent. Math.* 227, 1323–1429, 2022.

[8] A. Rout and V. Sohinger. A microscopic derivation of Gibbs measures for the 1D focusing quintic nonlinear Schrödinger equation. *SIAM J. Math. Anal.* 57, 4680–4755, 2025.

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FROM MANY-BODY QUANTUM GIBBS STATES TO SINGULAR GIBBS MEASURES

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Classification AMS 2020: 81V70, 82B10, 35Q40, 60H15.

Keywords: many-body quantum Gibbs states; nonlinear Gibbs measures; Bose gases; Φ^4_3 measure; fractional Bessel interaction; stochastic quantization.

This extended abstract is based on joint works with Phan Thành Nam and Rongchan Zhu [9, 10]. The talk discussed a program which derives singular classical Gibbs measures from grand-canonical many-body quantum Gibbs states in the high-temperature, or semiclassical, limit. The basic philosophy is that, after the rescaling $b_x = \sqrt{\lambda}\, a_x$ of annihilation operators, the canonical commutation relation becomes

$$[b_x, b_y^*] = \lambda\delta(x-y) \longrightarrow 0,$$

so the quantum field should converge to a classical random field. The difficulty is that the limiting field theories are singular and require renormalization, while the quantum model is formulated through non-commuting operators and nonlocal interactions.

The first result concerns the derivation of the complex Φ^4_3 measure on $\mathbb{T}^3$. Formally this measure is

$$\mathrm{d}\nu(\Phi) = \frac{1}{Z}\exp\left\{-\int_{\mathbb{T}^3}\left(|\nabla\Phi|^2 + m_0|\Phi|^2 + \frac{1}{2}|\Phi|^4\right)\mathrm{d}x\right\} D\Phi,$$

which has to be understood after renormalization since the Gaussian free field is a distribution of regularity $C^{-1/2-}$. On the quantum side one considers, on bosonic Fock space, an interacting Bose gas with a short-range two-body potential $v^\varepsilon = \varepsilon^{-3}v(\cdot/\varepsilon)$ and Gibbs state $\Gamma_\lambda = Z_\lambda^{-1}e^{-H_\lambda}$. The chemical potential is chosen as

$$\vartheta = \frac{\zeta(3/2)}{(4\pi)^{3/2}}\lambda^{-1/2} + C_0 + a_\varepsilon - 6b_\varepsilon + 2C_1 + 2C_2 - m_0,$$

where the ideal-gas term places the system just above the critical density, while $a_\varepsilon \sim \varepsilon^{-1}$, $b_\varepsilon \sim |\log\varepsilon|$, and the finite constants C_1, C_2 encode the mass renormalization needed in dimension three.

The main theorem of [9] states that, if $\lambda, \varepsilon \to 0$ and $\lambda^\eta \le \varepsilon$ for a sufficiently small $\eta > 0$, then the quantum field observables converge to those of the Φ^4_3 measure. More precisely, for every continuous $f : [0,\infty) \to \mathbb{R}$ satisfying

$$|f(x) - f(y)| \le C|x-y|(1 + x^5 + y^5)$$

and every fixed test function φ with finitely supported Fourier transform,

$$\mathrm{Tr}\left[f(\lambda a^*(\varphi)a(\varphi))\Gamma_\lambda\right] \longrightarrow \int f(|\langle\varphi, u\rangle|^2)\,\mathrm{d}\nu(u).$$

Consequently, for $1 \le n \le 6$,

$$n!\lambda^n\langle\varphi^{\otimes n}, \Gamma_\lambda^{(n)}\varphi^{\otimes n}\rangle \longrightarrow \int |\langle\varphi, u\rangle|^{2n}\,\mathrm{d}\nu(u).$$

Under the stronger logarithmic assumption $|\log\lambda|^{-\eta} \le \varepsilon \to 0$, the same convergence holds for all $n \ge 1$. This gives a rigorous bridge from a regular quantum Bose gas to the renormalized three-dimensional Euclidean field.

