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INDEX FORMULA FOR QUARTER-PLANE TOEPLITZ OPERATORS VIA EXTENDED SYMBOLS

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In this talk, we presented an index formula for some Toeplitz operators on a discrete quarter-plane of two-variable rational matrix function symbols.

Let N be a positive integer and $\mathbb{S}^1$ be the unit circle in the complex plane. Let $f\colon \mathbb{S}^1 \times \mathbb{S}^1 \to M(N, \mathbb{C})$ be a continuous map. We focus on cases where each entry of the matrices consists of two-variable rational matrix functions with respect to $(z, w) \in \mathbb{S}^1 \times \mathbb{S}^1$. We consider a bounded linear operator $T_f^{x,y}$ on $l^2(\mathbb{N}^2, \mathbb{C}^N)$ obtained as the compression of the multiplication operator M_f on $L^2(\mathbb{S}^1 \times \mathbb{S}^1, \mathbb{C}^N) \cong l^2(\mathbb{Z}^2, \mathbb{C}^N)$ onto its closed subspace $l^2(\mathbb{N}^2, \mathbb{C}^N)$, which we call the *quarter-plane Toeplitz operator* of symbol f. In the same way, two half-plane Toeplitz operators T_f^x on $l^2(\mathbb{Z} \times \mathbb{N}, \mathbb{C}^N)$ and T_f^y on $l^2(\mathbb{N} \times \mathbb{Z}, \mathbb{C}^N)$ are defined as the compressions of the multiplication operator M_f.

Index theory for quarter-plane Toeplitz operators has been investigated by Simonenko, Douglas–Howe, Park [18, 5, 15]. A necessary and sufficient condition for these operators to be Fredholm is stated as follows.

Theorem 0.1 (Douglas–Howe [5])**.** *The quarter-plane Toeplitz operator $T_f^{x,y}$ is Fredholm if and only if two half-plane Toeplitz operators T_f^x and T_f^y are invertible.*

Index formulas for Fredholm quarter-plane Toeplitz operators are obtained by Coburn–Douglas–Singer, Dudučava, Park [4, 6, 15]. Coburn–Douglas–Singer derived their formula by showing that there is a deformation to some quarter-plane Toeplitz operators of a standard form preserving Fredholm indices [4]. Dudučava employed *Wiener–Hopf factorizations* for matrix-valued functions on a circle developed by Gohberg–Kreĭn [8, 3, 9] and obtained a formula by using a construction of a parametrix [6]. Park obtained an index formula by constructing a cyclic cocycle and using a pairing between K-theory and cyclic cohomology [15].

A motivation of our work comes from an application to *higher-order topological insulators* [2], a topic in condensed matter physics. In [10], we introduced a characteristic for them (especially for (extrinsic) second-order topological insulators). by using index theory for quarter-plane Toeplitz operators. In this application, we want a method to compute Fredholm indices for qurater-plane Toeplitz operators for given matrix-valued functions, therefore investigate their index formulas. For that purpose, we revisit Dudučava's idea from geometric viewpoint.

The following is the main theorem of this talk.

Theorem 0.2 ([11]). *Let $f\colon \mathbb{S}^1 \times \mathbb{S}^1 \to GL(N,\mathbb{C})$ be a two-variable rational matrix function. Assume that the quarter-plane Toeplitz operator $T_f^{x,y}$ is Fredholm. Under this setup, the following holds.*

(1) *The symbol f canonically extends as a continuous invertible matrix-valued function onto a three sphere through Wiener–Hopf factorizations.*

$$f^E\colon \widetilde{\mathbb{S}}^3 \to GL(N,\mathbb{C}) \text{ satisfying } f^E|_{\mathbb{S}^1\times\mathbb{S}^1} = f.$$

(2) *The Fredholm index of $T_f^{x,y}$ coincides with the three-dimensional winding number of the extension f^E, that is,*

$$\mathrm{index} T_f^{x,y} = W_3(f^E).$$

Note that, when the quarter-plane Toeplitz operator $T_f^{x,y}$ is Fredholm, its symbol f takes values in invertible matrices, and two half-plane Toeplitz operators T_f^x and T_f^y are both invertible by Douglas–Howe's result. For (1) of our theorem, we investigate geometric implications of the invertibility of two half-plane Toeplitz operators by using Wiener–Hopf factorizations, which are introduced next.

Let $D_+ = \{z \in \mathbb{C} \mid |z| < 1\}$ and $D_- = \{z \in \mathbb{C} \mid |z| > 1\} \cup \{\infty\}$, which are open disks. We write $\mathbb{D}_\pm = \mathbb{S}^1 \cup D_\pm$ (the double sign corresponds) for closed disks whose union is the Riemann sphere $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$. For a (single variable) rational invertible matrix-valued function $g\colon \mathbb{S}^1 \to GL(N,\mathbb{C})$ (with poles off $\mathbb{S}^1$), the following decomposition, called the Wiener–Hopf factorization, exists:

$$g = g_- \Lambda g_+, \tag{0.1}$$

where $g_\pm$ and Λ are continuous maps $\mathbb{S}^1 \to GL(N,\mathbb{C})$ satisfying the following conditions.

- Λ is the diagonal matrix-valued function of the form $\Lambda(z) = \mathrm{diag}(z^{\kappa_1}, \ldots, z^{\kappa_n})$, where $\kappa_1 \geq \cdots \geq \kappa_n$ is a nonincreasing sequence of integers called *partial indices*.
- f_+ admits a continuous extension onto $\mathbb{D}_+$ that is holomorphic on D_+ as an invertible matrix-valued function.
- f_- admits a continuous extension onto $\mathbb{D}_-$ that is holomorphic on D_- as an invertible matrix-valued function.

Among many results known for Wiener–Hopf factorizations [8, 3, 9], we notice the followings.

Lemma 0.3. (1) *The partial indices are uniquely determined by g.*

(2) *The Toeplitz operator T_g is invertible if and only if all of the partial indices are zero (in this case, called the* canonical factorization*).*

(3) *If $g = g_- g_+ = h_- h_+$ are two canonical factorizations, there exists an invertible matrix $B \in GL(N,\mathbb{C})$ (considered a constant matrix-valued function) such that $g_+ = Bh_+$ and $g_- = h_- B^{-1}$.*

Under our setup, through the isomorphism $l^2(\mathbb{N}\times\mathbb{Z},\mathbb{C}^N) \cong l^2(\mathbb{N},\mathbb{C}^N) \otimes L^2(\mathbb{S}^1)$, the invertible half-plane Toeplitz operator T_f^y corresponds to a family of invertible Toeplitz operators $\{T_{f(\cdot,w)}\}_{w\in\mathbb{S}^1}$. Therefore, for each $w_0 \in \mathbb{S}^1$, there exists a canonical factorization,

$$f(z,w_0) = f_-(z,w_0) f_+(z,w_0).$$

f_- and f_+ admits a holomorphic extension onto $\mathbb{D}_-$ and $\mathbb{D}_+$ for which we write f_-^e and f_+^e, respectibely. For each $z \in \mathbb{D}_+$, we define an invertible matrix as follows:

$$f^E(z, w_0) = f^e(\bar{z}^{-1}, w_0) f_+^e(z, w_0).$$

By the above lemma, f^E is independent of the choice of Wiener–Hopf factorizations. Since f_-^e and f_+^e are extensions of f_- and f_+, and that $\bar{z}^{-1} = z$ for $z \in \mathbb{S}^1$, f^E is an extension of f. By considering this construction for families (with respect to $w \in \mathbb{S}^1$), we obtain an extension f^E of f onto $\mathbb{D}_+ \times \mathbb{S}^1$, which is an invertible matrix-valued function. By using the invertibility of another half-plane plane Toeplitz operator T_f^x, we obtain a similar extension onto $\mathbb{S}^1 \times \mathbb{D}_+$. Their continuity follows from Šubin's study of Wiener–Hopf factorizations for families of matrix-valued functions [19]. Summarizing, when the quarter-plane Toeplitz operator $T_f^{x,y}$ is Fredholm, there exists a canonical extension f^E of the symbol f, initially defined on the two-dimensional torus, onto the following three-sphere,

$$\widetilde{\mathbb{S}}^3 = \mathbb{D}_+ \times \mathbb{S}^1 \cup_{\mathbb{S}^1 \times \mathbb{S}^1} \mathbb{S}^1 \times \mathbb{D}_+ = \partial(\mathbb{D}_+ \times \mathbb{D}_+) \subset \mathbb{C}^2,$$

as an invertible matrix-valued function. This provides (1) of our main theorem.

In the rest of my talk, I briefly explained the ideas for the proof of (2) of our main theorem, which utilizes (mainly) topological K-theory relying both on Coburn–Douglas–Singer's topological study [4] and Park's C^*-algebraic study [15].

Note that our index formula can be generalized to families of quarter-plane Toeplitz operators and those preserving some real structures, which are contained in [11].

In this report, we add a comment on the applications to (higher-order) topological insulators. Mathematical studies of topological insulators were initiated by Bellissard [1] and Kellendonk–Richter–Schulz-Baldes provided a proof of the bulk-boundary correspondence, a characteristic for topological insulators, by using index theory for Toeplitz operators [13]. K-theory is employed to classify topological insulators [14], and K-theoretic study has been widely expanded (for some equivariant setup, in its relation to topological crystalline insulators, for example), see also [7, 17]. As for higher-order topological insulators, an index theoretic approach is presented in [10] which do not include any point group symmetry. For *intrinsic* higher-order topological insulators, which have attracted much interest for condensed matter physicists, point group symmetry should be incorporated into the framework. Such a framework was established by Ojito–Prodan–Stoiber [16]. An alternative approach for a specific setup based on extensions of symbols for quarter-plane Toeplitz operators, as presented in this talk, can be found in our recent preprint [12].

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NONCOMMUTATIVE GEOMETRY OF THE SATAKE COMPACTIFICATION

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This is a report on a joint project with Jacob Bradd and Robert Yuncken, about which further details may be found in [1] and [2] (this document borrows from those papers). The goal of our work has been to examine from the perspective of C^*-algebras and noncommutative geometry the following celebrated discovery of Harish-Chandra (see [7] or [14]):

Theorem 0.1. *Let G be a real reductive group. A tempered irreducible unitary representation of G is either square-integrable, modulo center, or embeddable into a principal series representation, meaning one that is unitarily parabolically induced from a square-integrable, modulo center, irreducible unitary representation of a Levi subgroup.*

Harish-Chandra's result played an important role in his pursuit of the Plancherel formula. In his review of Harish-Chandra's *Collected Works*, Robert Langlands [8] writes that

> Harish-Chandra discovered quite early on the principles which allowed him to do this [obtain an explicit Plancherel formula] . . . The critical notions are those of a Cartan subgroup, of a parabolic subgroup, of an induced, and of a square-integrable representation.
>
> . . . The first principle is that the representations [parabolically] induced from . . . square-integrable [representations] suffice for the Plancherel formula . . .
>
> . . . The second is that [a real reductive group] has square-integrable representations if and only if there are [compact] Cartan subgroups . . .

