SCIENTIFIC REPORTS

Arithmetic Dynamics and Diophantine Geometry

25 Aug 2025–29 Aug 2025

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WHEN DO TWO ITERATED RATIONAL FUNCTIONS HAVE FINITELY MANY COMMON ZEROS?

CHATCHAI NOYTAPTIM
(JOINT WORK WITH XIAO ZHONG)

Classification AMS 2020: 37P05, 37P30, 14G25

Keywords: arithmetic dynamics, diophantine geometry, arithmetic equidistribution, compositional independence

Let f be a rational function with complex coefficients with degree ≥ 1. We write $f^{\circ n}$ to denote the n-fold composition of f with itself $f \circ f \circ \cdots \circ f$. Given two rational functions f and g, we call them *compositional independence* if the semigroup generated by f and g under composition is isomorphic to the free semigroup with two generators. Otherwise, we call them *compositional dependence*. For instance, $f(x) = -2x$ and $g(x) = 2x + 1$ are compositionally independent. In joint work with Xiao Zhong [NZ25], we study a finiteness result of orbits collision for two rational functions with complex coefficients. We state the question which was originally posed by Hsia and Tucker as follows:

Question 0.1. [HT17, Question18] *Let f and g be two non-constant, compositionally independent rational functions with $\mathbb{C}$-coefficients. Let c be any rational function in $\mathbb{C}(x)$. Is it true that there must be at most finitely many $\lambda \in \mathbb{C}$ such that*

$$f^{\circ n}(\lambda) = g^{\circ n}(\lambda) = c(\lambda)$$

for some positive integer $n \geq 1$?

Question 0.1 is partly motivated by results in Diophantine geometry. Applying Schmidt subspace theorem, Bugeaud-Corvaja-Zannier [BCZ03] provided an upper bound for the greatest common divisor of two sequences $a^n - 1$ and $b^n - 1$ with a and b are multiplicatively independent integers ≥ 2. In 2004, Ailon-Rudnick replaced integers by polynomials in $\mathbb{C}[x]$ and established a function field analog of [BCZ03]. The main tool in [AR04] is a Lang's conjecture which was proved by Ihara-Serre-Tate. Question 0.1, thus, can be thought of as a dynamical analog of [BCZ03, AR04].

Hsia and Tucker examined Question 0.1 when $f, g,$ and c are polynomials with coefficients are in $\mathbb{C}$. Let us briefly discuss the strategy of Hsia and Tucker.

- When $\deg(f) = 1, \deg(g) = 1$, they employ deep results from diophantine geometry and a specialization technique.
- When $\deg(f) > 1, \deg(g) > 1$, they apply arithmetic equidistribution of points with small height, the classification of polynomials with the same Julia set, and isotriviality properties of polynomials.
- When $\deg(f) = 1, \deg(g) > 1$ (and vice versa), Northcott theorem is needed.

There are many obstacles we need to overcome when $f, g,$ and c are rational functions. Nevertheless, our approach follows closely that of Hsia and Tucker. We highlight two cases that are highly different from work of Hsia and Tucker.

- When $\deg(f) = 1, \deg(g) = 1$, it is necessarily to exclude some exceptional families of f and g in order to make the Question 0.1 true. We also provide a counter-example showing that

$$f^{\circ m}(x) = g^{\circ n}(x) = c(x)$$

$(m, n \geq 1)$ could have infinitely many common solutions where f and g are compositionally independent.
- When $\deg(f) > 1, \deg(g) > 1$, a Tits alternative for rational functions–in place of the classification of polynomials with identical Julia set–is needed to conclude the proof.

There are a couple interesting variant questions posed by Hsia and Tucker along this line of research. One is a positive characteristic version [HT17, Question17] and the other is a higher dimensional version [HT17, Question19] of Question 0.1. A progress towards a higher dimensional setting has recently been made by Noytaptim and Zhong [NZ24].

References

[AR04] N. Ailon and Z. Rudnick. Torsion points on curves and common divisors of $a^k - 1$ and $b^k - 1$. *Acta Arithmetica*, Vol. 113, no.1, 31–38, 2004.

[BCZ03] Y. Bugeaud, P. Corvaja, and U. Zannier. An upper bound for the G.C.D. of $a^n - 1$ and $b^n - 1$. *Mathematische Zeitschrift*, Vol. 243, no.1, 79–84, 2003.

[HT17] L.-C. Hsia and T. J. Tucker. Greatest common divisors of iterates of polynomials. *Algebra Number Theory*, Vol. 11, no.6, 1437–1459, 2017.

[NZ25] C. Noytaptim and X. Zhong. A finiteness result for common zeros of iterates of rational functions. *International Mathematics Research Notices*, Vol. 2025, no.11, 1–29, 2025.

[NZ24] C. Noytaptim and X. Zhong. *Towards Common Zeros of Iterated Morphisms*. Preprint arXiv:2412.15141, 2024.

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ON UNIFORM GENERALIZED SUM-PRODUCT BOUNDS

HARRY SCHMIDT
JOINT WORK WITH JOSEPH HARRISON AND AKSHAT MUDGAL

For each polynomial $f \in \overline{\mathbb{Q}}[T]$ of degree $d \geq 2$, we can define its canonical height

$$\hat{h}_f : \overline{\mathbb{Q}} \to \mathbb{R}_{\geq 0}, \quad \hat{h}_f(z) = \lim_{n\to\infty} \frac{h(f^{\circ n}(z))}{d^n},$$

where h is the Weil–height on the algebraic numbers that is defined by the famous sum

$$h(z) = \frac{1}{[F:\mathbb{Q}]} \sum_{v \in M_F} n_v \log\max\{1, |z|_v\}.$$

Here, F is a number field containing z, M_F the set of (equivalence classes of) non-trivial absolute values of F and $n_v = [F_v : \mathbb{Q}_v]$ is the local degree. The summands $\log\max\{1,|z|_v\}$ are often referred to as local heights and abbreviated by $\lambda_v(z) = \log\max\{1,|z|_v\}$. These notations go back at least to Néron's construction of heights attached to divisors. The Weil–height has the striking property that

$$h(z) = 0 \text{ if and only if } z \in \{0\} \cup \mu_\infty,$$

where μ_∞ is the set of roots of unity. The set $\{0\} \cup \mu_\infty$ is the set of pre–periodic points of the polynomial T^2. More generally, $\hat{h}_f$ vanishes exactly on the set of pre-periodic points, that is the points of finite orbit of f. This can be shown by using the pull–back property $\hat{h}_f \circ f = d\hat{h}_f$ and the Northcott property.