A central intermediate object is the Hartree approximation

$$\mathrm{d}\nu^{\varepsilon}(\Psi) = Z_{\varepsilon}^{-1} e^{-D_{\varepsilon}(\Psi)}\, \mathrm{d}\mu_0(\Psi),$$

where μ_0 is the Gaussian free field and

$$D_{\varepsilon}(\Psi) = \frac{1}{2}\int_{\mathbb{T}^3\times\mathbb{T}^3} :|\Psi(x)|^2 : v^{\varepsilon}(x-y) : |\Psi(y)|^2 : \, \mathrm{d}x\,\mathrm{d}y - (a_{\varepsilon} - 6b_{\varepsilon} - m + 1)\int_{\mathbb{T}^3} :|\Psi|^2 : \, \mathrm{d}x.$$

The proof has two layers. First, stochastic quantization and paracontrolled calculus show that ν^{ε} converges to the Φ^4_3 measure; new resonant terms generate the finite constants C_1 and C_2. Second, a variational comparison between quantum relative entropy and classical entropy derives the Hartree measure from Γ_{λ}. Low frequencies are handled by Husimi measures and quantitative de Finetti theorems; high frequencies are controlled by correlation inequalities and moment estimates. This also improves earlier two-dimensional shrinking-range results, since the same strategy yields all density matrices in $d = 2$ under the polynomial-type constraint $\lambda^{\eta} \le \varepsilon$.

The second result, proved in [10], treats a different singular regime: a fixed fractional Bessel interaction in two dimensions. On $\mathbb{T}^2$, let

$$\widehat{v}_{\beta}(k) = \langle k\rangle^{-\beta} = (1+|k|^2)^{-\beta/2}, \qquad \frac{3}{2} < \beta \le 2.$$

The case $\beta = 2$ is the Yukawa kernel, with the same short-distance singularity as the Coulomb potential in two dimensions. Unlike the local δ-limit, this is already singular at the quantum level: throughout the range $\frac{3}{2} < \beta \le 2$ one has $\sum_{k\in\mathbb{Z}^2} \widehat{v}_{\beta}(k) = \infty$. Therefore the usual density-square rewriting produces an infinite self-energy and cannot be used directly.

The quantum Hamiltonian must instead be renormalized at the operator level. If $N_0 = \mathrm{Tr}(N\Gamma_0)$ is the free expected particle number, the renormalized interaction is

$$W^{\mathrm{re}} = \frac{\lambda^2}{2(2\pi)^2}\sum_{k\neq 0}\widehat{v}_{\beta}(k)\sum_{p,q\in\mathbb{Z}^2} a^*_{p+k}a^*_{q-k}a_q a_p + \frac{\lambda^2}{2(2\pi)^2}(N-N_0)^2,$$

and the Gibbs state is $\Gamma^{\mathrm{re}}_{\lambda} = (Z^{\mathrm{re}}_{\lambda})^{-1} e^{-\lambda\,\mathrm{d}\Gamma(h) - W^{\mathrm{re}}}$, with $h = -\Delta + 1$. The limiting classical Gibbs measure is

$$\mathrm{d}\nu(u) = z_{\mathrm{cl}}^{-1} e^{-D(u)}\, \mathrm{d}\mu_0(u), \qquad D(u) = \frac{1}{2}\int_{\mathbb{T}^2\times\mathbb{T}^2} v_{\beta}(x-y) : |u(x)|^2 :: |u(y)|^2 : \, \mathrm{d}x\,\mathrm{d}y.$$

The main theorem is the convergence, as $\lambda \downarrow 0$, of both the relative free energy and the reduced density matrices:

$$-\log\frac{Z^{\mathrm{re}}_{\lambda}}{Z_0} \longrightarrow -\log z_{\mathrm{cl}},$$

and, for every fixed $k \ge 1$,

$$k!\lambda^k \Gamma^{(k)}_{\lambda} \longrightarrow \int |u^{\otimes k}\rangle\langle u^{\otimes k}|\, \mathrm{d}\nu(u) \quad \text{in } \mathfrak{S}^2(L^2_{\mathrm{sym}}(\mathbb{T}^2)^{\otimes k}).$$

Thus the full classical fractional-Bessel Gibbs measure is obtained from a renormalized Bose gas with the same singular fixed interaction.

