SCIENTIFIC REPORTS

Relative Langlands Program

29 Dec 2025–16 Jan 2026

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ON THE SURJECTIVITY OF THE LOCAL GENESTIER-LAFFORGUE PARAMETERIZATION

RAPHAEL BEUZART-PLESSIS

Genestier-Lafforgue and Fargues-Scholze have constructed a semi-simple local Langlands correspondence for reductive groups over local fields of positive characteristic.

In this talk, I will explain how, assuming a version of the stable (twisted) trace formula for base change over a function field, one can show the surjectivity of this parametrization for unramified groups and when the characteristic does not divide the order of the Weyl group. This is based on joint work with Michael Harris and Jack Thorne.

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AUTOMORPHIC UNFOLDING AND HIGHER TEICHMÜLLER THEORY

ERIC YEN-YO CHEN

In higher Teichmüller theory, one is interested in identifying special components in real character varieties of a closed oriented surface. We observe an analogy between this procedure and automorphic unfolding, which can be made precise under the nonabelian Hodge correspondence. Finally, based on recent joint work with Enya Hsiao and Mengxue Yang, we characterize and study an underlying ”codimension 2” relative Langlands duality in the Dolbeault setting.

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RELATIVE LANGLANDS DUALITY FOR $\mathfrak{osp}(2n+1|2n)$

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Classification AMS 2020: Primary 14F08; secondary 14D24, 14M15, 17B20.

Keywords: Satake equivalence, affine Grassmannian, Weyl algebra, Langlands duality.

1. RELATIVE LANGLANDS DUALITY FOR $\mathfrak{osp}(2n+1|2n)$

This is a report on a joint work with Alexander Braverman, David Kazhdan and Roman Travkin.

1.1. **Ring objects in the Satake category and relative Langlands duality.** Let G be a connected reductive group over $\mathbb{C}$. Let $\mathcal{K} = \mathbb{C}((t)) \supset \mathcal{O} = \mathbb{C}[\![t]\!]$. The affine Grassmannian ind-scheme $\mathrm{Gr}_G = G_{\mathcal{K}}/G_{\mathcal{O}}$ is the moduli space of G-bundles on the formal disc equipped with a trivialization on the punctured formal disc. One can consider the *derived Satake category* $D_{G_{\mathcal{O}}}(\mathrm{Gr}_G)$.[1] This is a factorization monoidal category which is monoidally equivalent to $D^{G^\vee}(\mathrm{Sym}^\bullet(\mathfrak{g}^\vee[-2]))$: the derived category of dg-modules over $\mathrm{Sym}^\bullet(\mathfrak{g}^\vee[-2])$ endowed with a compatible action of $G^\vee$ (the monoidal structure on this category is just given by tensor product over $\mathrm{Sym}^\bullet(\mathfrak{g}^\vee[-2])$); we shall denote the corresponding functor from $D_{G_{\mathcal{O}}}(\mathrm{Gr}_G)$ to $D^{G^\vee}(\mathrm{Sym}^\bullet(\mathfrak{g}^\vee[-2])$ by Φ_G.

In [4] we have attached to any $\mathbf{N}$ as above a certain ring object $\mathcal{A}_{G,\mathbf{N}}$ in $D_{G_{\mathcal{O}}}(\mathrm{Gr}_G)$. This construction was generalized in [6] to the case when $\mathbf{N}$ is an arbitrary smooth affine variety with a G-action. Moreover, it is argued in [4], [2] and [6] that the object $\mathcal{A}_{G,\mathbf{N}}$ should only depend on $\mathbf{M} = T^*\mathbf{N}$. One of the main conjectures of [6] says that if $\mathbf{N}$ is a spherical G-variety then $H^*(\Phi_G(\mathcal{A}_{G,\mathbf{N}})$ should be the algebra of functions on certain (*relative Langlands dual* or *S-dual*) hyper-spherical Poisson G-variety $\mathbf{M}^\vee$, and this construction is expected to be involutive in some reasonable generality, cf. also [12].

1.2. **The non-cotangent case.** The construction of $\mathcal{A}_{G,\mathbf{N}}$ should in principle make sense for any smooth affine symplectic G-variety $\mathbf{M}$. However, when $\mathbf{M}$ is not of cotangent type some extra care is needed. This is discussed in detail in [2] when $\mathbf{M}$ is a symplectic linear representation of G. In this case in *loc. cit.* the corresponding object $\mathcal{A}_{G,\mathbf{M}}$ was constructed, but in general it is not an object of $D_{G_{\mathcal{O}}}(\mathrm{Gr}_G)$ but rather of some twisted version of it. The twisting is by a square root of some line bundle on Gr_G which depends on $\mathbf{M}$; in the case when the twisting is indeed non-trivial we say that an *anomaly* is present. The derived Satake equivalence can be extended to such twisted categories (but one has to change the notion of the Langlands dual group $G^\vee$) so even in the anomalous case all of the above constructions go through.

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[1] In fact we are going to work with a renormalized version of it, which by definition is equal to the ind-completion of the corresponding subcategory of bounded complexes with constructible cohomology.

1.3. **The subject of** [5]. This note is a sequel to [5]. There we considered the symplectic group $\mathrm{Sp}(2n) \circlearrowright \mathbb{C}^{2n}_{-}$ and its Langlands dual group $\mathrm{SO}(2n+1) \circlearrowright \mathbb{C}^{2n+1}_{+}$. We also considered a symplectic vector space $\mathbf{M} = \mathbb{C}^{2n+1}_{+} \otimes \mathbb{C}^{2n}_{-}$ and the corresponding Weyl algebra $\mathcal{W}$ of $\mathbf{M}_{\mathcal{K}}$. Here $\mathcal{K} = \mathbb{C}(\!(t)\!) \supset \mathbb{C}[\![t]\!] = \mathcal{O}$. We studied the category $\mathcal{DW}\mathrm{-mod}^{\mathrm{Sp}(2n)_{\mathcal{O}},\mathrm{lc}}$ of locally compact $\mathrm{Sp}(2n)_{\mathcal{O}}$-equivariant objects in the tensor product cagegory $\mathrm{D\text{-}mod}_{1/2}(\mathrm{Gr}_{\mathrm{Sp}(2n)}) \otimes \mathcal{W}\mathrm{-mod}$ (D-modules on the affine Grassmannian of $\mathrm{Sp}(2n)$ twisted by the square root of the determinant line bundle).

We established the following algebraic description of $\mathcal{DW}\mathrm{-mod}^{\mathrm{Sp}(2n)_{\mathcal{O}},\mathrm{lc}}$ conjectured by D. Gaiotto. We consider the (infinite-dimensional) graded algebra $\mathrm{Sym}^{\bullet}(\Pi\mathbf{M}[-1])$ (Π assigns to $\mathbf{M}$ the odd parity) as a dg-superalgebra with trivial differential. We constructed an equivalence of categories

$$\mathcal{DW}\mathrm{-mod}^{\mathrm{Sp}(2n)_{\mathcal{O}},\mathrm{lc}} \xrightarrow{\sim} D^{\mathrm{SO}(2n+1)\times\mathrm{Sp}(2n)}_{\mathrm{perf}}(\mathrm{Sym}^{\bullet}(\Pi\mathbf{M}[-1])).$$

This is also a particular case of the general [6, Conjecture 7.5.1]. In other words, it means that the S-dual of the symplectic variety $T^{*}\mathrm{Sp}(2n) \times \mathbb{C}^{2n}_{-}$ equipped with the hamiltonian action of $\mathrm{Sp}(2n) \times \mathrm{Sp}(2n)$, is $\mathbf{M} \circlearrowleft \mathrm{SO}(2n+1) \times \mathrm{Sp}(2n)$. (Note that due to the twisting by the square root of the determinant line bundle, the second factor $\mathrm{Sp}(2n)$ corresponds under the S-duality to its metaplectic Langlands dual $\mathrm{Sp}(2n)$.)

1.4. **The subject of this paper.** In the present note we confirm the converse claim: the S-dual of $\mathbf{M} \circlearrowleft \mathrm{SO}(2n+1) \times \mathrm{Sp}(2n)$ is $T^{*}\mathrm{Sp}(2n) \times \mathbb{C}^{2n}_{-} \circlearrowleft \mathrm{Sp}(2n) \times \mathrm{Sp}(2n)$. That is, we construct an equivalence of categories

$$D^{\mathrm{Sp}(2n)\times\mathrm{Sp}(2n)}_{\mathrm{perf}}\big(\mathbb{C}[\mathrm{Sp}(2n)] \otimes \mathrm{Sym}^{\bullet}(\mathfrak{sp}(2n)[-2]) \otimes \mathrm{Sym}^{\bullet}(\Pi(\mathbb{C}^{2n}_{-})[-1])\big) \\ \xrightarrow{\sim} \mathcal{W}\mathrm{-mod}^{\mathrm{SO}(2n+1)_{\mathcal{O}}\times\mathrm{Sp}(2n)_{\mathcal{O}},\mathrm{lc}}.$$

1.5. **The global conjecture.** Another very important point of [6] is that the local relative Langlands duality is expected to give rise to certain global geometric ("period") representation of automorphic L-functions. The precise formulation of [6] is in the case when the spectral side is of cotangent type and when there is no anomaly. Hopefully, both assumptions can be overcome. We are not going to discuss how to do this in general, however, we are going to present a conjecture in the above case, which we would like to think of as a natural extension of the setting of [6, Section 12] to our setting.

Let C be a smooth projective irreducible curve over $\mathbb{C}$.[2] For an algebraic group G we denote by Bun_G the moduli stack of G-bundles on C; in the case when $G = \mathrm{Sp}(2n)$ we shall also consider the twisted version of $\mathrm{Bun}_{\mathrm{Sp}(2n)}$ which we shall denote by $\mathrm{Bun}^{\omega}_{\mathrm{Sp}(2n)}$ — it classifies vector bundles M on C of rank $2n$ equipped with a non-degenerate skew-symmetric form $\Lambda^2(M) \to \omega_C$. We have a natural morphism $\iota\colon \mathrm{Bun}^{\omega}_{\mathrm{Sp}(2n)} \times \mathrm{Bun}_{\mathrm{SO}(2n+1)} \to \mathrm{Bun}^{\omega}_{\mathrm{Sp}(2n(2n+1))}$. Let Θ denote the theta-sheaf of [9] on $\mathrm{Bun}^{\omega}_{\mathrm{Sp}(2n(2n+1))}$. This is actually a sheaf twisted by the square root of the natural determinant bundle on $\mathrm{Bun}^{\omega}_{\mathrm{Sp}(2n(2n+1))}$. Consider now $\iota^{!}\Theta$. This is a sheaf on $\mathrm{Bun}^{\omega}_{\mathrm{Sp}(2n)} \times \mathrm{Bun}_{\mathrm{SO}(2n+1)}$ twisted by the square root of the determinant line bundle along the first factor.

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[2] A similar discussion should make sense in the ℓ-adic setting when C is a curve over a finite field; however in that case we do not know how to formulate a precise conjecture.

It is expected (cf. [8]) that the twisted global geometric Langlands duality should assign to any sheaf $\mathcal{F}$ on $\mathrm{Bun}^{\omega}_{\mathrm{Sp}(2n)} \times \mathrm{Bun}_{\mathrm{SO}(2n+1)}$ twisted by the square root of the determinant bundle along the first factor an ind-coherent sheaf $\mathbb{L}(\mathcal{F})$ (with nilpotent singular support) on $\mathrm{LS}_{\mathrm{Sp}(2n)}(C) \times \mathrm{LS}_{\mathrm{Sp}(2n)}(C)$ where for an algebraic group H over $\mathbb{C}$ we denote by $\mathrm{LS}_H(C)$ the moduli stack of de Rham H-local systems on C.[3]

To formulate our conjecture we need to introduce the following notation. Let $\mathcal{E}$ be a symplectic local system on C. For simplicity let us assume that $H^0_{\mathrm{dR}}(C,\mathcal{E}) = H^2_{\mathrm{dR}}(C,\mathcal{E}) = 0$ where the subscript dR stands for de Rham cohomology. The space $H^1_{\mathrm{dR}}(C,\mathcal{E})$ has a canonical symmetric non-degenerate bilinear form; it also has a canonical maximal isotropic subspace $L_{\mathcal{E}}$ which is equal to the image of $H^0(C,\mathcal{E}\otimes\Omega_C)$ in $H^1_{\mathrm{dR}}(C,\mathcal{E})$. We set $S_{\mathcal{E}}$ to be the corresponding spinor representation of the Clifford algebra $\mathrm{Cliff}(H^1_{\mathrm{dR}}(C,\mathcal{E}))$ of $H^1_{\mathrm{dR}}(C,\mathcal{E})$ (by definition, it is induced from the trivial representation of $\Lambda(L_{\mathcal{E}})$) which is naturally a subalgebra of $\mathrm{Cliff}(H^1_{\mathrm{dR}}(C,\mathcal{E}))$).

The following conjecture appears in a slightly weaker form in [10, Conjecture 1.2.4] for $n = 1$:

Conjecture 1.1. (1) *There exists a sheaf $\Theta^\vee$ on $\mathrm{LS}_{\mathrm{Sp}(2n)}(C)$ such that $\mathbb{L}(\iota^!\Theta)$ is equal to $\Delta_*\Theta^\vee$ where $\Delta\colon \mathrm{LS}_{\mathrm{Sp}(2n)}(C) \to \mathrm{LS}_{\mathrm{Sp}(2n)}(C) \times \mathrm{LS}_{\mathrm{Sp}(2n)}(C)$ is the diagonal embedding.*

(2) *The fiber of $\Theta^\vee$ at $\mathcal{E}$ as above is equal to $S_{\mathcal{E}}$.*

Corollary 1.2. *Let Ψ denote the natural functor from $D(\mathrm{Bun}_{\mathrm{SO}(2n+1)})$ to $D_{-1/2}(\mathrm{Bun}^{\omega}_{\mathrm{Sp}(2n)})$ defined by the kernel $\iota^!\Theta$ (here $D_{-1/2}(\mathrm{Bun}^{\omega}_{\mathrm{Sp}(2n)})$ stands for the corresponding twisted category of D-modules). Let Ψ^* denote the functor in the opposite direction given by $\mathbb{D}(\iota^!\Theta)$ (Verdier duality). Let also $\mathcal{E}$ be as above and let $A_{\mathcal{E}}$ be a Hecke eigen-D-module on $\mathrm{Bun}_{\mathrm{SO}(2n+1)})$ with eigenvalue $\mathcal{E}$. Then $\Psi^*\circ\Psi(A_{\mathcal{E}}) \simeq A_{\mathcal{E}} \otimes \mathrm{Cliff}(H^1_{\mathrm{dR}}(C,\mathcal{E}))$.*

Note that $\mathrm{Cliff}(H^1_{\mathrm{dR}}(C,\mathcal{E}))$ can be thought of as a categorification of the value of the L-function of $\mathcal{E}$ at $1/2$, so the above corollary is a version of the correspondence between period sheaves and L-sheaves which is the main subject of [6].

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PERIODS AND SHIMURA CORRESPONDENCES

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Classification AMS 2020: Primary 11F70; Secondary 11F25; 11F27; 11F67; 11F72; 22E50; 22E55.

Keywords: Shimura correspondence, metaplectic group, cubic cover, relative trace formula, Hecke algebra, minimal representation, period.

This talk is based on joint work [3, 4] with **Omer Offen** (Brandeis University).

The classical Shimura lift [11] takes holomorphic Hecke eigen-cuspforms of weight $k+\frac{1}{2}$ to holomorphic Hecke eigenforms of weight $2k$. It was generalized by Gelbart and Pitatetski-Shapiro [5] to a map $\mathcal{A}_g^0(\widetilde{SL}_2^{(2)}) \longrightarrow \mathcal{A}(PGL_2)$, where $\widetilde{SL}_2^{(2)}$ is the metaplectic double cover of $SL_2(\mathbb{A})$, $\mathcal{A}$ is the space of automorphic representations of the adelic group shown, the superscript ^{0} means cuspidal and the subscript $_g$ means genuine. This map may also be constructed by a theta lift. Waldspurger [12] used this to characterize the image of the lift by a period condition: $\pi \in \mathcal{A}(PGL_2)$ is a lift if and only if an integral over a certain orthogonal group (related to the choice of additive character in the theta lift) is nonzero.

It is natural to try to replace the groups above by other reductive groups or their covers. In the 1980s, Kazhdan-Patterson [8] and Savin [10] established an unramified local Shimura correspondence based on an isomorphism of local Hecke algebras. This gives rise to a conjectural global Shimura lift that matches the local lift at almost all places. The formulation is analogous to the Langlands functoriality conjecture. As with functoriality, this is mostly not known and seems to be difficult.

One may also ask when the image of such a lift is characterized by means of a period; this question is a point of contact with the relative Langlands program. For the conjectural Shimura lift for the double cover of the general linear group, Jacquet [7] formulated such a conjecture in the 1990s, while for the triple cover of SL_2 Ginzburg, Rallis and Soudry [6] were able to use an exceptional theta correspondence to obtain such a map and also to characterize its image by the non-vanishing of a certain period. Bump, Friedberg and Ginzburg [1] conjectured a period with similar properties for the (itself conjectural) lift from the triple cover of SL_3 to PGL_3. There is no dual pair for this lifting. The period involves a pairing with an automorphic form in the space of the automorphic minimal representation Θ_{min} on the split special orthogonal group SO_8.

When one has a conjectural period, a natural way to study the situation is by means of a relative trace formula. For example, Mao and Rallis [9] developed a relative trace formula for the triple cover of SL_2 and used it to give another proof of the Shimura correspondence for this group. The key step in the relative trace formula approach is the comparison of distributions on two different groups. This talk concerns a body of work by Friedberg and Offen [3, 4] establishing such a comparison for the conjectural Shimura lift from the triple cover of SL_3 to PGL_3. With additional work, we expect to prove both the existence of the lift and the conjectured period characterization of its image.

We next describe the two global distributions that we study. Let $G = PGL_3$ considered as an algebraic group defined over a number field F. We suppose that F contains the cube roots of unity. Let N be the subgroup consisting of upper triangular 3×3 unipotent matrices, and ψ be a generic character of $[N]$. Let Φ a Schwartz-Bruhat function on $G(\mathbb{A})$. Form the automorphic kernel

$$K_\Phi(g_1, g_2) = \sum_{\gamma \in G(F)} \Phi(g_1^{-1} \gamma g_2), \qquad g_1, g_2 \in G(\mathbb{A}).$$

Introduce the *relative distribution*, depending on $\theta \in \Theta_{min}$, given by

$$I(\Phi, \theta) = \int_{[G] \times [N]} K_\Phi(g, n)\, \theta(\mathrm{Ad}(g))\, \psi(n)\ d(g, n).$$

Here $\mathrm{Ad} : PGL_3 \to SO_8$ is the Adjoint representation. The period conjectured by Bump, Friedberg and Ginzburg is built into this distribution.

We compare this to the Kuznetsov distribution on the 3-fold cover of SL_3. The groups $SL_3(F)$ and $N(\mathbb{A})$ split in the 3-fold cover $\widetilde{SL}_3^{(3)}(\mathbb{A})$ of $SL_3(\mathbb{A})$. For $\widetilde{\Phi}$ an anti-genuine Schwartz-Bruhat function on this group, form the automorphic kernel

$$K_{\widetilde{\Phi}}(g_1, g_2) = \sum_{\gamma' \in SL_3(F)} \widetilde{\Phi}(g_1^{-1} \gamma' g_2), \qquad g_1, g_2 \in \widetilde{SL}_3^{(3)}(\mathbb{A}).$$

Introduce the *metaplectic Kuznetsov distribution* given by

$$I'(\widetilde{\Phi}) = \int_{[N] \times [N]} K_{\widetilde{\Phi}}(n_1, n_2)\, \psi(n_1^{-1} n_2)\ d(n_1, n_2).$$

We show that each of these distributions can be written as a sum of orbital integrals $\mathcal{O}$ (resp. $\mathcal{O}'$) that are factorizable. Each sum is over certain families ξ (resp. ξ') of relevant orbits. We show that these orbits are in one-to-one correspondence and provide a specific correspondence between the big cell representatives $\xi(a, b)$ $(a, b \in F^*)$ for the relative distribution and representatives $\xi'(c, d)$ $(c, d \in F^*)$ for the metaplectic Kuznetsov distribution. Our main result then compares the local orbital integrals for matching test functions in the local Hecke algebras.

We pass to the local situation. Let F now denote a non-archimedean local field containing the cube roots of unity and of residual characteristic at least 5, and let $\mathcal{H}$ be the spherical Hecke algebra for $PGL_3(F)$. This is the algebra of bi-K_*-invariant Schwartz functions on $PGL_3(F)$, where $K_* = GL_3(\mathcal{O}_F)Z/Z$, Z the center of $GL_3(F)$. Fix a nontrivial additive character ψ of F. Using the dual pair (SL_2, SO_8) inside Sp_{16} and the Schrodinger model for the Weil representation ω_ψ, $f \in \mathcal{H}$ acts on the space of Schwartz functions ϕ on F^8 by:

$$(\omega(f)\phi)(x) = \int_{PGL_3(F)} f(g)\, (\omega_\psi(1, \mathrm{Ad}(g))\phi)(x)\, dg.$$

Let $\phi_\circ$ be the characteristic function of $\mathcal{O}_F^8$.

Similarly, let $\widetilde{G}$ be the triple cover of $SL_3(F)$, K' be the image of $SL_3(\mathcal{O}_F)$ in $\widetilde{G}$ (the cover splits over $SL_3(\mathcal{O}_F)$), and $\widetilde{\mathcal{H}}$ be the Hecke algebra of anti-genuine K'-invariant Schwartz functions on $\widetilde{G}$. Kazhdan and Patterson gave an explicit isomorphism Sh from $\mathcal{H}$ to $\widetilde{\mathcal{H}}$. We establish the following Fundamental Lemma for Hecke Correspondences.

Theorem 0.1. *For $f \in \mathcal{H}$ and all $a, b \in F^*$,*

$$\mathcal{O}(\xi(a,b), \omega(f)\phi_\circ) \;=\; (d,c)_3 \mathcal{O}'(\xi'(c,d), Sh(f)),$$

where $c = -54a$, $d = 54b$ and where $(d,c)_3$ denotes the local cubic Hilbert symbol. A similar matching holds for the other families of relevant orbits.

The proof involves number-theoretic input. Duke and Iwaniec [2] used the Hasse-Davenport relation to compare Kloosterman integrals with a cubic character

$$\int_{\mathcal{O}^*} (t,u)_3\, \psi(au + bu^{-1})\, du \qquad (t,u)_3 \not\equiv 1$$

and cubic exponential integrals

$$\int_{\mathcal{O}} \psi(cx + dx^3)\, dx \qquad \max(|c|, |d|) = q.$$

This comparison was used in the work of Mao and Rallis [9]. We generalize the relation of Duke and Iwaniec to all $c, d \in F^*$ using the method of stationary phase ([3]). We then use this systematically in working with the orbital integrals that appear in our relative trace formula. We match the orbital integrals for the unit elements of the two Hecke algebras in [3]. We then establish the comparison for the full Hecke algebras in [4]. For the big cell, there are 28 non-trivial cases and each is evaluated in terms of cubic exponential integrals and cubic Gauss sums.

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GENERALIZED FOURIER TRANSFORMS AND MINIMAL REPRESENTATIONS

NADYA GUREVICH

Classification AMS 2020: 22E50, 11F70

Keywords: minimal representation, generalized Fourier transform, Schrodinger model.

The talk is based on joint works with D. Kazhdan and with W.T. Gan.

Let F be a p-adic field and H be the group of F-points of a reductive algebraic group, or a central covering of such a group. The minimal representation Π of H, which exists and is unique for split simply-connected simply-laced groups, is a necessary ingredient for the theta correspondence for various dual pairs (G_1, G_2) in H. The theta correspondence, which studies the restriction of Π to $G_1 \times G_2$, is an efficient method for constructing functorial lifting between representations of G_1 and G_2, both locally and globally. To study the theta correspondence, it is useful (if not necessary) to have concrete models of the minimal representation. The unitary minimal representation is often realized on the Hilbert space $L^2(\Omega)$ of square integrable functions on a space Ω and its subspace $S(\Omega)$ of smooth vectors is the smooth minimal representation of H, also denoted by Π. Indeed, it is typical to have several such models for the representation Π with natural equivariant isomorphisms between them. For each model, there is usually a maximal parabolic subgroup $Q = MN$, with Levi subgroup M and unipotent radical N, whose action is given by explicit geometric formulas. To complete the description of the action of H, it is necessary and sufficient to determine the action of an additional element $s \notin Q$. In particular, for an involution $s \notin Q$ normalizing M, one may expect a nice formula for $\Pi(s)$.

For example, in the context of classical theta correspondence, a minimal representation is a Weil representation of the double cover $\widetilde{Sp}(W)$ of a symplectic group $Sp(W)$. Given a polarization $W = W^+ \oplus W^-$, with associated Siegel parabolic subgroup $Q = MN$ stabilizing W^+, the corresponding Schrodinger model of Π is realized on the space of Schwartz-Bruhat functions $S_c(W^-)$ and a representative of the longest element of the Weyl group s acts by the classical Fourier transform.

For a split simply-laced group H of type D_n, E_6, E_7 Savin [3] constructed the analog of the Schrodinger model for the minimal representation, on which a maximal parabolic subgroup $Q = MN$ with abelian radical N acts explicitly. The space of this Schrodinger model is a space of smooth functions on a cone Ω. The cone Ω is the minimal nonzero M-orbit in the Lie algebra $\bar{\mathfrak{n}}$ of the opposite unipotent radical $\bar{N}$. For some groups, such as those of type D_n, there are several choices for Q and hence several Schrodinger models. For the group E_8, there are no such parabolic subgroups with abelian unipotent radical and hence a Schrodinger model does not exist.

However, what has been lacking for a while is an explicit description of an additional element $s \notin Q$ (normalizing M). Thus, our knowledge of the Schrodinger model of the minimal representation has for some time been less complete than desired, compared to the classical case of the Weil representation. In particular, one would like to have the analog of the Fourier transform on the cone Ω.

We single out the involutive element s in $Aut(G)$, that conjugates the parabolic subgroup Q to its opposite $\bar{Q}$. This gives rise to the inner product on $\bar{\mathfrak{n}}$ given by $\langle c_1, c_2\rangle = -\kappa(c_1, sc_2s^{-1})$ where κ is the Killing form.

For groups of type D_n, E_6 and E_7 the action of s is given by an integral operator Φ, whose restriction to the dense space $S_c(\Omega)$ of smooth functions of compact support is given by

$$\Phi(f)(c_1) = \int\limits_{F^\times}\int\limits_{\Omega} f(c_2)\psi(r\cdot\langle c_1, c_2\rangle + r^{-1})\eta(c_2)|r|^k d^\times r,$$

where the value of k is given by

$$k = \begin{cases} n-3 & G = D_n \\ 2 & G = E_6 \\ 3 & G = E_7 \end{cases}$$

and η is an M-equivariant measure on Ω, that is unique up to multiplication by a non-zero constant.