For every $D \geq 1, \{z \in \overline{\mathbb{Q}};\ \hat{h}_f(z) \leq D, [\mathbb{Q}(z):\mathbb{Q}] \leq D\}$ is a finite set.

This property makes heights relevant for proving finiteness theorems in Diophantine geometry and the arithmetic of dynamical systems. A property used in various proofs, such as the proof of the uniform bounds on S-unit equations by Evertse–Schmidt–Schlickewei is that for every $\epsilon > 0$

$$h(z) \gg_\epsilon [\mathbb{Q}(z):\mathbb{Q}]^{-1-\epsilon}$$

or $h(z) = 0$. This follows from a theorem of Dobrowolski and a famous conjecture of Lehmer states that one can choose $\epsilon = 0$. In work with Philipp Habegger we establish lower bounds for the canonical height $\hat{h}_f$ that decays like the square of the field degree for certain polynomial f. Our results apply to centrered, postcritically finite, hyperbolic polynomials of prime power degree whose coefficients are algebraic integers. Our approach is inspired by an idea of Dimitrov that led him to prove that

$$\max_{v \in M_F} \log\max\{1, |z|_v\} \gg [F:\mathbb{Q}]^{-1}, \quad F = [\mathbb{Q}(z):\mathbb{Q}],$$

for any algebraic integer z that is not 0 or a root of unity. This inequality was conjectured by Schinzel and Zassenhaus. (It is amusing to note that in their first observation they wrote that they "can not disprove" the inequality). Our height lower bound is derived from a similar lower bound with $\lambda_v(z) = \log\max\{1,|z|_v\}$ replaced by the local height of f, $\hat{\lambda}_{f,v} = \lim_{n\to\infty} \frac{\lambda_v(f^{\circ n}(z))}{d^n}$ of the local canonical height at an archimedean place v. In previous work we used a result of Dubinin on the transfinite diameter of a star-shaped tree. We develop tools to bound the transfinite diameter of more general finite trees in the complex plane. The trees are constructed using the Hubbard tree of a postcritically finite polynomial. We also use Thurston's core entropy which encodes combinatorial data of the Hubbard tree.

A variation of our method allows us to prove Galois lower bounds for pre-periodic points of post-crticically finite maps. For a pre-periodic point z, we set

$$\text{preper}(z) = \min\{n \geq 0; f^{\circ n}(z) \text{ is periodic }\}, \text{per}(z) = \min\{k \geq 1; f^{\circ \text{preper}(z)+k}(z) = z\}.$$

For a number field F, we set $\mathcal{O}_F$ the set of algebraic integers. In my presentation I focused on the following theorem.

Theorem 0.1 Let F be a number field and let $f \in \mathcal{O}_F[T]$ be monic of degree $d \geq 2$ which is a power of p. Suppose f is postcritically finite. Suppose furthermore that the barycenter of f is an algebraic integer and that f is not $\overline{F}$-linearly conjugate to a Chebyshev polynomial or a negative Chebyshev polynomial. We set

$$\mathfrak{e} = \max\{e_v : v \in M_F, v \mid p\} \quad \text{and} \quad \mathfrak{f} = \text{lcm}\{f_v : v \in M_F, v \mid p\}$$

where e_v denotes the ramification index and f_v denotes the residue degree of the place v, respectively. Let $z \in \overline{F}$ be f-preperiodic, then

$$d^{\frac{\text{preper}(\alpha)}{k}-1} \leq [F(z) : F] \quad k \leq 1 + \left\lceil \frac{\log \mathfrak{e}}{\log d} \right\rceil \quad \text{and} \quad \text{per}(z) \leq p\mathfrak{f}[F(z) : F].$$

As an example, we can consider $f = T^2 + c$ that is pcf. By a result of Epstein and Poonen the prime 2 is unramified in $\mathbb{Q}(c)$ over $\mathbb{Q}$, if 0 is periodic. In that case, we can choose $k = 1$ and we obtain a uniform lower bound for $[\mathbb{Q}(b, c) : \mathbb{Q}(c)]$ for a pre-periodic point b independently of the period of 0. This resembles Galois properties of torsion points of elliptic curves with complex multiplication. The analogy between PCF maps and abelian varieties with complex multiplication inspired the dynamical André–Oort conjecture which has been partially proved. Our result adds further evidence to the power of this analogy.

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ARITHMETIC DEGREES ARE COHOMOLOGICAL LYAPUNOV MULTIPLIERS

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JOINT WORK WITH JIARUI SONG AND JUNYI XIE

Classification AMS 2020: 37P05, 37P30

Keywords: Arithmetic degrees, cohomological Lyapunov multipliers, height

Let $f : X \dashrightarrow X$ be a dominant rational self-map of a projective variety X defined over $\overline{\mathbb{Q}}$. Fix a Weil height h_H associated to an ample line bundle H on X. For a point $x \in X(\overline{\mathbb{Q}})$ whose f-orbit $\mathcal{O}_f(x) = \{x, f(x), f^2(x), \ldots\}$ is well-defined, the arithmetic degree of x is defined by

$$\alpha_f(x) = \lim_{n\to\infty} \max\{1, h_H(f^n(x))\}^{\frac{1}{n}},$$

provided the limit exists. It is conjectured by Kawaguchi and Silverman that this limit always exists, and furthermore, that $\alpha_f(x) = \lambda_1(f)$, the first dynamical degree of f, whenever the f-orbit $\mathcal{O}_f(x)$ is Zariski dense in X. See [4]. For recent advances on this conjecture, see [7]. If the f-orbit $\mathcal{O}_f(x)$ is generic, meaning that it intersects every proper closed subset of X in only finitely many points, then it was proved in [8, Theorem 2.2] that $\alpha_f(x)$ exists and coincides with one of the cohomological Lyapunov multipliers $\mu_i(f)$ of f (please see below for the definition).