The proof remains variational but requires new high-frequency analysis. After localizing to low frequencies $P = \mathbf{1}_{\{h \le \Lambda^2\}}$, the projected quantum interaction is compared with a finite-dimensional Hartree functional. The error is decomposed into a zero-mode part and a nonzero-mode part; the latter is further split into an intermediate shell and a genuine far tail. The zero-mode contribution is controlled by a second-order correlation inequality and a double-commutator estimate. The shell contribution is rewritten through polarized localized fluctuation observables and estimated mode by mode. The far tail is controlled by combining positivity of the full Bessel operator with small high-frequency particle number. These estimates are precisely what makes it possible to treat non-summable kernels and to recover the Hilbert–Schmidt convergence of all fixed reduced density matrices.

Together, [9] and [10] show two complementary ways in which singular Gibbs measures emerge from many-body quantum mechanics. In the Φ^4_3 problem the singularity comes from the shrinking interaction range and the local field-theory limit. In the fractional-Bessel problem the interaction is fixed but already ultraviolet singular, so the main challenge is the divergent quantum self-energy and the resulting high-frequency remainders. Both results strengthen the bridge between critical Bose gases, constructive quantum field theory, and singular stochastic PDEs, and suggest further questions on Bose–Einstein condensation, Gross–Pitaevskii scaling, Coulomb interactions, and dynamic correlation functions at positive temperature.

References

[1] N. Barashkov and M. Gubinelli. A variational method for Φ^4_3. *Duke Math. J.* 169, 3339–3415, 2020.

[2] B. Bringmann. Invariant Gibbs measures for the three-dimensional wave equation with a Hartree nonlinearity I: Measures. *Stoch. PDE: Anal. Comp.* 10, 1–89, 2022.

[3] R. Catellier and K. Chouk. Paracontrolled distributions and the 3-dimensional stochastic quantization equation. *Ann. Probab.* 46, 2621–2679, 2018.

[4] A. Deuchert, P. T. Nam and M. Napiórkowski. The Gibbs state of the mean-field Bose gas. Preprint arXiv:2501.19396, 2025.

[5] J. Fröhlich, A. Knowles, B. Schlein and V. Sohinger. Gibbs measures of nonlinear Schrödinger equations as limits of many-body quantum states in dimensions $d \le 3$. *Comm. Math. Phys.* 356, 883–980, 2017.

[6] J. Fröhlich, A. Knowles, B. Schlein and V. Sohinger. The Euclidean Φ^4_2 theory as a limit of an interacting Bose gas. *J. Eur. Math. Soc.* 27, 4399–4468, 2025.

[7] M. Lewin, P. T. Nam and N. Rougerie. Derivation of nonlinear Gibbs measures from many-body quantum mechanics. *J. Ècole Polytech. Math.* 2, 65–115, 2015.

[8] M. Lewin, P. T. Nam and N. Rougerie. Classical field theory limit of many-body quantum Gibbs states in 2D and 3D. *Invent. Math.* 224, 315–444, 2021.

[9] P. T. Nam, R. Zhu and X. Zhu. Φ^4_3 theory from many-body quantum Gibbs states. Preprint arXiv:2502.04884, 2025.

[10] P. T. Nam, R. Zhu and X. Zhu. Derivation of Gibbs measure from Gibbs state with the fractional Bessel interaction in two dimensions. Preprint arXiv:2604.21583, 2026.

[11] T. Oh, M. Okamoto and L. Tolomeo. Focusing Φ^4_3-model with a Hartree-type nonlinearity. *Mem. Amer. Math. Soc.* 304, no. 1529, 2024.

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NOISE SENSITIVITY AND DIRECTED POLYMERS

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Classification AMS 2020: Primary: 05D40; Secondary: 82B44, 60H15.

Keywords: Directed Polymer in Random Environment, Disordered System, Influence, Noise Sensitivity, Stochastic Heat Equation, Stochastic Heat Flow.