The second principle has long been studied from a geometric perspective, culminating in the work of Lafforgue [6], who recovered Harish-Chandra's classification of the discrete series using noncommutative geometry and K-theory. The theorem that we stated above is a precise version of the first principle. Combined, the two principles paint in broad outline a picture of the tempered dual of any real reductive group.

Our approach to the theorem proceeds via the (maximal) Satake compactification $\mathscr{X}$ of the Riemmanian symmetric space associated to a real reductive group G [13]. We incorporate the Satake compactification into an argument involving C^*-algebras by associating to $\mathscr{X}$ first a groupoid, and then the C^*-algebra of that groupoid.

The purpose of [1] is to describe the groupoid from three different points of view: those of topology, Lie theory and geometry. The fastest way to present the *Satake groupoid* (as we call it) is to use the following simple observation of Omar Mohsen [9] (which he has used to great effect in his own work): if G is any group, and if $\{S\}$ is any collection of subgroups of G that is closed under conjugation by elements of G, then the collection $\{C\}$ of all cosets of all the subgroups in $\{S\}$ carries the following structure of a groupoid over the object space $\{S\}$:

$$\mathrm{source}(C) = C^{-1}C, \quad \mathrm{target}(C) = CC^{-1} \quad \text{and} \quad C_1 \circ C_2 = C_1 C_2.$$

Now, the Satake compactification $\mathscr{X}$ of a real reductive group G may be defined to be the closure of the space of maximal compact subgroups of G within the compact space of all closed subgroups of G [5]. So Mohsen's observation immediately applies, and we obtain a locally compact Hausdorff topological groupoid. This is our Satake groupoid.

Although the above quickly characterizes the Satake compactification and the Satake groupoid, for computations it is much more convenient to construct both the compactification and the groupoid using Lie theory. This may be done following the approach of Toshio Ōshima [12] to the Satake compactification.

Ōshima's construction makes it clear that the Satake compactification has finitely many G-orbits, which may be described using an Iwasawa decomposition $G{=}KAN$, as follows. It is well-known in Lie theory that a standard parabolic subgroup $P_I{=}M_IA_IN_I$ of G may be associated to each subset I of the set Σ of simple restricted roots that is associated to the given Iwasawa decomposition, and that these are the only standard parabolic subgroups. The G-orbits in $\mathscr{X}$ are also in bijection with the subsets $I \subseteq \Sigma$, with the orbit $\mathscr{X}_I \subseteq \mathscr{X}$ being of the type

$$\mathscr{X}_I \cong G\big/K_IA_I\overline{N}_I,$$

where $K_I{=}K \cap M_I$, and where $\overline{N_I} = \theta[N_I]$, and where θ is the Cartan involution. It follows that the Satake compactification, viewed as a collection of closed subgroups of G, consists of all the conjugates in G of all the groups $H_I{=}K_I\overline{N}_I$.

The orbit $\mathscr{X}_I$ is contained in the closure of the orbit $\mathscr{X}_J$ if and only if $I \subseteq J$. It follows, for instance, that the orbit $\mathscr{X}_\Sigma$ is open and dense in $\mathscr{X}$. In addition, $K_\Sigma = K$, while the group A_Σ is the intersection of the center of G with A, and N_Σ is the trivial one-element group. The orbit

$$\mathscr{X}_\Sigma \cong G/KA_\Sigma,$$

therefore identifies, via the map $gKA_\Sigma \mapsto gKg^{-1}$, with the space of maximal compact subgroups of G.

As for the Satake groupoid, $\mathscr{G}_{\mathscr{X}}$, each of the orbits $\mathscr{X}_I$ above is a locally closed, saturated subset of $\mathscr{X}$, and also a smooth embedded submanifold, and the reduction of the Satake groupoid $\mathscr{G}_{\mathscr{X}}$ to $\mathscr{X}_I$ has the form

$$\mathscr{G}_I \cong G\big/K_I\overline{N}_I \underset{A_I}{\times} G\big/K_I\overline{N}_I$$

(quotient by the diagonal right action of A_I). For instance, the open and dense subgroupoid $\mathscr{G}_\Sigma$ is

$$\mathscr{G}_\Sigma \cong G/K \underset{A_\Sigma}{\times} G/K.$$

When G has compact center, the group A_Σ is trivial, and the above is simply the pair groupoid on $\mathscr{X}_\Sigma \cong G/K$.

Ōshima actually constructed a smooth, closed G-manifold $\mathscr{M}$ into which the variety of maximal compact subgroups of G embeds as an open subset, while the Satake compactification embeds smoothly as a compact submanifold with corners. We in fact construct a Lie groupoid $\mathscr{G}_{\mathscr{M}}$ over $\mathscr{M}$ that we call the *Ōshima groupoid*. It is a quotient of the transformation groupoid for the action of G on $\mathscr{M}$. The Satake compactification, viewed as a subset of $\mathscr{M}$, is a saturated subset for the Ōshima groupoid, and we prove that the reduction of $\mathscr{G}_{\mathscr{M}}$ to this subset is our Satake groupoid.

Now, the bounding submanifolds (of top dimension) of the Satake compactification within the Ōshima space $\mathscr{M}$ extend to smooth, closed hypersurfaces in $\mathscr{M}$ that cross one another normally. And to any manifold, such as $\mathscr{M}$, that is equipped with a finite family of normally crossing, closed hypersufaces there is associated a Lie groupoid [11]; the construction is an elaboration of ideas from the b-calculus of Richard Melrose; for instance the Lie algebroid is Melrose's b-tangent bundle. Our third view of the Satake groupoid identifies the Ōshima groupoid with this geometrically-defined *b-groupoid*.

Turning to Harish-Chandra's principle and the paper [2], C^*-algebras play two roles in our argument. First, Harish-Chandra's tempered irreducible unitary representations correspond precisely those irreducible unitary representations of G that integrate to irreducible representations of the reduced group C^*-algebra $C^*_r(G)$, and every irreducible representation of $C^*_r(G)$ is so-obtained [4]. Second, C^*-algebra theory provides a simple tool to separate the space of all these irreducible representations into two parts: indeed if A is any C^*-algebra, and if J is any ideal in A, then there is a partition the spectrum $\widehat{A}$ (the set of irreducible representations, up to equivalence) into those representations that vanish on all elements of J, and those that don't, and this partition takes the simple form

$$\widehat{A} = \widehat{A/J} \ \sqcup \ \widehat{J}.$$

In broad terms our proof of the theorem above goes as follows. We introduce an ideal I in $A{=}C^*_r(G)$ for which

$$\widehat{I} = \{\text{ discrete series representations of } G\,\}.$$

This is a very general construction that may be applied to any unimodular locally compact group. Then we define a second ideal $J \triangleleft A$ such that

$$\widehat{A/J} = \left\{ \begin{array}{c} \text{tempered irreducible representations of } G \text{ that} \\ \text{embed in a principal series representation} \end{array} \right\}.$$

The definition is specific to real reductive groups, of course, but it is otherwise very elementary, using only the definition of parabolic induction, as viewed from the perspective of C^*-algebra theory [3]. Harish-Chandra's principle amounts to the assertion that $I = J$.

We prove that the ideals I and J coincide using the Satake groupoid $\mathscr{G}_{\mathscr{X}}$. The reduced C^*-algebra of the groupoid, $C^*_r(\mathscr{G}_{\mathscr{X}})$ fits into an exact sequence

$$0 \longrightarrow C^*_r(\mathscr{G}_{\operatorname{int}\mathscr{X}}) \longrightarrow C^*_r(\mathscr{G}_{\mathscr{X}}) \longrightarrow C^*(\mathscr{G}_{\partial\mathscr{X}}) \longrightarrow 0$$

according to the decomposition of $\mathscr{X}$ into its interior and boundary. We prove that $I{=}J$ by relating I to the image of the inclusion morphism in the exact sequence, and J to the kernel of the quotient morphism; obviously the two ideals in the groupoid C^*-algebra are the same. Crucial to the argument is a C^*-algebra morphism

$$C^*_r(G) \longrightarrow C^*_r(\mathscr{G}_{\mathscr{X}})$$

that was introduced by Mohsen in [10].

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ALGEBRAIC TOPOLOGY OF 24 DIMENSIONAL STRING MANIFOLDS

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String manifolds of dimension 24 are of special interest in geometry and topology. In this dimension, by [5, Page 85-87] the famous Witten genus $W(M)$ [9] satisfies

$$W(M) = \widehat{A}(M)\bar{\Delta} + \widehat{A}(M,T)\Delta,$$

where $\widehat{A}(M)$ is the A-hat genus, $\widehat{A}(M,T)$ is the twisted A-hat by the tangent bundle, and $\bar{\Delta} = E_4^3 - 744 \cdot \Delta$ with E_4 being the Eisenstein series of weight 4 and Δ being the modular discriminant of weight 12.

Hirzebruch raised his prize question in [5] that whether there exists a 24 dimensional compact string manifold M such that $W(M) = \bar{\Delta}$ (or equivalently $\widehat{A}(M) = 1, \widehat{A}(M,T) = 0$) and the Monster group acts on M as self-diffeomorphisms. The existence of such manifold was confirmed by Mahowald-Hopkins [7]. Indeed, they determined the image of Witten genus at this dimension via tmf. However, the part of the question concerning the Monster group is still open.

In two joint works [3, 4] with Fei Han, we find representatives of an integral basis of the string cobordism group at dimension 24. Historically, Gorbounov-Mahowald [2] showed that

$$\Omega_{24}^{String} \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

Our main theorem is stated as follows:

Theorem 0.1. *The correspondence $\kappa : \Omega_{24}^{String} \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ defined by*

$$\kappa(M) = (\widehat{A}(M), \frac{1}{24}\widehat{A}(M,T), \widehat{A}(M,\Lambda^2), \frac{1}{8}\mathrm{Sig}(M))$$

is an isomorphism of abelian groups, where $\widehat{A}(M,\Lambda^2)$ is the twisted A-hat by the second exterior power of the tangent bundle and $\mathrm{Sig}(M)$ is the signature. Moreover, there exist two geometrically constructed manifolds $M_3, M_4 \in \ker W$ such that

$$K := \begin{pmatrix} \kappa(M_1) \\ \kappa(M_2) \\ \kappa(M_3) \\ \kappa(M_4) \end{pmatrix}^{\tau} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 2^3 \cdot 3^3 \cdot 5 & 2^2 \cdot 3 \cdot 17 \cdot 1069 & -1 & 0 \\ 2^8 \cdot 3 \cdot 61 & 2^8 \cdot 5 \cdot 37 & 2^2 \cdot 7 & 1 \end{pmatrix},$$

where $M_1, M_2 \in \mathrm{image}\, W$ are the two manifolds constructed by Mohowald-Hopkins [7].

A notable consequence is that the four manifolds M_i in Theorem 0.1 form a basis of the group Ω_{24}^{String}.

This theorem implies various Rokhlin type divisibility results of the characteristic numbers of 24 dimensional string manifolds. For instance, we have

$$32 \mid \mathrm{Sig}(M) - \big\langle (\nu_{16}^{HS})^2(M), [M] \big\rangle,$$

where $\nu_*^{HS}(M)$ is the spin integral Wu class of Hopkins-Singer [6].