In our work [11], we study the behavior of arithmetic degrees of endomorphisms under the weaker assumption that the orbit is Zariski dense in X, rather than generic. In the case where $f : X \to X$ is an endomorphism, it was shown in [3] that $\alpha_f(x)$ exists, and is equal to the modulus of an eigenvalue of the linear map $f^* : \mathrm{N}^1(X)_{\mathbb{R}} \to \mathrm{N}^1(X)_{\mathbb{R}}$ or 1, where $\mathrm{N}^1(X)$ is the numerical group of line bundles on X. We strengthen this result by proving that, if $\mathcal{O}_f(x)$ is Zariski dense in X, then $\alpha_f(x)$ must be one of the cohomological Lyapunov multipliers of f.

Previous works on arithmetic degrees (see for example [3, 4, 6]) have largely focused on the case where the base field is a number field, as there is a natural notion of Weil heights. In [5], a generalization to arbitrary fields of characteristic zero was introduced via spread-out techniques and the notion of Moriwaki heights [9]. This broader framework allows one to address problems in arithmetic dynamics over fields such as $\mathbb{C}$, including, for example, applications to the dynamical Mordell–Lang conjecture. For generality, our results are proved over finitely generated fields, or more generally, over arbitrary fields of characteristic zero.

Settings of our work.

We start with a normal projective variety $X_{\mathbb{C}}$, a surjective endomorphism $f_{\mathbb{C}} : X_{\mathbb{C}} \to X_{\mathbb{C}}$ over $\mathbb{C}$ and a point $x_{\mathbb{C}} \in X(\mathbb{C})$. Let K be a finitely generated field over $\mathbb{Q}$ such that the coefficients of defining equations of $X_{\mathbb{C}}$ and $f_{\mathbb{C}}$, as well as the coordinates of $x_{\mathbb{C}}$, are all contained in K. In other words, there exists a projective variety X, a surjective endomorphism $f : X \to X$ over K, and a point $x \in X(K)$ such that (X, f, x) is a model of $(X_{\mathbb{C}}, f_{\mathbb{C}}, x_{\mathbb{C}})$. We fix an ample line bundle L on X. Then we can measure the complexity of the dynamical system (X, f) and the orbit $\mathcal{O}_f(x)$.

(1) For $i \in \{0, \ldots, \dim(X)\}$, the i-th dynamical degree $\lambda_i(f)$ of (X, f) is defined as $\lim_{n\to\infty} ((f^n)^* L^i \cdot L^{\dim(X)-i})^{\frac{1}{n}}$. The limits exist and are independent of the choice of L. See [2, 1, 12].
(2) Fix a Moriwaki height function (see [9]) $h_L : X(\overline{K}) \to \mathbb{R}_{\geq 1}$. Then the arithmetic degree $\alpha_f(x)$ is defined as $\lim_{n\to\infty} h_L(f^n(x))^{\frac{1}{n}}$. The limit exists and is independent of the choice of L, h_L and the field K. See [3, 10, 5].

Remark 0.1. When K is a number field, the arithmetic degree is defined directly using a Weil height function.

In [13], Junyi Xie introduced a notion of cohomological Lyapunov multipliers of the dynamical system (X, f). For $i \in \{1, \ldots, \dim(X)\}$, the i-th cohomological Lyapunov multiplier $\mu_i(f)$ is defined as $\frac{\lambda_i(f)}{\lambda_{i-1}(f)}$.

Our main result is as follows. We continue with the previous notions.

Theorem 0.2. We have $\alpha_f(x) \in \{\mu_1(f), \ldots, \mu_{\dim(X)}(f)\} \cap \mathbb{R}_{\geq 1}$, provided that the orbit $\mathcal{O}_f(x)$ is Zariski dense in X.

From Theorem 0.2, we deduce the following corollary about the Kawaguchi–Silverman conjecture.

Corollary 0.3. Let X be a normal projective variety and $f : X \to X$ be a surjective endomorphism of X with $\lambda_1(f) > \lambda_2(f)$. Then the Kawaguchi–Silverman conjecture holds. In other words, if $\mathcal{O}_f(x)$ is Zariski dense in X, then $\alpha_f(x) = \lambda_1(f)$.

Remark 0.4. (1) We have assumed that X is normal and geometrically connected for simplicity. These conditions are mild and are usually easy to fulfill in the applications.
(2) It is proved in [3] that $\alpha_f(x)$ is the modulus of an eigenvalue of the linear map $f^* : \mathrm{N}^1(X)_{\mathbb{R}} \to \mathrm{N}^1(X)_{\mathbb{R}}$ or 1. In fact, all of the cohomological Lyapunov multipliers $\mu_i(f)$ are real and positive eigenvalues of this map (see [14, Theorem 1.4]). So our theorem is a strengthening of Kawaguchi–Silverman's result.
(3) According to the Kawaguchi–Silverman conjecture, one expects that $\alpha_f(x) = \mu_1(f) = \lambda_1(f)$ when $\mathcal{O}_f(x)$ is Zariski dense in X.

As an application, we can prove the following result of dynamical Mordell–Lang type. It is another example of applying height arguments toward the DML conjecture, following the spirit of [15]. The point in the corollary is that the speeds of height growth of $\mathcal{O}_f(x)$ and $\mathcal{O}_g(y)$ must be different. One could compare this corollary with [15, Proposition 4.1].

Corollary 0.5. Let X and Y be projective varieties over $\mathbb{C}$. Let f and g be surjective endomorphisms of X and Y, respectively. Let $x \in X(\mathbb{C})$ and $y \in Y(\mathbb{C})$ be closed points and let $V \subseteq X \times Y$ be a positive dimensional irreducible closed subvariety. Suppose that
(1) $\{\mu_1(f), \ldots, \mu_{\dim(X)}(f)\} \cap \{\mu_1(g), \ldots, \mu_{\dim(Y)}(g)\} \cap \mathbb{R}_{\geq 1} = \emptyset$;
(2) the orbits $\mathcal{O}_f(x)$ and $\mathcal{O}_g(y)$ are dense in X and Y, respectively; and
(3) both of the projection maps $V \to X$ and $V \to Y$ are generically finite onto their image.

Then $V \cap \mathcal{O}_{f\times g}((x, y))$ cannot be dense in V.