Given a function $f(\omega)$ that depends on independent random variables $\omega = (\omega_1, \omega_2, \ldots)$, the concept of noise sensitivity describes the phenomenon where a small perturbation of the variables ω_i significantly alters the function's output. This phenomenon is particularly relevant in the study of Boolean functions [16], where small changes in input bits can make the output unpredictable, and in statistical physics [12], where it describes the sensitivity of systems to random noise, especially in the proximity of a phase transition.

More precisely, we say that a sequence of functions $(f_N)_{N\in\mathbb{N}}$ is noise sensitive if for every $\varepsilon > 0$

$$\lim_{N\to\infty} \mathbb{C}ov(f_N(\omega), f_N(\omega^\varepsilon)) = 0\,, \tag{0.1}$$

where $\omega^\varepsilon = (\omega_1^\varepsilon, \omega_2^\varepsilon, \ldots)$ is the modified family where $\omega_i^\varepsilon = \omega_i$ with probability $1-\varepsilon$, while ω_i^ε is independently resampled with probability ε.

The literature on noise sensitivity has largely focused on binary variables ω_i taking two values, say x_+ and x_-:

$$\mathbb{P}(\omega_i = x_+) = p \quad \text{and} \quad \mathbb{P}(\omega_i = x_-) = 1-p \quad \text{for some } p \in (0,1)\,. \tag{0.2}$$

In [2], we extend the classical BKS criterion for noise sensitivity [1] to a general setting, allowing for a wide class of non-Boolean functions $f(\omega) \in \mathbb{R}$ of general random variables ω_i. More specifically, we consider independent random variables $\omega = (\omega_i)_{i\in\mathbb{T}}$, labeled by a finite or countable set $\mathbb{T}$, defined on a probability $(\Omega, \mathcal{A}, \mathbb{P})$ and taking values in measurable spaces $(E_i, \mathcal{E}_i)$:

$$\omega_i \colon \Omega \to E_i \qquad \text{with law} \qquad \mu_i(\cdot) = \mathbb{P}(\omega_i \in \cdot)\,.$$

We then require that the independent random variables $(\omega_i)_{i\in\mathbb{T}}$ and the function $f(\omega) \in L^2$ satisfy the following conditions:

- For any $i \in \mathbb{T}$, there is a closed and separable vector space $\mathcal{V}_i \subseteq L^2(E_i, \mu_i)$ containing the constants, i.e. $1 \in \mathcal{V}_i$, such that $\omega_i \mapsto f(\omega) \in \mathcal{V}_i$ a.s.
- There exist an exponent $q \in (2, \infty)$ and a constant $M_q < \infty$ such that, for any $i \in \mathbb{T}$,

$$\|g(\omega_i)\|_q \le M_q \, \|g(\omega_i)\|_2 \qquad \forall g \in \mathcal{V}_i \quad \text{with} \quad \mathbb{E}[g(\omega_i)] = 0\,,$$

  where $\|\cdot\|_q = \mathbb{E}[|\cdot|^q]^{1/q}$ denotes the usual L^q norm.

These assumptions are quite general. Useful examples for which these conditions hold include all functions $f \in L^2$, when $(\omega_i)_{i\in\mathbb{T}}$ have finite support, and any polynomial chaos $f \in L^2$, when $(\omega_i)_{i\in\mathbb{T}}$ are real-valued satisfying a suitable moment condition.

In this setting, we are able to prove a quantitative version (in the style of [13]) of the classical BKS criterion based on influences (see [1]). First, we extend the classical notion of influence to general functions $f(\omega)$ of general random variables $(\omega_i)_{i\in\mathbb{T}}$. For binary ω's, one can define the influence of f with respect to the k-th variable as

$$I_k(f) = \mathbb{E}[|(f(\omega_+^k) - f(\omega_-^k)|] \,,$$

where $\omega_\pm^k$ denotes the configuration $\omega = (\omega_1, \omega_2, \dots)$ with ω_k fixed to $x_\pm$. For independent variables $(\omega_i)_{i\in\mathbb{T}}$ satisying our assumptions, we define the influence of ω_k on f as

$$Inf_k[f] := \mathbb{E}\big[|f - \mathbb{E}_k[f]|\big] \qquad \text{where we set} \quad \mathbb{E}_k[\cdot] := \mathbb{E}[\cdot|\sigma((\omega_j)_{j\neq k})] \,,$$

which is a quantification of "how much f depends on ω_k".