Furthermore, we use Theorem 0.1 to show that the elliptic genus [8], a higher index theoretic invariant, determines 24 dimensional string cobordism, and thereby obtain an geometric application:

Theorem 0.2. *Given positive number λ, there exists some $\varepsilon = \varepsilon(\lambda) > 0$ such that if a compact 24 dimensional string Rimannian manifold (M, g) satisfies $\mathrm{diam}(M, g) \leq 1, \mathrm{Ric}(g) \leq \varepsilon$, sectional curvature $\geq -\lambda$ and has infinite isometry group, then M bounds a string manifold.*

This result corresponds to a higher version of a conjecture of Farrell-Zdravkovska [1] and Yau [10] which claims that every almost flat manifold is the boundary of a closed manifold.

Theorem 0.1 also provides potential clue for understanding a question of Weiping Zhang:

Question 0.3. *Is $\frac{1}{24}\widehat{A}(M, T)$ the index of a twisted Dirac operator?*

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TENSOR NETWORKS IN CONDENSED MATTER PHYSICS AND SUBFACTORS

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Keywords: tensor network, topological order, fusion category, subfactor, anyon

As studied in [2], [17], many researchers in two-dimensional topological order in condensed matter physics are interested in studies of (braided) fusion categories [3] using tensor networks recently. It has been well-known that subfactor theory of Jones [7], [8] in operator algebras gives useful and powerful tools to study structures of fusion categories. This approach is closely related to operator algebraic studies of quantum field theory [15], [16], [5].

In a usual operator algebraic study of fusion categories, we realize an object as a bimodule over (type II_1) factors or an endomorphism of a (type III) factor [4]. Another approach [1] based on bi-unitary connections [18], [20], [9] is less common, but contains the same information as these two methods and has an advantage that everything is finite dimensional. Recall that a bi-unitary connection gives a characterization [20], [4] of a non-degenerate commuting square [19]. It has been pointed out in [10], [12] that the 4-tensors in [2] are mathematically the same as bi-unitary connections, and identification of some objects in condensed matter physics and subfactor theory has been given in [11]. This shows that anyons [**?**] are studied with such 4-tensor networks [6]. We have Tables 1 and 2 for correspondences between these methods to represent fusion categories.

Our aim is now to complete this table by identifying tensors satisfying the zipper condition in [2], flat fields of strings in [1], [4], and elements in the higher relative

TABLE 1. Correspondence among endomorphisms, bimodules and connections

endomorphism	bimodules	connections
identity	identiry bimodule	trivial connection
direct sum	direct sum	direct sum
composition	relative tensor product	composition
cojugate endomorphism	dual bimodule	dual connection
dimension	(Jones index)$^{1/2}$	Perron-Frobenius eigenvalue
intertwiner	intertwiner	flat field of strings

TABLE 2. Correspondence between connections, commuting squares and 4-tensors

connections	commuting square	4-tensor
trivial connection	commuting square	trivial 4-tensor
direct sum	direct sum	direct sum
composition	composition	concatenation
dual connection	basic construction	complex conjugate tensor
Perron-Frobenius eigenvalue	(Pimsner-Popa index)$^{1/2}$	Perron-Frobenius eigenvalue
flat fields of strings	relative commutant	tensors with the zipper condition

commutants of a subfactor [1], [4] arising from the commuting square. We then prove the following, which is the main theorem in [13].

Theorem 0.1. *The following are equivalent for a 2-tensor F and the corresponding field f of strings in the setting of tensor networks of 4-tensors describing fusion categories..*

(1) [The zipper condition] *The 2-tensors F satisfies the invariance property, which gives an equivalent formulation of the zipper condition.*

(2) [Flatness] *The field f of strings satisfies the flatness in the usual meaning of subfactor theory.*

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STABLE HOMOTOPY THEORY OF INVERTIBLE GAPPED QUANTUM SPIN SYSTEMS

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Keywords: SPT phase, stable homotopy theory, coarse geometry

In his talk [5], A. Kitaev proposed that the set of d-dimensional invertible gapped quantum spin systems $\{IP_d\}_{d\in\mathbb{Z}_{\geq 0}}$ should form an Ω-spectrum. This talk outlines the framework developed in [7], which provides a rigorous formulation of quantum spin systems based on functional analysis in which Kitaev's conjecture holds.

For a Hamiltonian in quantum spin systems, we impose the three conditions: short-range, gapped, and invertible. At each point $\boldsymbol{x}$ in a discrete metric space Λ, namely a d-dimensional lattice, we place a matrix algebra $\mathcal{A}_{\boldsymbol{x}} = M_n(\mathbb{C})$. The full observable algebra is defined by their tensor product $\mathcal{A}_\Lambda := \bigotimes_{\boldsymbol{x}\in\Lambda}\mathcal{A}_{\boldsymbol{x}}$. A Hamiltonian is specified by a family of operators $\mathsf{H}_{\boldsymbol{x}}$, each supported on a the open ball $B_r(\boldsymbol{x})$ with radius $r > 0$, which define a $*$-derivation $[\mathsf{H}, a] = \sum_{\boldsymbol{x}}[\mathsf{H}_{\boldsymbol{x}}, a]$. In [7], we relax the locality assumption, following earlier work such as [8], and treat almost local Hamiltonians. Namely, we do not require each $\mathsf{H}_{\boldsymbol{x}}$ to be strictly supported on a ball, but instead assume that it is well approximated by operators that are supported on such balls. Ground state, the non-degeneracy, and the spectral gap for such H are formulated in terms of the GNS theory. A gapped Hamiltonian H is called invertible if there exists another Hamiltonian $\check{\mathsf{H}}$ such that the composite system $\mathsf{H} \boxtimes \check{\mathsf{H}}$ is homotopic to the trivial one.

An important motivation of this conjecture is that it provides a homotopy theoretic interpretation of the group-cohomology valued topological invariant of Hamiltonians that are invariant under an on-site action of a compact Lie group G, as studied in the literature for low dimensions (e.g., [3, 9, 10, 12]). Indeed, the invariant is obtained as the composition

$$[\mathrm{pt}, IP_d]^G \to [EG, IP_d]^G \to [BG, IP_d] \to [BG, K(\mathbb{Z}, d+2)].$$

The existence of the second and the third morphisms are guaranteed by Kitaev's conjecture.

Kitaev's conjecture was inspired by the work of Kitaev himself [4], and also of Schnyder–Ryu–Furusaki–Ludwig [11], on the topological classification of gapped free fermions by real or complex K-theory. In the functional-analytic formulation by [6], a free-fermion Hamiltonian is defined analogously to a quantm spin system, but with direct sums in place of tensor products. The space of gapped Hamiltonians is identified with the space of projection operators of the C*-algebra so-called the (uniform) Roe algebra. Kitaev's conjecture for these spaces is proved by Higson–Roe–Yu [2] in their early work. We emphasize that the short-range condition for IP_d is highly compatible with the concept of coarse geometry and our proof of Kitaev's conjecture run in parallel to [2].

The map $\kappa_d \colon IP_d \to \Omega IP_{d+1}$ is given in the following way. First, by the assumption of invertibility, we have a path connecting the trivial Hamiltonian, say h, and $\mathsf{H} \boxtimes \check{\mathsf{H}}$. By regarding an infinite stack of this homotopy as a $(d+1)$-dimensional layered Hamiltonian, we obtain a path connecting h and the infinite stack

$$\cdots \mathsf{H} \boxtimes \check{\mathsf{H}} \boxtimes \mathsf{H} \boxtimes \check{\mathsf{H}} \boxtimes \mathsf{H} \boxtimes \check{\mathsf{H}} \boxtimes \mathsf{H} \boxtimes \check{\mathsf{H}} \boxtimes \cdots .$$

We then return to the trivial Hamiltonian by the same homotopy, but with a rearranged pairing of H and $\check{\mathsf{H}}$. Such a 1-parameter family is called Kitaev's pump. A central part of [7] is to constructed a homotopy inverse of κ_d following the line of Kitaev and [2], with a careful treatment of the subtle analysis of spectral gap in quantum spin systems.

Replacing the tensor products with $\mathbb{Z}/2$-graded tensor products yields the fermionic version fIP_d. As the name suggests, a free fermion is a special case of a fermionic system, and this inclusion is ultimately realized as a morphism of Ω-spectra

$$\mathrm{Q} \colon \Sigma^{-2} KO \to fIP.$$

A rigorous construction of this morphism Q is provided by Araki's quasi-free second quantization [1]. Truncating these spectra in degrees $-1, \dots, 2$ gives a weak equivalence

$$\Sigma^{-2}(KO\langle 1, 4\rangle) \simeq fIP\langle -2, \infty\rangle,$$

which answers to a question by D. Freed: we obtain an explicit homotopy equivalence between $KO\langle 1, 4\rangle$ and the truncated Picard spectrum $\mathrm{pic}_0^3 KU$, which had previously been known to be abstractly homotopy equivalent by comparing their Postnikov k-invariants.

Finally, this talk also briefly discusses the main theme of latter part of [7], which studies quantum spin systems placed on spaces X more general than Euclidean space. The space '$IP(X)$' of invertible gapped Hamiltonians placed on X should form a coarse homology theory, rather than a homology theory, and therefore violates the local topology of X. For example, the cases $X = S^d$ and $X = \mathrm{pt}$ are not distinguished. In [7], inspired by early work by Yu [13], we introduce the notion of localization flow of gapped Hamiltonians.

Roughly speaking, a localization flow of gapped Hamiltonians on X is a family $\{\mathsf{H}(s)\}_{s \in [1,\infty)}$ of uniformly gapped Hamiltonians whose interaction range decays as $s \to \infty$. This notion is named after Yu's localization algebra, as well as matrix product renormalization group flow. We prove that the π_0-group of the space $IP_{\mathrm{loc}}(X)$ of localization flows on X forms a generalized homology theory that agrees with the one associated to the Ω-spectrum IP via the Spanier–Whitehead duality.

Moreover, one can also incorporate spatial symmetries given by a crystallographic group Γ acting on X into this framework. In this setting, the map

$$\mu_\Gamma \colon \pi_0(IP^\Gamma_{\mathrm{loc}}(X)) \to \pi_0(IP^\Gamma(X))$$

which forgets the parameter $s > 1$ can be regarded as a quantum-spin analogue of the Baum–Connes assembly map in noncommutative geometry. Following a well-known idea in coarse geometry, the split injectivity of our μ_Γ can be proved. That is, the group $\pi_0(IP^\Gamma(X))$ of our interest contains a subgroup $\pi_0(IP^\Gamma_{\mathrm{loc}}(X))$ as a direct summand, which is computable from the homotopy groups $\pi_n(IP_d)$ by algebraic topology.

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GENERALIZED POSITIVE SCALAR CURVATURE ON SPIN^c MANIFOLDS

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Classification AMS 2020: 53C21, 53C27, 58J22, 55N22, 19L41

Keywords: positive scalar curvature, spin^c manifold, bordism, K-theory, index, index difference, Stolz sequence

This is joint work with Boris Botvinnik (University of Oregon) and Paolo Piazza (La Sapienza University, Rome), which has appeared in [1, 2, 3].