The cases $G = D_n, (n \geq 4)$ and $G = E_7$ were treated in [1] and [2] respectively and the case $G = E_6$ is the work in progress.

The explicit formula for Φ allows to describe asymptotic behavior of functions in $S(\Omega)$ near 0, the unique singular point in the affine closer of Ω.

The uniform expression for the action of the element s in all these cases can be explained by its relation to a generalized Fourier transform on basic affine space Ω as defined by Braverman and Kazhdan in [4]. The minimal M orbit Ω can be identified with $[Q_2, Q_2]\backslash M_1$ where $M_1 = [M, M]$ and $Q_2 = M_2N_2$ is a maximal parabolic subgroup of M_1. We write $\bar{Q}_2$ for the opposite parabolic subgroup, sharing the same Levi subgroup M_2. Braverman and Kazhdan have defined an unitary $M_2^{ab} \times M_1$ equivariant operator $\mathcal{F}_{Q_2,\bar{Q}_2} : L^2(\Omega) \to L^2(\bar{\Omega})$. We show that $Ad(s) \circ \mathcal{F}_{Q_2,\bar{Q}_2} = \Phi$. This also implies that the Schwartz space on Ω defined by Braverman and Kazhdan is exactly the space $S(\Omega)$ of smooth vectors of the minimal representation Π.

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PERIODS OF AUTOMORPHIC FORMS AND MOTIVIC PERIODS

MICHAEL HARRIS

I will be reporting on recent results with Grobner, Lin, and Raghuram on the relation between the expressions of critical values of automorphic L-functions in terms of automorphic periods — integrals of automorphic forms over group-theoretic cycles — and the conjectural expression in terms of Deligne's motivic periods. We use Eisenstein cohomology and the Ichino-Ikeda identity for unitary groups to provide simultaneous proofs of automorphic versions of Deligne's conjecture on critical values of Rankin-Selberg L-functions, for sufficiently regular motives, and of predicted identities between periods of automorphic representations of different unitary groups with the same base change to GL(n).

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ON NORMS ON HARISH CHANDRA MODULES

BERNHARD KRÖTZ

I report on joint work [1] with Joseph Bernstein, Pritam Ganguly, Job Kuit and Eitan Sayag.

In its inception, the focus of representation theory was the study of unitary representations. In the case where G is a real reductive group, Harish-Chandra introduced the useful category of $(\mathfrak{g}, K)$-modules, where $\mathfrak{g} = \mathrm{Lie}(G)$ and $K \subset G$ is a maximal compact subgroup. Harish-Chandra used $(\mathfrak{g}, K)$-modules to effectively study unitary representations, and thus interest shifted to the study of these more algebraic objects.

The language of $(\mathfrak{g}, K)$-modules provided the correct notion of infinitesimal equivalence of representations that allowed to deal with non-unitary representations and turned out to be of extraordinary use.

To pass from a $(\mathfrak{g}, K)$-module V to a continuous representation of G on some complete locally convex space E there are two obstacles:

One needs to complete the vector space V to such a topological vector space E and provide a continuous action of G on E that is compatible with the given $(\mathfrak{g}, K)$-module structure on V. More precisely, the induced $(\mathfrak{g}, K)$-action on the space $E^{K-\mathrm{finite}}$ of K-finite vectors in E is required to coincide with $(\mathfrak{g}, K)$-structure on the dense submodule V. Such a G-representation on a completion E of V is called a *globalization* of V.

Globalizations exist provided that V is finitely generated as a $\mathfrak{g}$-module and admissible, i.e. with finite K-multiplicities. Those $(\mathfrak{g}, K)$-modules V are called Harish-Chandra modules, and all modules V from now on will be assumed to be of this type.

The most important globalizations are obtained using so-called G-continuous norms. A norm p on V is G-continuous if the Banach completion $E = V_p$ of the normed space (V, p) carries a globalization of V. We note that such a G-action on V_p, if it exists, is unique.

Although the choice of the norm p has a great effect on the geometry induced on V, it turns out that the representation of G on the space of smooth vectors $V_p^\infty \subset V_p$ is in a very precise sense independent of the choice of the norm p. We will make this precise.

We recall the foundational Casselman-Wallach theorem, which states that up to isomorphism there exists a unique smooth Fréchet globalization V^∞ of V with moderate growth. To introduce a quantitative version, we recall the more explicit version of the theorem:

- *V admits a G-continuous norm.*
- *Any two G-continuous norms p, q on V are Sobolev equivalent.*

Sobolev equivalence means that there exists a $k \in \mathbb{N}$ such that p is dominated by the k-th Sobolev norm of q and vice versa. Notice that the second assertion implies that the smooth vectors V_p^∞ and V_q^∞ are G-isomorphic as Fréchet representations and in particular yields the uniqueness statement mentioned above.

The natural quantitative question then is to provide a reasonable bound on k in terms of the two given G-continuous norms on the Harish-Chandra module V. From now on we shall denote the unique (up to isomorphism) smooth Fréchet globalization of moderate growth of a Harish-Chandra module V by V^∞.

It appears futile to try to compare two general G-continuous norms as the order of the Sobolev domination may be arbitrarily high. Our approach is to compare norms with similar growth. More precisely, to a G-continuous norm p on V we attach the basic growth invariant $w_p : G \to \mathbb{R}_{>0}$, by setting

$$w_p(g) := \sup_{p(v)\leq 1} p(g\cdot v) \qquad (g \in G).$$

Here $g \cdot v \in V_p$ is the action of $g \in G$ on $v \in V \subseteq V_p$.

This function is positive, locally bounded and submultiplicative. Any function $w : G \to \mathbb{R}_{>0}$ satisfying these properties will be called a *weight*.

The space of all G-continuous norms on a Harish-Chandra module V is thus filtered by the set of weights. For any w we let $\mathrm{Norm}(V, w)$ be the set of G-continuous norms p with growth function $w_p \leq Cw$, with $C > 0$. The most important case is $w = \mathbf{1}$, which corresponds to uniformly bounded norms. In this case we use the shortened notation $\mathrm{Norm}(V) = \mathrm{Norm}(V, \mathbf{1})$. Here we focus mainly on the case $w = \mathbf{1}$.

It is natural to introduce a partial order on the set of norms as follows. Write $p \lesssim q$ if p is dominated by a multiple of q. Mutual domination defines an equivalence relation and we write $\mathfrak{Norm}(V)$ for the set of equivalence classes of $\mathrm{Norm}(V)$. The equivalence class of a norm p is denoted by $[p]$. Notice that $\lesssim$ induces a partial order on $\mathfrak{Norm}(V)$. In the sequel, we assume that V is such that $\mathrm{Norm}(V) \neq \emptyset$ which, for instance, is guaranteed if V is unitarizable.

0.1. **Isometric norms in harmonic analysis on homogeneous spaces.** We now provide key examples for isometric norms. Let $H \subset G$ be a closed subgroup, and let $X = H\backslash G$ be the attached homogeneous space. Let V be a Harish-Chandra module that is distinguished with respect to H, i.e. there exists a non-zero continuous linear functional $\eta : V^\infty \to \mathbb{C}$ that is H-invariant. Frobenius reciprocity yields a map

$$i_\eta : V^\infty \to C^\infty(X), \quad i_\eta(v)(Hg) := m_{v,\eta}(Hg) := \eta(g\cdot v).$$

Assume that the space X carries a G-invariant positive Radon measure μ_X and let $1 \leq r \leq \infty$. We say that (V, η) is (X, r)-bounded provided that $\operatorname{im} i_\eta \subset L^r(X) = L^r(X, \mu_X)$.

In this case the L^r norm on functions on X induces an isometric G-continuous norm on V given by:

$$p_{\eta,X,r}(v) = \|m_{v,\eta}\|_{L^r(X)} \qquad (v \in V^\infty).$$

These norms are ubiquitous in harmonic analysis.

0.2. **Minimal and maximal norms.** A key observation is that the ordered space $\mathfrak{Norm}(V)$ has unique minimal and maximal elements $[p_{\min}]$ and $[p_{\max}]$.

Let us explain the construction of a standard representative for $[p_{\min}]$.

For this we need a key result about Harish-Chandra modules: Any such V can be realized in the space of analytic functions on G via matrix coefficients. To explain that we recall that the category of Harish-Chandra modules admits a duality functor $V \mapsto \widetilde{V}$, with the dual module $\widetilde{V}$ defined as the space of K-finite vectors of the algebraic dual V^*

of V. Let now $p \in \mathrm{Norm}(V)$. Then any element $\widetilde{v} \in \widetilde{V}$ uniquely defines a continuous functional on V_p. We can thus attach to any pair $(v, \widetilde{v}) \in V \times \widetilde{V}$ a continuous matrix coefficient

$$m_{v,\widetilde{v}} : G \to \mathbb{C}, \quad g \mapsto \widetilde{v}(g \cdot v).$$

Here $g \cdot v \in V_p$ is given by the action of g on $v \in V \subseteq V_p$. The resulting function is bounded and moreover real analytic. The right derivatives $R(u)m_{v,\widetilde{v}}(e)$ for $u \in \mathcal{U}(\mathfrak{g})$ at the origin coincide with $\widetilde{v}(uv)$. Therefore, the matrix-coefficients $m_{v,\widetilde{v}}$ do not depend on the globalization V_p.

For simplicity assume that $\widetilde{V}$ is cyclic. If $\widetilde{v} \in \widetilde{V}$ is a cyclic vector of $\widetilde{V}$, then a representative of $[p_{\min}]$ is given by

$$p_{\min}(v) = \sup_{g \in G} |m_{v,\widetilde{v}}(g)| \qquad (v \in V).$$

It is straightforward to see that this construction defines a minimal element in $\mathfrak{Norm}(V)$. This construction of $p_{\min}$ can be viewed as a special case of the construction in Section 0.1, with $X = \mathrm{diag}(G)\backslash G \times G \simeq G$ and $r = \infty$.

Using double duality $\widetilde{\widetilde{V}} \simeq V$ one obtains now a maximal element of $\mathfrak{Norm}(V)$: *a maximal norm is the dual norm of a minimal norm of the dual module.*

It is an important observation of the present paper that the existence of minimal and maximal elements in $\mathrm{Norm}(V)$ has striking consequences.

0.3. **The Sobolev gap.** Natural Sobolev norms p_s of any order $s \in \mathbb{R}$ for any G continuous norm p on V can be defined algebraically via the K-Laplacian. The family $(p_s)_{s \geq 0}$ then defines the Fréchet structure on V^∞.

We then define *Sobolev gap* $s(V)$ as an invariant of the Harish-Chandra module V by setting

$$s(V) := \inf\{s > 0 \mid p_{\max} \lesssim p_{\min,s}\}.$$

We repeat that the mere finiteness of this number encodes the Casselman-Wallach theorem. Similarly we define the *w-Sobolev gap* $s(V, w)$ with respect to general weights w with the convention that $s(V) = s(V, \mathbf{1})$.

0.4. **Main Results.** The paper offers two results on the invariants $s(V, w)$: One very general on uniform finiteness and valid for any real reductive group G and a second very explicit for the basic group $G = \mathrm{SL}(2, \mathbb{R})$.

We begin by describing our general results. To prepare the statements, write $\mathcal{HC}$ for the class of all Harish-Chandra modules and let $\mathcal{HC}_{\mathrm{d}}$ and $\mathcal{HC}_{\mathrm{mp}}$ denote the subclasses corresponding to discrete series and minimal principal series. Our main results then are:

Theorem 0.1. *Let G be a real reductive group.*

1. *The Sobolev gap for discrete series is uniformly bounded, i.e.,*

$$\sup_{V \in \mathcal{HC}_{\mathrm{d}}} s(V) < \infty.$$

2. *For every weight w the w-Sobolev gap for all minimal principal series admitting G-continuous norms with growth class at most w is uniformly bounded, i.e.,*

$$\sup_{\substack{V \in \mathcal{HC}_{\mathrm{mp}} \\ \mathrm{Norm}(V,w) \neq \emptyset}} s(V, w) < \infty.$$

Theorem 0.2. *Let V be a non-trivial unitarizable irreducible Harish-Chandra module of* $\mathrm{SL}(2,\mathbb{R})$*. Then the Sobolev gap $s(V)$ is one.*

The techniques which lead up to Theorem 0.2 features a remarkable

Corollary 0.3 (Abstract Convexity Bound)**.** *Let V be an irreducible unitarizable Harish-Chandra module for* $\mathrm{SL}(2,\mathbb{R})$*. Let q be a unitary norm on V. Let $S = \mathrm{spec}_K(V) \subset \widehat{K} = \mathbb{Z}$ and $(e_n)_{n\in S}$ an orthonormal basis consisting of K-types. Let p be any isometric G-continuous norm on V and $\epsilon > 0$. Then there exists a constant $C_\epsilon > 0$, such $p \leq C_\epsilon\, q_{\frac{1}{2}+\epsilon}$. Moreover, there exists a constant $C > 0$ such that*

$$p(e_n) \leq C(1+|n|)^{\frac{1}{2}} \qquad (n \in S).$$

Below, see Theorem 0.4, we exploit this bound providing an application to harmonic analysis on homogeneous spaces and in particular to the theory of automorphic forms.

0.5. **Applications to automorphic forms.** We continue with our discussion of homogeneous spaces and let now $H = \Gamma$ be a lattice and $X = \Gamma\backslash G$. The Γ-fixed functionals η are referred to as automorphic functionals. We will assume that (V,η) is (X,∞)-bounded (see Subsection 0.1), drop η from the notation, and define the automorphic sup-norm as $p_{\mathrm{aut}} = p_{\eta,X,\infty}$, i.e.

$$p_{\mathrm{aut}}(v) = \|m_{v,\eta}\|_{L^\infty(X)} \qquad (v \in V^\infty).$$

Notice that (X,∞)-boundedness for (V,η) is automatic if X is compact or η is cuspidal. For the remainder of this section we let $G = \mathrm{SL}(2,\mathbb{R})$. With the following result we complete an extensive extensive literature on p_{aut}:

Theorem 0.4. *Let Γ be a lattice in $G = \mathrm{SL}(2,\mathbb{R})$ and $\eta : V^\infty \to \mathbb{C}$ a Γ-invariant (automorphic) functional on some unitarizable Harish-Chandra module V with K-spectrum $S = S(V) \subset \mathbb{Z}$. Let $(e_n)_{n\in S}$ be an orthonormal basis of K-types and $\epsilon > 0$. Then for every $\epsilon > 0$ there exist a constant $C_\epsilon > 0$ such that the following assertions hold:*

1. *Suppose that η is cuspidal. Then $p_{\mathrm{aut}} \leq C_\epsilon\, q_{\frac{1}{2}+\epsilon}$. Moreover, there exists a constant $C > 0$ such that*

$$|\eta(e_n)| \leq C(|n|+1)^{\frac{1}{2}} \qquad (n \in S).$$

2. *Suppose that η gives a realization in $L^r(X)$ for some $1 \leq r \leq \infty$, then*

$$\|m_{v,\eta}\|_{L^r(X)} \leq C_\epsilon\, q_{\frac{1}{2}+\epsilon}(v) \qquad (v \in V^\infty).$$

Note that unitary Eisenstein series satisfy the assumption of the second item for $r > 2$.

Classification AMS 2020: 22F30, 22E46, 53C35, 22E40

Keywords: Harish-Chandra module, Sobolev norms for representations, Casselman-Wallach theorem, bounds on cusp forms

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TOWARDS BAR INVOLUTIONS ON DOUBLE AFFINE HECKE ALGEBRAS

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Classification AMS 2020: 20C08.

Keywords: Double affine Hecke algebras; bar involution; Steinberg variety.

Bar involutions on Hecke algebras of Coxeter groups were introduced by Kazhdan and Lusztig in their seminal paper [3]. They are of crucial importance in Geometric Representation theory, especially, in establishing various kinds of character formulas. The goal of this project, joint with Dougal Davis and Calder Morton-Ferguson, is to extend the bar involutions to double affine Hecke algebras introduced by Cherednik in [1].

Let G be an extended Kac-Moody group, W be its Weyl group, P the weight lattice, and $R := \mathbb{Z}[v^{\pm 1}]$. Then we can form the affine Hecke algebra $\mathcal{H}_v^a(W)$ for G. It contains the Iwahori-Hecke algebra $\mathcal{H}_v(W)$ and the group algebra RP as subalgebras; the commutation relation between the generators T_s of $\mathcal{H}_v(W)$ and the elements $X_\lambda \in RP$ with $\lambda \in P$ is given by the Bernstein relation. In the case when G is finite dimensional semisimple, we recover the usual affine Hecke algebra of the Langlands dual group $\check{G}$, while for G untwisted affine, we get a version of the double affine Hecke algebra.

In the case when G is finite dimensional semisimple, Lusztig in [4] discovered a formula for the bar involution on $\mathcal{H}_v^a(W)$ in terms of the Bernstein presentation (a usual formula would be in terms of the Coxeter presentation). The formula reads (up to some normalization):

$$\bar{a} = \Upsilon(T_{w_0}\delta(a)T_{w_0}^{-1}),$$

where Υ is the automorphism of $\mathcal{H}_v^a(W)$ induced by the Cartan involution of G, w_0 is the longest element of W, and δ is the ring automorphism of $\mathcal{H}_v^a(W)$ given by $\delta(v) = v^{-1}, \delta(T_w) = T_{w^{-1}}^{-1}, \delta(X_\lambda) = X_{-\lambda}$. Note that while δ is defined for the general Kac-Moody G, neither T_{w_0} nor Υ are (for the former, there's just no longest element in the infinite Coxeter group W).

Our first result is to explain in which sense $\Upsilon(T_{w_0}?T_{w_0}^{-1})$ still makes sense. Namely, we establish an Υ-twisted version of T_{w_0} as an element of some geometrically defined bimodule for $\mathcal{H}_v^a(W)$. In order to give a construction, we need some notation. Let B, B^- denote the usual and opposite Borel subgroups of G, a pro- and ind-algebraic groups, respectively. Consider the stack $B\backslash G/B^-$, a countable union of finite type quotient stacks, and set $\mathcal{B} := K^{\mathbb{G}_m}(T^*(B\backslash G/B^-))$, where $\mathbb{G}_m$ acts by dilations of cotangent fibers. One can equip $\mathcal{B}$ with commuting left and right $\mathcal{H}_v^a(W)$-actions making it a bimodule. Next, we consider the following elements $1_1^w \in \mathcal{B}$ for $w \in W$. Let $\dot{w}$ denote the $B \times B^-$-orbit in $B\backslash G/B^-$ corresponding to w (so that $w = 1$ gives an open orbit), let $j_w : \dot{w} \hookrightarrow B\backslash G/B^-$. Let $\underline{\mathbb{C}}_w$ denote the constant sheaf on this orbit. It has a natural mixed Hodge module structure, and $j_{w,!}\underline{\mathbb{C}}_w$ inherits a mixed Hodge structure. The associated graded sheaf with respect to the Hodge filtration, $\mathrm{gr}^H(j_{w,!}\underline{\mathbb{C}}_w)$ becomes an object of $\mathrm{Coh}^{\mathbb{G}_m}(T^*(B\backslash G/B^-))$. For 1_1^w we take the K_0-class of $\mathrm{gr}^H(j_{w,!}\underline{\mathbb{C}}_w)$.

It turns out that $\mathcal{B}$ is a bimodule over $\mathcal{H}^a_v(W)$. Here v is the equivariant parameter for the group $\mathbb{G}_m$, the generators T_s act by convolutions with usual correspondences (resulting in $T_w 1^1_! = 1^1_! T_w = 1^w_!$ for all $w \in W$), while X_λ act on the left and on the right by twisting with the corresponding characters of B and B^-, where the identification $\mathfrak{X}(B) \cong P$ is via the identity map, while the identification $\mathfrak{X}(B^-) \cong P$ is via -1. In the case when G is finite dimensional semisimple, $\mathcal{B}$ is the Υ-twisted regular bimodule, and $\Upsilon(T_{w_0} ? T_{w_0}^{-1})$ is the commutation past $1^1_!$. So, in the general case, for $a \in \mathcal{H}^a_v(W)$, the operation of commuting the element $\delta(a)$ past $1^1_!$ should be viewed as an analog of $a \mapsto \bar{a}$.

Defined in this way, the bar operation sends an element of $\mathcal{H}^a_v(W)$, a finite sum of the form $\sum_w F_w T_w$ with $F_w \in RP$, to an infinite sum of this form. In particular, it does not make sense to speak about compatibility of this operation with the algebra structure. To make sense of this, we need to modify $\mathcal{H}^a_v(W)$. Namely, define the *strict Tits cone*, $\mathcal{T}^> \subset P$ as the set of all elements λ such that only finitely many real roots α satisfy $(\lambda, \alpha) \leqslant 0$ (for the usual Tits cone one considers the strict inequality). Then the R-span of $X_\lambda T_w$ with $w \in W, \lambda \in \mathcal{T}^>$ is a non-unital subalgebra of $\mathcal{H}^a_v(W)$ to be denoted by $\mathcal{H}^>_v(W)$. We construct a completion $\hat{\mathcal{H}}^>_v(W)$ of $\mathcal{H}^>_v(W)$ consisting of suitable R-linear combinations of the elements $X_\lambda T_w, \lambda \in \mathcal{T}^>, w \in W$, such that:

- The product extends from $\mathcal{H}^>_v(W)$ to $\hat{\mathcal{H}}^>_v(W)$.
- The map $a \mapsto \bar{a}$ is a ring automorphism of $\hat{\mathcal{H}}^>_v(W)$.

Here are some future directions. With Dougal Davis, we interpret the bar operation on $\hat{\mathcal{H}}^>_v(W)$ in categorical terms. Namely, we are in the process of showing that (up to completions) there is a subcategory in $D\operatorname{Coh}^{\mathbb{G}_m}(T^*(B\backslash G/B^-))$ that carries a highest weight structure so that the bar operation becomes a transition matrix between standard and costandard objects. Another interesting direction is to compare our construction with the construction of R-polynomials for affine Kac-Moody groups, [5, 2]

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PERIOD INTEGRALS ATTACHED TO STRONGLY TEMPERED BZSV QUADRUPLES

ZHENGYU MAO (JOINT WORK WITH CHEN WAN AND LEI ZHANG)

Classification AMS 2020:Primary 11F67; 11F72

Keywords: relative Langlands duality, strongly tempered hyperspherical Hamiltonian spaces

In [1], Ben-Zvi, Sakellaridis, and Venkatesh introduced a striking relative Langlands duality for the so-called anomaly-free hyperspherical Hamiltonian spaces, which we will refer to as the BZSV duality. In Theorem 3.6.1 of [1], Ben-Zvi, Sakellaridis, and Venkatesh proved a structure theorem stating that each hyperspherical G-Hamiltonian spaces is associated to a quadruple $\Delta = (G, H, \rho_H, \iota)$ where H is a split reductive subgroup of G; ρ_H is a symplectic representation of H; and ι is a homomorphism from SL_2 into G whose image commutes with H. We will call such a quadruple the BZSV quadruple. Let $\hat{G}$ be the dual group of G. The BZSV duality is a conjectural duality between the set of anomaly-free hyperspherical BZSV quadruples of G and the set of anomaly-free hyperspherical BZSV quadruples of $\hat{G}$.

Despite its conceptual beauty, a major challenge in BZSV duality is the lack of a general algorithm to explicitly compute the dual of a given anomaly-free hyperspherical Hamiltonian space. In other words, for a given anomaly-free hyperspherical BZSV quadruple $\Delta = (G, H, \rho_H, \iota)$, there is currently no known systematic procedure to determine its dual $\hat{\Delta}$. This remains a fundamental open problem.

In Section 4 of [1], the authors devised an algorithm to compute the dual in a special case known as the polarized case, which is when the symplectic representation $\rho_{H,\iota}$ of H is of the form $\rho_{H,\iota} = \tau \oplus \tau^\vee$ for some representation τ of H. In particular this include the cases when $\Delta = (G, H, 0, 1)$ (i.e. the spherical variety case).

In this work, we focus on another fundamental case: when the Hamiltonian variety is a vector space, i.e. the case when $\Delta = (G, G, \rho, 1)$. We describe an algorithm to compute the dual in this case (which are called *strongly tempered BZSV quadruples*) and provide several evidence of the duality conjecture in this case.

1. PERIOD INTEGRAL CONJECTURE

For an automorphic form ϕ of $G(\mathbb{A})$ (resp. $\hat{G}(\mathbb{A})$), we can define the period integral $\mathcal{P}_{H,\iota,\rho_H}(\phi)$ (resp. $\mathcal{P}_{\hat{H}',\hat{\iota}',\rho_{\hat{H}'}}(\phi)$) of it associated to the quadruple. Let's briefly recall the definition. We have a symplectic representation $\rho_{H,\iota} : H \to \mathrm{Sp}(V)$. Let Y be a maximal isotropic subspace of V and Ω_ψ be the Weil representation of $\widetilde{\mathrm{Sp}}(V)$ on the Schwartz space $\mathcal{S}(Y(\mathbb{A}))$. The anomaly free condition on $\rho_{H,\iota}$ ensures $\widetilde{\mathrm{Sp}}(V)$ splits over $Im(\rho_{H,\iota})$ and Ω_ψ restricts to a representation of $H(\mathbb{A})$ on $\mathcal{S}(Y(\mathbb{A}))$. We define the theta series

$$\Theta_\psi^\varphi(h) = \sum_{X \in Y(k)} \Omega_\psi(h)\varphi(X),\ h \in H(\mathbb{A}), \varphi \in \mathcal{S}(Y(\mathbb{A})),$$

and we can define the period integral to be

$$\mathcal{P}_{\Delta}(\phi,\varphi)=\int_{H(k)\backslash H(\mathbb{A})}\mathcal{P}_{\iota}(\phi)(h)\Theta_{\psi}^{\varphi}(h)dh.$$

Here $\mathcal{P}_{\iota}$ is the degenerate Whittaker period associated to ι. To simplify the notation, we will omit the Schwartz function in the notion of the period and simply write it as $\mathcal{P}_{\Delta}(\phi)$.