With this definition, we can extend the Keller-Kindler bound [13] to our general setting, in the form of Van Handel (see [17]). In particular, we prove that for any $\varepsilon \in (0,1)$, there is an exponent $\gamma_{\epsilon,q} > 0$ such that

$$\frac{\mathbb{C}ov\big[f(\omega^\varepsilon), f(\omega)\big]}{\mathbb{V}ar[f]} \leq 4\left(\frac{\mathcal{W}[f]}{\mathbb{V}ar[f]}\right)^{\gamma_{\varepsilon,q}} \,, \tag{0.3}$$

where $\mathcal{W}[f] := \sum_k Inf_k[f]^2$ is the sum of squared influences. The exponent $\gamma_{\epsilon,q}$ depends on a hypercontractivity constant η_q and satisfies $\gamma_{\varepsilon,q} \sim \alpha_q\,\varepsilon$ as $\varepsilon \downarrow 0$ for an explicit $\alpha_q \in (0, \frac{1}{2})$.

When specialised to the binary setting, our result yields asymptotically optimal rates.

Finally, we can give a uniform lower bound on the exponent $\gamma_{\epsilon,q}$ and apply (0.3) to a sequence of functions $(f_N(\omega))_{N\in\mathbb{N}}$ such that $\limsup_{N\to\infty} \mathbb{V}ar[f_N] < \infty$. This yields the announced generalisation of the BKS criterion for noise sensitivity, i.e.

$$\lim_{N\to\infty} \mathcal{W}[f_N] = 0 \qquad \Longrightarrow \qquad \forall \varepsilon > 0\colon \quad \lim_{N\to\infty} \mathbb{C}ov\big[f_N(\omega^\varepsilon), f_N(\omega)\big] = 0\,.$$

The concept of noise sensitivity is usually formulated in terms of vanishing correlations which — for Boolean function $f(\omega) \in \{0,1\}$ — corresponds to asymptotic independence. For general real functions $f(\omega) \in \mathbb{R}$, however, it is natural to consider an enhanced form of noise sensitivity, where $f(\omega)$ is composed with suitable test functions to ensure asymptotic independence, rather than mere decorrelation. We show that the BKS criterion implies enhanced noise sensitivity, at least when the random variables ω_i take finitely many values.

Finally, we apply our results on noise sensitivity to the $2d$ directed polymer in random environment. We first establish an instance of enhanced noise sensitivity for the partition functions, in the critical regime where they converge to the Critical 2D Stochastic Heat Flow (SHF) [5]. Next, we prove the independence of the SHF from the white noise originating from the environment, which indicates that the SHF is not the solution of a stochastic PDE driven by white noise.

References

[1] I. Benjamini, G. Kalai and O. Schramm, *Noise sensitivity of Boolean functions and applications to percolation*, Inst. Hautes Études Sci. Publ. Math., 90 (1999), 5–43.

[2] F. Caravenna and A. Donadini, *Enhanced noise sensitivity, 2D directed polymers and Stochastic Heat Flow*, arXiv:2507.10379, 2025.

[3] F. Caravenna, R. Sun and N. Zygouras, *Universality in marginally relevant disordered systems*, Ann. Appl. Probab., 2017.