1. Generalized positive scalar curvature and the spin^c Dirac operator

If M is a closed spin manifold with a Riemannian metric g, then g determines a natural connection on the spinor bundle S and we have a (spin) Dirac operator $D = \sum_i c(e_i)\nabla_{e_i}$, where $\{e_i\}$ is a local orthonormal frame and c denotes Clifford multiplication on spinors. (One can check that D is independent of the choice of frame.) The *Schrödinger-Lichnerowicz Formula* says that

$$D^2 = \nabla^*\nabla + \frac{1}{4}R_g,$$

where R_g is the scalar curvature function.

Corollary 1.1. *If $R_g \geq 0$ everywhere, and R_g is not identically 0, then all index invariants of D vanish.*

This is the starting point for all work on *positive scalar curvature* (psc).

In dimension 2, R is just (twice) the Gaussian curvature and everything we need to know is given by Gauss-Bonnet. But in dimensions ≥ 3, we have

Theorem 1.2 (Kazdan-Warner, 1975)**.** *Every closed connected manifold M of dimension ≥ 3 falls into exactly one of the following three classes:*

(1) *Those admitting a psc metric, in which case every smooth function on M is the scalar curvature of some metric;*
(2) *Those admitting a metric g with $R_g \equiv 0$ but not a metric with $R_g \geq 0$ and R_g positive somewhere — in this case a metric with $R_g \equiv 0$ is necessarily Ricci-flat, and the possible scalar curvature functions on M are 0 and the functions negative somewhere;*
(3) *All other manifolds, those not admitting any metric with nonnegative scalar curvature — in this case, the possible scalar curvature functions of metrics on M are exactly those functions which are negative somewhere.*

We sought an analogue of the above results for closed spin^c manifolds, relating an analogue of the scalar curvature to index theory of the spin^c Dirac operator. A spin^c manifold M comes with a choice of a spin^c line bundle L that satisfies $c_1(L) \mod 2 =$

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$w_2(M)$. In addition to the metric g, we need to choose a hermitian metric and unitary connection A on L. Then we obtain a spinor bundle and Dirac operator D on M, and the Schrödinger-Lichnerowicz Formula now takes the form

$$D^2 = \nabla^*\nabla + \frac{1}{4}R^{\text{tw}},$$

where $R^{\text{tw}} = R_g + 2ic(\Omega_L)$, Ω_L the curvature 2-form of A.

The quantity R^{tw}, the *twisted scalar curvature*, is matrix-valued. It is easier to work with the *generalized scalar curvature* $R^{\text{gen}} = R_g - 2|\Omega_L|_{\text{op}}$, which is scalar-valued. It is not hard to show that $R^{\text{tw}} > 0 \iff R^{\text{gen}} > 0$, and thus positivity of R^{gen} implies vanishing of all index invariants of the spin^c Dirac operator D.

We have an analogue of the Kazdan-Warner trichotomy theorem in this context:

Theorem 1.3 ([2]). *Every closed connected spin^c manifold M of dimension ≥ 3 falls into exactly one of the following three classes:*

1. *Those admitting a metric and connection (g, A) with $R^{\text{gen}}_{(g,A)} > 0$, in which case every smooth function on M is the generalized scalar curvature of some metric and connection;*
2. *Those admitting a pair (g, A) with $R^{\text{gen}}_{(g,A)} \geq 0$ but not one with $R^{\text{gen}}_{(g,A)} > 0$ — in this case, the possible generalized scalar curvature functions on M are 0 and the functions negative somewhere;*
3. *All other manifolds, those not admitting any metric and connection with nonnegative generalized scalar curvature — in this case, a smooth function on M is $R^{\text{gen}}_{(g,A)}$ for some (g, A) if and only if it's negative somewhere.*

As in the classical case of scalar curvature, there is a rigidity phenomenon in case (2).

Theorem 1.4 ([2]). *Suppose M is a closed simply connected non-spin spin^c manifold with spin^c line bundle L. Assume M admits a pair (g, A) with $R^{\text{gen}}_{(g,A)} \equiv 0$ but not one with $R^{\text{gen}}_{(g,A)} > 0$. Then M admits a parallel spinor, and if M does not split as a product, then M is conformally Kähler, and L is either the canonical or the anti-canonical line bundle on M.*

2. Classification of spin^c manifolds with generalized positive scalar curvature

Parallel to many results about psc on spin manifolds, we have a classification theory of gpsc (generalized positive scalar curvature — maybe it would be better to say *positive generalized scalar curvature*) in the *totally non-spin case*. We say a connected manifold M is *totally non-spin* if its universal cover $\widetilde{M}$ does not admit a spin structure, i.e., $w_2(\widetilde{M}) \neq 0$.

Let ku denote *connective* complex K-theory, and let $\operatorname{per}\colon ku_* \to K_*$ denote the periodization map (inversion of the Bott element). For any group π, let $\mathbf{As}\colon K_*(B\pi) \to K_*(C^*(\pi))$ denote the *assembly map* (which appears in study of the Novikov Conjecture). The following theorem is parallel to a result of Stolz, Jung, and Führing [4], though the proof requires some additional homotopy theoretic techniques.

Theorem 2.1 ([3]). *Let π be a finitely presented group. Then for each $n \geq 5$, there is a subgroup $ku^+_n(B\pi)$ of $ku_n(B\pi)$, contained in the kernel of the composite $\mathbf{As} \circ \operatorname{per}\colon ku_n(B\pi) \to K_n(C^*(\pi))$, with the property that if M^n is a closed totally non-spin connected spin^c manifold with $n \geq 5$, with fundamental group π, with classifying map*

$f\colon M \to B\pi$, and with ku-fundamental class $[M]$, then M admits gpsc if and only if $c_*([M]) \in ku_n^+(B\pi)$.

Parallel to the so-called *Gromov-Lawson-Rosenberg (GLR) Conjecture* in the spin case, one can then formulate:

Conjecture 2.2 (GLR^c Conjecture)**.** *For M^n a connected closed totally non-spin spin^c manifold with $n \geq 5$ and fundamental group π and classifying map $f\colon M \to B\pi$, M admits gpsc iff $\mathbf{As} \circ \operatorname{per}(f_*([M])) = 0$ in $K_n(C^*(\pi))$.*

As for the GLR Conjecture in the spin case, this holds if $\mathbf{As} \circ \operatorname{per}\colon ku_n(B\pi) \to K_n(C^*(\pi))$ is injective, for example if π is free abelian or a surface group. Here is another one of our major results:

Theorem 2.3. *The GLR^c Conjecture holds if π is finite with periodic cohomology.*

However, by the same method used by Schick [5], we have constructed counterexamples to the GLR^c Conjecture with $\pi = \mathbb{Z}^4 \times \mathbb{Z}/p$.

By mimicking Stolz's theory of the "R-group" for concordance classes of psc metrics, we are able to construct an analogous theory for classification of gpsc pairs:

Theorem 2.4 ([3])**.** *Fix a finitely presented group π. There is a long exact sequence*

$$\cdots \to R_{n+1}^{\mathsf{spin}^c}(B\pi) \xrightarrow{\partial} \operatorname{Pos}_n^{\mathsf{spin}^c}(B\pi) \to \Omega_n^{\mathsf{spin}^c}(B\pi) \to R_n^{\mathsf{spin}^c}(B\pi) \to \cdots .$$

Here the groups $\operatorname{Pos}^{\mathsf{spin}^c}$ and Ω^{spin^c} are spin^c bordism groups, the former also keeping track of a gpsc pair. R^{spin^c} is a relative group of equivalence classes of spin^c manifolds with boundary, with gpsc on the boundary, and ∂ comes from restriction to the boundary.

Theorem 2.5 ([3])**.** *If M^n is a closed connected totally non-spin spin^c manifold with fundamental group π, admitting gpsc, and $n \geq 5$, then $R_{n+1}^{\mathsf{spin}^c}(B\pi)$ acts simply transitively on the concordance classes of gpsc pairs on M. In particular, if $R_{n+1}^{\mathsf{spin}^c}(B\pi) \neq 0$, then the space of gpsc pairs on M is disconnected.*

In some cases one can map the R-group sequence to the Higson-Roe analytic surgery sequence to conclude that the space of concordance classes is quite complicated.

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AUTOMORPHIC FORMS AND THE SUP NORM PROBLEM: A SURVEY

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In this talk we will review automorphic forms starting from the classical holomorphic forms on the Poincare upper half plane and discuss the sup norm problem associated with them.

Notations.$G = SL(2, R),\ \Gamma = SL(2, Z).\ k \in \aleph$ is even. Recall that the quotient G/Γ has a finite G invariant measure. This induces a measure on $\mathbb{H}/\Gamma$,

This measure is finite and with appropriate normalisation is $\frac{3}{\pi}$. Recall that $\mathbb{H}$ has the G invariant Riemannian measure which in Cartesian coordinates is $d\mu(z) = \frac{dxdy}{y^2}$.

Definition 1.

Modular form of weight k for Γ

Let $f : \mathbb{H} \longrightarrow \mathbb{C}$ be holomorphic and satisfy the following properties.

1. $f(\gamma z) = f(\frac{az+b}{cz+d}) = (cz+d)^k f(z)\ \forall\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\ and\ z \in \mathbb{H}$

2. f is holomorphic at ∞.

Definition 2:

Cusps of Γ These are the subset $Q \cup\ i\infty$ of $P^1(\mathbb{R}) = \mathbb{R} \cup\ i\infty$ where a point $z \in \mathbb{H}$ approaches $i\infty$ if x stays bounded and y tends to ∞. We are interested in inequivalent cusps i.e. (representatives) of the various Γ orbits in $Q \cup\ i\infty$. For our Γ as above, there is only one Γ orbit , the orbit of $i\infty$. A modular form f as defined above is a cusp form if it vanishes at the cusp $i\infty$. Equivalently its zeroth Fourier coefficient at the cusp $i\infty$ is 0.

Examples of modular forms.

1. Holomorphic Eisenstein series of weight $k, k \geq 4$ is even.

$$E_k(z) = \frac{1}{2} \sum_{(c,d)=1} \frac{1}{(cz+d)^k}$$

This series converges absolutely and uniformly on compact subsets of $\mathbb{H}$ and is a modular form of weight k. It is <u>not</u> a cusp form. We have $E_k(i\infty) = 1$.

2. **Poincare Series.**

Let $m \in \mathbb{N}$ be fixed but arbitary. The m th holomorphic Poincare series of weight P_m is defined by

$$P_m(z) = \frac{1}{2} \sum_{(c,d)=1} e^{i2\pi m(\frac{az+b}{cz+d})}\ (cz+d)^k$$

Date: 5th August 2025.

Here $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ is any completion of the last row (c, d). It is well defined since any two completions of (c, d), differ by an element of the form $\begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$ for some $l \in \mathbb{Z}$ and the function $e^{i2\pi mz}$. is translation invariant.

Propostion

P_m is a cusp form. In fact, since the space of modular forms of weight k (and hence a fortiori the subspace of cusp forms) S_k is finite dimensional, the first d_k Poincare series i.e. $P_1 \dots P_{d_K}$ (where $d_k = \dim S_k$) is a basis of S_k.