Conjecture 1.1. *(Ben-Zvi–Sakellaridis–Venkatesh, Conjecture 14.3.5 and (14.6.4) of* [1]*)*

(1) *Let π be an irreducible discrete automorphic representation of $G(\mathbb{A})$ and let $\nu:\pi\to L^2(G(k)\backslash G(\mathbb{A}))_{\pi}$ be an embedding. Then the period integral*

$$\mathcal{P}_{\Delta}(\phi),\ \phi\in Im(\nu)$$

is nonzero only if the Arthur parameter of π factors through $\hat{\iota}':\hat{H}'(\mathbb{C})\times \mathrm{SL}_2(\mathbb{C})\to \hat{G}(\mathbb{C})$. If this is the case, π is a lifting of a global tempered Arthur packet Π of $H'(\mathbb{A})$ (the Langlands dual group of $\hat{H}'$). Then we can choose the embedding ν so that

$$\frac{|\mathcal{P}_{\Delta}(\phi)|^2}{\langle\phi,\phi\rangle}\text{“}=\text{”}\frac{L(1/2,\Pi,\rho_{\hat{H}'})\cdot\Pi_{k\in\hat{I}}L(k/2+1,\Pi,\hat{\rho}_k)}{L(1,\Pi,Ad)^2},\ \phi\in Im(\nu).$$

Here $\langle,\rangle$ is the L^2-norm.

(2) *Same statement holds with the role of G and $\hat{G}$ reversed.*

2. DUAL OF SYMPLECTIC VECTOR SPACES

Knop and Losev independent gave a classification of the symplectic vector spaces. Let $\hat{\Delta}=(\hat{G},\hat{G},\hat{\rho},1)$ be a BZSV quadruple arising from a symplectic vector space, we find its dual with the following observations:

- Clearly G is the dual of $\hat{G}$.
- In Knop's work [2], for each symplectic vector space he associates a Weyl group which determines H, and a Levi subgroup which determines ι.
- The last data ρ_H is assigned in an ad hoc way, motivated by known Rankin-Selberg integrals and unramified relative character calculations.

When the above cases also fall within the polarized quadruples studied in [1], then the dual quadruple matches with their description.

A detailed description of the dual quadruples is given in [4].

Some period integrals associated to the dual quadruple appeared in previous Rankin-Selberg constructions:

- Integrals for exterior square $L-$functions by Bump-Friedberg.
- Integrals for Spin $L-$function by Bump-Ginzburg.
- Integrals for standard $L-$functions of exceptional groups E_6 by Ginzburg.
- Multivariable Rankin-Selberg integrals by Ginzburg-Hundley and Pollack-Shah.
- Rankin-Selberg convolution by Jacquet-Piatetski-Shapiro-Shalika.
- Integrals for exterior square $L-$functions by Jacquet-Shalika.
- Some (about 10) new Rankin-Selberg constructions.
- If we allow the generic stabilizer to be not connected, then some examples lead to Rankin-Selberg construction on covering groups, for example Bump-Ginzburg's construction for symmetric square L function of GL_n.

There are cases where the period integral has been studied before, such as Gan-Gross-Prasad period, Ginzburg-Rallis period. If we can relate the period integrals to Rankin-Selberg integrals or previous studied periods, then we can provide evidence of the duality with the prior knowledge on these period integrals.

There are about 30 period integrals that have not been studied before. In these cases the evidence of the duality comes from the following conjecture of BZSV: (we will not define the reduced quadruple and inflated quadruple in this note).

Conjecture 2.1. *If $\Delta_{red} = (L, H, \rho_H \oplus \rho_\iota, 1)$ is the reduced quadruple of $\Delta = (G, H, \rho_H, \iota)$, and the dual of Δ_{red} is $\widehat{\Delta_{red}}$, then $\hat{\Delta}$ is the inflated quadruple of $\widehat{\Delta_{red}}$.*

Example: When $\hat{\Delta} = (\mathrm{SL}_6, \mathrm{SL}_6, \wedge^3, 1)$, its dual is $\Delta = (\mathrm{GL}_6/Z, \mathrm{GL}_2/Z, 0, \iota)$. Here the reduced quadruple is

$$\Delta_{red} = (\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2, \mathrm{GL}_2, 0, 1)$$

whose attached period is the trilinear period. $\widehat{\Delta_{red}}$ is

$$(\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2, \mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2, \mathrm{GL}_2^1 \otimes \mathrm{GL}_2^2 \otimes \mathrm{GL}_2^3, 1).$$

Then $\hat{\Delta}$ is an inflated quadruple of $\widehat{\Delta_{red}}$.

Finally we remark that one motivation to study the strongly tempered BZSV quadruples is the following conjecture, which connects all period integrals of BZSV quadruples to period integrals attached to the strongly tempered ones.

Conjecture 2.2. *Mao-Wan-Zhang* [3] *Let $\Delta = (G, H, \rho_H, \iota)$ be a BZSV quadruple. We conjecture a RTF comparison between $\hat{G}$ and $\hat{H}$ (dual of H) which reflects the functorial lift from $\hat{H}$ to $\hat{G}$.*

Let $\hat{\Delta}$ be the dual of Δ. Let Δ' be the quadruple dual to $(H, H, \rho_\Delta, 1)$ (where ρ_Δ can be determined from ρ_H and ι with an explicit algorithm, $\rho_\Delta = \rho_H \oplus \cdots$).

- *Let $I(f)$ on $\hat{G}$ be the distribution defined by $\mathcal{P}_{\hat{\Delta}}$ and (U, ψ_U)-coefficient (determined by ι).*
- *Let $J(f')$ on $\hat{H}$ be the distribution defined by $\mathcal{P}_{\Delta'}$ and $(N_{\hat{H}}, \psi)$-coefficient (generic Whittaker coefficient on $\hat{H}$.)*
- *We expect a relative trace identity $I(f) = J(f')$.*

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LOCAL-GLOBAL PRINCIPLES FOR PERIODS OF AUTOMORPHIC REPRESENTATIONS

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Classification AMS 2020: 11F70

Keywords: period integrals, automorphic representations, distinguished representations

These notes summarize some of the main results of [7] which is a joint work with Omer Offen and Chang Yang.

Let F be a number field with ring of adeles $\mathbb{A}$, let D/F be an F-central division algebra of dimension d^2, and let m be a positive integer. We denote by G_m the F-reductive group such that $G_m(F) := \mathrm{GL}_m(D)$. We denote by E/F a separable quadratic algebra of dimension 2, i.e. either E is a quadratic field extension of F, or $E \simeq F \times F$. We write $E = F[\alpha]$ with $\mathrm{tr}_{E/F}(\alpha) = 0$. From now on we make the following assumption.

Assumption 0.1. *E embeds in $\mathcal{M}_m(D)$ in such a way that $\mathrm{tr}_{\mathcal{M}_m(D)/F}(\alpha) = 0$.*

This implies that md is even, and furthermore that m is even when $E = F \times F$. We denote by θ the inner involution of G_m defined by α, and denote by H_m its fixed points group. We let $\pi = \otimes'_v \pi_v$ be a cuspidal (irreducible) automorphic representation of $G_m(\mathbb{A})$.

Definition 0.2. (1) *We say that π is E-distinguished if its central character is trivial, and the period integral*

$$\int_{\mathbb{A}^\times H_m(F)\backslash H_m(\mathbb{A})} \varphi(h)dh$$

does not vanish on π.

(2) *We say that π_v is E_v-distinguished if $\mathrm{Hom}_{H_m(F_v)}(\pi_v, \mathbb{C})$ is not reduced to zero.*

It is clear that if π is E-distinguished, then each of its local components π_v is E_v-distinguished. We want to understand under which extra conditions the converse holds. From now on we make a second assumption.

Assumption 0.3. *The Jacquet-Langlands transfer $\mathrm{JL}(\pi)$ of π to $\mathrm{GL}_n(\mathbb{A})$ is also cuspidal.*

Each local component π_v of π can be written in a unique (up to re-ordering) way as a commutative Bernstein-Zelevinsky product

$$\pi_v \simeq \delta_{1,v} \times \cdots \times \delta_{r_v,v},$$

where the representations $\delta_{i,v}$ are essentially square-integrable. The following result is proved in [1], building on the works of many.

Theorem 0.4. *The representation π_v is E_v-distinguished if and only if:*

1. *there exists an involution ι_v of $\{1, \dots, r_v\}$ such that $\delta_{\iota_v(k),v} = \delta_{k,v}^\vee$ for all $k \in \{1, \dots, r_v\}$.*

(2) *The representation* $\delta_{k,v}$ *is* E_v*-distinguished whenever* $\iota_v(k) = k$.

If π_v *is* E_v*-distinguished, then* $\mathrm{Hom}_{H_m(F_v)}(\pi_v, \mathbb{C})$ *has dimension one.*

Let us call the multiset $\{\delta_{1,v}, \ldots, \delta_{r_v,v}\}$ the ESI support of π_v. We denote by JL_v the local Jacquet-Langlands transfer to the split form.

Definition 0.5. *If* π_v *is* E_v*-distinguished, we say that it is* E_v*-compatible if there exists no non* E_v*-distinguished* δ_v *in the ESI support of* π_v *such that* $\mathrm{JL}_v(\delta_v)$ *is* $F_v \times F_v$*-distinguished.*

We define η to be the quadratic character of $F^\times\backslash\mathbb{A}^\times$ attached to the quadratic extension E/F by class field theory when E/F is quadratic, and to be the trivial character of $F^\times\backslash\mathbb{A}^\times$ when $E \simeq F \times F$. The following result is the main theorem [7] for linear and twisted linear periods.

Theorem 0.6. *Let* $L(s, \mathrm{JL}(\pi))$ *be the standard L-function of* $\mathrm{JL}(\pi)$*, and let* $L(s, \mathrm{JL}(\pi), \wedge^2)$ *be its exterior-square L-function. Then* π *is* E*-distinguished if and only if:*

1. *Each local component* π_v *is* E_v*-distinguished and* E_v*-compatible.*
2. *The completed L-function* $L(s + 1/2, \mathrm{JL}(\pi))L(s + 1/2, \eta \otimes \mathrm{JL}(\pi))L(2s, \mathrm{JL}(\pi), \wedge^2)$ *has a necessarily simple pole at* $s = 0$.

Moreover when $E \simeq F \times F$ *and* d *is odd, Condition* (1) *can be ignored.*

This theorem naturally extends a well-known result of Waldspurger ([9, Théorème 2]) for E/F quadratic and $n = 2$. When $D = F$ and $E = F \times F$, it reduces to a well-known result of Jacquet-Friedberg [5]. The implication (1) $\Longrightarrow$ (2) proves a conjecture of Chong Zhang ([10]) when $E \simeq F \times F$, and the direct implication of the Guo-Jacquet conjecture stated in [6] when E/F is quadratic. When $D = F$ the Guo-Jacquet conjecture was previously partially established in [2] using the Guo-Jacquet trace formula, and later fully established in [8] via the residue method. When $D = F$ or D/F is quaternionic, it was established in a more general form involving a character twist via a new global trace formula approach, under local and global constraints, in [11]. The method in [7] is based on a careful study of local and global intertwining periods. The crucial global input is the Maass-Selberg relations, and the key local input is the computation of the order of the pole at $s = 0$ of certain local intertwining periods. The paper [7] also contains similar statements for Galois periods, generalizing results of Flicker [3] and Flicker-Hakim [4].

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SYMMETRIC BOW VARIETIES AND EXAMPLES OF S-DUAL PAIRS

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Classification AMS 2020: Primary: 14D21; Secondary: 22E57, 81T13

Keywords: S-duality, relative Langlands duality, Coulomb branches, Bow varieties

Let us first briefly recall the relation between S-duality for 3-dimensional supersymmetric (SUSY) quantum field theories (QFTs) [6] and relative Langlands duality [1]. An exposition can be found in [8]. We use the symbols G, H, etc. for either compact Lie groups in the QFT context or complex reductive groups in the Langlands context.

Kapustin-Witten [7] interpreted geometric Langlands conjecture as a consequence of S-duality of 4-dimensional SUSY Yang-Mills (YM) theory, compactified over a Riemann surface Σ: there are two twists of SUSY YM for the gauge group G, the A-twist and the B-twist, corresponding to the automorphic and Galois sides in the usual Langlands correspondence. They assign categories $\mathcal{A}_G(\Sigma)$, $\mathcal{B}_G(\Sigma)$ to $\Sigma \times \mathrm{pt}$, respectively. The S-duality predicts an equivalence between SUSY YM theories for G and for $G^\vee$, the Langlands dual group. Under this, A-twist and B-twist are exchanged, hence we have equivalences $\mathcal{A}_G(\Sigma) \simeq \mathcal{B}_{G^\vee}(\Sigma)$, $\mathcal{B}_G(\Sigma) \simeq \mathcal{A}_{G^\vee}(\Sigma)$. This should be the statement of geometric Langlands correspondence.

Subsequently, Gaiotto-Witten [6] studied the role of a 3d SUSY QFT $\mathscr{T}$ with $G \times H$ symmetry, as an *interface* between SUSY YM theories for G and H in this context. As a consequence, it gives functors $\mathcal{A}_{\mathscr{T}}(\Sigma)\colon \mathcal{A}_G(\Sigma) \to \mathcal{A}_H(\Sigma)$ for A-twist, and $\mathcal{B}_{\mathscr{T}}(\Sigma)\colon \mathcal{B}_G(\Sigma) \to \mathcal{B}_H(\Sigma)$ for B-twist. There is an S-dual 3d QFT $\mathscr{T}^\vee$ with $G^\vee \times H^\vee$ symmetry, which gives a commutative diagram

$$\begin{tikzcd}
\mathcal{A}_G(\Sigma) \arrow[r] \arrow[d, "\mathcal{A}_{\mathscr{T}}(\Sigma)"'] & \mathcal{B}_{G^\vee}(\Sigma) \arrow[d, "\mathcal{B}_{\mathscr{T}^\vee}(\Sigma)"] \\
\mathcal{A}_H(\Sigma) \arrow[r] & \mathcal{B}_{H^\vee}(\Sigma),
\end{tikzcd} \tag{0.1}$$

where horizontal arrows are geometric Langlands equivalences. This is relevant for Langlands functoriality, but we do not assume that $\mathscr{T}$ or $\mathscr{T}^\vee$ come from $G \to H$. In the special case when $G = \{1\}$, $\mathcal{A}_G(\Sigma)$, $\mathcal{B}_{G^\vee}(\Sigma)$ are trivial categories with trivial objects. Their images under $\mathcal{A}_H(\Sigma)$, $\mathcal{B}_{H^\vee}(\Sigma)$ are nontrivial objects, which are exchanged under geometric Langlands correspondence for H.

Standard examples of $\mathscr{T}$ come from a $G \times H$-hamiltonian space $\mathbf{M}$, i.e., a holomorphic symplectic manifold $\mathbf{M}$ with $G \times H$ symplectic action together with the moment map $\mu\colon \mathbf{M} \to \mathfrak{g}^* \times \mathfrak{h}^*$.[1] Conversely, any 3d SUSY QFT $\mathscr{T}$ can be *approximated by* (or has a low-energy effective field theory given by) a $G \times H$-hamiltonian space. This

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[1] If the reader is familiar with BFN construction of Coulomb branches of gauge theories [2], it is a gauge theory for the trivial group $\{1\}$ with a representation $\mathbf{M}$ (which is assumed to be $T^*\mathbf{N}$ in [2]), and $G \times H$ is the flavor symmetry group. Thus $\mathbf{M}$ is the Higgs branch, while the Coulomb branch is a single point.

hamiltonian space is called the Higgs branch of $\mathscr{T}$, denoted by $\mathcal{M}_H(\mathscr{T})$. It should be emphasized that $\mathcal{M}_H(\mathscr{T})$ gives only an approximation of $\mathscr{T}$. The Higgs branch $\mathcal{M}_H(\mathscr{T})$ is usually singular, and the corresponding QFT and what we mean by *approximation* must be discussed with care.

Let us suppose $\mathscr{T}_{\mathbf{M}_{12}}$, $\mathscr{T}_{\mathbf{M}_{23}}$ are QFTs associated with $G_2 \times G_1$-hamiltonian space $\mathbf{M}_{12}$ and $G_3 \times G_2$-hamiltonian space $\mathbf{M}_{23}$. The composite of functors $\mathcal{A}_{\mathscr{T}_{\mathbf{M}_{12}}} \circ \mathcal{A}_{\mathscr{T}_{\mathbf{M}_{23}}}$ is given by a QFT $(\mathscr{T}_{\mathbf{M}_{12}} \times \mathscr{T}_{\mathbf{M}_{23}}) \mathbin{\#} G_2$, obtained from $\mathscr{T}_{\mathbf{M}_{12}} \times \mathscr{T}_{\mathbf{M}_{23}}$ by *gauging* with respect to the diagonal G_2.[2] This gauging process yields a new QFT by the path integral over all G_2-connections. The Higgs branch of $(\mathscr{T}_{\mathbf{M}_{12}} \times \mathscr{T}_{\mathbf{M}_{23}}) \mathbin{\#} G_2$ is the hamiltonian reduction $(\mathbf{M}_{12} \times \mathbf{M}_{23})/\!\!/\!\!/G_2$, which certainly does not contain any information about G_2-connections. This is OK, as $(\mathbf{M}_{12} \times \mathbf{M}_{23})/\!\!/\!\!/G_2$ is usually singular, and the singularity is related to the nontriviality of the *Coulomb branch* of $(\mathscr{T}_{\mathbf{M}_{12}} \times \mathscr{T}_{\mathbf{M}_{23}}) \mathbin{\#} G_2$.

Because of (0.1), we would like to understand how to compute $\mathscr{T}^\vee$ from $\mathscr{T}$, say if $\mathscr{T}$ is coming from $\mathbf{M}$. The answer was given in [6]. But their answer

$$\mathscr{T}^\vee = ((\mathscr{T} \times \mathscr{T}[G]) \mathbin{\#} G)^* \tag{0.2}$$

involves a *mysterious* SUSY QFT $\mathscr{T}[G]$ with $G \times G^\vee$ symmetry and an operation $*$ on 3d SUSY QFTs called 3d mirror symmetry, as well as the less mysterious operation $\#G$, gauging with respect to G. The former two concepts are difficult to make mathematically rigorous.

In [4], a mathematically rigorous proposal for *Higgs branch* of $\mathscr{T}^\vee$ was given under the cotangent type assumption $\mathbf{M} = T^*\mathbf{N}$. For details of the proposal, please refer to [8], where the Higgs branch of $\mathscr{T}^\vee$ is called the S-dual of $\mathbf{M}$, and denoted by $\mathbf{M}^\vee$. Although the definition is mathematically rigorous, it is rather unsatisfactory: the proposed definition usually gives a singular $\mathbf{M}^\vee$. Then it is not possible to consider the S-dual of $\mathbf{M}^\vee$. On the other hand, $\vee$ is an involution at the level of 3d SUSY QFTs, i.e., $\mathscr{T}^{\vee\vee} = \mathscr{T}$.

Ben-Zvi, Sakellaridis and Venkatesh [1] formulated the S-duality as pairs of functors with the commutativity in (0.1), as well as versions with Σ replaced by other 'spaces' arising in other contexts, such as a smooth projective curve over a finite field, a number field, and also their local counterparts, etc. Moreover, they considered the case when $\mathbf{M}$ is *hyperspherical*, for whose definition we refer to the original paper [1]. They proposed that the S-dual theory $\mathcal{T}^\vee$ is given by another hyperspherical hamiltonian space $\mathbf{M}^\vee$. The definition of $\mathbf{M}^\vee$ in [4] naturally arises from (0.1) with the 'local' Σ, hence there is no risk of confusion about the notation $\mathbf{M}^\vee$. In this case, $\mathbf{M}^\vee$ is also hyperspherical, hence it is meaningful to ask why $\mathbf{M}^{\vee\vee} = \mathbf{M}$. This is checked in many examples by computation, but remains mysterious as there is no clue as to how to go in the opposite direction in the definition of [4].

Basic examples of S-dual pairs of hamiltonian spaces are given by branes in string theory, as explained in [6]. They can be understood in the context of Coulomb branches

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[2] We twist the G_2 action on $\mathbf{M}_{12}$ by Chevalley involution. We denote the twist symbolically by $G_1 \curvearrowright \mathbf{M}_{12} \curvearrowleft G_2$.

of quiver gauge theories of type A and Cherkis bow varieties (see [9]):

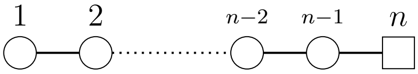
$$\begin{aligned} \mathrm{GL}_m \curvearrowright \mathbf{M} &= T^*\mathrm{Hom}(\mathbb{C}^m,\mathbb{C}^n) \curvearrowleft \mathrm{GL}_n,\\ \mathrm{GL}_m \curvearrowright \mathbf{M}^\vee &= \begin{cases} T^*(\mathrm{GL}_m\times\mathbb{C}^m) & \text{if } m=n\\ \mathrm{GL}_m\times\mathcal{S}(m-n,1^n) & \text{if } m>n\\ \text{above with } m\longleftrightarrow n & \text{if } m<n \end{cases} \curvearrowleft \mathrm{GL}_n, \end{aligned} \tag{0.3}$$

where $\mathcal{S}(m-n,1^n)$ is the Slodowy slice associated with the nilpotent orbit corresponding to the partition $(m-n,1^n)$.

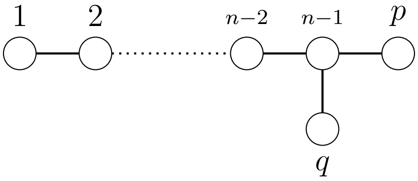
We can compute $\mathrm{M}^\vee$ from M, which was essentially done in [9]: The bifundamental M appears as a basic ingredient for quiver gauge theories of type A. The Coulomb branches of those gauge theories are Cherkis bow varieties, whose basic ingredients are bifundamentals and $\mathrm{M}^\vee$. On the other hand, we do not have a definition of $\mathrm{M}^{\vee\vee}$ for general $\mathrm{M}^\vee$ as it is not a cotangent bundle. Nevertheless, hypothetical compatibility of the S-duality with Hanany-Witten transition allows us to compute $\mathrm{M}^{\vee\vee}$, and gives M. See [8].

We can give more examples by considering quiver gauge theories of type D. This was given in [6], but can also be explained in the context of symmetric bow varieties, introduced in [5]. As an illustration, let us first consider the following quiver gauge theory:

(0.4)



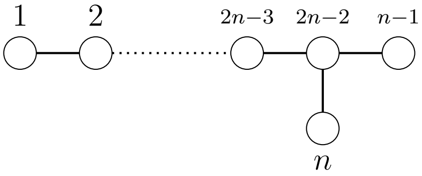
By [6], this gives the theory $\mathscr{T}[\mathrm{GL}_n]$, used in the definition of $\mathscr{T}^\vee$. Then a modified quiver gauge theory



with $n=p+q$ is $\big(\mathscr{T}[\mathrm{GL}_n]\times\mathscr{T}_{T^*(\mathrm{GL}_n/\mathrm{GL}_p\times\mathrm{GL}_q)}\big) /\!\!/\!\!/ \mathrm{GL}_n$. Comparing with (0.2), we find that the Coulomb branch of this theory, which is the Higgs branch of the 3d mirror theory, is nothing but the S-dual of $T^*(\mathrm{GL}_n/\mathrm{GL}_p\times\mathrm{GL}_q)$. This Coulomb branch was computed in [3] as the moduli space of based maps from $\mathbb{P}^1$ to a flag variety of type D with degree given by numbers assigned to vertices. It can be computed ([5]) as the fixed point locus in a Cherkis bow variety with respect to an involution. The answer we get is

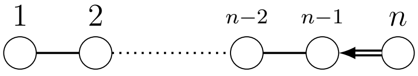
$$\mathrm{GL}_n \curvearrowright \begin{cases} (\mathrm{GL}_n\times\mathcal{S}(p-q,2q)) /\!\!/\!\!/ \mathrm{Sp}_{2q} & \text{if } p>q\\ T^*(\mathrm{GL}_n\times\mathbb{C}^n) /\!\!/\!\!/ \mathrm{Sp}_n & \text{if } p=q=\frac{n}{2}. \end{cases} \tag{0.5}$$

A slightly more involved example is given by the following quiver gauge theory:



Note that $n+(n-1) \neq 2n-2$, hence it is different from the above example. Nevertheless, we can compute this theory as $\left(\mathscr{T}[\mathrm{GL}_{2n}] \times \mathscr{T}_{\mathrm{GL}_{2n}\times \mathcal{S}(2^n)/\!\!/\!\!/\mathrm{GL}_n}\right) /\!\!/\!\!/ \mathrm{GL}_{2n}$ by an application of Hanany-Witten transition. Here, $\mathcal{S}(2^n)$ is Slodowy slice to the nilpotent orbit for (2^n), and GL_n acts as the stabilizer of the corresponding $\mathfrak{sl}_2$ triple. Therefore the Coulomb branch of this quiver gauge theory computes the S-dual of $\mathrm{GL}_{2n} \curvearrowright \mathrm{GL}_{2n} \times \mathcal{S}(2^n)/\!\!/\!\!/\mathrm{GL}_n$. The Coulomb branch is the fixed point locus in a Cherkis bow variety with respect to an involution as above. The answer is $T^*(\mathrm{GL}_{2n}/\mathrm{Sp}_{2n})$. It has a natural GL_{2n}-action induced from the left multiplication $\mathrm{GL}_{2n} \curvearrowright \mathrm{GL}_{2n}$. We can also check that it is the GL_{2n}-action corresponding to the S-dual in [4]. In this example, $\mathrm{GL}_{2n} \curvearrowright T^*(\mathrm{GL}_{2n}/\mathrm{Sp}_{2n})$ and the hamiltonian spaces in (0.5) are hyperspherical.

Let us finish this article by giving a puzzling example. We consider a quiver gauge theory of type C:



The Coulomb branch can be computed by the method above, and the answer is $T^*(\mathrm{GL}_n/\mathrm{O}_n)$. It carries GL_n-action induced by left multiplication. It is tempting to understand (0.5) as $(\mathscr{T}[\mathrm{GL}_n] \times \mathscr{T}_{\mathbf{M}}) /\!\!/\!\!/ \mathrm{GL}_n$ for a GL_n-hamiltonian space $\mathbf{M}$, as (0.5) is similar to (0.4). However, this is not obvious as there is no good understanding of the Higgs branch of (0.5), which is a *non symmetric* quiver.

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EQUIDISTRIBUTION AND MOMENTS OF L-FUNCTIONS

PAUL NELSON

There are many works relating subconvexity to equidistribution (see e.g. [3, 7]). In joint work in progress with Subhajit Jana, we aim to reduce the general case of subconvexity to an equidistribution problem for unipotent shears on $(\mathrm{GL}_{2n}, \mathrm{GL}_n)$.