References

[1] Nguyen-Bac Dang. Degrees of iterates of rational maps on normal projective varieties. Proc. Lond. Math. Soc. (3), 121(5), 1268–1310, 2020.

[2] Tien-Cuong Dinh and Nessim Sibony. Une borne supérieure pour l'entropie topologique d'une application rationnelle. Ann. of Math. (2), 161(3), 1637–1644, 2005.

[3] Shu Kawaguchi and Joseph H. Silverman. Dynamical canonical heights for Jordan blocks, arithmetic degrees of orbits, and nef canonical heights on abelian varieties. Trans. Amer. Math. Soc., 368(7), 5009–5035, 2016.

[4] Shu Kawaguchi and Joseph H. Silverman. On the dynamical and arithmetic degrees of rational self-maps of algebraic varieties. J. Reine Angew. Math., 2016(713), 21–48, 2016.

[5] Wenbin Luo and Jiarui Song. Arithmetic degrees of dynamical systems over fields of characteristic zero. arXiv:2401.11982.

[6] Yohsuke Matsuzawa. On upper bounds of arithmetic degrees. Amer. J. Math., 142(6), 1797–1820, 2020.

[7] Yohsuke Matsuzawa. Recent advances on Kawaguchi–Silverman conjecture. arXiv:2311.15489.

[8] Yohsuke Matsuzawa. Existence of arithmetic degrees for generic orbits and dynamical Lang–Siegel problem. J. Reine Angew. Math., 2025(825), 305–335, 2025.

[9] Atsushi Moriwaki. Arithmetic height functions over finitely generated fields. Invent. Math., 140(1), 101–142, 2000.

[10] Tomoya Ohnishi. Arakelov geometry over an adelic curve and dynamical systems. PhD thesis, Kyoto University, 2022.

[11] Jiarui Song, Junyi Xie, and She Yang. Arithmetic degrees are cohomological Lyapunov multipliers. arXiv:2507.17643.

[12] Tuyen Trung Truong. Relative dynamical degrees of correspondences over a field of arbitrary characteristic. J. Reine Angew. Math., 2020(758), 139–182, 2020.

[13] Junyi Xie. Algebraic dynamics and recursive inequalities. arXiv:2402.12678.

[14] Junyi Xie. Numerical action for endomorphisms. arXiv:2502.04779.

[15] Junyi Xie and She Yang. Height arguments toward the dynamical Mordell–Lang problem in arbitrary characteristic. arXiv:2504.01563.

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QUANTITATIVE MORDELL CONJECTURE

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Classification AMS 2020: 11G30, 14H25

Keywords: Mordell conjecture; quantativity; Bogomolov conjecture

This talk is based on the joint work with Xinyi Yuan and Shengxuan Zhou.
The celebrated Faltings theorem, also known as Mordell conjecture, is as follows.

Theorem 0.1 (Faltings [4]). *Let C be an algebraic curve of genus $g \geq 2$ over a number field K, then $C(K)$ is finite.*

It was proved by Faltings in 1983. Later, Vojta [13] gave another proof with Diophantine approximation. Based on Vojta's proof, Dimitrov–Gao–Habegger [3] and Kühne [6] proved the following uniform result, which answers a question by Mazur [8, p. 234].

Theorem 0.2 (Vojta [13], Dimitrov–Gao–Habegger [3], Kühne [6]). *Let $g \geq 2$ be an integer. Then there exist two positive constants $c_1(g)$ and $c_2(g)$ depending only on g with the following property. Let C be an algebraic curve of genus g over a number field K. Then*

$$\#C(K) \leq c_1(g)c_2(g)^r.$$

Here $r = \mathrm{rank}(\mathrm{Jac}(C)(K))$ is the Mordell-Weil rank.

It is natural to ask for explicit constants. We give the following result.

Theorem 0.3 (Yu–Yuan–Zhou). *Let C be an algebraic curve of genus $g \geq 2$ over a number field K. Then*

$$\#C(K) \cap \Gamma \leq 3 \cdot 10^3 g^8 \left(1 + \frac{3 \log g}{g}\right)^r.$$

Here $c_2(g) = 1 + \frac{3\log g}{g}$ converges to 1 as $g \to \infty$. This confirms a conjecture of Gao and Habegger (cf. [5]).

We sketch the idea of proof. Let $J = \mathrm{Jac}(C)$ be the Jacobian variety of C. There is a canonical height function $\hat{h} : J(K) \to \mathbb{R}$. Assume for simplicity that there is a line bundle α on C an isomorphism $(2g-2)\alpha \cong \omega$. Via the Abel-Jacobi map with repect to α, the canonical height function is pulled back to $C(K)$. We also take a height $h(C)$ of the curve C. For example, it could be $\max\{h_{\mathrm{Fal}(J)}, 1\}$. The rational points are divided into two parts and count separately.

For points with large height, Vojta [13] proved an inequality between them. Based on that, Rémond [10] and de Diego [1] gave uniform bounds for large points. By applying Arakelov theory, we make constants explicit.

Although the small points is automatically finite by the Northcott property. It is the difficult part in proving uniform results. We use a uniform version of the following theorem, which is conjectured by Bogomolov and proved by Ullmo [12].

Theorem 0.4 (Bogomolov conjecture, Ullmo [12])**.** *Let C be an algebraic curve of genus $g \geq 2$ over a number field K. Let $J = \mathrm{Jac}(K)$ be the Jacobian variety. Let α be a line bundle of degree 1 on C. Then there exists a constant $\epsilon > 0$ such that*

$$\#\{P \in C(\bar{K}) : \hat{h}(P - \alpha) < \epsilon\} < \infty.$$

Ullmo's proof is based on the equidistribution theorem by Szpiro–Ullmo–Zhang [11]. It is generalized by Zhang [17] to subvarieties of abelian varieties.

Dimitrov–Gao–Habegger [3] developed a height inequality. Kühne [6] generalize the equidistribution to subvarity of abelian schemes. The following uniform Bogomolov conjecture is obtained by combining their results.