Petersson inner product on cusp forms. We have $< f, g > \triangleq \int_{\mathcal{F}} y^k f(z) \overline{g(z)} \frac{dxdy}{y^2}$ where $\mathcal{F}$ is any fundamental domain for Γ in $\mathbb{H}$. We now turn to the sup norm problem itself.

Proposition: $f \in S_k \Leftrightarrow$ the Γ invariant function $y^{k/2} \mid f(z) \mid$ on $\mathbb{H}$ is bounded.

Definition 3. The sup norm of f, $\| f \|_\infty = \sup_{z \in \mathbb{H}} y^{k/2} \mid f(z) \mid = \sup_{z \in \mathcal{F}} y^{k/2} \mid f(z) \mid$ by Γ invariance.

The sup norm problem is to obtain (as sharp as possible) upper and lower bounds for $\| f \|_\infty$ in terms of the weight k which is the spectral parameter. It suffices to do this for f which is a normalised eigenfunction for all the Hecke operators $T(n), n \in \mathbb{N}$.

The Hecke operators $T(n)$ acting on M_k.

$$(T(n)f)(z) = n^{\frac{k}{2}-1} \sum_{d|n} \sum_{b=0}^{d-1} f(\frac{\frac{n}{d}z + b}{d}); \; z \in \mathbb{H}$$

Facts 1. $T(n)$ maps M_k into M_k and leaves S_K invariant.

2. $T(n) : S_k \to S_k$ is Hermitian w.r.t. the Petersson inner product on S_k.

3. $T(n) \mid n \in \mathbb{N}\}$ is a commuting family of Hermitian operators on S_k.

By 3, S_k has an orthonormal basis $\{f_j; 1 \leq j \leq d_k\}$ consisting of simultaneous eigenfunctions of the various $T(n)$. Let $f \in S_k$ be a simultaneous eigenfunction of the $T(n)$ and let $f(z) = \sum_{n=1}^{\infty} a(n) e^{i2\pi nz}$ be the Fourier expansion of f. Then we have $a(1) \neq 0$ and $f(z) = a(1) \sum_{n=1}^{\infty} \lambda(n) e^{i2\pi nz}$ where $T(n)f = \lambda(n)f$.

We are now in a position to state Xia's result.

Theorem (Xia, 20) $\epsilon > 0$, we have

$$k^{\frac{1}{4}-\epsilon} \ll \| f_j \|_\infty \ll k^{\frac{1}{4}+\epsilon}$$

Recall that f_j is a L^2 normalised Hecke eigenform. Thus we have a sharp result in this case.

Method of proof: A direct approach using the Fourier expansion of f_j i.e.

$f_j(z) = \sum_{n=1}^{\infty} a_j(n)e^{i2\pi nz}$

Automorphic forms (not so classical) There are forms of weight $k = 0$ i.e. they are Γ invariant functions having the following additional properties.

(1) f is C^{∞}.

(2) f is an eigenfunction of the hyperbolic Laplacian $\Delta_H = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$

(3) f has polynomial growth at $i\infty$ i.e. $\mid f(z) \mid \ll y^l$ for some $l \in \mathbb{N} \cup \{0\}$ as $y \to \infty$.

Examples of nonholomorphic modular forms.

Nonholomorphic Eisenstein series, $E(z, s)$ which is defined by

$E(z, s) = \frac{1}{2} \sum_{(c,d)=1)} \frac{y^s}{|(cz+d)|^{2s}}$ $Re\ s > 1$

Note that since Δ_H is an $SL(2, \mathbb{R})$ invariant differential operator on $\mathbb{H}$ and the power function $p(z) = (Imz)^s = y^s$ is an eigenfunction of Δ_H with eigenvalue $\lambda = s(1-s), s \in \mathbb{C}$ it follows that $E(z, s)$ is an eigenfunction of Δ_H with eigenvalue λ since all necessary convergence conditions on the series above are satisfied for $Re\ s > 1$. However, $E(z, s)$ is not a cusp form since it does not vanish at the cusps.

Definition (special for the modular group Γ)

A Maass cusp form is a non-constant eigenfunction of Δ_H in $L^2(\Gamma \backslash \mathbb{H}$

Example of a Maass cusp form for Γ. Not known!
Let f be a Maass cusp form with Laplacian eigenvalue λ. Since f decays exponentially at ∞, it is a bounded function.

Definition. Let f be as above. The L^{∞} norm of f is $\| f \|_{\infty} = \sup_{z\in\mathbb{H}} |f(z)| = \sup_{z\in\mathcal{F}} |f(z)|$ by Γ invariance.

Results. Here the baseline bound is $\| f \| \ll \lambda^{1/4}$ We have here that f is L^2 normalised.

The improved result by Iwaniec-Sarnak (1995) is $\| f \| \ll \lambda^{\frac{5}{24}+\epsilon}$.

The above result of Iwaniec and Sarnak remains unbeatable to this day.

Conjecture: Iwaniec and Sarnak conjectured that $\epsilon > 0$ we have $\| f \|_{\infty} \ll \lambda^{\epsilon}$ for f an L^2 normalised. Hecke-Maass eigencusp as in their theorem. This conjecture was shown to be false by Brumley and Templier who showed that for $z = \frac{1}{4} + \frac{ir_j}{2\pi} + o(1)$ we have $\lambda^{\frac{1}{6}-\epsilon} \ll f_j(z)$ $\lambda = \frac{1}{4} + r_j^2$

In view of this Iwaniec-Sarnak modified their conjecture.

Modified Conjecture: $\forall \epsilon > 0,\ f_j(z) \ll_{\epsilon,z} \lambda_j^{\epsilon}$

This conjecture is open.

Modular forms of higher level.

These are modular forms which are automorphic with respect to congruence subgroups of Γ. A typical example of such a subgroup is the Hecke congruence subgroup $\Gamma_0(N)$ where $N \in \mathbb{N}$

Definition: $\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, N \mid c$ i.e. $c \equiv 0 \ mod\ N\}$.

Note that $\Gamma_0(1) = \Gamma$.
Definition: Let $f : \mathbb{H} \to \mathbb{C}$ be holomorphic and have the following properties $(k \in 2N)$.

1. $f(\gamma z) = (cz + d)^k f(z),\ \forall \gamma \in \Gamma_0(N)$
2. f is holmorphic at the cusps of Γ.

Definition of sup norm of $f, \parallel f \parallel_\infty$ is the same except that now $\mathcal{F}$ has to be replaced by $\mathcal{F}_n =$ any fundamental domain of $\Gamma_0(N)$.

Newforms and oldforms: Roughly speaking oldforms are those which are automorphic forms for overgroups of $\Gamma_0(N)$ inside Γ. In fact the space $S_k(\Gamma_0(N)), N > 1$ splits into the orthogonal direct sum of two subspaces, oldforms and its orthocomplement in $S_K(\Gamma_0(N))$ which are newforms. Their definition is given inductively.

Now the supnorm problem (for newforms) on $\Gamma_0(N)$ acquires 2 different aspects,

(1) the level N varies with the weight k remaining fixed.
(2) hybrid i.e. the weight k and the level N both vary.

Results for cases 1 and 2.
Case 1. (Blomer and Halowinsky). $\parallel f \parallel_\infty \ll_k N^{-\frac{1}{37}}$ for N squarefree; f is L^2 normalised newform .
Case 2. $\parallel f \parallel_\infty \ll_\in (kN)^{\frac{1}{4}+\in}, \forall \in > 0$. This is the result of Y. Hu and A. Saha.

We now turn to the theory of oldforms and newforms in the case of Maass cusp forms. This is exactly the same as that of holomorphic form. Furthermore the Hecke operators $T(n)$ for $(n, N) =$ all commute with them Laplacian Δ_H and with each other. Thus we can find a $o.n$ basis of newforms for any fixed value of the eigengvalue λ of Δ_H which are all eigenfunctions of the $T(n)$ also.

We can now state the supnorm result for Maass cusp forms (newforms, L^2 normalised). Case(1) $\| f \|_\infty \ll_\lambda N^{\frac{1}{2}-\frac{1}{37}}$. This is due to Blomer-Holowinsky for squarefree level N.

Remark: In all of the results above the authors first prove the result for Maass cusp forms for $\Gamma_0(N)$ and then observe that the same holds for holomorphic cusp forms (newforms, L^2 normalised) without too much work.

More general automorphic forms. Here there is more, than one avenue of generalisation. In the more modern theory of automorphic forms the group $SL(2,\mathbb{R})$ is replaced by $GL(2,\mathbb{R})$.

We have $\mathbb{H} = \frac{G}{KZ}$ where now $G = (GL2,\mathbb{R})=$ all $2x2$ real invertible matrices by representing $\mathbb{H}$ as the homogenous space G/KZ where $G = GL(2,\mathbb{R})$ $K = 0(2,\mathbb{R}),\ Z = Z(G) =$ centre of $G = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, $\lambda \in \mathbb{R}^*$.

Note that $Z(G) \subset GL^+(2,\mathbb{R})$ = the group of $2x2$ real matrices having positive determinant. In this way, one can regard Maass forms as functions on the group $GL(2,\mathbb{R})$ which are invariant under the subgroup KZ of G.

There are Iwasawa coordinates on $\mathbb{H}_n$; each $z \in \mathbb{H}$ is uniquely of the form $z = x.y$ where (see the board)/

Let $\nu = (\nu_1, \nu_2, , \nu_{n-1}) \in \mathbb{C}^{n-1}$.

The analogue of the power function $p(y)(n = 2)$ is the function $I_\nu(z) = \Pi_{i=1}^{n-1}\Pi_{j=1}^{n-1} y_i^{b_{ij}}\nu_j$ with

$$b_{ij} \begin{cases} ij \ \ if \ \ i+j \leq n \\ (n-1)(n-j) \ if \ i+j \geq n \end{cases}$$

is an eigenfunction of every G invariant differential operator D on $\mathbb{H}_n$ Let us write

$$DI_\nu(Z) = \lambda_D \cdot I_\nu(z)$$

The map $D \mapsto \lambda_D$ is a character of $\mathcal{D}^n$ = commutative algebra of G invariant differential operators on $\mathbb{H}_n$.

Definition An $SL(n,\mathbb{Z})$ Maass cusp form on $\mathbb{H}_n$ is a C^∞ function $\phi \in L^2(SL(n,2)/\mathbb{H}_n)$ having the following properties

(1) ϕ is an eigenfunction of every $D \in \mathcal{D}$ with eigenhomorphism $D \mapsto \lambda_D$ for a suitable λ.
(2) ϕ is cuspidal .

Condition (2) implies that ϕ is a bounded, $SL(n,\mathbb{Z})$ invariant C^∞ function on $\mathbb{H}_n$.

Upper Bounds (Blomer-Harcos-Maga) Let ϕ be as above. Then we have , $\parallel \phi \parallel_{\infty} \ll \lambda^{\frac{(n^2-1)(n+1)}{16}+\epsilon}$

Lower bounds (Brumley-Templier). Here the spectral parameter of ϕ is very regular.