Let us recall what is known towards the subconvexity problem, focusing for simplicity on the archimedean aspect over $\mathbb{Q}$. Let π be a unitary cuspidal automorphic representation of GL_n over $\mathbb{Q}$ of level one. Its L-function $L(s,\pi)$, defined initially by an Euler product over the finite primes, admits a meromorphic continuation to the whole complex plane. There are archimedean parameters $\mu_1, \dots, \mu_n \in \mathbb{C}$ for which the completed L-function $L(s,\pi)\prod_{j=1}^{n} \Gamma_{\mathbb{R}}(s+\mu_j)$ satisfies a functional equation relating s and $1-s$. The analytic conductor is defined by $C(s,\pi) := \prod_{j=1}^{n}(1+|s+\mu_j|)$, and the convexity bound asserts that for $\Re s = 1/2$, one has $L(s,\pi) \ll_C (s,\pi)^{1/4+}$ for each > 0. The subconvexity problem is to improve the exponent to $1/4-\delta$ for some fixed $\delta > 0$. This has been done for $n \leq 2$ [4] in general and for $n \geq 3$ [6, 5] when π is restricted to a family satisfying *uniform parameter growth*, meaning that the ratios $|s+\mu_i|/|s+\mu_j|$ are bounded from above and below by positive constants. Establishing subconvexity beyond the uniform parameter growth case remains a long-standing open problem, with applications to quantum unique ergodicity for $n = 3, 6$ (see e.g. [10, 9]).

We briefly recall the method employed by Michel–Venkatesh [4] (compare with [8]) in their proof of a uniform subconvex bound for GL_1 (in all aspects, over number fields). It consists of taking a suitable automorphic form φ_0 on GL_2 (an Eisenstein series), translating it on the right by a unipotent element

$$u_T = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}$$

for a large adelic parameter T, and studying in two ways the L^2-norm of the restriction of that translate to the upper-left copy of GL_1. This L^2-norm expands, via Parseval's identity, to a weighted fourth moment of GL_1 L-functions. The weights concentrate on analytic conductors in a dyadic range cut out by T. An essentially sharp upper bound for such a moment recovers the convexity bound for each individual L-value, while an asymptotic formula delivers, via amplification, a subconvex bound.

We present a higher-rank variant of this method. To describe it, we first recall Rankin–Selberg theory for $\mathrm{GL}_m \times \mathrm{GL}_n$ [1] for $n < m$. We work over a number field F with adele ring $\mathbb{A}$, and write $[G] := G(F)\backslash G(\mathbb{A})$. Given $\varphi \in \pi$ on GL_m, $\Psi \in \sigma$ on GL_n, we have

$$\int_{[\mathrm{GL}_n]} P\varphi \cdot \Psi = L(\tfrac{1}{2}, \pi \times \sigma) \int W_\varphi W_\Psi, \tag{0.1}$$

where GL_n is embedded in the upper-left corner of GL_m and the projection operator P is given by

$$P\varphi(g) := |\det g|^{-\frac{m-n-1}{2}} \int_{[Y_{m,n}]} \varphi(ug)\psi^{-1}(u)\, du,$$

with $Y_{m,n}$ the unipotent radical of the standard upper-triangular parabolic subgroup of signature $(n+1,1,\dots,1)$. In (0.1), the L-value is the partial L-function outside some finite set of places S, and the integral of the Whittaker functions on the right hand side is taken over places inside S. We now fix a suitable automorphic form φ_0 on GL_{2n} and form its right translate $\varphi_0(\bullet u_Q)$ by the unipotent element u_Q, given, e.g., for $n = 2,3$ by

$$u_Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & Q & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & Q & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $Q \in \mathbb{A}$ is a large adelic parameter. We then consider the L^2-norm of the restriction of $P(\varphi_0(\sqrt{Q}\bullet u_Q))$ to $[\mathrm{GL}_n]$; here $\sqrt{Q} \in \mathbb{A}^\times$ denotes any idele with $|\sqrt{Q}|^2 = |Q|$, identified with a scalar matrix inside $\mathrm{GL}_m(\mathbb{A})$. We write P_{n+1} for the mirabolic subgroup, e.g., for $(m,n) = (4,2)$,

$$P_{n+1} = \begin{pmatrix} * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We make the following conjecture concerning such restricted L^2-norms:

Conjecture 0.1 ($\mathcal{C}(m,n)$). *Fix $\varphi \in C_c^\infty([G_m])$. Let $Q \in \mathbb{A}$ vary, where $|Q|_\mathfrak{p} \geq 1$ for each place $\mathfrak{p}$. Then*

$$\int_{[\mathrm{GL}_n]} |P\varphi(\sqrt{Q} g u_Q)|^2 \, dg = \int_{[P_{n+1}]} |P\varphi(\sqrt{Q} g u_Q)|^2 \, dg + \operatorname{O}(|Q|^{-\delta}), \tag{0.2}$$

Similar estimates hold for the corresponding bilinear estimates attached to a pair of test functions φ_1 and φ_2, and the implied constant may be taken to be a product of suitable Sobolev norms of φ_1 and φ_2.

We note that on the right-hand side of (0.2), the shifted element u_Q may be omitted (since it lies in $P_{n+1}(\mathbb{A})$). In practice, the first term on that side may be understood as an "explicit main term". The conjecture may thus be understood informally as asserting the equidistribution of the translates $\sqrt{Q}[\mathrm{GL}_n]u_Q$ inside $[P_{n+1}]$ when tested against the projections $|P\varphi|^2$. When $(m,n) = (2,1)$, such equidistribution is known and forms a key ingredient in the work Michel–Venkatesh [4] (see also Kelmer–Kontorovich [2]).

In general, via spectral expansion on GL_n, the L^2-norm on the left-hand side of (0.2) is a weighted moment of $\mathrm{GL}_m \times \mathrm{GL}_n$ L-functions, where the weight turns out to concentrate on GL_n forms with analytic conductor in a dyadic range cut out by Q. An essentially sharp upper bound recovers the convexity bound for each individual L-value when $m = 2n$, so it is not surprising that the conjecture implies a subconvex bound via amplification. Moreover, applying the above conjecture to suitable Eisenstein series yields subconvex bounds beyond the case $m = 2n$. In this way, we aim to show that if $\mathcal{C}(2n,n)$ holds for all n, then subconvexity holds (at least in the depth aspect, i.e., freezing ramification outside some fixed finite set of places) for:

- (standard L-functions) $L(\frac{1}{2},\sigma)$, for any σ on GL_n.

- (Rankin–Selberg L-functions with one factor fixed) $L(\frac{1}{2}, \pi \times \sigma)$, for π fixed (and possibly tempered) on GL_m, σ varying on GL_n, and any (m, n).

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RELATIVE BEYOND ENDOSCOPY AND SEPARATION OF THE RAMANUJAN SPECTRUM

YIANNIS SAKELLARIDIS

In the development of the Beyond Endoscopy program to date, a lot of effort has been focused on finding a geometric expression for the ”Ramanujan”' spectrum of the Arthur–Selberg trace formula, i.e., removing the non-tempered Arthur packets. An early suggestion by Sarnak was to circumvent this problem altogether, by using the Kuznetsov formula instead, where only the Ramanujan spectrum appears.

I will report on joint work with Chen Wan, where we directly compare the two trace formulas for GL(n), including explicit calculations of the ”transfer operators” that isolate the non-Ramanujan spectrum in low rank. For GL(2), our work generalizes Rudnick's PhD thesis, and this operator coincides with the Fourier transform on the affine parameter space of orbital integrals, which has already appeared in works of Frenkel–Langlands–Ngô and Altuğ; but in higher rank it has a different form. Further motivation for this work comes from the Relative Langlands program, where there is a hope that non-standard comparisons of (relative) trace formulas could be used to prove conjectural relationships between periods of automorphic forms and special values of L-functions.

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ARCHIMEDEAN MODULAR SYMBOLS AND AUTOMORPHIC PERIODS

BINYONG SUN

Modular symbols are cohomological interpretations of period integrals for cohomological automorphic representations. The Archimedean modular symbols, which are linear functionals on certain relative Lie algebra cohomology spaces, capture the Archimedean behavior of modular symbols. We define automorphic periods by investigating the rationality of these Archimedean modular symbols. This construction is an analogue of Deligne's periods for critical pure motives. By using Archimedean modular symbols and automorphic periods, together with the rationality of certain Eisenstein cohomology spaces and cuspidal cohomology spaces, we obtain rationality results for critical values of standard L-functions and Rankin-Selberg L-functions. These results align with Blasius's conjecture. The talk is based on some recent works joint with Dihua Jiang, Yubo Jin, Jiang-Shu Li, Dongwen Liu, and Fangyang Tian.

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THETA LIFTINGS IN HIGHER CHOW GROUPS FOR ORTHOGONAL SHIMURA VARIETIES

LIANG XIAO

In the celeborated theory of arithmetic theta liftings, a.k.a. the Kudla program, a key component is to prove the modularity of the generating series with values in (arithmetic) Chow cycles. In this talk, we report on a joint work in progress with Haocheng Fan, Wenxuan Qi, Peihang Wu, and Yichao Zhang, in which we propose a parallel story for higher Chow cycles. In the talk, I will focus on explaining the basic framework, which involves a (somewhat) new proof of the modularity in the cycles case and a mild generalization of Borcherds products. Assuming a yet-missing technical input on cohomology vanishing, this is expeted to construct a modular generating series with values in higher Chow groups of orthogonal Shimura varieties, and to relate their regulators with the special values of L-functions, as predicted by Beilinson's conjecture.

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NONTEMPERED LOCAL GAN–GROSS–PRASAD CONJECTURE FOR UNITARY COHOMOLOGICAL REPRESENTATIONS OF U(N, 1)

HANG XUE

Classification AMS 2020:

Restriction problems, real unitary groups, local Gan–Gross–Prasad conjectures

The local GGP conjecture seeks to characterize the nonvanishing of the space

$$\mathrm{Hom}_{\mathrm{U}(n)}(\pi, \sigma)$$

where π and σ are irreducible representations of (not necessarily compact) $\mathrm{U}(n+1)$ and $\mathrm{U}(n)$ repsectively. General conjectures characterizing of the nonvanishing of this Hom space is proposed in [GGP12] (for all classical groups, not necessary unitary groups). Representations considered in [GGP12] are *tempered*, or at least lie in a *generic* Vogan packet. Further conjectures for nontempered representations are proposed in [GGP20]. The local GGP conjectures for generic Vogan packets are now fully proved, thanks to the work of many people. For real unitary groups, the proof is contained in [Xue23, Xuea].

In this work, we give a simple characterization of the above Hom space when π and σ are irreducible unitary cohomological representations (with trivial coefficients) of $\mathrm{U}(n,1)$ and $\mathrm{U}(n-1,1)$ respectively. These representations are generally not tempered, nor lie in any generic packets. When interpreted in terms of the Langlands parameterization, this characterization is compatible with the prediction of the local GGP conjecture. The main body of this work is contained in [Xueb].

0.1. **Cohomological representations.** Let V be a hermitian space of dimension $n+1$ and signature $(n,1)$. Let a, b be two nonnegative integers with $m = a+b \leq n$ and

$$x_1 > x_2 > \cdots > x_a > 0 > x_{a+1} > \cdots > x_{a+b}$$

be a sequence of numbers. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u} \subset \mathfrak{g}$ be a θ-stable parabolic subalgebra given by the cocharacter

$$(x_1, \ldots, x_a, x_{a+1}, \ldots, x_{a+b}, 0, \ldots, 0). \tag{0.1}$$

Let $\mathfrak{l}_0 = \mathfrak{l} \cap \mathfrak{u}(V) = \mathfrak{u}(1)^m \times \mathfrak{u}(n-m, 1)$ be its Levi subalgebra. Then we have the cohomological representations $A_{\mathfrak{q}}$. It is characterized by the following two properties, cf. [VZ84].

- The infinitesimal character is
$$\left(\frac{n}{2}, \frac{n-2}{2}, \ldots, -\frac{n}{2}\right).$$
- Its minimal K-type is an irreducible representation of $\mathrm{U}(n) \times \mathrm{U}(1)$. The highest weight of the representation of $\mathrm{U}(n)$ is given by
$$(\underbrace{1, \ldots, 1}_{a}, \underbrace{0, \ldots, 0}_{n-m}, \underbrace{-1, \ldots, -1}_{b}),$$

As $\mathfrak{q}$ varies, these $A_{\mathfrak{q}}$'s are all the unitary cohomological representations whose infinitesimal character is the same as that of the trivial representation, and different $A_{\mathfrak{q}}$'s are not isomorphic to each other. In other words, the integers a, b completely characterize the representation.

0.2. **The theorem.** Consider the following setup.

- Let V_{n+1} and V_n be hermitian spaces of signature $(n, 1)$ and $(n-1, 1)$ respectively.
- Let $x_1 > x_2 > \cdots > x_a > 0 > x_{a+1} > \cdots > x_{a+b}$ be a sequence of numbers. Let $\mathfrak{q}_{n+1}$ be a θ-stable parabolic subalgebra of $\mathfrak{gl}_{n+1}(\mathbb{C})$, given by the cocharacter

$$(x_1, \ldots, x_a, x_{a+1}, \ldots, x_{a+b}, 0, \ldots, 0).$$

  We consider the representation $A_{\mathfrak{q}_{n+1}}$ of $\mathrm{U}(V_{n+1})$.
- Similarly let $y_1 > y_2 > \cdots > y_c > 0 > y_{c+1} > \cdots > y_{c+d}$ be a sequence of numbers. Let $\mathfrak{q}_n$ be a θ-stable parabolic subalgebra of $\mathfrak{gl}_n(\mathbb{C})$, given by the cocharacter

$$(y_1, \ldots, y_c, y_{c+1}, \ldots, y_{c+d}, 0, \ldots, 0). \tag{0.2}$$

  We consider the representation $A_{\mathfrak{q}_n}$ of $\mathrm{U}(V_n)$
- Let

$$\mathrm{Hom}_{\mathrm{U}(V_n)}(A_{\mathfrak{q}_{n+1}}, A_{\mathfrak{q}_n}). \tag{0.3}$$

  be the space of continuous $\mathrm{U}(V_n)$-equivariant maps from $A_{\mathfrak{q}_{n+1}}$ to $A_{\mathfrak{q}_n}$.

Theorem 0.1. *Let the notation and assumptions be as above. Then* (0.3) *is nonzero if and only if either $a = c$ and $b = d$, or $a = c + 1$ and $b = d + 1$.*

0.3. **Proof of Theorem 0.1.** We give a sketch of the proof of the main theorem in this subsection. The argument splits into two cases depending on whether $a + b \leq c + d + 1$ or not.

0.3.1. *Case* $a + b \leq c + d + 1$. Let us introduce the following additional notation.

- Let W be a skew-hermitian space of dimension $m = a + b$ and signature (a, b). Decompose $V_{n+1} = V_n \oplus^{\perp} L_+$. We will work with Weil representations for $\mathrm{U}(W) \times \mathrm{U}(V_{n+1})$, $\mathrm{U}(W) \times \mathrm{U}(V_n)$ and $\mathrm{U}(W) \times \mathrm{U}(L_+)$. Denote by ξ_n the character of $\mathbb{C}^\times$ given by $z \mapsto (z/\sqrt{z\overline{z}})^n$. Choose the characters $\chi_W = \xi_{b-a}$, $\chi_{V_{n+1}} = \xi_{n+1}$, $\chi_{V_n} = \xi_n$, and $\chi_{L_+} = \xi_1$ to split the metaplectic covers. We suppress these characters from the notation.
- Let σ be an irreducible discrete series representation of $\mathrm{U}(W)$ whose Harish-Chandra parameter is of the form

$$\left(\frac{n-m+2a}{2}, \frac{n-m+2a-2}{2}, \ldots, \frac{n-m+2}{2}; -\frac{n-m+2}{2}, \ldots, -\frac{n-m+2b}{2}\right)$$
$$+\left(\frac{n+1}{2}, \ldots, \frac{n+1}{2}\right).$$

It can be proved that the full theta lift $\Theta_{W,V}(\sigma)$ is irreducible, and by [Li90] it equals $A_{\mathfrak{q}_{n+1}}$. So the basic idea of the proof of Theorem 0.1 when $a + b \leq c + d + 1$ is to use the seesaw diagram

$$\begin{tikzcd}
\mathrm{U}(W) \times \mathrm{U}(W) \arrow[d, dash] \arrow[dr, dash] & \mathrm{U}(V_{n+1}) \arrow[d, dash] \arrow[dl, dash] \\
\mathrm{U}(W) & \mathrm{U}(V_n) \times \mathrm{U}(L_+)
\end{tikzcd}$$

and transform the problem to the local GGP conjecture for the Fourier–Jacobi model $\mathrm{U}(W) \times \mathrm{U}(W)$. The representations involved in the Fourier–Jacobi models are tempered (in fact discrete series), and the relevant local GGP conjecture has been established in [Xue23]. The condition $a + b \leq c + d + 1$ is imposed to ensure that $A_{\mathfrak{q}_n}$ is not a theta lift from a smaller unitary group, and hence its theta lift to $\mathrm{U}(W)$ is tempered, if nonzero.

From the above seesaw diagram we know that (0.3) is not zero if and only if $\Theta_{V_n,W}(A_{\mathfrak{q}_n}) \neq 0$ and

$$\mathrm{Hom}_{\mathrm{U}(W)}\left(\Theta_{V_n,W}(A_{\mathfrak{q}_n}) \widehat{\otimes}\, \omega_{W,L_+}, \sigma\right) \neq 0 \tag{0.4}$$

We will need the following proposition on $\Theta_{V_n,W}(A_{\mathfrak{q}_n})$. This is indeed the main technical point solved in [Xueb].

Proposition 0.2. *We have $\Theta_{V_n,W}(A_{\mathfrak{q}_n}) = 0$ unless $a = c$ and $b = d$. In this case $\Theta_{V_n,W}(A_{\mathfrak{q}_n})$ is irreducible, and is a discrete series representation with the Harish-Chandra parameter*

$$\left(\frac{n-m+2a-1}{2}, \frac{n-m+2a-3}{2}, \dots, \frac{n-m+1}{2}; -\frac{n-m+1}{2}, \dots, -\frac{n-m+2b-1}{2}\right)$$
$$+\left(\frac{n}{2}, \dots, \frac{n}{2}\right).$$

If $a = c$ and $b = d$, the local GGP conjecture for tempered representations of $\mathrm{U}(W) \times \mathrm{U}(W)$, proved in [Xue24], implies that (0.4) holds. To summarize, when $a+b \leq c+d+1$, the space (0.3) is nonzero if and only if $a = c$ and $b = d$.

0.3.2. *Case* $a + b \geq c + d + 2$. In this case, Proposition 0.2 does not hold for $A_{\mathfrak{q}_n}$ directly, because it could come from a theta lift of a much smallest unitary group. To deal with this, we borrow a “going-up” trick from [Xue23].

Let W be a skew-hermitian space of signature (c, d) and put $m = c + d$ (note that this W and m are different from the W and m in the previous subsection). Let ρ be the irreducible discrete series of $\mathrm{U}(W)$ of Harish-Chandra parameter

$$\left(\frac{n-m+2c-1}{2}, \frac{n-m+2c-3}{2}, \dots, \frac{n-m+1}{2}; -\frac{n-m+1}{2}, \dots, -\frac{n-m+2d-1}{2}\right)$$
$$+\left(\frac{n}{2}, \dots, \frac{n}{2}\right).$$

Then $A_{\mathfrak{q}_n} = \Theta_{W,V_n}(\rho)$.

Let $V^+ = V_{n+1} \oplus^\perp L_-$ be the hermitian space of signature $(n, 2)$. Let Q^+ be a parabolic subgroup of $\mathrm{U}(V^+)$ whose Levi subgroup is $\mathbb{C}^\times \times \mathrm{U}(V_n)$. Define an induced representation

$$A^+_{\mathfrak{q}_n} = \mathrm{Ind}_{Q^+}^{\mathrm{U}(V^+)} |\cdot|^s_{\mathbb{C}} \otimes A_{\mathfrak{q}_n}$$

of $\mathrm{U}(V^+)$. The same proof as [Xuea, Lemma 4.2] gives that there is a countable subset of $\sqrt{-1}\mathbb{R}$, such that if $s \in \sqrt{-1}\mathbb{R}$ is not in this subset, then $A^+_{\mathfrak{q}_n}$ is irreducible and

$$\mathrm{Hom}_{\mathrm{U}(V_n)}(A_{\mathfrak{q}_{n+1}}, A_{\mathfrak{q}_n}) = \mathrm{Hom}_{\mathrm{U}(V_{n+1})}(A^+_{\mathfrak{q}_n}, A_{\mathfrak{q}_{n+1}}).$$

Let $W^+ = W \oplus^\perp \mathbb{H}$ be an $(m+2)$-dimensional skew-hermitian space where $\mathbb{H}$ is the two-dimensional split skew-hermitian space. We consider theta lifts between $\mathrm{U}(W^+) \times \mathrm{U}(V^+)$ and $\mathrm{U}(V_{n+1})$ and $\mathrm{U}(W^+)$. Choose characters $\chi_{W^+} = \xi_{d-c}$, $\chi_{V^+} = \xi_{n+2}$, and $\chi_{V_{n+1}}$, χ_{V_n} χ_{L_+} as before to split the metaplectic covers.

Let $P^+ \subset \mathrm{U}(W^+)$ be the parabolic subgroup of $\mathrm{U}(W^+)$ whose Levi subgroup is isomorphic to $\mathbb{C}^\times \times \mathrm{U}(W)$, i.e. stabilizing an isotropic line in $\mathbb{H}$. Let ρ^+ be the parabolic induction

$$\rho^+ = \mathrm{Ind}_{P^+}^{\mathrm{U}(W^+)} \xi_{n+2-d-c}^{-1} |\cdot|^s \otimes \rho.$$

Proposition 0.3. *There is a countable subset of* $\sqrt{-1}\mathbb{R}$*, such that if* s *is away from this subset, then* ρ^+ *is irreducible and we have* $\Theta_{W^+,V^+}(\rho^+)$ *is irreducible and equals* $A_{\mathfrak{q}_n}^+$*.*

Argue as (1), we have that

$$\mathrm{Hom}_{\mathrm{U}(V_{n+1})}(A_{\mathfrak{q}_n}^+, A_{\mathfrak{q}_{n+1}}) = \mathrm{Hom}_{\mathrm{U}(W^+)}\left(\Theta_{V_{n+1},W^+}(A_{\mathfrak{q}_{n+1}}) \widehat{\otimes}\, \omega_{W^+,L_-}, \rho^+\right) \tag{0.5}$$

Therefore if

$$\mathrm{Hom}_{\mathrm{U}(V_{n+1})}(A_{\mathfrak{q}_n}^+, A_{\mathfrak{q}_{n+1}}) \neq 0,$$

then $\Theta_{V_{n+1},W^+}(A_{\mathfrak{q}_{n+1}}) \neq 0$. Use Proposition 0.2 again, for $A_{\mathfrak{q}_{n+1}}$ in place of $A_{\mathfrak{q}_n}$, we conclude that $a = c+1$ and $b = d+1$. Note that the assumption $a+b > c+d+1$ implies $(c+1)+(d+1) \leq a+b+1$, so Proposition 0.2 does apply. Again in this case we conclude using the local GGP conjecture for Fourier–Jacobi models proved in [Xue24] that the right hand side of (0.5) is not zero. Therefore $\mathrm{Hom}_{\mathrm{U}(V_n)}(A_{\mathfrak{q}_{n+1}}, A_{\mathfrak{q}_n}) \neq 0$.

Combining the above two cases finishes the proof of Theorem 0.1.

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THETA CORRESPONDENCE AND SPRINGER CORRESPONDENCE

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The exact form of the theta correspondence between representations of an orthogonal group and a symplectic group over a finite field was conjectured by Aubert-Michel-Rouquier, and proved by S-Y. Pan, and then by Ma–Qiu–Zou using different methods. In this talk I will give a geometric description of the theta correspondence between unipotent principal series representations in terms of the Springer correspondence. The construction fits well into the general framework of relative Langlands. This is joint work with Jiajun Ma, Congling Qiu and Jialiang Zou.

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GROSSZAGIER FORMULA IN HIGH DIMENSIONS

WEI ZHANG

In their 1986 paper, Gross and Zagier proved a formula relating the height of Heegner point on an elliptic curve to the first derivatives of the L-function of that elliptic curve. Since then, the problem of generalizing this fundamental result to higher dimensional algebraic varieties has been of great interest. In this talk we will present some of the generalizations with relatively recent results, with an emphasis on Kudla's program and the arithmetic Gan-Gross-Prasad conjecture.

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SHIMURA VARIETIES AND TWISTED PERIOD INTEGRALS

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Classification AMS 2020: 11G18, 11F67, 11F70, 11R39.

Keywords: L-functions, Shimura varieties, special cycles, orbital integrals.

L-functions are fundamental yet mysterious objects in number theory. For a nice polynomial equation X over a number field F, the *Beilinson-Bloch conjecture* predicts that L-functions of X shall encode ranks of Chow groups of algebraic cycles on X. It recovers the *Birch and Swinnerton-Dyer* (BSD) conjecture of X, when X is an elliptic curve. For a global Galois representation ρ, there is a similar and closely-related conjecture called the *Bloch-Kato conjecture*.

The BSD conjecture is proved in major cases for $F = \mathbb{Q}$ using the *Gross–Zagier formula* of modular curves in 1980s. The Gross-Zagier formula may be regarded as an arithmetic analog of Waldspurger formula on twisted period integrals, and is firstly proved via direct computations of Fourier coefficents. However, the Beilinson-Bloch conjecture for general X is still widely open. There are at least three reasons:

(1) Formulations of arithmetic L-functions are conditional on analytic extensions (global), weight-monodromy conjecture and ℓ-independence (local).
(2) The relations of L-functions and automorphic invariants are not proved.
(3) The conjectural Gross-Zagier formula using special cycles are not proved.

Over number fields, the only known tool to relate automorphic representations and Galois representations seems to be Shimura varieties and its explicit algebraic geometry. Therefore, our goal is to study the Beilinson-Bloch conjecture and corresponding Gross-Zagier formula for a Shimura variety

$$X = \mathrm{Sh}_G.$$

More precisely, we use the Hecke geometry on Sh_G, and consider conjectures for the localization

$$\rho_\pi = [X]_\pi$$

given by a cuspidal tempered automorphic representation π appearing in X.

Now we consider (1). Then we know all such results on projective unitary Shimura varieties, at least when wild ramifications only occur in split places.

The relative Langlands program [1] give some hopes for (2). It predicts that L-functions of global Galois representations may be realized as certain period integrals of corresponding automorphic forms. Here a classical period integral is given by integration along a subgroup $H \subseteq G$:

$$P_H(f) = \int_{H(F)\backslash H(\mathbb{A})} f(h)d.$$

Fundamental cases involving classical period integrals are proved for cuspidal tempered forms, such as the Hecke period formulas, Rallis inner product formulas (for several theta liftings), and the Rankin-Selberg formulas for $GL_n \times GL_m$.

However, the general cases are still unknown, and it is important to study more explicit examples. There are two new things:

- It turns out that most examples require the use of non-classical period integrals, involving complicated kernel functions. It seems mysterious to classify all such Langlands kernel functions, which is also related to explicit realizations of Langlands functoriality. However, we know some examples from Weil representations and theta series.
- To study arithmetic analogs, it is also necessary to consider twisted period integrals where H and G are non-split in general (e.g. to ensure the existence of related Shimura varieties). For instance, the Waldspurger formula is a twisted version of Hecke period formula, which has more arithmetic applications.