Theorem 0.5 (Uniform Bogomolov conjecture, Dimitrov–Gao–Habegger [3], Kühne [6])**.** *Let $g \geq 2$ be an integer. Then there exist two positive constants $c_1(g)$ and $c_2(g)$ depending only on g with the following property. Let C be an algebraic curve of genus g over a number field K. Let $J = \mathrm{Jac}(K)$ be the Jacobian variety. Let α be a line bundle of degree 1 on C. Then*

$$\#\{P \in C(\bar{K}) : \hat{h}(P - \alpha) < c_1(g)h(C)\} < c_2(g).$$

There is another approach to the Bogomolov conjecture for curves. Zhang [16] introduced the admissible volume $\overline{\omega}^2_{C,a}$, which is an arithmetic invariant of the curve C. He also reduced the Bogomolov conjecture to $\overline{\omega}^2_{C,a} > 0$. Robin de Jong [2] proved this positivity. Based on the theory of Yuan–Zhang [15], Yuan [14] developed these theories on relative curves to gave another proof of the uniform Bogomolov conjecture and strengthened it by adding an extra term. He also showed that $\overline{\omega}^2_{C,a}$ is a height of C.

Looper–Silverman–Wilms [7] proved a quantitative result on Bogomolov conjecture over function fields. We transfer their proof to the number field case. The canonical height function $\hat{h}(\cdot)$ on $J(K)$. So we define the norm $|x| = (\hat{h}(x))^{\frac{1}{2}}$ and the angle $\angle(x, y) = \arccos(\frac{|x+y|^2 - |x|^2 - |y|^2}{2|x||y|})$.

Theorem 0.6. *Let C be an algebraic curve of genus g over a number field K. Let $J = \mathrm{Jac}(K)$ be the Jacobian variety.*

(1) *Let α be a line bundle of degree 1 on C. Then*

$$\#\left\{P \in C(\bar{K}) : \hat{h}(P - \alpha) < \frac{\overline{\omega}^2_{C,a}}{32g}\right\} < 7 \cdot 10^{11} g^{\frac{17}{3}}.$$

(2) *Let $x \in J(K)$. Assume $|x| \neq 0$. Then*

$$\#\left\{P \in C(\bar{K}) : 1 \leq \frac{|P|}{|x|} \leq 2, \angle(x, P) \leq \arccos\sqrt{\frac{2.13}{g+1}}\right\} < 4 \cdot 10^{11} g^{\frac{17}{3}}.$$

Here (2) is inspired by Yuan's extra term and helps to prove $c_2(g) \to 1$.

References

[1] T. de Diego, *Points rationnels sur les familles de courbes de genre au moins 2*, J. Number Theory, 67 (1997), pp. 85–114, `https://doi.org/10.1006/jnth.1997.2146`.

[2] R. de Jong, *Néron-Tate heights of cycles on Jacobians*, J. Algebraic Geom. 27 (2018), pp. 339–381, `https://doi.org/10.1090/jag/700`.

[3] V. Dimitrov, Z. Gao, and P. Habegger, *Uniformity in Mordell-Lang for curves*, Ann. of Math. (2) 194 (2021), no. 1, 237–298.

[4] G. Faltings. *Endlichkeitssätze für abelsche Varietäten über Zahlkörpern*, Invent. Math., 73 (1983), pp. 349–366, https://doi.org/10.1007/BF01388432.
[5] P. Habegger, *The number of rational points on a curve of genus at least two*, ICM–International Congress of Mathematicians 3. Sections 1–4, EMS Press, Berlin, pp. 1838–1869.
[6] L. Kühne, *Equidistribution in Families of Abelian Varieties and Uniformity*, preprint, https://arxiv.org/abs/2101.10272v2, 2021.
[7] N. Looper, J. Silverman, and R. Wilms, *A uniform quantitative Manin-Mumford theorem for curves over function fields*, J. Reine Angew. Math. 828 (2025), pp. 127–147.
[8] B. Mazur, *Arithmetic on curves. Bulletin of the American Mathematical Society*, 14 (1986), pp. 207–259, https://doi.org/10.1090/S0273-0979-1986-15430-3.
[9] L. J. Mordell, *On the rational solutions of the indeterminate equations of the third and fourth degrees*, Mathematical Proceedings of the Cambridge Philosophical Society 21 (1922), pp. 17–192.
[10] G. Rémond, *Inégalité de Vojta en dimension supérieure*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 29 (2000), pp. 101–151, https://doi.org/10.1016/S0764-4442(00)88135-5.
[11] L. Szpiro, E. Ullmo, and S. Zhang, *Équirépartition des petits points*, Invent. Math., 127 (1997), pp. 337–347, https://doi.org/10.1007/s002220050123.
[12] E. Ullmo, *Positivité et discrètion des points algébriques des courbes*, Ann. of Math. (2), 147(1998), pp. 167–179, https://doi.org/10.2307/120987.
[13] P. Vojta, *Siegel's theorem in the compact case*, Ann. of Math. (2), 133 (1991), pp. 509–548, https://doi.org/10.2307/2944318.
[14] X. Yuan, *Arithmetic bigness and a uniform Bogomolov-type result*, preprint, https://arxiv.org/abs/2108.05625, 2021, to appear in the Annals of Mathematics.
[15] X. Yuan and S. Zhang, *Adelic line bundles on quasi-projective varieties*, preprint, https://arxiv.org/abs/2105.13587v6, 2021, to appear in the Annals of Mathematics Studies.
[16] S. Zhang, *Admissible pairing on a curve*, Invent. Math. 112 (1993), pp. 171–193, https://doi.org/10.1007/BF01232429.
[17] S. Zhang, *Equidistribution of small points on abelian varieties*, Ann.of Math. (2), 147 (1998), pp. 159–165, 1998, https://doi.org/10.2307/120986.

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ALGEBRAIC FAMILIES OF WEAKLY POLARIZED ENDOMORPHISMS

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Classification AMS 2020: 37P55, 37P30, 37P45

Keywords: Northcott property, complex function fields, geometric height functions, algebraic dynamics

The classical *Northcott property* over a number field $\mathbb{K}$ asserts that, for a height function associated with a big and nef divisor, there exist only finitely many $\overline{\mathbb{K}}$-points of bounded degree and bounded height outside a certain Zariski closed subset (which is empty when the divisor is ample). This finiteness property is one of the most fundamental features of height functions in arithmetic geometry and dynamics. Let $f : X \to X$ be a polarized endomorphism of degree $d \geq 2$ of a projective variety X, defined over K. Let h be an ample height function on X. Call and Silverman ([1]) constructed the canonical height function $\hat{h}_f$ associated with f by $\hat{h}_f(x) := \lim_n \frac{1}{d^n} h(f^n(x))$, which is independent of the choice of the ample height function h. The canonical height is f-invariant in the sense that $\hat{h}_f \circ f = d\hat{h}_f$ and there exists a constant $C > 0$, which does not depend on the rational point x, such that $|h(x) - \hat{h}_f(x)| < C$. These two properties uniquely characterize $\hat{h}_f$, and, together with the Northcott property, imply that there are only finitely many preperiodic points of f of bounded degree.