$\parallel \phi \parallel_{\infty} \gg \lambda^{\frac{c(n)}{2}-\epsilon}$ where $\frac{c(n)}{2} = \frac{n(n-1)(n-2)}{24}$.

Note that for $n \geq 6$, $\frac{c(n)}{2} > \frac{d-1}{4}$ where $d = \dim \mathbb{H}_n$. Thus Hormander's local bound (which is very general) doesn't hold globally for ϕ when $n \geq 6$.

Another avenue of generalisation.
Siegel modular forms

Recall that $\mathbb{H}$ = upper half-plane in $\mathbb{C} = SL(2,(R)/SO(2,\mathbb{R})$ but $SL(2,\mathbb{R}) = Sp(1,\mathbb{R}) =$ the symplectic group in dimension two. Therefore we define $\mathcal{H}_n$ by $Sp(n,\mathbb{R})/U(n)$ where $U(n) = nxn$ unitary group = maximal compact subgroup of $Sp(n,\mathbb{R})$ via its*real* embedding. We have $Sp(n,\mathbb{R}) = \{g \in M_{2n}(\mathbb{R}) \mid t_g J\ g = J\} \mid J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, I_n = nxn$ identity matrix.

Just like $Sp(1,\mathbb{R}), Sp(n,\mathbb{R})$ acts on the Siegel upper half-space $\mathcal{H}_n = \{Z \in S_n(\mathbb{C}) \mid Z = X + iY,\ Y > 0$ (positive definite). Here $S_n(\mathbb{C}) = n \times n$ complex symmetric matrices. The action is $(g,z) \mapsto g \cdot z = (AZ+B)(CZ+D)^{-1}$ where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$.

Let $\Gamma_n = Sp(n,\mathbb{Z})$ be the modular group of degree n.

Definition: Let $k \in \mathbb{N}$. A Siegel modular form F of degree n and weight k is a holomorphic function on $\mathcal{H}_n$ having the property

F((AZ+B)(CZ+D)$^{-1}$) $= det\ (CZ+D)^k F(z) \ldots (*)$
$\forall \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$

Fourier expansion of Siegel Modular forms.
$F(z) = \sum_{T \geq 0\ T\ \frac{1}{2}}$ integral $A_F(T)\ e^{2\pi i Tr(TZ)}\ Tr =$ trace

$T\ \frac{1}{2}$ integral means that $2T_{ij} \in \mathbb{Z}$ for $i \neq j$ and $t_{i,i} \in \mathbb{Z}$ for $1 \leq i \leq n$.

Definition: A Siegel modular form F is a cusp form if $A_F(T) \neq 0 \Rightarrow T > 0$.

Definition: Let F and G be a Siegel cusp forms of degree n. Their Petersson inner product is $< F, G > \triangleq \int_{F^n} (det\ Y)^k F(z)\overline{G(Z)} d\mu(z)$ where $d\mu(z) = Sp(n,\mathbb{R})$ invariant measure on $\mathcal{H}_n$ and F^n = any fundamental domain for Γ_n in $\mathcal{H}_n$

Hecke operators : $T(n)$ on Siegel modular forms are defined in a similar fashion with the set of integral matrices of determinant n being replaced by integral matrices which are symplectic similitudes with factor n.

Definition: The sup norm of $F, \parallel F \parallel_\infty$ is

$$sup_{z\in\mathcal{H}_n}\ (det\ Y)^{k/2} \mid F(z) \mid$$

Some sample results for $n = 2, k$ even ≥ 10,
Here there is a distinguished subspace, S_k^* of the space of S_k = Siegel cusp forms of weigh k which is Hecke stable. It is called the Maass subpsace and for F a Hecke eigenform in S_k^* with $\parallel F \parallel_\infty = 1$ we have $\forall\epsilon > 0$, .
Theorem (Blomer) $\parallel F \parallel_\infty \ll_\epsilon k^{\frac{3}{4}+\epsilon}$

This is conditional on the Generalized Lindeloff hypothesis for $L(\frac{1}{2}, f \times \chi_D)$ for all negative fundamental discriminants with F being the lift of $f \in S_{2k-2}(SL,(2,\mathbb{Z})$, a normalised Hecke eigenform.

Unconditionally we have $\parallel F \parallel_\infty \ll_\epsilon k^{\frac{5}{4}+\epsilon}$

Theorem (Das-Sengupta). A simplified proof of the above result with a slightly larger value of the exponent namely $\parallel F \parallel_\infty \ll_\in k^{\frac{17}{12}+\epsilon}$. Our proof uses the theory of Jacobi forms.

Das and his collaborators mainly P. Anamby and H. Krishna have studied the sup norm problem for Siegel cusp forms extensively. They use a 'new' tool namely the Bergman Kernel. We quote some of their results.

Let $\mathbb{B}_k^n$ denote an orthonormal basis of S_k^n. The Bergman Kernel $B_k(z, W))$ for $z, W \in \mathcal{H}^n$.. is $\sum_{G\in\mathbb{B}_k^n} G(z)G(W)$.

The quantity of interest here is

$$B_k(Z,Z) = \sum_{F\in\mathbb{B}_\neg{}^n} \mid F(Z) \mid^2 (det\ Y)^k = ((det\ Y)^k\mathbb{B}_k(Z,Z))$$

and by abuse of notation call this the Bergman Kernel as well. We measure the size of S_k^n by the quantity $sup_{z\in\mathcal{H}^n}\ \mathbb{B}_k(Z,Z)$

Conjecture: With the above notation and setting the following is true

$$k^{\frac{3n(n+1)}{4}} \ll \sup_{z\in\mathcal{H}^n}\ \mathbb{B}_k(Z,Z) \ll k^{\frac{3n(n+1)}{4}}$$

Theorem (Das-Krishna). Let $n \in \mathbb{N}$ be given. Put $l(n) = \frac{3n(n+1)}{4}$. Then with the above notation and setting we have

$$\left.\begin{matrix} k^{l(1)} \\ k^{l(2)} \\ k^{l(n)} \end{matrix}\right\} \overset{\ll}{n} \quad \sup_{z\in\mathcal{H}^n} \mathbb{B}_k(z,z) \overset{\ll}{n,\epsilon} \begin{cases} k^{l(1)} \\ k^{l(2)+\epsilon} \\ k^{\frac{5l(n)}{3}-\frac{3(n+1)}{4}+\epsilon}. \end{cases}$$

Epilogue.

Where does the future lie ?

All indications are that it lies in " The orbit method in the analysis of automorphic forms " created and developed by Paul Nelson and Akhshay Venkatesh.

In a nutshell this is microlocal analysis on coadjoint orbits of real reductive groups coupled with representation theory, which takes care of the needs of number theory. Microlocalisation of test vectors lying in various representation (spaces) has to be done, both for representation of real groups as well as for representations of p-adic groups.

Using this method, Assing and Toma have obtained the following result for Hecke-Maass newforms of powerful level.

Theorem (Assing-Toma) Let ϕ be an L^2 normalised Hecke-Maass newform of level $N = p^{4n}$, p a prime with Laplacian eigenvalue λ. If λ is sufficiently large, then we have

$$\| \phi \|_\infty \ll_{p,\epsilon} (\lambda N)^{\frac{5}{24}+\epsilon}$$

This is a **new, hybrid** result for powerful levels !

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LLARULL'S THEOREM ON ODD DIMENSIONAL MANIFOLDS AND SPECTRAL FLOW

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Keywords: Dirac operator, scalar curvature, spectral flow

Llarull's rigidity theorem [6] states that for a closed spin Riemannian manifold (M, g^{TM}) of dimension n such that the associated scalar curvature k^{TM} verifies that $k^{TM} \geq n(n-1)$, then any (non-strictly) area decreasing smooth map $f : M \to S^n(1)$ of nonzero degree is an isometry, where $S^n(1)$ is the standard unit n-sphere. In [3], we gave a direct proof of Llarull's theorem in odd dimensions by the spectral flow. In [3], we also proved the following spin-area convex extremality theorem in odd dimensions, which compares with [1] and [5] for even dimensional case.

Theorem 0.1 ([3]). *Let M be a closed spin manifold of odd dimension $2k-1 (k \geq 2)$ equipped with a Riemannian metric g^{TM}, and $X \subset \mathbb{R}^{2k}$ be a smooth strictly convex closed hypersurface equipped with the metric g_0 induced by the Euclidean metric in $\mathbb{R}^{2k}$. Suppose that there exists a $(1, \Lambda^2)$-contracting map $f : (M, g^{TM}) \to (X, g_0)$ of nonzero degree. Then, either there exists a point $x \in M$ where the scalar curvature $k^{TM}(x) < k^{TX}(f(x))$, or f is an isometry.*

In [7], Zhang proved that for an even dimensional noncompact complete spin Riemannian manifold (M, g^{TM}) and a smooth (non-strictly) area decreasing map $f : M \to S^{\dim M}(1)$ which is locally constant near infinity and of nonzero degree, if the associated scalar curvature k^{TM} verifies

$$k^{TM} \geq (\dim M)(\dim M - 1) \text{ on } \mathrm{Supp}(\mathrm{d}f), \tag{0.1}$$

then $\inf(k^{TM}) < 0$. When $\dim M$ is odd, Zhang [7] proved that $\inf(k^{TM}) < 0$ still holds if the inequality in (0.1) is strict, by using the standard trick of passing M to $M \times S^1$. In [4], we improved Zhang's result in the odd dimensional case so that one gets a complete answer to Gromov's question ([2]).

Theorem 0.2 ([4]). *Let (M, g^{TM}) be an odd dimensional ($\dim M \geq 3$) connected oriented noncompact complete spin Riemannian manifold. Let k^{TM} be the associated scalar curvature. Let $f : M \to S^{\dim M}(1)$ be a smooth area decreasing map which is locally constant near infinity and of nonzero degree. Suppose*

$$k^{TM} \geq (\dim M)(\dim M - 1) \quad \text{on} \quad \mathrm{Supp}(\mathrm{d}f), \tag{0.2}$$

then $\inf(k^{TM}) < 0$.

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SUPERCONNECTION AND ORBIFOLD CHERN CHARACTER

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Keywords: orbifold Chern character, proper étale groupoid, antiholomorphic superconnection

Let M be a closed complex manifold. Consider the d-bar operator

$$\Omega^{(0,0)}(M) \xrightarrow{\bar\partial} \Omega^{(0,1)}(M) \xrightarrow{\bar\partial} \Omega^{(0,2)}(M) \cdots .$$

Let E be a holomorphic vector bundle on M. Consider the generalized d-bar operator

$$\Omega^{(0,0)}(M,E) \xrightarrow{\bar\partial} \Omega^{(0,1)}(M,E) \xrightarrow{\bar\partial} \Omega^{(0,2)}(M,E) \cdots .$$

Let $H^k(M,E)$ be the k-th cohomology group of E.

The Riemann-Roch-Hirzebruch theorem computes the holomorphic Euler characteristic of M.

Theorem 0.1 (Riemann-Roch-Hirzebruch). *Let E be a holomorphic vector bundle on a closed complex manifold M.*

$$\chi(X,E) = \sum_i (-1)^i H^i(M,E) = \int_M \mathrm{ch}(E)\,\mathrm{Td}(M),$$

where $\mathrm{ch}(E)$ is the Chern character of E and $\mathrm{Td}(M)$ is the Todd class of M.