One new example is the twisted Gan-Gross-Prasad (TGGP) conjecture for unitary groups [3] involving Weil representations and Asai L-functions. We develop a new (non-classical) relative trace formula approach towards this example, and partially answer (2).

Theorem 0.1 ([4],[6]). *The twisted Gan-Gross-Prasad conjecture is true for cuspidal tempered representations, under certain assumptions on ramifications of field extensions and automorphic forms, and existences of supercuspidal split places.*

In particular, this theorem relates central L-values of twisted triple products of elliptic curves with automorphic period integrals. This would have potential applications to Bloch-Kato conjecture for twisted triple products.

Arithmetic analogs of relative Langlands program [1] give some hopes for (3). There are two fundamental examples where Gross-Zagier formulas are studied in detail, namely arithmetic analogs of Rallis inner product (AIP) formulas for unitary groups, and arithmetic Gan-Gross-Prasad (AGGP) conjecture for unitary groups. In particular, the local analog of AGGP conjecture called the arithmetic fundamental lemma (AFL) [7, 8, 5, 9] are formulated and proved, with some ramified versions called arithmetic transfers (AT) proved in [9]. These AFL and AT are used to prove p-adic Gross-Zagier formula [2] in the set up of AGGP conjecture under mild assumptions, partially answering (3).

Motivated by applications to arithmetic of twisted triple product and Asai motives, in [11, 10] we formulate an arithmetic analog of TGGP conjecture for (3). Even formulation of this ATGGP conjecture requires the introduction of some non-reductive cycles, beyond classical cycles coming from Shimura subvarieties $Sh_H \subseteq Sh_G$.

Moreover, we establish the local analogs of such conjecture, namely the twisted AFL and twisted AT in this set up.

Theorem 0.2. *The twisted arithmetic fundamental lemma, as formulated in* [11] *is true on integral local Shimura varieties for $GL_{n,\mathbb{Q}_{p^2}/\mathbb{Q}_p}$ over $\mathbb{Q}_p$ for $p > 2$.*

The proof is based on a global induction method, using modularity of arithmetic theta series and its non-singular Fourier coefficients. However, there is a serious obstruction on the local induction process. We have to use a so-called mirabolic special cycle to do an extra local induction, which seems of independent interest. The name of mirabolic special cycles come from the mirabolic subgroup of GL_n. Moreover, using the double induction method and the almost modularity result in [9], the twisted arithmetic transfers are also proved.

In summary, we expect that the use of non-reductive / non-classical period integrals and non-reductive / non-classical special cycles on Shimura varieties, would be helpful to understand more general L-functions, in particular proving (2)(3). The author hopes to have a more precise framework in a future work. In particular, geometric structures of these non-reductive cycles may have applications in other areas, such as arithmetic groups.

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ON THE WALDSPURGER'S IDENTITY

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Classification AMS 2020: Primary 22E50 22E45; Secondary 20G20.

Keywords: Endoscopy, Smooth transfer, Fundamental lemma, Waldspurger's identity

1. ENDOSCOPY AND TRANSFER

Let F be a local field with $\mathrm{char}(F)=0$. We denote $\Gamma=\mathrm{Gal}(\bar{F}/F)$.

For a reductive group G/F and its **endoscopic triple** (H,s,ξ)

- H is quasi-split over F with dual group $\widehat{H}$ and Langlands group ${}^LH$;
- $s\in Z_{\widehat{H}}^{\Gamma}$ and $\xi:{}^LH\to{}^LG$ satisfying

$$\xi(\widehat{H})=(\mathrm{Cent}_{\widehat{G}}\xi(s))^0.$$

This triple is called **elliptic** if ξ does not factor through any proper Levi, or say

$$\xi([Z(\widehat{H})^{\Gamma}]^0)\subset Z(\widehat{G}).$$

For $X\in\mathfrak{g}(F)$, we recall that the orbital integral is defined as

$$\mathrm{J}_G(X,f):=\mathrm{D}^G(X)^{1/2}\int_{G_X(F)\backslash G(F)}f(g^{-1}Xg)dg,\quad f\in\mathcal{S}(\mathfrak{g}(F)).$$

- dg is the Haar measure, and
- $\mathrm{D}^G(X)=|\det(\mathrm{ad}_X|_{\mathfrak{g}/\mathfrak{g}_X})|$ is the absolute value of the Weyl discriminant of X.

By Harish-Chandra's regularity theorem, the generalized function

$$f(X)dX\mapsto J_G(X,\widehat{f})$$

is a function that we denote by $\widehat{j}^G(X,\cdot)$ and set

$$\widehat{i}^G(X,\cdot)=D^G(X)^{1/2}\widehat{j}^G(X,\cdot).$$

For $X_G,X'_G\in\mathfrak{g}(F)$, X'_G is called **a stable conjugate** to X_G if there is a $g^*\in G(\bar{F})$ such that

$$X'_G=Ad(g^*)X_G.$$

- We denote by X_G^{st} the set of elements in $G(F)$ that are stably conjugated to X_G, and
- We denote by $X_G^{\mathrm{st}}/conj$ the set of conjugacy classes in X_G^{st}.

We define the **stable orbital integral** as

$$J_G^{\mathrm{st}}(X_G,f_G):=\sum_{X'_G\in X_G^{\mathrm{st}}/conj}J_G(X'_G,f_G).$$

The Langlands-Shelstad transfer factor is defined in [6], which is a function

$$\Delta_{G,H}(X_H, X_G) : \mathfrak{h}_{G-\mathrm{reg}}(F)/\mathrm{stconj} \times \mathfrak{g}_{\mathrm{rss}}(F)/\mathrm{conj} \to \mathbb{C}^{\times}.$$

Define

$$J_{G,H}(X_H, f_G) := \sum_{X_G \in \mathfrak{g}(F)/conj} \Delta_{G,H}(X_H, X_G) J_G(X_G, f_G).$$

In particular,

$$J_{G,G}(X_G, f_G) = J^{\mathrm{st}}(X_G, f_G).$$

A function $f_H \in \mathcal{S}(\mathfrak{h}(F))$ is called a **smooth transfer** of $f_G \in \mathcal{S}(\mathfrak{g}(F))$ if and only if

$$J_{G,H}(X_H, f_G) = J^{\mathrm{st}}(X_H, f_H).$$

2. WALDSPURGER'S IDENTITY

When F is non-Archimedean, the proof of the existence of smooth transfer uses Waldspurgers identity $\mathrm{D}_{G,H} = \widetilde{\mathrm{D}}_{G,H}$, where $\mathrm{D}_{G,H}$ and $\widetilde{\mathrm{D}}_{G,H}$ are defined as follows.

$$\mathrm{D}_{G,H}(X_H, X_G) = \gamma_\psi(\mathfrak{g}) \sum_{X'_G \in \mathfrak{g}_{\mathrm{reg}}(F)/conj} \Delta_{G,H}(X_H, X'_G) \widehat{i}^G(X'_G, X_G),$$

$$\widetilde{\mathrm{D}}_{G,H}(X_H, X_G) = \gamma_\psi(\mathfrak{h}) \sum_{\substack{X'_H \in X_H^{\mathrm{st}}/conj \\ X''_H \in \mathfrak{h}_{G-\mathrm{reg}}(F)/conj}} w(X''_H)^{-1} \Delta_{G,H}(X''_H, X_G) \cdot \widehat{i}^H(X'_H, X''_H),$$

where

- $w(X''_H) = |\mathfrak{s}_H^{-1}(X''_H)|$, which is the number of $H(F)$-conjugacy classes within the stable conjugacy class of X''_H;
- $\gamma_\psi(\mathfrak{g})$ and $\gamma_\psi(\mathfrak{h})$ are the Weil constants associated to the quadratic spaces $(\mathfrak{g}(\mathbb{R}), \langle\cdot,\cdot\rangle_{\mathfrak{g}})$ and $(\mathfrak{h}(\mathbb{R}), \langle\cdot,\cdot\rangle_{\mathfrak{h}})$.

Theorem 2.1 (Waldspurger over p-adic and over $\mathbb{C}$ in [8], C.-Luo $\mathbb{R}$ purely local method in [4])**.**

$$\mathrm{D}_{G,H}(X_H, X_G) = \widetilde{\mathrm{D}}_{G,H}(X_H, X_G), \quad \forall X_H \in \mathfrak{h}_{G-reg}(F), X_G \in \mathfrak{g}_{\mathrm{reg}}(F).$$

Corollary 2.2. *The Fourier transform of a stable orbital integral is also stable.*

Corollary 2.3 (Fourier transform and smooth transfer are compatible)**.** *Let $f_G \in \mathcal{S}(\mathfrak{g}(F))$ and $f_H \in \mathcal{S}(\mathfrak{h}(F))$ satisfy the smooth transfer condition*

$$\mathrm{J}_{G,H}(X_H, f_G) = \mathrm{J}_H^{\mathrm{st}}(X_H, f_H), \quad X_H \in \mathfrak{h}_{G-\mathrm{reg}}(F).$$

Then

$$\gamma_\psi(\mathfrak{g}) \mathrm{J}_{G,H}(X_H, \mathcal{F}_{\psi,\mathfrak{g}}(f_G)) = \gamma_\psi(\mathfrak{h}) \mathrm{J}_H^{\mathrm{st}}(X_H, \mathcal{F}_{\psi,\mathfrak{h}}(f_H)).$$

Remark 2.4. (1) *When F non-Archimedean,*

- *Waldspurger proved that the fundamental lemma for the unit element (at almost all places) implies $D = \widetilde{D}$.*
- *Kazhdan and Varshavsky proved in* [5] *that $D = \widetilde{D}$ implies the fundamental lemma for the unit element.*
- *Waldspurger proved that $D = \widetilde{D}$ implies the existence of smooth transfer.*

(2) *When F is Archimedean, the existence of smooth transfer on groups was proved by Shelstad in* [7]*, and the existence on Lie algebras was proved by C.-Luo for both smooth compactly supported functions and Schwartz functions in* [4].

Remark 2.5. *There are similar results for weighted Orbital Integral. When F is non-Archimedean,*

- *if H is an inner form of G, $D_v = \widetilde{D}_v$ was proved by Chaudouard in* [1] *($G_v = H_v$ at almost all places with trivial Weighted FL);*
- *The same method applies to the general case with the weighted fundamental lemma for unit elements (proved by Chaudouard and Laumon for split groups in* [2][3]*).*

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BEYOND ENDOSCOPY VIA THE TRACE FORMULA

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Classification AMS 2020: 11F72.

Keywords: beyond endoscopy, trace formula

At the beginning of this century, Langlands introduced a strategy known as Beyond Endoscopy to attack the principle of functoriality. Altuğ studied GL_2 over $\mathbb{Q}$ in the unramified setting. In the first paper, he isolated the trace of the trivial representation in the elliptic part of the trace formula, and used this isolation to recover the Kuznetsov bound in the second paper. In the last paper, he gave a new proof of the bound of the average of the trace of Hecke operator. We will generalize the three papers to the case over $\mathbb{Q}$ with ramification. In particular, we isolate the non-tempered part in the elliptic part, prove the Kuznetsov bound in the ramified case, and bound of the average of the trace of Hecke operator for modular forms with arbitrary level. Moreover, we present an asymptotic formula for the elliptic part in the general case.

Theorem 0.1 (C., 2025+, [Che25a]). *If $S = \{\infty, q_1, \dots, q_r\}$ such that $2 \in S$, then*

$$\begin{aligned} &I_{\mathrm{ell}}(f^n) \\ =&\sum_{\mu} \mathrm{Tr}(\mu(f^n)) - \frac{1}{2}\sum_{\mu} \mathrm{Tr}((\xi_0 \otimes \mu)(f^n)) - \Sigma^n(\square) + \Sigma^n(0) + \Sigma^n(\xi \neq 0), \end{aligned}$$

where μ runs over all 1-dimensional representations of $Z_+\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathring{A})$.

Theorem 0.2 (C., 2025+, [Che25b]). *For any $\varepsilon > 0$, we have*

$$I_{\mathsf{cusp}}(f^n) \ll_{f_\infty, f_{q_i}, \varepsilon} n^{1/4+\varepsilon}.$$

Theorem 0.3 (C., 2025+, [Che25c]).

$$\begin{aligned} \sum_{\substack{n<X \\ \gcd(n,S)=1}} I_{\mathrm{ell}}(f^n) = \textit{hyperbolic contribution} \\ + AX^{3/2} + BX\log X + CX + o(X). \end{aligned}$$

Conclusion:

$$\lim_{X\to+\infty} \frac{1}{X} \sum_{\substack{n<X \\ \gcd(n,S)=1}} I_{\mathrm{cusp}}(f^n) = 0$$

for

- The (Deligne-Kazhdan & Arthur) simple trace formula cases (which can not happen in the unramified case $S = \{\infty\}$);
- The modular form case (generalization of Altuğ's final result).

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COMPUTING MICROSTALKS IN THE BETTI AUTOMORPHIC CATEGORY

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Classification AMS 2020: 14D24, 32S60

Keywords: characteristic cycle, vanishing cycles, microstalk, Whittaker coefficient, global nilpotent cone, geometric Langlands

Let G be a complex reductive group with a fixed Borel B, unipotent N, and maximal torus T compatible with B and N. We let Bun_G (similarly for N, B, T) be the stack of G-bundles on a fixed curve.

For D a divisor on the curve valued in dominant coweights of T, consider the diagram

$$\begin{tikzcd}
\mathrm{Bun}_N^{\Omega(-D)} \arrow[r, hookrightarrow] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Bun}_B \arrow[r] \arrow[d] & \mathrm{Bun}_G \\
\{\Omega(-D)\} \arrow[r, hookrightarrow] & \mathrm{Bun}_T &
\end{tikzcd}$$

where Ω is the T-bundle $\rho^\vee(\omega)$, for ω the canonical bundle on our curve and $\rho^\vee$ the T-coweight that is the half-sum of positive coweights. Let $i\colon \mathrm{Bun}_N^{\Omega(-D)} \to \mathrm{Bun}_G$ be the composition of the horizontal maps in the upper row of the diagram.

Let f be the composition

$$\mathrm{Bun}_N^{\Omega(-D)} \to \mathrm{Bun}_{N/[N,N]}^{\Omega(-D)} = \prod H^1(\omega\langle\alpha, -D\rangle) \to \prod H^1(\omega) = \mathbb{A}^r \xrightarrow[\Sigma]{} \mathbb{A}^1.$$

On the category $\mathrm{Sh}_{\mathcal{N}}(\mathrm{Bun}_G(C))$ (where $\mathcal{N}$ is the global nilpotent cone), the Whittaker functional coeff_D is defined as the map $\phi_{f,\Omega(-D)}i^![\dim \mathrm{Bun}_{B^-}^{\Omega(-D)}]$. Here (by abuse of notation) $\Omega(-D)$ is the trivial B-bundle induced from $\Omega(-D)$, so one can think of it as having "zero extensions", and $\phi_{f,\Omega(-D)}i^!$ means to take the stalk at the point $\Omega(-D)$ of vanishing cycles with respect to f.

It was shown in [NT24] that for $D = 0$ (i.e. the usual Whittaker coefficient) the functor coeff is t-exact (using the perverse t-structure on $\mathrm{Sh}_{\mathcal{N}}(\mathrm{Bun}_G)$). Their proof involves writing $\phi_{f,\Omega(-D)}i^!$ as $\phi_{F,\Omega(-D)}$ for a suitably chosen function F on Bun_G, and using the exactness properties of microstalks. (Actually since Bun_G is a stack, they first need to pullback to a scheme via a smooth map and then take vanishing cycles, but the idea is to show that the functor is a microstalk.)

Meanwhile, [FR25] showed via algebraic methods that in fact all coeff_D are t-exact. In this talk we explain why this result is surprising from the geometric point of view, and explain how to calculate functionals like coeff_D in terms of the characteristic cycle of a sheaf.

Date: March 2026.

1. THE QUADRIC CONE IN $\mathbb{A}^3$

We can already see why the result is surprising in the simplest case of nonzero D for SL_2. If we take our curve to be $\mathbb{P}^1$, then $\omega = \mathcal{O}(-2)$ and $\rho^\vee$ is the coweight $\begin{bmatrix} 1/2 & \\ & -1/2 \end{bmatrix}$, so the T-bundle Ω is $\mathcal{O}(-1) \oplus \mathcal{O}(1)$. The smallest dominant coweight is $[\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}]$, and twisting Ω by this coweight at a point yields $\mathcal{O}(-2) \oplus \mathcal{O}(2)$.

Now we have $\mathrm{Bun}_N^{\Omega(-D)} = \mathrm{Ext}^1(\mathcal{O}(2), \mathcal{O}(-2)) = \mathbb{A}^3$ and the map $\mathrm{Bun}_N^{\Omega(-D)} \to \mathrm{Bun}_G$ is smooth (this is a general feature of the $\mathbb{P}^1$ case). On $\mathbb{A}^3$ we have a stratification by extensions in $\mathrm{Ext}^1(\mathcal{O}(2), \mathcal{O}(-2))$ (i.e. splitting loci for this $\mathbb{A}^3$-family of $\mathbb{P}^1$-bundles). The bundles $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ form a quadric cone (which we can write in coordinates as $\{x^2 - yz = 0\}$). In this case the function f is such that $\{f = 0\}$ is tangent to the quadric cone (e.g. we have $f = y$), so there is a line of intersection between Γ_{df} and the conormal to the quadric cone in $T^*\mathbb{A}^3$.

The quadric cone has fundamental group $\mathbb{Z}/2\mathbb{Z}$ and so it has two local systems. Let $\mathcal{L}$ be the nontrivial one. Then $\phi_{f,0}(\mathrm{IC}_{\mathrm{cone}}) = 0$, but $\phi_{f,0}(\mathrm{IC}(\mathcal{L}[2])) = \mathbb{C}[1]$. So in particular $\phi_{f,0}$ is not exact on $\mathrm{Bun}_N^{\Omega(-D)}$, even though in this case it is a smooth cover of Bun_G! The nontrivial local system $\mathcal{L}$ is not seen on Bun_G since the whole stratum is one stacky point.

2. WHITTAKER COEFFICIENTS IN TERMS OF CHARACTERISTIC CYCLES FOR THE BETTI AUTOMORPHIC CATEGORY

In the above example, more specifically we have $\mathrm{CC}(\mathrm{IC}_{\mathrm{cone}}) = [\overline{T^*_{\mathrm{cone}}\mathbb{A}^3}] + [\overline{T^*_0\mathbb{A}^3}]$, and $\mathrm{CC}(\mathcal{L}) = [\overline{T^*_{\mathrm{cone}}\mathbb{A}^3}]$. Let χ be the Euler characteristic map (sending complexes of finite dimensional vector spaces to $\mathbb{Z}$). It turns out that $\chi \circ \phi_{f,0}$ is a functional that is linear in the characteristic cycle; it has a value of 1 on $[\overline{T^*_0\mathbb{A}^3}]$ and -1 on $[\overline{T^*_{\mathrm{cone}}\mathbb{A}^3}]$.

Fix the curve to be $\mathbb{P}^1$. We generalize this computation to arbitrary coeff_D, using the following result:

Proposition 2.1. *[FGV01; FR25, Theorem 5.3.0.1]* $\mathrm{coeff}_D(-) \simeq \mathrm{coeff}(H_D * (-))$.

Over $\mathbb{P}^1$ all Hecke operators are equivalent (non-canonically). So given a divisor $D = \sum \lambda_i[p_i]$ for points p_i, the operation of H_D is the same as that of all the H_{λ_i} at e.g. a single point. Let V_{λ_i} be the $\check{G}$-representation corresponding to the G-coweight λ_i. Then if $\bigotimes V_{\lambda_i} = \bigoplus V_{\mu_i}$, we have

$$\mathrm{coeff}(H_D * (-)) \simeq \mathrm{coeff}(\bigoplus H_{\mu_i}(-)) = \bigoplus \mathrm{coeff}_{\mu_i}(-)$$

where coeff_λ is the coefficient for the divisor that is λ at a single point.

Now let $G = PGL_2$ (note that this case subsumes $G = SL_2$). Then we can identify dominant coweights with positive integers, and each divisor D can be represented as a multiset of positive integers. We can also identify the regular irreducible components of the nilpotent cone (i.e. the components that are not the zero section) with the nonnegative integers.

We can rephrase the result in [NT24] as follows:

Theorem 2.2. *The functor* $\mathrm{coeff}(-)$ *calculates microstalk at the* 0*th component of* $\mathcal{N}^{\mathrm{reg}}$.

A small modification of the argument gives the following extension:

Theorem 2.3. *The functor* $\operatorname{coeff}_{1^n}(-)$ *calculates microstalk at the* n*th component of* $\mathcal{N}^{\text{reg}}$*, where* 1^n *corresponds to the divisor with* 1 *at* n *points. In other words,* $\operatorname{coeff}((H_1)^{*n} * (-))$ *calculates microstalk at the* n*th component of* $\mathcal{N}^{\text{reg}}$.

Now both representations of the form V_{1^n} and of the form V_n form a basis for the ring of $\check{G}$-representations. If in the ring we have $V_n = \sum a_{n,m} V_{1^m}$, then we get that $\chi \circ \operatorname{coeff}_n = \sum a_{n,m} \chi \circ \operatorname{coeff}_{1^m}$, i.e. $\chi \circ \operatorname{coeff}_n$ is a sum over taking the microstalk at the mth component, times a weight $a_{n,m}$. The matrix $(a_{n,m})$ is the inverse of the matrix writing repeated tensor powers of V_1 in terms of irreducible representations (see the two matrices below; the right one has $a_{n,m}$ and the left has coefficients of tensor powers of V_1).

$$\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ 1 & & 1 & & & \\ & 2 & & 1 & & \\ 2 & & 3 & & 1 & \\ \vdots & & & & & \ddots \end{pmatrix} \qquad \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ -1 & & 1 & & & \\ & -2 & & 1 & & \\ 1 & & -3 & & 1 & \\ \vdots & & & & & \ddots \end{pmatrix}$$

Despite the alternating positive and negative values in coeff_n, the sum over the characteristic cycle of an actual perverse sheaf in $\operatorname{Sh}_{\mathcal{N}} \operatorname{Bun}_G$ must be nonnegative due to t-exactness of coeff_D.

We can use this result to compute characteristic cycles of sheaves in $\operatorname{Sh}_{\mathcal{N}} \operatorname{Bun}_G$. Let δ be the sheaf $j_! \mathbb{C}_U[-\dim U]$ for the open orbit U on Bun_G. By [Laf09], we know that δ is perverse and that $\chi \circ \operatorname{coeff}_n(\delta) = \delta_{0,n}$. It is then clear that the characteristic cycle of $H_n * \delta$ is the nth column (using 0 indexing) of the first matrix above. (Note that the characteristic cycle of a perverse sheaf must have nonnegative values for each irreducible component of its singular support.)

Remark 2.4. *This method computing* $\chi \circ \operatorname{coeff}_n$ *generalizes to other reductive groups assuming an appropriate version of 2.3, where we use the fundamental weights of the Lie algebra and the adjoint group corresponding to it.*

The generalization to higher rank curves requires a more careful accounting of the regular irreducible components of $\mathcal{N}$ *(as the center of* G *and a choice of* $\omega^{1/2}$ *come into play), as well as a notion of local constancy for the Hecke operators* H_D.

There is a Radon transform relating sheaves on $\operatorname{Bun}_G(\mathbb{P}^1)$ and sheaves on the affine Grassmannian for G. By [EM99], the μ-contribution to the characteristic cycle of $\operatorname{IC}_\lambda$ is the dimension of the μ-weight space of V_λ. The Radon transform of $\operatorname{IC}_\lambda$ is $H_\lambda * \delta$, so it would be interesting to find an explanation for the characteristic cycle calculation above using this fact. Note that the simplest case of Radon transform between $T^*\mathbb{P}^n$ and $T^*\mathbb{P}^{n,\vee}$ is a symplectomorphism outside the zero sections.

3. Vanishing cycle stalks in general

In general, computing the stalk of vanishing cycles $\phi_{f,x}(\mathcal{F})$ is akin to computing the microlocal hom between Γ_{df} and $\operatorname{CC}(\mathcal{F})$. More precisely, we have the following fact.

Proposition 3.1. *Given (complex algebraic) Lagrangian cycles α, β and a point $x = (y, \xi)$, all in $T^*\mathbb{C}^n$ (everything is local), there is an integer $I_x(\alpha, \beta)$ (the "local intersection number") with the following properties:*

- *I_x is symmetric bilinear in α, β*
- *I_x is invariant under (local) symplectomorphism*
- *For $\alpha = \mathbb{C}^n$ and $\beta = \Gamma_{df}$, I_x is the stalk of vanishing cycles $\chi(\phi_{f,y}(\underline{\mathbb{C}}_{\mathbb{C}^n}))$*
- *For $\alpha = \mathrm{CC}(\mathcal{F})$ and $\beta = \mathrm{CC}(\mathcal{G})$, I_x is the stalk of $\mu\mathrm{hom}(\mathcal{F}, \mathcal{G})$*
- *For $\alpha = \mathrm{CC}(\mathcal{F})$ and $\beta = \Gamma_{df}$, I_x is the stalk of vanishing cycles $\chi(\phi_{f,y}(\mathcal{F}))$.*

Note that this function is an extension of the local intersection number from [Gin86, Section 11]. The function $I_x(\alpha = \mathbb{C}^n, \beta = \Gamma_{df})$ is equivalent to the Behrend function ν at $y \in \mathrm{Crit}(f) = \Gamma_{df} \cap \mathbb{C}^n \subset T^*\mathbb{C}^n$.

Theorem 3.2. *[Beh09; PP01, though possibly known earlier] The function ν only depends on (the (underived) scheme-theoretic structure of) $\mathrm{Crit}(f)$, defined to be the intersection of Γ_{df} and $\mathbb{C}^n$.*

The function is defined as follows:

1. Take the "deformation to the normal cone" for Γ_{df} with respect to $\mathbb{C}^n$, which is equivalent to the "Macpherson graph construction". The result is a cycle which is a sum of cones equivalent to the "**intrinsic** normal cone" of $\mathrm{Crit}(f)$.
2. For each cone, take the "local Euler obstruction". This is equivalent to determining the Euler characteristic of a constructible sheaf that has that cone as its characteristic cycle.

The same procedure works for generalized vanishing cycles, essentially by [KS90, Section 9] and [Gin86, Sections 6 and 11], with three caveats:

1. We must remember the two Lagrangians.
2. We need to reduce to the diagonal; instead of intersecting α with β, we intersect $\alpha \times \beta$ with $T^*_{\Delta_X}(X \times X)$ via deformation to the normal cone.
3. Usual intersection theory allows one to directly intersect reducible cycles; however performing deformation to the normal cone and taking distinguished varieties does not respect bilinearity in this case. See [Ful84, Example 6.1.2].