We now consider the case where K is a complex function field $\mathbb{C}(B)$, where B is a smooth complex projective variety. In this setting, the Northcott property fails in general. In this talk, we will present a natural weaker form of the Northcott property over complex function fields in the dynamical setting. An important invariant that measures the complexity of an algebraic dynamical system is the first dynamical degree [3]. Recall that it is defined by $\lambda_1(f) := \lim_n \left((f^n)^*(A) \cdot A^{\dim X - 1}\right)^{1/n}$.

Our main result is Theorem 0.9 ([10]). To avoid technical definitions at first, we begin by presenting the surface case in Theorem 0.1.

Theorem 0.1 (Surfaces case). *Let S be a smooth projective surface over a complex function field K and $g : S \to S$ an automorphism with first dynamical degree $\lambda_1 > 1$. Let $E \subset S$ be the (reducible) invariant curve. Then there exist only finitely many periodic K-rational points outside E.*

In the remainder of the text, we develop a more general framework that encompasses not only surfaces, but also hyperkähler varieties and varieties admitting abelian group actions of maximal dynamical rank [3, 4]. Moreover, our finiteness result applies to rational points of sufficiently small height.

Let $\pi : X \to \Lambda$ be a flat, surjective and projective morphism of smooth complex quasi-projective varieties. Denote by $\overline{X}$ and B smooth compactifications of X and Λ such that $\overline{X} \to B$ restricts to π over Λ. Let $f := (f_1, \cdots, f_m)$ be an m-tuple of automorphisms of X such that $f_i \circ \pi = \pi$ for all $i = 1, \cdots, m$. On each fiber $X_t := \pi^{-1}(t)$, $f_{i,t}$ acts on it as an automorphism. Thus, we obtain a family $(f_t)_{t\in\Lambda} = (f_{1,t}, \cdots, f_{m,t})_{t\in\Lambda}$ of automorphisms parameterized by Λ.

Let $D_i, 1 \leq i \leq m$ be $\mathbb{R}$-divisors on $\overline{X}$ such that

- for any $1 \leq i \leq m$, there exists $\lambda_i > 1$ such that

$$f_i^* D_i|_\Lambda \sim_{\mathbb{R}} \lambda_i D_i|_\Lambda; \tag{0.1}$$

- for any $1 \leq i \neq j \leq m$, there exists $\mu_{i,j} < 1$ such that

$$f_i^* D_j|_\Lambda \sim_{\mathbb{R}} \mu_{i,j} D_j|_\Lambda; \tag{0.2}$$

- D_i is π-nef; that is, for every $t \in \Lambda$, the restriction $D_{i,t}$ is nef on X_t;
- the sum $D := \sum_i D_i$ is π-big and nef; equivalently, for every $t \in \Lambda$, the restriction D_t is big and nef on X_t.

where $\sim_{\mathbb{R}}$ means $\mathbb{R}$-linear equivalence of $\mathbb{R}$-divisors.

For any $1 \leq i \leq m$, denote by $\mathfrak{S}_{f_i}$ the set of points $x \in X(\mathbb{C})$ such that x is an $f_{i,\pi(x)}$-periodic point of saddle type in the fiber $X_{\pi(x)}$. Here, an $f_{i,t}$-periodic point $x \in X(\mathbb{C})$ above $t = \pi(x)$ of exact period k is of *saddle type* if no eigenvalues of the differential of $f_{i,t}^k$ at x are on the unit circle.

Definition 0.2. We encode the above data by (π, f, D), and we call it a *family of hyperbolic automorphisms (of smooth complex projective varieties)*. This family is said to be *good* if moreover $\mathfrak{S}_{f_i}$ is Zariski dense in X for some $1 \leq i \leq m$.

Definition 0.3. Let $d \geq 1$ be a positive integer. A *d-marked point (of π)* is an irreducible subvariety σ of $\overline{X}$ such that the projection $\pi|_\sigma$ to B is generically finite of degree d. Denote by $\deg(\sigma)$ the degree d and Σ the set all of marked points.

Definition 0.4. *A d-marked point is periodic if it is periodic for all f_i, that is, there exists $n \in \mathbb{N}$ such that $f_i^n(\sigma) = \sigma$ for all i.*

For good family, σ is periodic if and only if it is periodic for some f_i by Theorem 0.9.

By considering the above family of dynamical systems at the generic fiber of π, we can reformulate the preceding notions as follows.

Let K be the function field of a smooth complex quasi-projective variety Λ, and let X be a K-variety (i.e., a separated, geometrically integral scheme of finite type over K). Let $f_1, \ldots, f_m$ be K-automorphisms of X. Then a family of hyperbolic automorphisms consists of the data (X, f_i, D_i), where the D_i are nef $\mathbb{R}$-divisors on X such that

$$f_i^* D_i \sim_{\mathbb{R}} \lambda_i D_i, \qquad f_i^* D_j \sim_{\mathbb{R}} \mu_{i,j} D_j,$$

and the sum $D = \sum_i D_i$ is big and nef. Moreover, d-marked points correspond exactly to rational points of X of degree d.

In fact, given such a variety X/K, there always exists a smooth quasi-projective variety $\mathcal{X}$ and a projection $\pi : \mathcal{X} \to \Lambda$, flat, surjective and projective, such that the generic fiber is $X \to \operatorname{Spec} K$. We refer to $\mathcal{X}$ or π as a *model* of X/K.