In complex geometry, Hirzebruch's Riemann-Roch theorem represents a substantial advance beyond the case of Riemann surfaces, the classical Riemann-Roch theorem. It is the generalization of Hirzebruch's result, dating back to the work of Borel and Serre in the 1950s, that led Grothendieck to his ingenious introduction of the K-theory and the Grothendieck-Riemann-Roch theorem, which are fundamental objects in the study of both differential geometry and algebraic geometry. Substantial developments appeared in the works of Baum, Fulton, MacPherson and in SGA 6 (led by Grothendieck). Beyond algebraic schemes, a notable advance is Toen's result on Riemann-Roch for algebraic stacks of Deligne-Mumford type. In the 1980s, the Grothendieck-Riemann-Roch theorem was successfully introduced into Arakelov geometry, which led to exciting progress in arithmetic geometry.

Unlike projective varieties in algebraic geometry, not all coherent sheaves on a general complex manifold have a resolution by holomorphic vector bundles. This key difference from algebraic geometry was for a long time an obstruction to a Grothendieck-Riemann-Roch theorem on general complex manifolds. A recent major breakthrough was obtained by Bismut, Shen, and Wei [BSW23] by integrating new ideas from derived geometry, antiholomorphic superconnections, and geometric analysis, in particular the hypoelliptic Laplacian.

Let $K_0(M)$ be the Grothendieck group of $\mathcal{O}_M$-coherent sheaves on M, and let $H^{\bullet,\bullet}_{\mathrm{BC}}(M,\mathbb{R})$ be the Bott-Chern Cohomology of M.

Theorem 0.2 (Bismut-Shen-Wei). *There is a (unique) Chern character map*

$$\mathrm{ch}_{\mathrm{BC}} : K_0(M) \to \oplus_p H^{p,p}_{\mathrm{BC}}(M, \mathbb{R})$$

satisfying the following property, for a holomorphic map $f : M \to N$,

$$\mathrm{Td}_{\mathrm{BC}}(TM)\,\mathrm{ch}_{\mathrm{BC}}(f_!\mathcal{E}) = f_*[\mathrm{Td}_{\mathrm{BC}}(TN)\,\mathrm{ch}_{\mathrm{BC}}(\mathcal{E})],$$

for any coherent sheaf $\mathcal{E}$ on M.

In this talk, we reported the recent attempt to generalize the above Grothendieck-Riemann-Roch theorem on complex manifolds to complex orbifolds.

We study a complex orbifold through its representation by a proper étale groupoid with an invariant complex structure. Such a groupoid is called a complex orbifold groupoid.

Definition 0.3. *Let $\mathcal{G}$ be a complex orbifold groupoid. A sheaf $\mathcal{F}$ of $\mathcal{O}_{\mathcal{G}}$-modules is called a* coherent *sheaf if it satisfies the following conditions.*

1. *$\mathcal{F}$ is finite type, i.e. for every $x \in G_0$ there exists an invariant neighborhood (U, G) of x and a $\mathcal{G}$-sheaf $\mathcal{M}$, a finite rank free sheaf on G_0, such that there exists a $\mathcal{G}|_U$-equivariant surjective map $\mathcal{M}|_U \twoheadrightarrow \mathcal{F}|_U$;*
2. *For every (U, G) and any $\mathcal{G}|_U$-equivariant map $\phi\colon \mathcal{M}|_U \to \mathcal{F}|_U$, the kernel of ϕ is also finite type.*

We denote the category of coherent sheaves on $\mathcal{G}$ by $\mathrm{coh}(\mathcal{G})$, and the derived category of coherent G-sheaves by $\mathrm{D}^b_{\mathrm{coh}}(\mathcal{G})$.

Inspired by [BD10], we generalize the approach [BSW23] to coherent sheaves on complex orbifolds via *antiholomorphic flat superconnections*. Let $B(\mathcal{G})$ be the dg-category of antiholomorphic flat superconnections on $\mathcal{G}$ and $\underline{B}(\mathcal{G})$ be the associated homotopy category. Given a complex orbifold X, we establish the following equivalence of dg-categories

$$D^b_{\mathrm{coh}}(X) \simeq \underline{B}(X). \tag{0.1}$$

Such an equivalence (0.1) allows us to introduce the *orbifold Chern character*, which is a group homomorphism

$$\mathrm{ch}_{\mathrm{BC}} : K(X) \to H^{(=)}_{\mathrm{BC}}(IX, \mathbb{C}), \tag{0.2}$$

from the K-group of coherent sheaves on X to the Bott-Chern cohomology $H^{(=)}_{\mathrm{BC}}(IX, \mathbb{C})$ of the inertia orbifold IX. We show that the orbifold Chern character has the following property.

Theorem 0.4. *Let $i_{X,Y}\colon X \hookrightarrow Y$ be an embedding of a compact complex orbifold groupoid. Let $\mathcal{F} \in D^b_{\mathrm{coh}}(X)$ and $i_{X,Y,*}\mathcal{F} \in D^b_{\mathrm{coh}}(Y)$ be its direct image. We have*

$$\mathrm{ch}_{\mathrm{BC}}(i_{X,Y,*}\mathcal{F}) = Ii_{X,Y,*}\left(\frac{\mathrm{ch}_{\mathrm{BC}}(\mathcal{F})}{\mathrm{Td}_{\mathrm{BC}}(N_{X/Y})}\right) \text{ in } H^{(=)}_{\mathrm{BC}}(IY, \mathbb{C}), \tag{0.3}$$

where $Ii_{X,Y}$ is the induced morphism between inertia groupoids.

With Theorem 0.4 we establish the following property about the orbifold Chern character:

Theorem 0.5. *The orbifold Chern character $\mathrm{ch}_{\mathrm{BC}}\colon K(X) \to H^{(=)}_{\mathrm{BC}}(IX, \mathbb{C})$ in (0.2) is the unique group homomorphism satisfying the following properties.*

($\star$1) *For complex vector bundles E on complex orbifolds, our definition of* $\mathrm{ch}_{\mathrm{BC}}(E)$ *agrees with the one in* [Ma05, Section 1.2].
($\star$2) $\mathrm{ch}_{\mathrm{BC}}$ *is functorial under pullbacks.*
($\star$3) $\mathrm{ch}_{\mathrm{BC}}$ *satisfies the Riemann-Roch-Grothendieck formula for orbifold embeddings, Equation (0.3).*

After our paper [MTTW25] appeared on the arXiv, we became aware of the paper [Xu25] by Guangzhe Xu. The paper [Xu25] establishes the main results of [BSW23] in the setting of equivariant geometry of a finite group acting on a complex manifold. On one hand, [Xu25] establishes our Theorems 0.4 and 0.5 in the (more restrictive) setting of equivariant geometry with respect to *finite* group actions. On the other hand, [Xu25] establishes Riemann-Roch-Grothendieck for proper morphisms between complex manifolds equivariant with respect to finite group actions, which is more general than what is available in this paper (our Theorem 0.4 is only valid for embeddings).

In literature, the Grothendieck-Riemann-Roch type results refer to transformations from K-theory to suitable cohomology theories that commute with pushforwards of proper morphisms.

For algebraic orbifolds, more precisely Deligne-Mumford stacks, a Riemann-Roch theorem was proved by Toen [Toe99].

We aim to establish a Riemann-Roch-Grothendieck theorem for complex orbifolds, which will calculate the orbifold Chern character $\mathrm{ch}_{\mathrm{BC}}(f_*\mathcal{F})$ of the pushforward of a coherent sheaf $\mathcal{F}$ under a holomorphic map f.

A holomorphic map

$$f\colon X \to Y$$

between complex orbifolds can be decomposed as the composition of the embedding

$$i_f\colon X \to X \times Y$$

and the projection

$$p\colon X \times Y \to Y.$$

Our plan to establish the Riemann-Roch-Grothendieck theorem for f is by proving the Riemann-Roch-Grothendieck theorems for i_f and p separately. In Theorem 0.4, we have solved the case of embeddings. In the sequel, we will prove the case that covers p and thus complete the proof of the Riemann-Roch-Grothendieck for f.

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LIE GROUPOID STRUCTURES ON DONALDSON MODULI SPACES

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1. Introduction

Donaldson moduli spaces [3] of instantons on four-manifolds are fundamental in differential geometry and gauge theory, but their structure is often complicated by quotient singularities and bubbling phenomena in the compactitications. We will provide Lie groupoid structures on Donaldson moduli spaces and their bubble tree compactifications, providing a smooth and categorical framework that captures their stratified and singular nature. This construction lays the groundwork for future applications, including the definition of K-theoretical Donaldson invariants, the geometric realization of μ-maps. Some applications will be discussed. This is based on joint work with Bohui Chen and Shuauge Qiao.

Gauge theory provides a profound link between the differential geometry of four-manifolds and topological invariants. Let $P \to X$ be a principal G-bundle over a closed, oriented Riemannian four-manifold (X, g), with $G = SU(2)$ or $SO(3)$. A connection A on P has curvature $F_A \in \Omega^2(X, \mathfrak{g}_P)$, and the Yang-Mills functional

$$YM(A) = \int_X |F_A|^2 \,\mathrm{dvol}_g$$

measures its total curvature energy. The Euler-Lagrange equation of YM is the Yang?Mills equation $d_A^* F_A = 0$, whose critical points are the Yang-Mills connections. In four dimensions, the Hodge decomposition of two-forms

$$\Omega^2(\mathfrak{g}_P) = \Omega^{2,+}(\mathfrak{g}_P) \oplus \Omega^{2,-}(\mathfrak{g}_P)$$

splits curvature into self-dual and anti-self-dual components $F_A = F_A^+ + F_A^-$. The energy identity

$$\|F_A\|_{L^2}^2 = \|F_A^+\|_{L^2}^2 + \|F_A^-\|_{L^2}^2 = 8\pi^2 k + 2\|F_A^+\|_{L^2}^2$$

(where $k = c_2(P)$ or $-p_1(P)/4$) shows that the absolute minima of the Yang?Mills functional are those connections satisfying

$$F_A^+ = 0, \qquad \text{equivalently} \quad * F_A = -F_A,$$

the *anti-self-dual (ASD) Yang-Mills equations*. Such solutions are called *instantons*, and their moduli spaces

$$\mathcal{M}_k(X, g) = \{A \in \mathcal{A}(P) \mid F_A^+ = 0\}/\mathcal{G}(P)$$

are finite?dimensional quotients of the infinite-dimensional affine space $\mathcal{A}(P)$ of connections by the gauge group $\mathcal{G}(P)$. For a generic Riemannian metric g, the linearized operator $d_A^+ : \Omega^1(\mathfrak{g}_P) \to \Omega^{2,+}(\mathfrak{g}_P)$ is surjective, and $\mathcal{M}_k(X, g)$ is a smooth manifold of dimension

$$\dim \mathcal{M}_k(X, g) = -2p_1(P) - 3(1 - b_1(X) + b_2^+(X)) = 8k - 3(1 - b_1 + b_2^+)$$

given by the index of the Atiyah-Hitchin-Singer deformation complex

$$0 \longrightarrow \Omega^0(\mathfrak{g}_P) \xrightarrow{d_A} \Omega^1(\mathfrak{g}_P) \xrightarrow{d_A^+} \Omega^{2,+}(\mathfrak{g}_P) \longrightarrow 0.$$

Points of $\mathcal{M}_k(X, g)$ correspond to gauge-equivalence classes of irreducible ASD connections, while reducible ones (with stabilizer $U(1)$) form lower-dimensional singular strata. The compactness problem for $\mathcal{M}_k$ is governed by bubbling phenomena: sequences of instantons with bounded energy may develop curvature concentration at finitely many points, where the energy lost is carried by instantons on S^4. The *Uhlenbeck compactification* $\overline{\mathcal{M}}_k^U(X)$ augments the moduli space by such ideal connections, but its corner structure is only stratified topologically.