[4] F. Caravenna, R. Sun and N. Zygouras, *The Two-Dimensional KPZ Equation in the Entire Subcritical Regime*, Ann. Probab., 2020.
[5] F. Caravenna, R. Sun and N. Zygouras, *The critical 2D Stochastic Heat Flow*, Invent. Math. 233, 325–460, 2023.
[6] C. Cosco, F. Cottini and A. Donadini, *A central limit theorem for two-dimensional directed polymers with critical spatial correlation*, arXiv:2509.16694, 2025.
[7] C. Cosco and A. Donadini, *On the Central Limit Theorem for the log-partition function of 2D directed polymers*, Electron. Commun. Probab., 2025.
[8] C. Cosco and O. Zeitouni, *Moments of partition functions of 2D Gaussian polymers in the weak disorder regime – I*, Comm. Math. Phys., 403:417–450, 2023.
[9] F. Comets, *Directed polymers in random environments*, Lecture Notes in Mathematics 2175, Springer, 2017.
[10] A. Dunlap, M. Hairer and X. Li, *A critical stochastic heat equation with long-range noise*, arXiv:2509.23790, 2025.
[11] P. Erdös and S. J. Taylor, *Some problems concerning the structure of random walk paths*, Acta Math. Acad. Sci. Hungar., 11:137–162, 1960.
[12] C. Garban and J. E. Steif, *Noise Sensitivity of Boolean Functions and Percolation*, Cambridge University Press, 2014.
[13] N. Keller and G. Kindler, *Quantitative relation between noise sensitivity and influences*, Combinatorica, 2013.
[14] H. Lacoin, *Influence of spatial correlation for directed polymers*, Ann. Probab., 2011.
[15] C. Mueller and R. Tribe, *A Singular Parabolic Anderson Model*, Electronic Journal of Probability, 9(none):98 – 144, 2004.
[16] R. O'Donnell, *Analysis of Boolean Functions*, Cambridge University Press, 2014.
[17] G. Rosenthal. Ramon van Handel's Remarks on the Discrete Cube. Notes from a series of lectures (2020). `https://www.cs.toronto.edu/~rosenthal/RvH_discrete_cube.pdf`

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A STATISTICAL ANALOGUE OF SOLITON RESOLUTION FOR THE FOCUSING SCHRÖDINGER EQUATION

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Classification AMS 2020: 60H30, 35Q55, 81T08

Keywords: Focusing Gibbs measures, nonlinear Schrödinger equation, phase transition, soliton resolution, Boué-Dupuis formula.

1. The phase transition

We study the infinite-volume limit of focusing Gibbs measures associated with the one-dimensional nonlinear Schrödinger equation. Let $L\mathbb{T} = \mathbb{R}/(L\mathbb{Z}) \cong [-L/2, L/2]$, let $2 < p < 6$, and set

$$M(u) = \int_{L\mathbb{T}} |u|^2 \,\mathrm{d}x.$$

For $\alpha > 0$, let μ_L^α be the complex Gaussian measure on $L\mathbb{T}$ with inverse covariance $\alpha - \Delta$. Given $K > \alpha^{-1/2}$, $\sigma > 0$, and $\gamma > 0$, we consider

$$\mathrm{d}\rho_L(u) = \frac{1}{Z_L} \exp\left(\frac{\sigma}{pL^\gamma} \int_{L\mathbb{T}} |u|^p \,\mathrm{d}x\right) \mathbf{1}_{\{M(u) \le KL\}} \mathrm{d}\mu_L^\alpha(u). \tag{1.1}$$

The mass cutoff is needed to make the measure normalisable, while the assumption $K > \alpha^{-1/2}$ means that the cutoff lies above the typical mass of the Gaussian field. For $\beta > 0$, we write μ_{OU}^β for the Gaussian measure on $\mathbb{R}$ with inverse covariance $\beta - \Delta$.

Lebowitz, Rose, and Speer [4] introduced these measures with mass cutoff for $L = 1$ and conjectured a phase transition in the infinite-volume limit. Subsequently, Rider [5] studied the infinite volume limit when $\gamma = 0$ and $p = 4$ and showed that the measure concentrates around a single soliton which is uniformly centred. In the limit, the soliton escapes to infinity and the measures converge locally weakly to the trivial measure $\delta_{u=0}$. More recently, Tolomeo and Weber [6] considered the coefficient $L^{-\gamma}$ in front of the nonlinearity and identified

$$\gamma_0(p) = \frac{p}{2} - 1$$

as the critical scaling exponent. If $\gamma < \gamma_0(p)$, the measure concentrates around a single soliton carrying essentially all the available mass KL and the local limit is δ_0. If $\gamma > \gamma_0(p)$, there is no soliton visible and the Gibbs measures converge locally to μ_{OU}^α. At $\gamma = \gamma_0(p)$, they proved convergence to μ_{OU}^α for sufficiently small σ and showed that no limit point is equal to μ_{OU}^α when σ is sufficiently large. In our joint work [3], we consider the critical scaling $\gamma = \gamma_0(p)$, determine a critical value σ_0 at which the phase transition occurs, and describe the limiting measure on both sides of it.