Let (π, f, D) be a family of hyperbolic automorphisms. Fix an ample divisor M on B, and let $d_B := \dim B$. The *i-th geometric height function* $h_{i,M} \colon \Sigma \longrightarrow \mathbb{R}$ associated with D_i and M is defined by

$$h_{i,M}(\sigma) := \sigma \cdot D_i \cdot \pi^*(M)^{d_B - 1}. \tag{0.3}$$

The corresponding *geometric canonical height function* associated with f_i is defined as

$$\hat{h}_{f_i}(\sigma) := \lim_{n\to+\infty} \frac{1}{\lambda_i^n}\, h_{i,M}(f_i^n \circ \sigma). \tag{0.4}$$

This limit is well-defined, f_i-invariant and non-negative.

A subvariety Y of X is called *horizontal* if $\pi(Y) = \Lambda$.

Definition 0.5. Let (π, f, D) be a family of hyperbolic automorphisms. The *maximal f-invariant subvariety* $E \subset X$ is the (possibly reducible) subvariety satisfying:

- every irreducible component of E is horizontal;
- $f_i(E) = E$ for all $1 \le i \le m$;

and moreover, if $E' \subset X$ is an irreducible subvariety such that

- $\pi(E') = \Lambda$;
- $d_B < \dim E' < \dim X$;
- E' is f_i-periodic for all $1 \le i \le m$,

then $E' \subset E$.

The maximal f-invariant subvariety E always exists. If D is relatively ample (that is, ample on the generic fiber), then $E = \emptyset$.

In the study of dynamical families, there are certain *trivial examples* that one needs to exclude in order to obtain meaningful results.

Example 0.6. Let $f\colon X \to X$ be a morphism of a smooth complex projective variety X. We may view it as a trivial family by setting $f_t = f$ for all $t \in B(\mathbb{C})$. For each point $x \in X(\mathbb{C})$, there is an associated constant marked point $\sigma_x(t) = (t, x)$.

Definition 0.7. Let (π, f, D) be a family of hyperbolic automorphisms. We say that it is *D-isotrivial* if, for any two general parameters $t_1, t_2 \in \Lambda(\mathbb{C})$, there exists an isomorphism

$$\Psi_{t_1,t_2}\colon (X \setminus E)_{t_1} \longrightarrow (X \setminus E)_{t_2}$$

such that, for all $1 \le i \le m$,

$$\Psi_{t_1,t_2}^{-1} \circ f_{i,t_2} \circ \Psi_{t_1,t_2} = f_{i,t_1}$$

on $(X \setminus E)_{t_1}$.

As in the number field case, we must exclude a certain Zariski closed subset in order to obtain a meaningful finiteness statement. Recall that Σ denotes the set of all d-marked points.

Definition 0.8. *We denote by Σ_E the subset of Σ consisting of d-marked points not contained in E.*

Theorem 0.9. *Let (π, f, D) be a non-D-isotrivial good family of hyperbolic automorphisms. For any integer $N \ge 2$, there exists a constant $\varepsilon_f > 0$ (depending on N) such that the set*

$$\left\{\, \sigma \in \Sigma_E \mid \deg(\sigma) < N,\ \hat{h}_f(\sigma) < \varepsilon_f \,\right\}$$

is finite.

Analogous results for families on $\mathbb{P}^n$ are known. In dimension one, Benedetto [5] first proved such a statement for families of polynomials; the general case was established by Baker [6], and DeMarco [7] gave another proof using complex dynamics. In higher dimensions, the result is due to Gauthier and Vigny [8] and to the author [9] (for the gap property).

References

[1] Call, Gregory S. and Silverman, Joseph H. Canonical heights on varieties with morphisms. *Compositio Math.*, 89, 163–205, 1993.

[2] Dinh, Tien-Cuong and Sibony, Nessim. Une borne supérieure pour l'entropie topologique d'une application rationnelle. *Ann. of Math. (2)*, 161, 1637–1644, 2005.

[3] Dinh, Tien-Cuong and Sibony, Nessim. Groupes commutatifs d'automorphismes d'une variété kählérienne compacte. *Duke Math. J.*, 123, 311–328, 2004.

[4] Hu, Fei and Zhong, Guolei. Canonical heights for abelian group actions of maximal dynamical rank. *Forum Math. Sigma*, 13, 25, 2025.

[5] Benedetto, Robert L. Heights and preperiodic points of polynomials over function fields. *Int. Math. Res. Not.*, 62, 3855–3866, 2005.

[6] Matthew Baker. A finiteness theorem for canonical heights attached to rational maps over function fields. *J. Reine Angew. Math.*, 626:205–233, 2009.

[7] Laura DeMarco. Bifurcations, intersections, and heights. *Algebra Number Theory*, 10:1031–1056, 2016.

[8] Thomas Gauthier and Gabriel Vigny. The geometric dynamical Northcott and Bogomolov properties. *Ann. Sci. Éc. Norm. Supér. (4)*, 58:231–273, 2025.

[9] Yugang Zhang. Gap for geometric canonical height functions. *Math. Z.*, 307(2):Paper No. 30, 8, 2024.

[10] Yugang Zhang. Marked points of families of hyperbolic automorphisms of smooth complex projective varieties. *Preprint*, 2025.

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PREIMAGES QUESTION AND DYNAMICAL CANCELLATION

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Classification AMS 2020: 37P55, 14G05

Keywords: preimages question, invariant subvarieties, arithmetic dynamics, rational points on varieties

In arithmetic geometry, one of the central problems is to study the distribution of rational points on a given geometric object. The dynamical analogue of this problem concerns the distribution of rational points in the orbit of a dynamical system. On the other hand, torsion points and algebraic subgroups (or abelian subvarieties) play a central role in understanding the arithmetic of an abelian variety. Dynamically, these correspond to invariant subvarieties or their preimages under a morphism

$$[m]\colon A \to A,$$

where A is an abelian variety and $m > 1$ is an integer. More generally, studying invariant subvarieties—those $Y \subset X$ satisfying $f(Y) \subseteq Y$—is a key step in understanding the dynamics of a pair (X, f).

Let X be a projective variety and f a surjective endomorphism of X. For a subvariety $Y \subset X$ that is invariant under f, we can form the tower of iterated preimages

$$Y \subseteq f^{-1}(Y) \subseteq f^{-2}(Y) \subseteq \cdots \subseteq f^{-n}(Y) \subseteq \cdots .$$

For a generically finite morphism $g\colon V \to V'$ between projective varieties, the pullback g^* induces an injective map between canonical rings, implying that the Kodaira dimension of V is at least that of V'. In other words, V is geometrically more complex than V'. Returning to the above tower, this suggests that the geometric complexity of the difference $f^{-n-1}(Y) \setminus f^{-n}(Y)$ tends to increase as n grows.