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BEILINSON–BLOCH–KATO CONJECTURE FOR POLARIZED MOTIVES

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Classification AMS 2020: 11F67, 11G40

Keywords: Bloch–Kato conjecture; polarized motives; Bloch–Kato Selmer groups; theta correspondence

Let E be an elliptic curve over $\mathbb{Q}$. The (rank part) of the Birch–Swinnerton-Dyer conjecture [5] states that

$$\operatorname{rank} E(\mathbb{Q}) = \operatorname{ord}_{s=1} L(s, E).$$

Here $E(\mathbb{Q})$ is naturally a f.g. Abelian group by Mordell–Weil theorem, and

$$L(s, E) = \prod_{p \nmid \Delta_E} \frac{1}{1 - a_p p^{-s} + p^{1-2s}}, \quad a_p = 1 + p - \# E(\mathbb{F}_p).$$

Here the right hand side is well-defined by the modularity theorem of Wiles et al. [13, 12, 1]. This conjecture is known when $\operatorname{ord}_{s=1} L(s, E)$ is at most one, by work of Kolyvagin and Gross–Zagier [8, 6]. When $\operatorname{ord}_{s=1} L(s, E)$ is at least 2, not much is known besides numerical evidences.

The Birch–Swinnerton-Dyer conjecture generalizes to higher dimensions (Bloch–Kato conjecture [3]): Let X/F be a smooth variety over a number field. For any rational prime ℓ, the ℓ-adic etále cohomologies of X are equipped with Galois representation of Gal_F. Then the Bloch–Kato conjecture predicts that

$$\dim_{\mathbb{Q}_\ell} \mathsf{H}^1_f(F, \mathsf{H}^{2r-1}(X_{\overline{F}}, \mathbb{Q}_\ell(r))) = \operatorname{ord}_{s=0} L(s, \mathsf{H}^{2r-1}(X_{\overline{F}}, \mathbb{Q}_\ell(r))).$$

Here we recall the definition of Bloch–Kato Selmer group

$$\mathsf{H}^1_f(F, V) = \ker\left(\mathsf{H}^1(F, V) \to \prod_{v \nmid \ell} \mathsf{H}^1(I_v, V) \times \prod_{v \mid \ell} \mathsf{H}^1(F_v, V \otimes \mathbb{B}_{\mathrm{crys},\ell})\right),$$

where ℓ is the underlying rational prime of λ and $\mathbb{B}_{\mathrm{crys},\ell}$ is the ℓ-adic crystalline period ring. In other words, the Bloch–Kato Selmer group is the subspace of "everywhere unramified/crystalline" Galois cohomology classes.

For example, if A is an Abelian variety over F of dimension g, then

$$\dim_{\mathbb{Q}_\ell} \mathsf{H}^1_f(F, \mathsf{H}^{2g-1}(A_{\overline{F}}, \mathbb{Q}_\ell)) = \operatorname{rank} A(\mathbb{Q}) + r_\ell(\text{Ш}(A))$$

The following is one of my main result. Let F/F_+ be CM extension.

Theorem 0.1 ([10]). *Let $r \geq 1$ and let A be a non-CM modular elliptic curve over F_+. Then for all large ℓ, we have*

$$L\left(r, \mathrm{Sym}^{2r-1} A_F\right) \neq 0 \implies \mathsf{H}^1_f\left(F, \mathrm{Sym}^{2r-1} \mathsf{H}^1(A, \mathbb{Q}_\ell(r))\right) = 0.$$

As a corollary, we obtain an approach to find arbitrarily high dimensional varieties over number fields where all Tate conjecture, Hodge conjecture, and Bloch–Kato conjecture are known for ℓ large, namely, certain $X = A_F^n$.

We explain the relation of the theorem to algebraic cycles. For any projective smooth variety X over F, we have an Abel–Jacobi map

$$\mathrm{AJ}_\ell : \mathrm{CH}^r(X)^0_{\mathbb{Q}} \to \mathsf{H}^1\left(F, \mathsf{H}^{2r-1}_\ell(X, \mathbb{Q}_\ell(r))\right),$$

which factors through the Selmer group if purity conjectures hold (for example, for Abelian varieties and many Shimura varieties). Here $\mathrm{CH}^r(X)^0_{\mathbb{Q}}$ is the rational Chow group of cohomologically trivial cycles of codimension r. Moreover, The Abel–Jacobi map is conjectured to be injective by Beilinson [2].

If we assume this injectivity for large ℓ, then we have a more popular corollary:

Corollary 0.2.

$$L(r, \mathsf{h}^{2r-1}(A_F^n)) \neq 0 \implies \mathrm{CH}^r(A_F^n)^0_{\mathbb{Q}} = 0,$$

for any r and n.

We now state a more general theorem in the automorphic form language.

Theorem 0.3 ([10]). *Let $\Pi/\mathrm{GL}_{2r}(\mathsf{A}_F)$ be a conjugate self-dual cuspidal automorphic representation of $\mathrm{GL}_{2r}(\mathsf{A}_F)$ such that Π_w is minimally regular algebraic (Archimedean weights $(\frac{1-2r}{2}, \dots, \frac{2r-1}{2})$). Let E be number field where all associated Galois representations of Π are defined. Then for all admissible prime λ of E, we have*

$$L(\frac{1}{2}, \Pi) \neq 0 \implies \mathsf{H}^1_f(F, \rho_{\Pi,\lambda}(r)) = 0.$$

I will not recall the definition of being admissible. It is expected that all large ℓ is admissible, for example, when Π has a supercuspidal place and a Steinberg place, or Π comes from odd symmetric power of non-CM elliptic curves.

The Archimedean condition is used to descent Π to unitary groups $\mathsf{U}(\mathsf{W}_{2r})$. In particular, at every Archimedean place w of F, Π_w is the base change of the trivial representation of the definite unitary group U_{2r}.

The proof is reduced, via the theta correspondence and the seesaw identity, to computations arising in the proof of the BlochKato conjecture in the Rankin–Selberg case [9]. The key is the following Burger–Sarnak argument for Fourier–Jacobi periods, in the spirit of Harris–Li, Prasad, and Wei Zhang [4, 7, 11, 14].

Proposition 0.4 ([10]). *Let W_{2r} be skew Hermitian with signature (r, r) at every infinite place, and $\sigma_0/\mathsf{U}(\mathsf{W}_{2r})(\mathsf{A}_{F_+})$ cuspidal, then there exists $\sigma_1/\mathsf{U}(\mathsf{W}_{2r})(\mathsf{A}_{F_+})$ cuspidal and cusp forms $\varphi_0 \in \sigma_0, \varphi_1 \in \sigma_1$, such that*

$$\mathcal{FJ}(\varphi_0, \varphi_1; \phi) := \int_{[\mathsf{U}(\mathsf{W}_{2r})]} \varphi_0(g)\varphi_1(g)\theta(g; \phi)\mathrm{d}g \neq 0$$

Here $\theta(g; \phi)$ is certain theta function. Moreover, we can fix local components of σ_1, as long as they are supercuspidal at finite places or holomorphic discrete series with scalar lowest K-type at infinite places.

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TWISTED OSBORNE CONJECTURE AND $\mathfrak{n}$-HOMOLOGY OF UNIPOTENT ARTHUR PACKETS

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Classification AMS 2020: 22E45

Keywords: twisted Osborne conjecture, unipotent Arthur packets.

1. Statement of the twisted Osborne conjecture

Let $\mathbf{G}$ be a connected real reductive group, and let τ be a semisimple $\mathbb{R}$-automorphism of $\mathbf{G}$. We assume that τ preserves a splitting of G, and that τ is of finite order d. Suppose π is a τ-stable irreducible Casselman-Wallach representation of $G := \mathbf{G}(\mathbb{R})$. Fix an intertwining operator $\tau_\pi : \pi \to \pi^\tau$ with $\tau_\pi^d = \mathrm{Id}$. Then π can be extended to a representation π^+ of $G^+ := G \rtimes \langle \tau \rangle$, with $\pi^+(\tau) = \tau_\pi$. The character $\Theta(\pi^+)$ is a locally integrable function on G^+, and is analytic on regular elements. We define the twisted character of π by setting $\Theta^\tau(\pi)(g) = \Theta(\pi^+)(g\tau)$.

Let $\mathfrak{g}$ be the Lie algebra of $\mathbf{G}$, and let K be a τ-stable maximal compact subgroup of G. Denote the underlying $(\mathfrak{g}, K)$-module of π by V. It is well-known that the category of Harish-Chandra modules is equivalent to the category of Casselman-Wallach representations. Thus, we also denote $\Theta^\tau(\pi)$ by $\Theta^\tau(V)$.

Let $P = MN$ be a τ-stable Levi decomposition of a standard real parabolic subgroup of G. Then the $\mathfrak{n}$-homology groups $H_q(\mathfrak{n}, V)$ of V are τ-stable Harish-Chandra modules of M, and consequently have well-defined twisted characters $\Theta^\tau(H_q(\mathfrak{n}, V))$.

Theorem 1.1 ([Hua25]). Let $M^- \subset M$ be the set of elements $m \in M$ such that m is τ-regular in G, and all eigenvalues of $\mathrm{Ad}(m) \circ \tau$ on $\mathfrak{n}$ have modulus < 1. Then the following identity holds on M^-:

$$\Theta^\tau(V) = \frac{\sum_q (-1)^q \Theta^\tau\left(H_q(\mathfrak{n}, V)\right)}{D_\mathfrak{n}^\tau}, \tag{1.1}$$

where $D_\mathfrak{n}^\tau(m) = \det_\mathfrak{n}(1 - m\tau)$ is the determinant of the endomorphism $\mathrm{Id}_\mathfrak{n} - \mathrm{Ad}(m) \circ \tau$ on $\mathfrak{n}$.

Two special cases have been studied before. When τ is trivial, (1.1) reduces to the ordinary Osborne conjecture proved in [HS83, Theorem 3.6]. When P is a minimal parabolic subgroup of G, [BC13, Theorem 4.5] asserts that (1.1) holds on $T^{\tau,\circ} \cap M^-$, where $T \subseteq P$ is a maximally compact Cartan subgroup. Both of their proofs are based on fundamental results in representation theory of real reductive groups: the coherent continuation, Langlands' classification, Harish-Chandra's estimates of tempered characters, and Hirai's induced character formula. All these results and arguments extend to the setting of Theorem 1.1, as presented in [Hua25].

2. Construction on unipotent Arthur packets

In a joint work [DHSXss] with Taiwang Deng, Binyong Sun, and Bin Xu, we study the effect of theta lift on unipotent Arthur packets of real symplectic groups and special even orthogonal groups, and give an inductive construction of these packets. Here we state our main result.

Let $H = \mathrm{Sp}_{2n}(\mathbb{R})$, or $\mathrm{SO}(p,q)$ with $p+q=2m$ even. Let $\widehat{H}$ be its complex dual group, and ${}^L H = \widehat{H} \rtimes \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ its L-group. In either case, there is a natural homomorphism $\mathrm{Std}_H : {}^L H \to \mathrm{GL}_N(\mathbb{C})$, which identifies H as an elliptic endoscopic group of $(\mathrm{GL}_N(\mathbb{R}), \tau)$ with

$$\tau(g) = J_N g^{-\mathrm{T}} J_N^{-1}, \quad J_N = \begin{pmatrix} & & 1 \\ & \iddots & \\ (-1)^{N-1} & & \end{pmatrix}.$$

Under an appropriate choice of Whittaker datum, we have the twisted endoscopic transfer $\mathrm{Tran}_H^\tau : \mathcal{D}^{\mathrm{st}}(H) \to \mathcal{D}^\tau(\mathrm{GL}_N(\mathbb{R}))$.

Let $W_\mathbb{R} = \langle \mathbb{C}^\times, j \rangle$ be the Weil group of $\mathbb{R}$. An Arthur parameter of H is a ($\widehat{H}$-conjugacy class of) homomorphism $\psi : W_\mathbb{R} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L H$ which is compatible with the projections to $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ and algebraic on $\mathrm{SL}_2(\mathbb{C})$. The Arthur parameter ψ of H gives rise to an N-dimensional representation $\mathrm{Std}_H(\psi)$ of $W_\mathbb{R} \times \mathrm{SL}_2(\mathbb{C})$, and hence to a τ-stable irreducible representation π_ψ^{GL} of $\mathrm{GL}_N(\mathbb{R})$ via the local Langlands correspondence. By [Art13, Theorem 2.2.1], there exists a unique stable distribution $\Theta_\psi \in \mathcal{D}^{\mathrm{st}}(H)$ such that $\mathrm{Tran}_H^\tau(\Theta_\psi) = \Theta^\tau(\pi_\psi^{\mathrm{GL}})$. Moreover, the irreducible representations occurring in Θ_ψ, which form the so-called Arthur packets $\Pi_\psi(H)$, are known to be Archimedean components of square integrable automorphic representations.

The parameter ψ is called unipotent if it is trivial on $\mathbb{C}^\times \subset W_\mathbb{R}$. In this case,

$$\mathrm{Std}_H(\psi) = \bigoplus_{i=1}^{r} \mathrm{sgn}^{\epsilon_i} \boxtimes S_{m_i}, \tag{2.1}$$

as a representation of $W_\mathbb{R} \times \mathrm{SL}_2(\mathbb{C})$, where $\mathrm{sgn} : W_\mathbb{R} \to \{\pm 1\}$ is the non-trivial quadratic character, $\epsilon_i \in \{0,1\}$, and S_{m_i} is the irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension m_i. We may assume $m_1 \geqslant \cdots \geqslant m_r$. If all m_i occurring in this decomposition are odd, we say that ψ has good parity. According to the work of Mœglin and Renard, the construction of the general Arthur packet $\Pi_\psi(H)$ can be reduced to the case where ψ is unipotent and has good parity.

Theorem 2.1 (Deng-H.-Sun-Xu). Suppose ψ is a unipotent Arthur parameter of H with good parity, and has decomposition (2.1). When $H = \mathrm{SO}(p,q)$, define $\Pi_\psi(\mathrm{O}(p,q))$ to be the set of irreducible representations τ of $\mathrm{O}(p,q)$ such that $\tau|_{\mathrm{SO}(p,q)} \in \Pi_\psi(\mathrm{SO}(p,q))$. Let $\chi_{p,q}$ denote the character of $\mathrm{O}(p,q)$ whose restriction to its maximal compact subgroup $\mathrm{O}(p) \times \mathrm{O}(q)$ is $1 \boxtimes \det$. For the theta lift in the following construction, we fix the additive character given by $x \mapsto e^{2\pi\sqrt{-1}x}$ of $\mathbb{R}$.

(1) If $H = \mathrm{Sp}_{2n}(\mathbb{R})$, then the following map is well defined:

$$\bigsqcup_{p+q=2n+1-m_1,\ \frac{p-q}{2} \equiv \epsilon_1 \mod 2} \Pi_{\psi_-}(\mathrm{O}(p,q)) \to \Pi_\psi(\mathrm{Sp}_{2n}(\mathbb{R})) \sqcup \{0\}, \quad \tau \to \theta_{\mathrm{O}(p,q)}^{\mathrm{Sp}_{2n}(\mathbb{R})}(\tau),$$

where ψ_- is the Arthur parameter of $H_- = \mathrm{SO}(p,q)$ with $\mathrm{Std}_{H_-}(\psi_-) = \bigoplus_{i=2}^r \mathrm{sgn}^{\epsilon_i+\epsilon_1} \boxtimes S_{m_i}$. Moreover, this map is bijective over $\Pi_\psi(\mathrm{Sp}_{2n}(\mathbb{R}))$.

(2) If $H = \mathrm{SO}(p,q)$, then denote $\epsilon = \frac{p-q}{2} \mod 2$. The following map is well-defined:

$$\Pi_{\psi_-}(\mathrm{Sp}_{2n_-}(\mathbb{R})) \to \Pi_\psi(\mathrm{SO}(p,q)) \sqcup \{0\}, \quad \sigma \to \left(\theta^{\mathrm{O}(p,q)}_{\mathrm{Sp}_{2n_-}(\mathbb{R})}(\sigma) \otimes \chi^{\epsilon_1}_{p,q}\right)|_{\mathrm{SO}(p,q)},$$

where $2n_- = p+q-m_1-1$, and ψ_- is the Arthur parameter of $H_- = \mathrm{Sp}_{2n_-}(\mathbb{R})$ with $\mathrm{Std}_{H_-}(\psi_-) = \bigoplus_{i=2}^r \mathrm{sgn}^{\epsilon_i+\epsilon_1+\epsilon} \boxtimes S_{m_i}$. Moreover, this map is bijective over $\Pi_\psi(\mathrm{SO}(p,q))$.

Remark 2.2. Our result is a refinement of the construction in [BMSZ23].

The proof of this result consists of three parts: well-definedness, injectivity, and surjectivity. The injectivity follows easily from the conservation relation in [SZ15]. The surjectivity is deduced from a result of Mœglin [Mœg17] together with certain induction principles. The main technical part is the well-definedness of the map in (1): for $\tau \in \Pi_{\psi_-}(\mathrm{O}(p,q))$, if $\theta^{\mathrm{Sp}_{2n}(\mathbb{R})}_{\mathrm{O}(p,q)}(\tau) \neq 0$, we need to verify that it belongs to $\Pi_\psi(\mathrm{Sp}_{2n}(\mathbb{R}))$.

We explain this in more detail. According to Arthur's global result on endoscopic classification, one can construct a square-integrable automorphic representation $\dot{\tau}$ of some $\mathrm{O}(V)$ whose Archimedean component is τ. By [Li97], we know that for sufficiently large integers T, the global theta lift $\dot{\sigma}_T$ of $\dot{\tau}$ to $\mathrm{Sp}_{2(n+T)}$ is also square-integrable. The relation between (global) Arthur parameters of $\dot{\tau}$ and $\dot{\sigma}_T$ follows easily from unramified theta correspondence. Thus, the Archimedean component $\sigma_T = \theta^{H_T}_{\mathrm{O}(p,q)}(\tau)$ of $\dot{\sigma}_T$ belongs to $\Pi_{\psi_T}(H_T)$, where $H_T = \mathrm{Sp}_{2(n+T)}(\mathbb{R})$, and $\mathrm{Std}_{H_T}(\psi_T) = \mathrm{sgn}^{\epsilon_1} \boxtimes S_{m_1+2T} \oplus \bigoplus_{i=2}^r \mathrm{sgn}^{\epsilon_i} \boxtimes S_{m_i}$. According to the explicit induction principle in [Fan17], σ_T is the (unique) irreducible subrepresentation of $\mathrm{Ind}^{H_T}_{Q_T}\left(\mathrm{sgn}^{\epsilon_1} |\cdot|^{-\frac{m_1-1}{2}-T} \boxtimes \sigma_{T-1}\right)$, where Q_T is the standard parabolic subgroup of H_T with Levi subgroup $\mathrm{GL}_1(\mathbb{R}) \times H_{T-1}$. It remains to deduce that $\sigma_{T-1} \in \Pi_{\psi_{T-1}}(H_{T-1})$ from $\sigma_T \in \Pi_{\psi_T}(H_T)$, which will be explained in the next section. Then, by induction on T, we conclude that $\sigma = \theta^H_{\mathrm{O}(p,q)}(\tau) \in \Pi_\psi(H)$.

3. Compatibility between endoscopic transfer and $\mathfrak{n}$-homology

We reformulate the remaining step in our proof of well-definedness for the map in Theorem 2.1(1). Let $Q = LU$ be the standard parabolic subgroup of $H = \mathrm{Sp}_{2n}(\mathbb{R})$ with Levi subgroup $L = \mathrm{GL}_1(\mathbb{R}) \times \mathrm{Sp}_{2(n-1)}(\mathbb{R})$. Suppose σ_- is an irreducible representation of $\mathrm{Sp}_{2(n-1)}(\mathbb{R})$ such that $\mathrm{Ind}^H_Q(\mathrm{sgn}^{\epsilon_1} |\cdot|^{-\frac{m_1-1}{2}} \boxtimes \sigma_-)$ has an irreducible subrepresentation $\sigma \in \Pi_\psi(H)$, where ψ is a unipotent Arthur parameter of H with decomposition (2.1) and $m_1 > m_2 \geqslant \cdots \geqslant m_r$ are odd integers. We have to verify $\sigma_- \in \Pi_{\psi_-}(H_-)$, where $H_- = \mathrm{Sp}_{2(n-1)}(\mathbb{R})$, and $\mathrm{Std}_{H_-}(\psi_-) = \mathrm{sgn}^{\epsilon_1} \boxtimes S_{m_1-2} \oplus \bigoplus_{i=2}^r \mathrm{sgn}^{\epsilon_i} \boxtimes S_{m_i}$.

According to Frobenius reciprocity, σ_- is a constituent of a certain isotypic component of $\mathfrak{u}$-homology of σ. For simplicity, we now pass to the category of Harish-Chandra modules. Let $Z \cong \mathbb{R}_{>0}$ denote the central split torus in L. For a Harish-Chandra module V of H, let $\mathrm{D}^x_q(V)$ denote the generalized $|\cdot|^x$-isotypic component of $H_q(\mathfrak{u}, V) \otimes |\det_{\mathfrak{u}}|^{-\frac{1}{2}}$ with respect to the action of $Z \subset L$. Our final step is completed by the following result:

Proposition 3.1. For any $V \in \Pi_\psi(H)$, the irreducible factors of $\mathrm{D}_0^{-\frac{m_1-1}{2}}(V)$ lie in $\Pi_{\psi_-}(H_-)$.

To establish this proposition, it suffices to verify the following compatibility between twisted endoscopic transfer and $\mathfrak{n}$-homology:

$$\begin{tikzcd}
\mathcal{D}^{\mathrm{st}}(H) \arrow[r, "\mathrm{Tran}^{\tau}_{H}"] \arrow[d, "\mathrm{D}^{-\frac{m_1-1}{2}}"'] & \mathcal{D}^{\tau}(\mathrm{GL}_{2n+1}(\mathbb{R})) \arrow[d, "\mathrm{D}^{\tau,-\frac{m_1-1}{2}}"] \\
\mathcal{D}^{\mathrm{st}}(H_-) \arrow[r, "\mathrm{Tran}^{\tau}_{H_-}"] & \mathcal{D}^{\tau}(\mathrm{GL}_{2n-1}(\mathbb{R}))
\end{tikzcd}, \tag{3.1}$$

where $\mathrm{D}^{-\frac{m_1-1}{2}} : \mathcal{D}^{\mathrm{st}}(H) \to \mathcal{D}^{\mathrm{st}}(H_-)$ is induced by the alternating sum $\sum_q (-1)^q \mathrm{D}_q^{-\frac{m_1-1}{2}}$ on the Grothendieck group, and $\mathrm{D}^{\tau,-\frac{m_1-1}{2}} : \mathcal{D}^{\tau}(\mathrm{GL}_{2n+1}(\mathbb{R})) \to \mathcal{D}^{\tau}(\mathrm{GL}_{2n-1}(\mathbb{R}))$ is defined later. Based on this commutative diagram, one can deduce the above proposition from the following two observations: $\mathrm{D}^{\tau,-\frac{m_1-1}{2}}(\Theta^{\tau}(\pi_{\psi}^{\mathrm{GL}})) = \Theta^{\tau}(\pi_{\psi_-}^{\mathrm{GL}})$, and $\mathrm{D}_q^{-\frac{m_1-1}{2}}(V)$ vanishes for $q > 1$ and $V \in \Pi_{\psi}(H)$.

The map $\mathrm{D}^{\tau,-\frac{m_1-1}{2}} : \mathcal{D}^{\tau}(\mathrm{GL}_{2n+1}(\mathbb{R})) \to \mathcal{D}^{\tau}(\mathrm{GL}_{2n-1}(\mathbb{R}))$ is defined as follows. Let $P = MN$ be the τ-stable standard parabolic subgroup of $\mathrm{GL}_{2n+1}(\mathbb{R})$ with Levi subgroup $M = \mathrm{GL}_1(\mathbb{R}) \times \mathrm{GL}_{2n-1}(\mathbb{R}) \times \mathrm{GL}_1(\mathbb{R})$, and let $A \cong \mathbb{R}^3$ be the central split torus of M. For $q \in \mathbb{N}$ and $x \in \mathbb{C}$, let $\mathrm{D}_q^{\tau,x}$ denote the generalized $(x, 0, -x)$-isotypic component of $H_q(\mathfrak{n}, -) \otimes |\det_{\mathfrak{n}}|^{-\frac{1}{2}}$ with respect to the action of $A \subset M$. Then $\mathrm{D}_q^{\tau,x}$ sends τ-stable Harish-Chandra modules of $\mathrm{GL}_{2n+1}(\mathbb{R})$ to τ-stable Harish-Chandra modules of $\mathrm{GL}_{2n-1}(\mathbb{R})$. Our Theorem 1.1 is necessary for (3.1) in two aspects:

- the alternating sum $\sum_q (-1)^q \mathrm{D}_q^{\tau,x}$ induces a well-defined linear map $\mathrm{D}^{\tau,x} : \mathcal{D}^{\tau}(\mathrm{GL}_{2n+1}(\mathbb{R})) \to \mathcal{D}^{\tau}(\mathrm{GL}_{2n-1}(\mathbb{R}))$, although twisted character does not necessarily distinguish representations;
- we verify the compatibility in (3.1) from the explicit formulas for Tran^{τ} and D.

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THE LOCAL TWISTED GAN-GROSS-PRASAD CONJECTURE

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The Gan-Gross-Prasad (GGP) conjecture studies a family of restriction problems for classical groups and proposes precise answers to these problems using the local and global Langlands correspondences. It also has a twisted variant in the equal-rank Fourier-Jacobi case, which is called the twisted Gan-Gross-Prasad conjecture. In this talk, motivated by the works of J.-L. Waldspurger and R. Beuzart-Plessis in Bessel models, I will introduce a local trace formula approach and how to use it to prove the twisted Gan-Gross-Prasad conjecture for tempered representations over nonarchimedean fields.

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AUTOMORPHIC PERIODS AND SUM OF L-VALUES

WEIXIAO LU

The relative Langlands duality posed by Ben-Zvi-Sakellaridis-Venkatesh predicts that periods of automorphic form can sometimes equal to sum of several special values of automorphic L-functions rather than a single one. In this talk, I will present explicit evidence for this phenomenon. I will discuss families of automorphic periods on general linear groups and classical groups that are designed to detect specific types of Eisenstein series. These periods indeed evaluate to finite sums of L-values. This is joint work with Guodong Xi.