This report is based on joint work with Justin Forlano and Leonardo Tolomeo.

Define

$$A_0 := \sup_{\substack{u\in H^1(\mathbb{R})\\ \|u\|_{L^2(\mathbb{R})}=1}} \left\{ \frac{1}{p}\int_{\mathbb{R}} |u|^p \, \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}} |\partial_x u|^2 \, \mathrm{d}x \right\}$$

and, for $0 < x \le K$,

$$\mathscr{F}_\sigma(x) := A_0 \sigma^{\frac{4}{6-p}} (K-x)^{\frac{p+2}{6-p}} - \frac{\alpha}{2}(K-x) - \frac{(x\sqrt{\alpha}-1)^2}{2x}. \tag{1.2}$$

Set

$$\sigma_0 := \inf\left\{ \sigma > 0 : \sup_{x\in(0,K]} \mathscr{F}_\sigma(x) > 0 \right\}.$$

This threshold has a closed form. Namely, with $c_p := \frac{p+2}{6-p} > 1$ and

$$\xi := \frac{2}{1 + c_p + \sqrt{(1+c_p)^2 + 4c_p K\sqrt{\alpha}(K\sqrt{\alpha}-2)}},$$

one has

$$\sigma_0 = \left(\frac{1}{2A_0 c_p}\right)^{\frac{6-p}{4}} \frac{1}{K^2 \xi^{\frac{6-p}{2}} (1-\xi)^{\frac{p-2}{2}}}.$$

Theorem 1.1. *Let $2 < p < 6$, $\alpha > 0$, $\sigma > 0$, and $K > \alpha^{-1/2}$, and assume that $\gamma = \gamma_0(p) = \frac{p}{2} - 1$. Then the following hold.*

(i) **Weak nonlinearity:** *If $\sigma < \sigma_0$, then*

$$\rho_L \to \mu^{\alpha}_{\mathrm{OU}},$$

weakly as $L \to \infty$.

(ii) **Strong nonlinearity:** *If $\sigma > \sigma_0$, then $\mathscr{F}_\sigma$ has a unique maximiser $\bar{x} \in (0, \alpha^{-\frac{1}{2}})$. With $\widetilde{\alpha} := \bar{x}^{-2} > \alpha$ we have*

$$\rho_L \to \mu^{\widetilde{\alpha}}_{\mathrm{OU}},$$

weakly as $L \to \infty$. Furthermore, for fixed $K > 1/\sqrt{\alpha}$ and $2 < p < 6$, we have $\widetilde{\alpha}(K,p,\sigma) \to \infty$ as $\sigma \to \infty$.

Moreover, for every $\sigma > 0$, the partition function satisfies

$$\lim_{L\to\infty} \frac{1}{L}\log Z_L = \max\left(0, \sup_{x\in(0,K]} \mathscr{F}_\sigma(x)\right) = \begin{cases} 0, & \sigma \le \sigma_0, \\ \mathscr{F}_\sigma(\bar{x}), & \sigma > \sigma_0. \end{cases} \tag{1.3}$$

The parameter x can be interpreted as the mass density of the Gaussian background, so that $L(K-x)$ is the mass available for a localised soliton.

The subcritical regime $\sigma < \sigma_0$ is similar to the subcritical case $\gamma > \gamma_0$ from [6]. Here, the nonlinear effects are weak and it is energetically inconvenient for a soliton to form. Thus, the measure looks more and more like the background Gaussian, and we obtain the limiting measure $\mu^{\alpha}_{\mathrm{OU}} = \lim_{L\to\infty} \mu^{\alpha}_L$. In the supercritical regime $\sigma > \sigma_0$, the Gibbs measures concentrate around a single soliton Q_L over a Gaussian background $\mu^{\widetilde{\alpha}}_L$. This soliton is again uniformly centred on the torus $L\mathbb{T}$ and thus escapes to infinity, which gives the local limit $\mu^{\widetilde{\alpha}}_{\mathrm{OU}}$. More precisely, we have the following.