A guiding principle in arithmetic geometry is that the geometric structure of a variety heavily influences its arithmetic properties. For instance, a conjectural extension of Faltings' theorem due to Bombieri and Lang (see [2, Conjecture F.5.2.1]) predicts that if an irreducible variety X over a number field K has Kodaira dimension equal to its dimension, then $X(K)$ is contained in a proper closed subset.

In light of this, one expects that $f^{-n-1}(Y) \setminus f^{-n}(Y)$ should contain fewer K-points as n increases, and eventually none at all. That is, the tower of K-points

$$Y(K) \subseteq f^{-1}(Y)(K) \subseteq \cdots \subseteq f^{-n}(Y)(K) \subseteq \cdots$$

should stabilize. This expectation was formalized in [3, Question 8.4(1)]:

Question 0.1. *Let $f\colon X \to X$ be a surjective endomorphism of a projective variety X defined over a number field K, and let $Y \subset X$ be a closed subscheme invariant under f. Does there exist $s_0 \geq 0$ such that*

$$(f^{-(s+1)}(Y) \setminus f^{-s}(Y))(K) = \emptyset$$

for all $s \geq s_0$? In other words, for $x \in X(K)$, if $f^s(x) \in Y(K)$ for some $s \geq 0$, then $f^{s_0}(x) \in Y(K)$.

Initial progress on this question was made in [1], where the authors considered split maps $f = (g, g)\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$, with g a rational function of degree > 1 and $Y = \Delta$ the diagonal subvariety. In this setting, the statement becomes a dynamical analogue of a cancellation theorem:

Theorem 0.2. *Let $f\colon X \to X$ be a surjective self-morphism of a projective curve X defined over a number field K. Then there exists $s_0 \geq 0$ such that for all $x, y \in X(K)$, if $f^s(x) = f^s(y)$ for some $s \geq 0$, then $f^{s_0}(x) = f^{s_0}(y)$.*

Later, I provided a complete affirmative answer to the preimages question for surjective endomorphisms of $(\mathbb{P}^1)^n$ for any $n \geq 2$.

Theorem 0.3 ([5])**.** *Let K be a number field, $n \geq 1$, and $f = (f_1, \ldots, f_n)\colon (\mathbb{P}^1_K)^n \to (\mathbb{P}^1_K)^n$ a split rational map defined over K with at least one f_i of degree > 1. If $V \subseteq (\mathbb{P}^1)^n$ is a subvariety defined over K invariant under f, then there exists $s_0 \geq 0$ such that*

$$(f^{-s-1}(V) \setminus f^{-s}(V))(K) = \emptyset$$

for all $s \geq s_0$.

Since the dynamics of polynomial maps are comparatively better understood, [1] also established a generalized cancellation result for multiple polynomial maps:

Theorem 0.4. *Let K be a number field, and let $\phi_1, \ldots, \phi_r$ be polynomial maps on $\mathbb{P}^1_K$ of degree at least two. Suppose that none of the indecomposable factors of $(\phi_i)_{\overline{K}}$ are linearly related to a Chebyshev polynomial T_d with d odd or a cyclic polynomial x^m. Then there exists a finite set $Z \subset (\mathbb{P}^1 \times \mathbb{P}^1)(K)$ such that for any $a, b \in \mathbb{P}^1(K)$ with $(a, b) \notin Z$, if*

$$\phi_{i_n} \circ \cdots \circ \phi_{i_1}(a) = \phi_{i_n} \circ \cdots \circ \phi_{i_1}(b)$$

for some $n \geq 0$ and indices $i_1, \ldots, i_n \in \{1, \ldots, r\}$, then

$$\phi_{i_2} \circ \phi_{i_1}(a) = \phi_{i_2} \circ \phi_{i_1}(b).$$

Subsequently, I obtained necessary and sufficient conditions for when such dynamical cancellation results hold in the setting of multiple polynomial maps, thus completely characterizing the phenomenon:

Theorem 0.5 ([4])**.** *Let S be a finite set of polynomials of degree at least two defined over a number field K, and let $\langle S \rangle$ be the monoid generated by S under composition. Then there exist $N \in \mathbb{N}^+$ and a finite set $Z \subset \mathbb{P}^1_K \times \mathbb{P}^1_K$ such that if*

$$\phi_k \circ \cdots \circ \phi_1(a) = \phi_k \circ \cdots \circ \phi_1(b) \tag{0.1}$$

with $\phi_j \in S$, $k > N$, and $(a, b) \notin Z$, then

$$\phi_N \circ \cdots \circ \phi_1(a) = \phi_N \circ \cdots \circ \phi_1(b), \tag{0.2}$$

if and only if $\langle S \rangle^2$ does not contain a special pair of polynomials (h_1, h_2).

Remark 0.6. *In my paper, I gave a complete classification of such special pairs (h_1, h_2). An important point is that N can be computed from S, and if special pairs exist, they can be chosen as compositions of at most N elements in S. Thus, one only needs to check all compositions of length up to N to determine whether a special pair exists, making the verification process finite and decidable.*

References

[1] Jason Bell, Yohsuke Matsuzawa and Matthew Satriano. On dynamical cancellation. *Int. Math. Res. Not. IMRN*, no. 8, 7099–7139, 2023.

[2] Marc Hindry, Joseph Silverman. *Diophantine geometry*, Graduate Texts in Mathematics, 201, Springer, New York, 2000.

[3] Yohsuke Matsuzawa, Sheng Meng, Takahiro Shibata, De-Qi Zhang. Non-density of points of small arithmetic degrees. *J. Geom. Anal.* 33, no. 4, Paper No. 112, 41 pp., (2023).

[4] Xiao Zhong. Dynamical cancellation of polynomials. *Bull. Lond. Math. Soc.* 55, no. 6, 2948–2962, 2023.

[5] Xiao Zhong. Preimages question for surjective endomorphisms on $(\mathbb{P}^1)^n$. *New York J. Math.* 31, 633–649, 2025.

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