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GEOMETRIC REALIZATIONS OF LOCAL ARTHUR PACKETS

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Classification AMS 2020: 22E50, 11F70, 32S30

Keywords: Arthur packets, Vogan's conjecture, ABV packets, automorphic representations, vanishing cycles

Local Arthur packets are sets of representations of a connected reductive algebraic group over a local field F that arise as local components of important classes of automorphic representations. Jeff Adams, Dan Barbasch, and David Vogan proposed a geometric realization of a local Arthur packet when F is archimedean in [1], and Vogan did the same for nonarchimedean F in [7]. We now assume F is nonarchimedean of characteristic 0. The aforementioned geometric realization involves Vogan's geometric perspective on the local Langlands correspondence from [7], which we now explain.

Let $\Pi(G)^{\text{pure}}$ be the set of equivalence classes of smooth irreducible representations of G along with its pure inner forms. In [7], we see that to every π, we assign an enhanced Langlands parameter (ϕ, τ), where $\phi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG$ and $\tau \in \mathrm{Irrep}\,(A_\phi)$ for $A_\phi = \pi_0(Z_{\widehat{G}}(\phi))$. To establish the geometric perspective, we fix an infinitesimal parameter $\lambda : W_F \to {}^LG$. The following is taken from [3], which adapts Vogan's work and formulates Vogan's conjecture on Arthur packets using a normalised vanishing cycles functor. Given a Langlands parameter ϕ, its infinitesimal parameter λ_ϕ is given by the formula $\lambda_\phi(w) = \phi\left(w, \begin{pmatrix} |w|^{1/2} & \\ & |w|^{-1/2} \end{pmatrix}\right)$. For a fixed infinitesimal parameter λ, one defines the Vogan variety V_λ with the group action of H_λ, as in Chapter 4 of [3]. There is a bijection $\phi \mapsto C_\phi$ between equivalence classes of ϕ and H_λ-orbits C_ϕ in V_λ. To every enhanced parameter (ϕ, τ), one attaches a simple equivariant perverse sheaf $\mathcal{P} = \mathcal{IC}(\mathcal{L}^\tau) \in \mathrm{Per}_{H_\lambda}(V_\lambda)^{\text{simple}}_{/\text{iso}}$, where $\mathcal{L}^\tau$ is the local system on C_ϕ corresponding to the irrep τ of A_ϕ, now interpreted as the equivariant fundamental group attached to C_ϕ. Let $\Pi^{\text{pure}}_\lambda(G)$ consist of elements of $\Pi(G)^{\text{pure}}$ which correspond (ϕ, τ) so that $\lambda_\phi = \lambda$. One may summarize the geometric perspective on the local Langlands correspondence as follows.

$$\begin{gathered} \Pi^{\text{pure}}_\lambda(G) \longrightarrow \Phi^e_\lambda(G) \longrightarrow \mathrm{Per}_{H_\lambda}(V_\lambda)^{\text{simple}}_{/\text{iso}} \\ \pi \mapsto (\phi, \tau) \mapsto \mathcal{P}(\pi) = \mathcal{IC}(\mathcal{L}^\tau_{C_\phi}). \end{gathered}$$

This identification determines a pairing between Grothendieck groups

$$\langle \cdot, \cdot \rangle : K\Pi^{\text{pure}}_\lambda(G) \times K\mathrm{Per}_{H_\lambda}(V_\lambda) \to \mathbb{C},$$

defined in [3, Equation 8.6] (adapted from its equivalent form in [7]). Now suppose M is a Levi subgroup of G and let λ factor through a parameter λ_M of M. Normalized parabolic induction gives a linear map $\mathrm{Ind}^G_M : K\Pi_{\lambda_M}(M) \to K\Pi_\lambda(G)$. The equivariant inclusion $V_{\lambda_M} \hookrightarrow V_\lambda$ induces the linear map $\varepsilon^* : K\,\mathrm{Per}_{H_\lambda}(V_\lambda) \to K\,\mathrm{Per}_{H_{\lambda_M}}(V_{\lambda_M})$ via pullback. In collaboration with Chi-Heng Lo, we show the following.

Theorem 0.1. *(In progress; Lo-R) Assuming the local Langlands correspondence with some desiderata* for a connected reductive group G over a nonarchimedean local field F, we have the nice property that*

$$\langle \operatorname{Ind}_M^G(\pi), \mathcal{P}\rangle = \langle \pi, \varepsilon^*(\mathcal{P})\rangle$$

where $\mathcal{P} \in K \operatorname{Per}_{H_\lambda}(V_\lambda)$ and $\pi \in K\Pi_{\lambda_M}(M)$.

One can update this statement to $\pi \in K\Pi_{\lambda_M}(M)^{\mathrm{pure}}$. There is a specific list of desiderta we must assume for the this result. The special case of Theorem 0.1 for $G = \mathrm{GL}_n$ is used to prove Theorem 0.2 for $G = \mathrm{GL}_n$ in [5].

We return to the question of Vogan's geometric version of a local Arthur packet. A local Arthur packet is parametrized by an Arthur parameter $\psi : W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG$. The corresponding Langlands parameter of Arthur type is $\phi_\psi(w,x) = \psi(w, x, \begin{pmatrix} |w|^{1/2} & \\ & |w|^{-1/2}\end{pmatrix})$. Vogan's work is adapted and extended in [3] to define an ABV-packet for any Langlands parameter ϕ using a normalized vanishing cycles functor on the category of perverse sheaves, denoted NEvs_ϕ:

$$\Pi_\phi^{\mathrm{ABV}}(G) = \{\pi \in \Pi_{\lambda_\phi}(G) : \mathrm{NEvs}_\phi(\mathcal{P}(\pi)) \neq 0\}.$$

A pure version of an ABV-packet can be similarly defined if one replaces $\Pi_{\lambda_\phi}(G)$ with $\Pi_{\lambda_\phi}(G)^{\mathrm{pure}}$ to include pure inner forms. In the Arthur-type case, i.e., $\phi = \phi_\psi$, the image of the functor $\mathrm{NEvs}_{\phi_\psi}(\mathcal{P}(\pi))$ can be identified with a representation of Arthur's component group $\pi_0(Z_{\widehat{G}}(\psi))$. Vogan's proposal is that the local Arthur packet $\Pi_\psi(G)$ coincides with the ABV-packet $\Pi_{\phi_\psi}^{\mathrm{ABV}}(G)$. In joint work with Cunningham [4, 5], we prove this conjecture for p-adic GL_n. The proof of the conjecture is in progress for p-adic classical groups in a joint project with Cunningham, Hazeltine, Liu, Lo, and Xu.

Theorem 0.2. *(In progress, Cunningham-Hazeltine-Liu-Lo-R-Xu) Let $G = \mathrm{SO}_n, \mathrm{Sp}_{2n}, U_n$. Assuming the twisted p-adic Kazhdan-Lusztig hypothesis for GL_{2n}, we have $\Pi_{\phi_\psi}^{\mathrm{ABV}}(G) = \Pi_\psi(G)$. Furthermore, for every $\pi \in \Pi_\psi(G)$, Arthur's function $\langle\ \cdot\ ,\pi\rangle_\psi : Z_{\widehat{G}}(\psi)^{ss} \to \mathbb{C}$ appearing in* [2, Theorem 2.2.1] *is given by*

$$\langle s, \pi\rangle_\psi = \operatorname{trace}\left(s, \mathrm{NEvs}_{\phi_\psi} \mathcal{P}(\pi)\right).$$

In fact, one can define this function $\langle s, \pi\rangle_\psi$ using $\mathrm{NEvs}_{\phi_\psi}$ for any $\pi \in \Pi_\psi^{\mathrm{ABV,\,pure}}$.

In order to establish the result on Arthur's function, we work with virtual representations in lieu of distributions. For demonstration purposes, let us assume the simplified case when G is split SO_{2n+1}. Let $\eta_{\psi,s} = \sum_{\pi\in\Pi_\psi(G)} (-1)^{d(\psi)}\langle s,\pi\rangle_\psi \cdot \pi \in K\Pi_\lambda(G)$ where for $s = 1$, $\eta_\psi := \eta_{\psi,1}$ is a stable distribution. Here $d(\psi)$ is the dimension of the H_λ-orbit of ϕ_ψ in V_λ. Now define,

$$\eta_{\phi,s}^{\mathrm{ABV}} := (-1)^{d(\phi)} \sum_{\pi\in\Pi_\phi^{\mathrm{ABV}}(G)} (-1)^{d(\pi)} \operatorname{trace}\left(s, \mathrm{NEvs}_\phi \mathcal{P}(\pi)\right)\cdot \pi, \tag{0.1}$$

where $d(\phi)$ (resp. $d(\pi)$) is the dimension of the H_λ-orbit C_ϕ (resp. C_{ϕ_π}). in V_λ. Like before, we set $\eta_\phi^{\mathrm{ABV}} := \eta_{\phi,1}^{\mathrm{ABV}}$. Our aim boils down to showing that for any Arthur parameter ψ of G,

$$\eta_{\psi,s} = \eta_{\phi_\psi,s}^{\mathrm{ABV}}.$$

We carry out the proof in the following fashion.

(1) We show that the space of stable virtual representations $K\Pi_\lambda(G)^{\mathrm{st}}$ is spanned by $\{\eta_\phi^{\mathrm{ABV}} : \lambda_\phi = \lambda\}$.

(2) Noting $\widehat{G} = \mathrm{GL}_{2n}^\theta$, where $\theta(X) = J^t X^{-1} J^{-1}$ for a suitable matrix J, we write a slightly modified Kottwitz-Shelstad transfer map $\mathrm{Trans}_{G,\theta} : K\Pi_\lambda(G)^{\mathrm{st}} \to K\Pi_{\lambda^+}(\mathrm{GL}_{2n} \rtimes \langle\theta\rangle)$ as $\eta_\psi \mapsto \pi_\psi^+ - \pi_\psi^-$. Here $\pi_\psi^\pm$ are irreducible components of the induced representation $\mathrm{Ind}_{\mathrm{GL_{2n}}}^{\mathrm{GL_{2n}} \rtimes \langle\theta\rangle}(\pi_\psi)$, where $\pi_\psi \in \Pi(\mathrm{GL_{2n}})$, once ψ is interpreted as a parameter of this general linear group; likewise $\lambda^+ = \lambda \rtimes 1$ is a parameter of the disconnected group $\mathrm{GL}_{2n} \rtimes \langle\theta\rangle$. We show
$$\mathrm{Trans}_{G,\theta}(\eta_\psi^{\mathrm{ABV}}) = \pi_\psi^+ - \pi_\psi^- = \mathrm{Trans}_{G,\theta}(\eta_\psi).$$
Using injectivity of $\mathrm{Trans}_{G,\theta}$, $\eta_\psi = \eta_{\phi_\psi}^{\mathrm{ABV}}$. The key here is to understand $\mathrm{Trans}_{G,\theta}$ geometrically, a generalization of 0.1, and use the result from [5] for GL_n.

(3) For an endoscopic datum $({}^LG', G', s, \xi)$, the embedding $\xi : {}^LG' \to {}^LG$ gives us the parameters ψ', λ' for G'. Let $\mathrm{Trans}_{G',s}^G : K\Pi_{\lambda'}^{\mathrm{st}}(G') \to K\Pi_\lambda(G)$ denote Langlands-Shelstad transfer. We show
$$\eta_{\phi_\psi,s}^{\mathrm{ABV}} = \mathrm{Trans}_{G',s}^G(\eta_{\phi_{\psi'}}^{\mathrm{ABV}}) = \mathrm{Trans}_{G',s}(\eta_{\psi,s}) = \eta_{\psi,s}.$$
The key here is to again understand $\mathrm{Trans}_{G',s}^G$ geometrically, another generalization of 0.1.

For steps (2) and (3) we write down the transfer maps between Grothendieck groups using a basis given by virtual representations written in terms of standard representations, interpret them geometrically, and then base change back to virtual representations written in terms of irreducible representations. The p-adic analogue of the Kazhdan-Lusztig hypothesis allows us to do the latter. It is known for classical groups due to [6](used in Step (3)) and is open in the twisted general linear group case (used in step (2)). Once again, the above method can easily be upgraded to include pure inner forms.

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RELATIVE KAZHDAN LUSZTING ISOMORPHISM FOE THE PAIR GL_{2n}, Sp_{2n}

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Classification AMS 2020: 22E50, 20C08

Keywords: Relative Langlands duality, Iwahori-Hecke algebra, equivariant K theory

Let F be a non-archimedean local field, $\mathcal{O}$ its ring of integers and k its residue field. Denote $q_r = \#k$.

Let $\mathbf{G}$ be a reductive connected group split over F and let $G = \mathbf{G}(F)$ be its F points.

Let B be a Borel subgroup of G and let I be an Iwahori subgroup of G. Let $H(G, I)$ be the affine Hecke algebra of G, it is the algebra of I bi-invariant compactly supported functions on G.

Let $G^\vee$ be the complex Langlands dual group of G. Let $\mathfrak{g}^\vee$ be the Lie algebra of $G^\vee$ and let $\mathfrak{g}^{*\vee}$ be its dual. Denote by Ad the natural action of $G^\vee$ on $\mathfrak{g}^\vee$. Let $\mathcal{B}$ be the flag variety of $G^\vee$. For $B^\vee \in \mathcal{B}$ we denote by $\mathfrak{b}^\vee$ its Lie algebra.

The Deligne Langlands conjecture gives a classification of the irreducible representations of G with an I fixed vector, in terms of $G^\vee$.

Theorem 0.1. [KL87],[Ree02] *The irreducible smooth representations of G with an I fixed vector are parametrized by conjugacy classes of triples (t, n, χ) with $t \in G^\vee$ semi simple and $n \in \mathfrak{g}^\vee$ such that $Ad_t(n) = q_r n$. The element χ is an irreducible representation of the finite group $C_{G^\vee}(t, n)/C^0_{G^\vee}(t, n)$ that appears in the representation $H_\bullet(\mathcal{B}^t_n)$. Here, $C_{G^\vee}(t, n)$ is the common centralizer of t, n and $C^0_{G^\vee}(t, n)$ is the connected component of the identity in $C_{G^\vee}(t, n)$. The variety $\mathcal{B}^t_n$ is the variety of Borel subgroups $B^\vee \in \mathcal{B}$ such that $n \in \mathfrak{b}^\vee$ and $t \in B^\vee$.*

We refer to (t, n, χ) as the Deligne Langlands parameter of π.

A key step in the proof of The Deligne Langlands conjecture is a geometric description of $H(G, I)$ using equivariant K theory.

Let $\tilde{N} = T^*\mathcal{B}$ be the cotangent bundle of the flag variety of $G^\vee$. It can be described as $\tilde{N} = \{(B^\vee, \phi) | B^\vee \in \mathcal{B}, \phi \in \mathfrak{g}^{\vee *}, \phi|_{\mathfrak{b}^\vee} = 0\}$.

Let $St = \tilde{N} \times_{\mathfrak{g}^{\vee *}} \tilde{N} = \{(B_1^\vee, B_2^\vee, \phi) | B_1^\vee, B_2^\vee \in \mathcal{B}, \phi \in \mathfrak{g}^{\vee *}, \phi|_{\mathfrak{b}_1^\vee} = \phi|_{\mathfrak{b}_2^\vee} = 0\}$ be the Steinberg variety of $G^\vee$. The group $G^\vee \times \mathbb{C}^\times$ acts on St, $G^\vee$ acts in the obvious way and $\mathbb{C}^\times$ acts on $\mathfrak{g}^{\vee *}$ by $z, \phi \mapsto z^2\phi$.

The three projection maps from $\tilde{N} \times_{\mathfrak{g}^{\vee *}} \tilde{N} \times_{\mathfrak{g}^{\vee *}} \tilde{N}$ to St, give a convolution product on homology theories of St.

Let W be the Weyl group of G and let W_{aff} be the extended affine Weyl group of G.

The proofs of the following results can be found in [KL87] and [CG97] .

Theorem 0.2.
(1) *The number of irreducible components of St is equal to the size of the Weyl group W.*
(2) *For top Borel Moore homology, there is an isomorphism $\Phi_f : H^{BM}_{top}(St) \xrightarrow{\sim} \mathbb{C}[W]$.*
(3) *For equivariant K theory, there is an isomorphism $\Phi_a : K^{G^\vee}(St) \xrightarrow{\sim} \mathbb{C}[W_{aff}]$.*

(4) *For equivariant K theory, there is an isomorphism* $\Phi_{KL} : K^{G^\vee \times \mathbb{C}^\times}(St) \xrightarrow{\sim} H(G, I)$.

In [Sht26a] we prove a relative version of Theorem 0.2 for $G = GL_{2n}(F)$ and the symmetric space $X = GL_{2n}(F)/Sp_{2n}(F)$.

Let $C^\infty(X)$ be the space of locally constant functions on X. Let $S(X)$ be the space of compactly supported locally constant functions on X, and let $S(X)^I$ be the space of I invariant compactly supported functions on X. The algebra $H(G, I)$ acts on $S(X)^I$ by convolution.

In [BZSV24], a generalization of Langlands duality is suggested. For every spherical variety X with a G action, we can consider $M = T^*X$, a Hamiltonian space. Under some conditions, a dual Hamiltonian space $M^\vee$ with a $G^\vee$ action is defined. It comes with a $\mathbb{G}_m$ action that commutes with the $G^\vee$ action.

For the GL_{2n} space $X = GL_{2n}/Sp_{2n}$ the attached dual space $M^\vee$ is described as follows. Let $H \subset GL_{2n}$ be the group of block matrices of the form

$$H = \{\begin{pmatrix} g & a \\ 0 & g \end{pmatrix} | g \in GL_n, a \in M_n\}$$

Denote by $\mathfrak{h}$ the Lie algebra of H and by $\mathfrak{h}^*$ its dual. We fix $\psi \in (\mathfrak{h}^*)^H$, an H invariant element of $\mathfrak{h}^*$, $\psi(\begin{pmatrix} g & a \\ 0 & g \end{pmatrix}) = tr(a)$.

Let $M^\vee = T^*_\psi(GL_{2n}/H)$ be the twisted cotangent bundle attached to ψ.

It can be described explicitly as $M^\vee = \{(gH, \phi), gH \in G^\vee/H, \phi \in \mathfrak{g}^{\vee *}, (g^{-1}\phi)|_{\mathfrak{h}} = \psi\}$.

The Weyl group W acts on the set of Borel orbits, $B\backslash X$ (see [Kno95]). The extended affine Weyl group W_{aff} acts on the set of Iwahori orbits, $I\backslash X$ (see [Sht26b]).

We consider the space $\Lambda = M^\vee \times_{\mathfrak{g}^*} \tilde{N}$, this is a relative analogue of St. There are three projection maps from $M^\vee \times_{\mathfrak{g}^*} \tilde{N} \times_{\mathfrak{g}^*} \tilde{N}$, two to Λ and one to St. These maps give a module structure on homology theories of Λ over homology theories of St as in Subsection 5.2.20 of [CG97].

Let sgn be the sign representation of W, and let sgn_f be its extension to W_{aff}.

Let $IM : H(G, I) \to H(G, I)$ be the Iwahori Matsumoto involution, it induces an involution on $H(G, I)$ modules. For each $H(G, I)$ module V we denote by $IM(V)$ the $H(G, I)$ module obtained from V by twisting the $H(G, I)$ action by this involution.

Let $\mathbb{C}[B\backslash X]$ be the vector space spanned by $B\backslash X$. Similarly, let $\mathbb{C}[I\backslash X]$ be the vector space spanned by $I\backslash X$.

We prove the following results.

Theorem 0.3. *(*[Sht26a]*)*

(1) *The number of Borel orbits on X is equal to the number of irreducible components of Λ.*

(2) *There is a module isomorphism $\Phi_{f,X} : H^{BM}_{top}(\Lambda) \otimes sgn \xrightarrow{\sim} \mathbb{C}[B\backslash X]$ compatible with Φ_f.*

(3) *There is a module isomorphism $\Phi_{a,X} : K^{G^\vee}(\Lambda) \otimes sgn_f \xrightarrow{\sim} \mathbb{C}[I\backslash X]$ compatible with Φ_a.*

(4) *There is a module isomorphism $\Phi_{KL,X} : IM(K^{G^\vee \times \mathbb{C}^\times}(\Lambda)) \xrightarrow{\sim} S(X)^I$ compatible with Φ_{KL}.*

Remark 0.4. *The first part of the above theorem is a special case of a conjecture made in* [FGT25].

We use Theorem 0.3 to prove a result about irreducible G representations with an I fixed vector that are X distinguished. To state it we first introduce additional notions.

Definition 0.5. *An irreducible representation π of G is called X distinguished if* $Hom_G(\pi, C^{\infty}(X)) \neq 0$.

For an irreducible representation π of G we denote by $Z(\pi)$ its Zelevinsky dual (defined in Section 9 of [Zel80]).

We use a Killing form on $\mathfrak{g}^{\vee}$ to identify $\mathfrak{g}^{\vee} \cong \mathfrak{g}^{\vee *}$. We can consider an element $n \in \mathfrak{g}^{\vee}$ as $n \in \mathfrak{g}^{\vee *}$.

Theorem 0.6. *Let π be an irreducible representation of G with an I fixed vector. Let (t, n, χ) be the Deligne Langlands parameter of π. Let $a = (t, \sqrt{q_r}) \in G^{\vee} \times \mathbb{C}^{\times}$. If $Z(\pi)^{\vee}$ is X distinguished then $n \in (\mathfrak{g}^{*\vee})^a$ is in the image of the moment map $\mu : (M^{\vee})^a \to (\mathfrak{g}^{*\vee})^a$. Here, $(M^{\vee})^a$ and $(\mathfrak{g}^{*\vee})^a$ denote the fixed points of a on $M^{\vee}$ and $\mathfrak{g}^{*\vee}$ respectivly.*

Remark 0.7. *A completely different proof of an equivalent formulation of Theorem 0.6 appears in* [MOS17].

Remark 0.8. *This does not solve the classification problem of X distinguished representations with an I fixed vector because not all representations which satisfy the condition given by Theorem 0.6 are X distinguished.*

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FARGUES'S CATEGORICAL CONJECTURE FOR ELLIPTIC PARAMETERS FOR $\mathrm{SL}(n)$

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Classification AMS 2020: 11S37, 14D24, 22E57

Keywords:

Let $p \neq \ell$ be primes and let $F/\mathbb{Q}_p$ be a finite extension. For any split reductive group G over F, the Local Langlands Correspondence (LLC) predicts a map

$$\mathrm{Irr}_{\overline{\mathbb{Q}}_\ell} G(F) \to \{W_F \to \check{G}(\overline{\mathbb{Q}}_\ell)\}/^{\text{conj.}}\check{G}(\overline{\mathbb{Q}}_\ell)$$

from the set of irreducible smooth $\overline{\mathbb{Q}}_\ell$-representations of $G(F)$ to the set of semisimple homomorphisms $W_F \to \check{G}(\overline{\mathbb{Q}}_\ell)$ modulo $\check{G}(\overline{\mathbb{Q}}_\ell)$-conjugacy, where W_F is the Weil group and $\check{G}$ is the Langlands dual. Moreover, the map is expected to be a surjection with finite fibers. Recently Fargues and Scholze [3] provided a general construction of such a map, but currently one cannot say anything in general about the fibers or the surjectivity of the map. More precisely, Fargues and Scholze organize the datum of excursion and Hecke operators [3, Corollary X.1.3] into the $\overline{\mathbb{Q}}_\ell$-linear action of the category $\mathrm{Perf}(\mathrm{Par}_{\check{G}})$ of perfect complexes on the stack of L-parameters on the category $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega$ of sheaves on Bun_G.

Properties such as finite fibers or surjectivity are often proved by comparing Fargues-Scholze's construction to classical constructions. For example when $G = \mathrm{GL}_n$, Fargues and Scholze proved compatibility of their map with the Local Langlands Correspondence constructed in [4, 5, 7]. For $G = \mathrm{SL}_n$, Harris and Taylor's correspondence, togetheer with the work of Gelbart and Knapp [6] analyzing how representations of $\mathrm{GL}_n(F)$ split upon restriction to $\mathrm{SL}_n(F)$ gives a LLC. As a formal consequence of the compatibility of Fargues-Scholze and Harris-Taylor's correspondence for GL_n, together with the compatibility of Fargues-Scholze with central isogenies [3, Theorem IX.6.1], there is an analogous compatibility for Fargues-Scholze's correspondence and the classical construction of LLC of Haris-Taylor and Gelbart-Knapp. We go further and prove the compatibility of the *internal parametrization of the L-packets* in Fargues-Scholze and Gelbart-Knapp's correspondences.

The data of Fargues and Scholze's spectral action allows one to relate coherent sheaves on $\mathrm{Par}_{\check{G}}$ to certain kinds of ℓ-adic sheaves on Bun_G. More precisely, a *Whittaker datum* is a choice of a Borel $B \subset G$ with unipotent radical U together with a generic character $\psi \colon U(F) \to \overline{\mathbb{Q}}_\ell^\times$. Let $\mathcal{W}_\psi \in \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ be the Whittaker sheaf, and consider the functor

$$\mathrm{Perf}(\mathrm{Par}_{\check{G}}) \to \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell) : M \mapsto M \star \mathcal{W}_\psi. \tag{0.1}$$

Fargues and Scholze conjecture that this functor decategorifies to the usual Local Langlands Correspondence, and prove their conjecture for $G = \mathrm{GL}_n$.

They furthermore conjecture in [3, Conjecture X.2.2] that their functor is an equivalence when restricted to supercuspidal parameters:

Conjecture 0.1 (Fargues's conjecture on elliptic parameters)**.** *Fix a Whittaker datum on a quasi-split group G whose connected split center is trivial. Let π be a generic representation with respect to the Whittaker datum with elliptic L-parameter $\varphi_\pi = \varphi$. Then the functor*

$$\mathrm{Perf}([\mathrm{Spec}\,\overline{\mathbb{Q}}_\ell/S_\varphi]) \to \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega_\varphi : W \mapsto \mathrm{Act}_W(\pi)$$

is an equivalence which is t-exact with respect to the standard t-structures on both sides. Here, $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega_\varphi$ denotes the localization of the category $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega$ at the Bernstein block corresponding to the L-parameter φ.

In particular, a corollary of the conjecture is:

Corollary 0.2. *Suppose Conjecture 0.1 holds for G. Then for any elliptic parameter φ, the functor* (0.1) *induces a bijection*

$$\mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(S_\varphi) \simeq \bigsqcup_{b \in B(G)_{\mathrm{basic}}} \big\{\pi \in \mathrm{Irr}_{\overline{\mathbb{Q}}_\ell} G_b(F) : \varphi_\pi = \varphi\big\}.$$

Remark 0.3. *To formulate Fargues's conjecture on elliptic parameters for arbitrary quasi-split groups, e.g.,* GL_n*, see* [8]*. The issue is that when G has a nontrivial connected split center, there are nontrivial unramified twists of φ, so the connected component C_φ of* $\mathrm{Par}_{\check{G}}$ *containing φ is no longer isomorphic to* $[\mathrm{Spec}(\overline{\mathbb{Q}}_\ell)/S_\varphi]$.