Statistical analogue of soliton resolution. In the *supercritical* regime $\sigma > \sigma_0$, the picture changes dramatically. Here, $\sup_{x\in(0,K]} \mathscr{F}_\sigma(x) > 0$ and it is energetically convenient for a soliton to form. More precisely, there exist couplings Γ_L of ρ_L and $\mu_L^{\widetilde{\alpha}}$ such that, for every $\delta > 0$,

$$\Gamma_L\left(\left\{(u,X) : \inf_{x_0\in L\mathbb{T},\theta\in[0,2\pi)} \|u - X - e^{i\theta}Q_L(\cdot - x_0)\|^2_{H^1(L\mathbb{T})} > \delta L\right\}\right) \to 0, \tag{1.4}$$

as $L \to \infty$. Here Q_L denotes a sequence of solitons with height of order $\sqrt{L}$ and width of order one. At leading order, the soliton carries mass $L(K - \bar{x})$, whereas the Gaussian background carries mass $\frac{L}{\sqrt{\widetilde{\alpha}}} = L\bar{x}$. Thus, unlike in the concentrating regimes previously studied by [5] and [6], not all of the available mass KL is carried by the soliton. The Gaussian background plays the role of the radiation term, since $\mu_L^{\widetilde{\alpha}}$ is invariant under the linear Schrödinger flow on $L\mathbb{T}$, and its infinite-volume limit $\mu_{\mathrm{OU}}^{\widetilde{\alpha}}$ is invariant under the linear Schrödinger flow on $\mathbb{R}$. We call this phenomenon *soliton resolution* of the Gibbs measure ρ_L. Since the soliton is uniformly centred on the torus $L\mathbb{T}$, it escapes to infinity in the local limit as $L \to \infty$. Hence the local limit only sees the background radiation $\mu_{\mathrm{OU}}^{\widetilde{\alpha}}$, with renormalised mass parameter $\widetilde{\alpha} > \alpha$. The fact that $\widetilde{\alpha} = \widetilde{\alpha}(K, p, \sigma)$ is independent of the original Gaussian mass parameter α suggests that an analogous result may hold for the corresponding canonical ensemble at the critical scaling.

2. Proof strategy

The main tool in the proof is the Boué-Dupuis variational formula [2], popularised by Barashkov and Gubinelli [1] in the context of constructive quantum field theory. It rewrites $\log Z_L$ as a variational problem over progressively measurable drifts. Moreover, almost optimisers of this variational problem give a quantitative description of the Gibbs measure. We decompose an almost optimiser into a localised component and a non-localised component. The localised component is controlled by a deterministic variational problem whose non-zero optimisers are solitons. Together with a large-deviation estimate for the mass of the Gaussian field, this reduces the leading-order analysis to the maximisation of $\mathscr{F}_\sigma$. A further analysis of the almost optimisers yields the coupling statement, while translation invariance gives the local limits.

References

[1] N. Barashkov and M. Gubinelli. A variational method for Φ^4_3. *Duke Math. J.*, 169(17):3339–3415, 2020.

[2] M. Boué and P. Dupuis. A variational representation for certain functionals of Brownian motion. *Ann. Probab.*, 26(4):1641–1659, 1998.

[3] J. Forlano, F. Höfer, and L. Tolomeo. Soliton resolution in the high temperature limit of the Gibbs measure for the nonlinear Schrödinger equation. *In preparation*, 2026+.

[4] J. L. Lebowitz, H. A. Rose, and E. R. Speer. Statistical mechanics of the nonlinear Schrödinger equation. *J. Statist. Phys.*, 50(3–4):657–687, 1988.

[5] B. C. Rider. On the ∞-volume limit of the focusing cubic Schrödinger equation. *Comm. Pure Appl. Math.*, 55(10):1231–1248, 2002.

[6] L. Tolomeo and H. Weber. Phase transition for invariant measures of the focusing Schrödinger equation. *Comm. Math. Phys.*, 407(4):Paper No. 82, 55 pp., 2026.

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