Our main theorem is:

Theorem 0.4. *Conjecture 0.1 holds for $G = \mathrm{SL}_n$. In particular as in Corollary 0.2, for any elliptic parameter φ and fixed choice of Whittaker datum there is a bijection*

$$\mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(S_\varphi) \simeq \big\{\pi \in \mathrm{Irr}_{\overline{\mathbb{Q}}_\ell} \mathrm{SL}_n(F) : \varphi_\pi = \varphi\big\}. \tag{0.2}$$

Moreover, the bijection matches Gelbart and Knapp's construction.

We give *two methods* of proofs for Theorem 0.4, which we will sketch below. One approach is character-theoretic, using recent results of Fu [9] on the stability of Fargues-Scholze L-packets. However, this only proves the first half of Theorem 0.4. In other words, it can prove the existence of a bijection (0.2) but it *may not* match Gelbart and Knapp's construction. By elementary group theory considerations there is a natural surjection $F^\times/(F^\times)^n \to \mathrm{Hom}(S_\varphi, \overline{\mathbb{Q}}_\ell^\times)$, so $F^\times/(F^\times)^n$ acts on $\mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(S_\varphi)$. On the other hand, the conjugation action of $\mathrm{PGL}_n(F)$ on $\mathrm{SL}_n(F)$ gives an action of $F^\times/(F^\times)^n$ on $\mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(\mathrm{SL}_n(F))$.

Definition 0.5. *Let $\pi\colon \mathrm{SL}_n(F) \to \mathrm{GL}(V)$ be a smooth representation and let $\gamma \in F^\times/(F^\times)^n$. Then let π^γ denote the twist by the outer automorphism $\mathrm{SL}_n(g)$:*

$$\pi^\gamma\colon \mathrm{SL}_n(F) \xrightarrow{\mathrm{ad}(g)} \mathrm{SL}_n(F) \xrightarrow{\pi} \mathrm{GL}(V),$$

where $g \in \mathrm{GL}_n(F)$ is an element with $\det(g) = \gamma$.

Remark 0.6. *The above is well-defined up to isomorphism because the determinant map identifies the quotient $\mathrm{GL}_n(F)/F^\times\mathrm{SL}_n(F)$ with $F^\times/(F^\times)^n$.*

The $F^\times/(F^\times)^n$-actions on both sides of the bijection (0.2) are transitive. Thus, proving the compatibility of the Fargues-Scholze and Gelbart-Knapp's internal parametrization of L-packets amounts to proving (0.2) intertwines the $F^\times/(F^\times)^n$-actions.

The second more categorical approach, closely following Gaitsgory and Raskin's method to deduce the Geometric Langlands Conjecture for arbitrary reductive groups from the same the Geometric Langlands Conjecture for groups with connected center [1, §8]. In particular this proves (0.2) intertwines the $F^\times/(F^\times)^n$-actions.

We emphasize that the two methods are *completely independent of each other*.

0.1. **Character-theoretic proof.** For any irreducible supercuspidal representation π of $\mathrm{SL}_n(F)$ with L-parameter φ, we can find an irreducible representation Π of $\mathrm{GL}_n(F)$ such that π is a direct summand of $\Pi|_{\mathrm{SL}_n(F)}$. For any $\chi \in \mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(S_\varphi)$ by [8, Lemma 2.28] the object $\chi\pi$, which is a priori only a sheaf on $\mathrm{Bun}_{\mathrm{SL}_n}$, is supported on the unique basic Harder-Narasimhan-stratum, and can be viewed as an irreducible smooth representation of $\mathrm{SL}_n(F)$.[1] By the compatibility of Fargues and Scholze's correspondence with isogenies [3, Theorem IX.6.1], the irreducible representation $\chi\pi$ is still a direct summand of $\Pi|_{\mathrm{SL}_n(F)}$. Proving Conjecture 0.1 for $G = \mathrm{SL}_n$ amounts to proving $\chi\pi$ exhausts all summands of $\Pi|_{\mathrm{SL}_n(F)}$. This follows from Fu's stability result [9].

0.2. **Categorical proof.** We hope to prove that Fargues-Scholze's correspondence intertwines the $F^\times/(F^\times)^n$-actions.

First of all, $\mathrm{Bun}_{\mathrm{SL}_n}$ classifies vector bundles $\mathcal{V}$ together with a trivialization of the determinant $\varphi\colon \det(\mathcal{V}) \simeq \mathcal{O}$. Any element $t \in F^\times$ acts on $\mathrm{Bun}_{\mathrm{SL}_n}$ by changing the trivialization φ to $t\varphi$. This defines an auto-equivalence Γ_t of $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathrm{SL}_n})$.

Remark 0.7. *The action of t takes the trivial vector bundle $\mathcal{O}^n$ with the trivial trivialization $\det(\mathcal{O}^n) \simeq \mathcal{O}$ to the trivial vector bundle $\mathcal{O}^n$ with the* nontrivial *trivialization $t\colon \det(\mathcal{O}^n) \simeq \mathcal{O}$. These are isomorphic under the automorphism $g\colon \mathcal{O}^n \simeq \mathcal{O}^n$ for any $g \in \mathrm{GL}_n(F)$ with determinant t. Thus, on the full subcategory $\mathrm{Rep}\,\mathrm{SL}_n(F)$ of $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathrm{SL}_n})$, the auto-equivalence is by the outer automorphism of $\mathrm{SL}_n(F)$ given by conjugation by $-^t$ of Definition 0.5.*

On the other hand, any $w \in W_F$ we may define a μ_n-bundle $\mathcal{L}_w$ on $\mathrm{Par}_{\mathrm{PGL}_n}$, whose total space parametrizes parameters $\varphi\colon W_F \to \mathrm{PGL}_n$ together with a lift of $\varphi(w) \in \mathrm{PGL}_n$ to SL_n. Now, the key statement is the following analog of [1, Theorem 5.1.7]:

Theorem 0.8. *For any $w \in W_F$, let $t \in F^\times$ be the image under the map of class field theory $W_F \to F^\times$. Then there is a natural equivalence $\mathcal{L}_w \star - \simeq \Gamma_t(-)$ of auto-equivalences of $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathrm{SL}_n}, \overline{\mathbb{Q}}_\ell)$, where $\mathcal{L}_w \in \mathrm{Perf}(\mathrm{Par}_{\mathrm{PGL}_n})$ acts by the spectral action. In particular, on the subcategory $\mathcal{D}(\mathrm{SL}_n(F), \overline{\mathbb{Q}}_\ell)$ of $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathrm{SL}_n}, \overline{\mathbb{Q}}_\ell)$, the spectral action is given by $-^t$ using notation from Definition 0.5.*

To prove Theorem 0.8, we follow the same strategy as in [1], which we briefly review here.

In [2], Gaitsgory defines the notion of a sheaf of categories on higher stacks. To deduce the Geometric Langlands equivalence for arbitrary semisimple G from G_{ad}, Gaitsgory and Raskin [1, §8.5] define two enhancements of $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$. One is an enhancement purely on the automorphic side, denoted by $\underline{\text{D-mod}}^{\mathrm{Z}_G}_{\frac{1}{2}}(\mathrm{Bun}_{G_{\mathrm{ad}}})$, to an an object of $\mathrm{ShvCat}(\mathrm{Ge}_{\mathrm{Z}_G}(X))$. The other enhancement is an enhancement purely on the

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[1] Here it is essential that χ is one-dimensional. Otherwise, $\chi\pi$ need not be an irreducible representation.

spectral side, denoted by $\underline{\text{D-mod}}_{\frac{1}{2}}^{\pi_1(\check{G})}(\mathrm{Bun}_G)$, to an object of $\mathrm{ShvCat}(\mathrm{Ge}_{\pi_1(\check{G})}(X))$. They furthermore prove the 2-Fourier-Mukai transform, which is an equivalence

$$\text{2-FM}\colon \mathrm{ShvCat}(\mathrm{Ge}_{\mathrm{Z}_G}(X)) \simeq \mathrm{ShvCat}(\mathrm{Ge}_{\pi_1(\check{G})}(X))$$

via the pairing $\mathrm{Ge}_{\mu_n}(X) \times \mathrm{Ge}_{\pi_1(\check{G})}(X) \to B^2\mathbb{G}_m$. Then, everything follows from the equivalence

$$\text{2-FM}(\underline{\text{D-mod}}_{\frac{1}{2}}^{\mathrm{Z}_G}(\mathrm{Bun}_{G_{\mathrm{ad}}})) \simeq \underline{\text{D-mod}}_{\frac{1}{2}}^{\pi_1(\check{G})}(\mathrm{Bun}_G).$$

For us, we want to let Ge_{μ_n} be the 2-stack classifying μ_n-gerbes on the Fargues-Fontaine curve, but for technical reasons we define it as a 2-stack over $\overline{\mathbb{Q}}_\ell$. On the other hand, we let $\mathrm{Par}_{B\mu_n}$ be the 2-stack of group homomorphisms $W_F \to B\mu_n$ up to $B\mu_n$-conjugation. Then we have:

Lemma 0.9. *There is an equivalence of* 2*-categories*

$$\text{2-FM}\colon \mathrm{ShvCat}(\mathrm{Ge}_{\mu_n}) \simeq \mathrm{ShvCat}(\mathrm{Par}_{B\mu_n}). \tag{0.3}$$

Now, analogous to [1], we define an enhancement of the category $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathrm{SL}_n}, \overline{\mathbb{Q}}_\ell)$ to an object $\underline{\mathcal{D}}_{\mathrm{lis}}^{\mathrm{aut}}(\mathrm{Bun}_{\mathrm{SL}_n}, \overline{\mathbb{Q}}_\ell)$ of $\mathrm{ShvCat}_{\mathrm{Perf}}(\mathrm{Ge}_{\mu_n}, \overline{\mathbb{Q}}_\ell)$ purely on the automorphic side. We also define a purely spectral enhancement of the category $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathrm{SL}_n}, \overline{\mathbb{Q}}_\ell)$ to an object $\underline{\mathcal{D}}_{\mathrm{lis}}^{\mathrm{spec}}(\mathrm{Bun}_{\mathrm{SL}_n}, \overline{\mathbb{Q}}_\ell)$ of $\mathrm{ShvCat}(\mathrm{Par}_{B\mu_n}, \overline{\mathbb{Q}}_\ell)$. We prove the following analog of [1, Theorem 8.5.8]:

Theorem 0.10. *Under the equivalence* (0.3)*, we have*

$$\text{2-FM}\left(\underline{\mathcal{D}}_{\mathrm{lis}}^{\mathrm{aut}}(\mathrm{Bun}_{\mathrm{SL}_n}, \overline{\mathbb{Q}}_\ell)\right) \simeq \underline{\mathcal{D}}_{\mathrm{lis}}^{\mathrm{spec}}(\mathrm{Bun}_{\mathrm{SL}_n}, \overline{\mathbb{Q}}_\ell).$$

Theorem 0.8 will then be a straightforward consequence.

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CLASSIFICATION OF HYPERSPHERICAL DATA AND THEIR DUALS

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Classification AMS 2020: 11F67; 11F72

Keywords: relative Langlands duality, hyperspherical Hamiltonian varieties, BZSV duality, spherical varieties

The theory of BZSV duality, introduced by Ben-Zvi, Sakellaridis, and Venkatesh in [1], focuses on a correspondence between hyperspherical Hamiltonian varieties and their duals. In their paper, they raised the period conjecture, which states that the period integral associated with a hyperspherical Hamiltonian variety should be equal to the L-function associated with its dual, and vice versa.

By the structure theorem in [1], these hyperspherical Hamiltonian spaces $\mathcal{M}_\Delta$ are captured by BZSV quadruples $\Delta = (G, H, \iota, \rho_H)$, such that H is a reductive subgroup of G, $\iota : \mathrm{SL}_2 \to G$ is a homomorphism whose image commutes with H, and ρ_H is a symplectic representation of H. Furthermore, when such quadruple satisfies specific hyperspherical and anomaly-free conditions, it uniquely defines an anomaly-free hyperspherical G-Hamiltonian space.

In this talk, we focus on the classification of all hyperspherical data $\Delta = (G, H, \iota, \rho)$ for simple reductive groups G and the construction of their conjectural dual data $\Delta^\vee = (G^\vee, G^\vee_\Delta, \iota^\vee_\Delta, \rho^\vee_\Delta)$, as presented in our paper [8].

The classification proceeds in three steps:

(1) Identify all SL_2-homomorphisms ι (equivalently, nilpotent orbits of G) that are Levi-spherical: the weight-two space $\mathfrak{u}/\mathfrak{u}^+ \subset \mathfrak{g}$ is a spherical variety of the Levi subgroup L associated with ι;
(2) For each such ι, enumerate all spherical subgroups $H \subset G_\iota$ that the generic stabilizer H' of $\mathcal{M}_\Delta$ in G is connected;
(3) Determine all anomaly-free multiplicity-free symplectic representations ρ_H of H [4, 5] such that $(\rho_H \oplus \mathfrak{u}/\mathfrak{u}^+)|_{H'}$ remains multiplicity-free.

For the construction of the dual data $\Delta^\vee = (G^\vee, G^\vee_\Delta, \iota^\vee_\Delta, \rho^\vee_\Delta)$, three major approaches are applied based on the geometric or representation-theoretic properties of the hyperspherical data:

(1) Distinguished Polarized Case: When the data takes the form $\Delta = (G, H, 1, T(\rho_+))$, the dual group $G^\vee_\Delta$ is derived from the dual group of the spherical variety $X = G \times^H \rho_+$, as defined in [2, 3, 7]. The dual representation $\rho^\vee_\Delta$ can be described in terms of the colors of X.
(2) Vector Space Case: For strongly tempered data $\Delta = (G, G, 1, \rho)$, the dual data is conjectured in [6] that should satisfy the period conjecture. The dual group $G^\vee_\Delta$ has a root type dual to the little Weyl group of the vector space.

(3) Whittaker Induction: For other cases, Whittaker induction [1] reduces the dual construction of Δ to the reductive data $\Delta_{red} = (L, H, 1, \rho_{H,\iota})$. The dual group $G_\Delta^\vee$ is generated by $L_{\Delta_{red}}^\vee$ and $\mathrm{SL}_{2,\alpha}$ for simple roots outside $L^\vee$.

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VOLUME OF SHTUKAS AND DERIVATIVES OF ZETA FUNCTIONS

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Classification AMS 2020: 11G40, 11F70, 14G35, 20G40

Keywords: Shtukas, zeta functions of curves, arithmetic volumes, special values of L-functions, Langlands dual group

In the works [1], [2], and [3], it is shown that the arithmetic volumes of certain Shimura varieties are related to special values of the first derivatives of certain zeta functions.

In [4], a parallel result over function fields is proved. Let G be a split reductive group and C a geometrically connected smooth projective curve defined over the finite field $\mathbb{F}_q$. Consider the r-leg Shtuka moduli space $\mathrm{Sht}^r_{G,\leq\mu}$, where $\mu \in X_*(T)$ is a dominant coweight of G. For a suitable choice of line bundle $\mathcal{L} \in \mathrm{Pic}(\mathrm{Sht}^r_{G,\leq\mu})$ involving the determinant line bundle, one can define the volume

$$\mathrm{vol}(\mathrm{Sht}^r_{G,\leq\mu}, \mathcal{L}) \in \mathbb{Q}.$$

Theorem 0.1 ([4])**.** *Assuming that the coweight μ is minuscule, one has*

(0.1)

$$\mathrm{vol}(\mathrm{Sht}^r_{G,\leq\mu}, \mathcal{L}) = q^{(g-1)\dim(G)}(-\log q)^{-r} \left(\frac{d}{ds}\right)^r \Big|_{s=0} \left(q^{-(2g-2)b_\mu s} \prod_{i=1}^{l} \zeta_C(\epsilon_{\mu,i} s + d_i) \right).$$

Here,

- *l is the rank of G,*
- *$b_\mu, \epsilon_{\mu,i} \in \mathbb{Q}$ are certain constants,*
- *g is the genus of C,*
- *d_i are the degrees of the fundamental invariants of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$.*

Among the constants appearing in the formula above, the numbers $\epsilon_{\mu,i}$ are the most interesting and are called the *eigenweights*. An explicit formula for the eigenweights is obtained in [5] for classical groups.

In this talk, we remove the assumption that μ is minuscule. More precisely, we prove the following:

Theorem 0.2. *Without the assumption that μ is minuscule, Theorem 0.1 still holds.*

Moreover, we obtain more conceptual and explicit formulas for the eigenweights $\epsilon_{\mu,i}$. Let $\check{G}_{\mathbb{Z}}$ be the Chevalley form of the Langlands dual group, and let $\Delta(\mu) \in \mathrm{Rep}(\check{G}_{\mathbb{Z}})$ be the standard module of highest weight μ. Denote by $\check{\mathfrak{g}} = \mathrm{Lie}(\check{G}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q})$.

Fix the minimal invariant bilinear form $\kappa_{\min} : \check{\mathfrak{g}} \otimes \check{\mathfrak{g}} \to \mathbb{Q}$, normalized so that $\kappa_{\min}(\alpha_{\mathrm{s}}, \alpha_{\mathrm{s}}) = 2$ for any short root α_{s}. Let $(x_i)_{i=1}^{l}$ be the Chevalley generators of $\check{\mathfrak{g}}_{\mathbb{Z}}$, and define

$$e = \sum_{i=1}^{l} \frac{\kappa_{\min}(\alpha_i, \alpha_i)}{2} x_i \in \check{\mathfrak{g}}_{\mathbb{Z}}.$$

Let $f, h \in \check{\mathfrak{g}}$ complete the principal $\mathfrak{sl}_2$-triple containing e.

Choose a generator $u_- \in \Delta(\mu)_{-w_0(\mu)}$ of the lowest weight space and a generator $u_+^* \in (\Delta(\mu)_\mu)^*$ of the dual of the highest weight space, normalized so that

$$\left\langle e^{\langle 2\rho, \mu \rangle} \cdot u_-,\ u_+^* \right\rangle \geq 0.$$

Define a bilinear form $\kappa_\mu : U(\check{\mathfrak{g}}) \otimes U(\check{\mathfrak{g}}) \to \mathbb{Q}$ by

$$\kappa_\mu(X \otimes Y) = \Big\langle \sum_{s=0}^{\langle 2\rho,\mu\rangle} e^s XY\, e^{\langle 2\rho,\mu\rangle - s} u_-,\ u_+^* \Big\rangle.$$

We have the following result:

Theorem 0.3. *When G is not of type $D_{2\cdot\frac{l}{2}}$, for each $1 \leq i \leq l$, we have*

$$\epsilon_{\mu,i} = \frac{\kappa_\mu|_{\check{\mathfrak{g}}_{e,\,2d_i-2} \otimes \check{\mathfrak{g}}_{f,\,2-2d_i}}}{\kappa_{\min}|_{\check{\mathfrak{g}}_{e,\,2d_i-2} \otimes \check{\mathfrak{g}}_{f,\,2-2d_i}}}. \tag{0.2}$$

Here $\check{\mathfrak{g}}_e$ is the centralizer of e in $\check{\mathfrak{g}}$, equipped with the principal grading $\check{\mathfrak{g}}_e = \bigoplus_k \check{\mathfrak{g}}_{e,k}$; similarly for $\check{\mathfrak{g}}_f$.

In particular, when G is of classical type and μ is minuscule, we obtain more explicit and elementary formulas for the eigenweights, compared with those in [5].

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HAUSDORFFNESS OF CERTAIN NILPOTENT COHOMOLOGY SPACES

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Classification AMS 2020: 22E46, 17B56

Keywords:smooth representation, nilpotent cohomology

This talk is based on joint work with Fabian Januszewski and Binyong Sun.

Let G be a real Lie group with complexified Lie algebra $\mathfrak{g}$. Let V be a smooth representation of G on a quasi-complete, locally convex, Hausdorff complex topological vector space.

Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$, so that V is an $\mathfrak{h}$-module via differentiation. We compute the Lie algebra homology $\mathrm{H}_\bullet(\mathfrak{h}, V)$ using the standard chain complex $\wedge^\bullet \mathfrak{h} \otimes V$. The homology space is then equipped with the topology naturally induced by the subquotient construction from the topology on the complex.

Question 0.1. *Whether the homology space* $\mathrm{H}_\bullet(\mathfrak{h}, V)$ *is a Hausdorff space?*

The above question is equivalent to asking whether the image of the boundary map ∂ is a closed subspace of $\wedge^\bullet \mathfrak{h} \otimes V$. The analogous question can be posed for the cohomology space.

Our main theorem is stated as bellow.

Theorem 0.2 ([6]). *Suppose that G is compact. Let $\mathfrak{q}$ be a parabolic subalgebra of $\mathfrak{g}$ with nilpotent radical $\mathfrak{u}$ and Levi subgroup $L := N_G(\mathfrak{q})$.*
(a) Denote by

$$\mathrm{d} : \wedge^\bullet(\mathfrak{u}^*) \otimes V \to \wedge^\bullet(\mathfrak{u}^*) \otimes V \quad \textit{and} \quad \partial : \wedge^\bullet \mathfrak{u} \otimes V \to \wedge^\bullet \mathfrak{u} \otimes V$$

the coboundary and boundary maps respectively, both of which are L-equivariant with respect to the natural actions of L. Then both of the inclusion maps

$$\operatorname{Im} \mathrm{d} \hookrightarrow \ker \mathrm{d} \quad \textit{and} \quad \operatorname{Im} \partial \hookrightarrow \ker \partial$$

admit a degree-preserving L-equivariant continuous linear splitting.
(b) The Lie algebra cohomology space $\mathrm{H}^\bullet(\mathfrak{u}, V)$ and the Lie algebra homology space $\mathrm{H}_\bullet(\mathfrak{u}, V)$ are Hausdorff and quasi-complete. Moreover, with the natural actions of L, $\mathrm{H}_\bullet(\mathfrak{u}, V)$ and $\mathrm{H}^\bullet(\overline{\mathfrak{u}}, V)$ are smooth representations of L that are isomorphic to each other. Here $\overline{\mathfrak{u}}$ is the complex conjugation of $\mathfrak{u}$.

We next review some related work on Question 0.1.

From now on, we assume that G is a real reductive group. Fix a Cartan involution θ of G. Denote by $K := G^\theta$ the fixed point subgroup of θ, which is a maximal compact subgroup of G. Let $\mathfrak{q}$ be a θ-stable or real parabolic subalgebra of $\mathfrak{g}$. Let $\mathfrak{u}$ be the nilpotent radical of $\mathfrak{q}$ and $L := N_G(\mathfrak{q}) \cap N_G(\theta\mathfrak{q})$ be the Levi subgroup.

Given a $(\mathfrak{g}, K)$-module M of G, a globalization of M is defined to be a representation V of G together with a $(\mathfrak{g}, K)$-module isomorphism between M and the underlying $(\mathfrak{g}, K)$-module of V.

When M has finite length (in this case M is called a Harish-Chandra module), it has four canonical globalizations: the minimal globalization $M^{\min}$, the Casselman-Wallach globalization M^{∞}, the distribution globalization $M^{-\infty}$, and the maximal globalization $M^{\max}$. These globalizations are smooth representations of G on Fréchet or dual Fréchet spaces, and they fit into a sequence of inclusions

$$M \subset M^{\min} \subset M^{\infty} \subset M^{-\infty} \subset M^{\max}.$$

Conjecture 0.3. *Let M be a Harish-Chandra module, and let V be one of the canonical globalizations of M.*
(a) The homology $\mathrm{H}_\bullet(\mathfrak{u}, V)$ is Hausdorff. In particular, it is a smooth representation of L.
(b) As a smooth representation of L, $\mathrm{H}_\bullet(\mathfrak{u}, V)$ is the corresponding canonical globalization of $\mathrm{H}_\bullet(\mathfrak{u}, M)$.

When $\mathfrak{q}$ is a real parabolic subalgebra, the above conjecture may be referred to as Casselman's comparison conjecture. When $\mathfrak{q}$ is θ-stable, it may be referred to as Vogan's conjecture; see [9, Conjecture 10.3].

Remark 0.4. *One can formulate an analogous conjecture for cohomology spaces. Note that for any $\mathfrak{g}$-module M, there are natural isomorphisms* [3, Equation (2.18)]

$$\mathrm{H}_p(\mathfrak{u}, M) \cong \mathrm{H}^{d-p}(\mathfrak{u}, M) \otimes \wedge^d \mathfrak{u}.$$

Here, $d = \dim \mathfrak{u}$. Therefore, the two conjectures are essentially equivalent.

Theorem 0.5. [4, 5, 8] *Suppose that $\mathfrak{q}$ is a real parabolic subalgebra.*
(a)Let V be the minimal globalization of M. Then Conjecture 0.3 holds.
(b)Let V be the Casselman-Wallach globalization of M. If $\mathfrak{q}$ is a minimal real parabolic subalgebra. Then Conjecture 0.3 holds.

Theorem 0.6. [1, 2] *Suppose that $\mathfrak{q}$ is a θ-stable parabolic subalgebra. Let V be the minimal or maximal globalization of M. Then Conjecture 0.3 holds.*

Finally, we discuss the relation between our results and Conjecture 0.3.

Suppose that $\mathfrak{q}$ is θ-stable. Recall that V is a smooth representation of a real reductive group G. Note that $\mathfrak{u} \cap \mathfrak{k}$ is a nilpotent subalgebra of $\mathfrak{k}$, where $\mathfrak{k} = \mathfrak{g}^\theta$ is the complexified Lie algebra of K.

Ignoring the topology, there exists a convergent spectral sequence

$$\{E_r^{p,q}\}_{r\geq 0} \Rightarrow \mathrm{H}_{p+q}(\mathfrak{u}, V)$$

in the category of $\mathfrak{l} \cap \mathfrak{k}$-modules, which is called the Hochschild-Serre spectral sequence. Here $\mathfrak{l}$ is the complexified Lie algebra of L. The E_1-terms are given by

$$E_1^{p,q} = \mathrm{H}_{n(p,q)}(\mathfrak{u} \cap \mathfrak{k}, V) \otimes X_p.$$

Here X_p is a finite-dimensional vector space and $n(p, q)$ is a certain integer depending on p and q. For more details on the Hochschild-Serre spectral sequence, we refer to [7, Chapter V, Section 10].

Equipped with natural subquotient topology on E_r, the Hochschild-Serre spectral sequence suggests that one may prove Vogan's conjecture by establishing the Hausdorffness of E_r-terms for every $r \geq 0$. Along this strategy, we expect that the total homology space $\mathrm{H}_\bullet(\mathfrak{u}, V)$ is Hausdorff for certain smooth representations V whose underlying $(\mathfrak{g}, K)$-module are not necessarily of finite length. The Hausdorffness of

E_0-terms is clear from the construction, and our results show that the E_1-terms are indeed Hausdorff. However, for $r \geq 2$, the Hausdorffness of E_r-terms is still unresolved.

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