For a smooth, closed, oriented 4-manifold (X, g), the Donaldson invariants are defined via the moduli spaces $\mathcal{M}_k(X, g)$ of anti-self-dual (ASD) $SU(2)$ or $SO(3)$ connections on a principal bundle $P \to X$. When $b_2^+(X) > 1$, these invariants are independent of the metric. The case $b_2^+(X) = 1$ is subtler: the invariants depend on the chamber structure of the positive cone in $H^2(X; \mathbb{R})$.

Kotschick and Morgan [5] conjectured that for $b_2^+(X) = 1$ and $b_1(X) = 0$, the wall-crossing difference

$$\delta_P(\alpha) = D_X^+(P) - D_X^-(P)$$

arising when the self-dual harmonic form crosses the hyperplane $\alpha^\perp$ is a polynomial in α and the intersection form Q_X, with coefficients depending only on α^2, $p_1(P)$, and the homotopy type of X.

Earlier approaches, including algebraic-geometric calculations for rational surfaces [4] and topological computations [6] for partial cases. Our work establishes the conjecture for all walls, including the obstructed case $\alpha^2 = -1$, by constructing a smooth compactification of the moduli space with an explicit local model around reducible and bubbling configurations.

In our work we employ the stronger *bubble tree compactification* of Chen [2], which records not only the positions of bubbling points but also the entire hierarchical configuration of bubbles and the gluing parameters between them. Each stratum is indexed by a weighted rooted tree T, whose vertices represent the base manifold and successive S^4 bubbles, and whose edges record gluing scales λ_e and group elements $\rho_e \in G$. This yields a smooth orbifold structure near all lower strata and provides a precise analytic framework for the wall-crossing analysis required to prove the Kotschick-Morgan conjecture.

2. Bubble Tree Compactification and Lie groupoid structure

The key analytic tool is the *bubble tree compactification* of the instanton moduli space $\mathcal{M}_k(X)$ introduced by Chen [2]. Unlike the Uhlenbeck compactification, which records only bubbling points, the bubble tree compactification keeps track of the entire hierarchy of bubbled spheres (S^4) and the gluing data between them.

For each weighted rooted tree T with total weight k, the associated stratum

$$S_T(X) = \mathcal{M}_{w(v_0)}(X) \times \Big(\prod_{v_i \in \mathrm{child}(v_0)} \mathcal{P}_{v_i}(X)\Big) \Big/ S_{m_{v_0}}$$

parametrizes configurations where bubbling occurs according to T. The Taubes gluing construction produces an orbibundle

$$\mathcal{GL}_T(X) \to S_T(X),$$

whose fibre encodes the gluing parameters $\rho_e \in G$ and scales $\lambda_e > 0$ for each edge e of T.

A smooth orbifold structure on $\overline{\mathcal{M}}_k(X)$ is then obtained by gluing these local models via perturbed gluing maps Ψ_T^b, compatible across overlapping strata. Ghost vertices (zero-energy bubbles) are resolved by a *flip resolution* procedure that replaces singular corners by exceptional divisors, ensuring smoothness throughout the compactification.

The orbifold structure on $\overline{\mathcal{M}}_k(X)$ admits a natural and more precise description in terms of a *Lie groupoid*. This point of view clarifies the analytic gluing construction and provides a canonical language for encoding gauge symmetries and local isotropy.

Let $\mathcal{A}(P)$ denote the Fréchet manifold of smooth connections on the principal G?bundle $P \to X$, and $\mathcal{G}(P)$ the corresponding gauge group. The classical configuration groupoid

$$\mathcal{C} = (\mathcal{G}(P) \times \mathcal{A}(P) \rightrightarrows \mathcal{A}(P))$$

has source and target maps $s(g, A) = A$ and $t(g, A) = g \cdot A$, with composition $(h, g \cdot A) \circ (g, A) = (hg, A)$. Its orbits are the gauge equivalence classes of connections. Restricting $\mathcal{C}$ to the submanifold of anti-self-dual connections $\mathcal{A}_{\mathrm{ASD}} \subset \mathcal{A}(P)$ yields the *ASD groupoid*

$$\mathcal{G}_{\mathrm{ASD}} = (\mathcal{G}(P) \times \mathcal{A}_{\mathrm{ASD}} \rightrightarrows \mathcal{A}_{\mathrm{ASD}}),$$

whose orbit space is precisely the moduli space $\mathcal{M}_k(X, g)$. Each object $A \in \mathcal{A}_{\mathrm{ASD}}$ has isotropy group $\Gamma_A \subset G$ equal to its stabilizer under the gauge action; for irreducible A, $\Gamma_A = \{\pm 1\}$, while for reducible connections $\Gamma_A \simeq S^1$.

The local slice theorem for the gauge action implies that $\mathcal{G}_{\mathrm{ASD}}$ is a smooth, étale Lie groupoid: near any $[A] \in \mathcal{M}_k(X, g)$ there exists a local slice $U_A \subset \Omega^1(\mathfrak{g}_P)$ on which $\mathcal{G}(P)$ acts smoothly with finite isotropy, and the quotient U_A/Γ_A provides an orbifold chart. Transition functions between such slices are encoded by the morphisms of the groupoid and are smooth on overlaps, giving $\mathcal{M}_k(X, g)$ a natural differentiable stack structure.

The bubble tree compactification $\overline{\mathcal{M}}_k(X)$ inherits a compatible groupoid description: for each weighted tree T, the local gluing model $\mathcal{GL}_T(X) \to S_T(X)$ is endowed with a groupoid

$$\mathcal{G}_T \rightrightarrows S_T(X),$$

whose arrows correspond to gauge transformations on the bubble components and whose isotropy groups Γ_{A_T} record the residual symmetries of the glued configurations. The compatibility of the gluing maps Ψ_T across adjacent trees extends to a morphism of Lie groupoids $\Psi_T : \mathcal{G}_T \to \mathcal{G}_{\mathrm{ASD}}$, and the collection $\{\mathcal{G}_T\}$ assembles into a global *groupoid atlas*

$$\mathfrak{G}_k(X) = \bigcup_T \mathcal{G}_T \rightrightarrows \bigsqcup_T S_T(X),$$

whose differentiable stack quotient $[\mathfrak{G}_k(X)]$ defines the smooth orbifold structure on $\overline{\mathcal{M}}_k(X)$. The isotropy groups of $\mathfrak{G}_k(X)$ describe the local symmetry type of each boundary point (trivial for irreducible bubbles, S^1 for reducibles, and higher tori for multi-bubble collisions).

3. Reducible Instantons and Equivariant Localization

When $b_2^+ = 1$, wall-crossing arises from the change in orientation of $\mathcal{M}_k(X, g)$ as the metric crosses a wall W_α. The contribution is concentrated near reducible instantons corresponding to line bundles $L \to X$ with $c_1(L) = \alpha$. The neighbourhood of a reducible solution A is described by a Kuranishi model

$$\phi: \ H_A^1 \to H_A^2, \qquad \mathcal{M}_k(X, g) \simeq \phi^{-1}(0)/\Gamma_A,$$

where $\Gamma_A \simeq S^1$ acts by complex multiplication on $H_A^1 \simeq \mathbb{C}^N$.

For $\alpha^2 < -1$, the moduli space is smooth near A; when $\alpha^2 = -1$, obstructions appear and are resolved by passing to a *thickened moduli space* $\mathcal{M}_A^{\text{thicken}}$. This yields a global virtual cycle amenable to equivariant localization.

Equivariant de Rham theory [1] provides a natural framework: the relevant S^1-action induces an equivariant differential d_{S^1}, and the localization formula expresses integrals over $\partial(\mathcal{M}_k(X, \lambda))$ in terms of fixed-point data on the reducible loci. For a local model $U_A \simeq \mathbb{C}^N$, one obtains

$$\int_{\partial U_A/S^1} (2\mu(\Sigma))^{N-1} = (-2\pi)^{N-1} \tfrac{1}{2\pi} \langle c_1(L), [\Sigma] \rangle^{N-1},$$

reproducing the wall-crossing coefficient predicted by Kotschick-Morgan.

4. Wall-Crossing Formula and Main Theorem

Let X be a simply connected 4-manifold with $b_2^+ = 1$, $P \to X$ an $SO(3)$-bundle, and $\alpha \in H^2(X; \mathbb{Z})$ an integral lift of $w_2(P)$ satisfying $p_1(P) \leq \alpha^2 < 0$. Denote by $\delta_P(\alpha)$ the difference of Donaldson invariants across the wall W_α. We prove:

Theorem 0.1. *For all α as above, including the obstructed case $\alpha^2 = -1$, the wall-crossing term is a universal polynomial*

$$\delta_P(\alpha) = \sum_{i=0}^{r} a_i(r, d, X)\, Q_X^{r-i}\, \alpha^{d-2r-2i}, \qquad r = \frac{\alpha^2 - p_1(P)}{4},$$

where Q_X is the intersection form and the coefficients $a_i(r, d, X)$ depend only on r, d, and the homotopy type of X.

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QUASI-REPRESENTATIONS AND A K-THEORETIC INVARIANT

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Voiculescu's classical example [6] of almost commuting unitary matrices that cannot be approximated by exactly commuting ones revealed a fundamental obstruction in the stability theory of operator algebras. This phenomenon is naturally formulated in terms of finite-dimensional *quasi-representations*: maps from a group (or a C^*-algebra) into matrices that satisfy multiplicativity only up to small error.

A central problem is to determine when such quasi-representations can be perturbed (in the operator norm) to genuine representations, and when this fails due to topological obstructions. Earlier work connects quasi-representations of $\pi_1(M)$ to almost-flat vector bundles over M and thus to K-theory; however, these invariants are often not fine enough to detect all quasi-representations.

In joint work with Weinberger and Yu, we introduce a refined invariant—a *character map*—taking values in equivariant K-theory of the universal proper G-space. For amenable groups, this invariant completely classifies quasi-representations up to stable equivalence.

Quasi-representations and Stability. Let G be a discrete group and $F \subseteq G$ finite.

Definition 0.1. *An (F, ε)-representation is a map*

$$\rho : F^2 \to U(n)$$

such that

$$\|\rho(ab) - \rho(a)\rho(b)\| < \varepsilon \qquad \text{for any } a, b \in F.$$

A group is said to be *matricially stable* if approximate multiplicativity on a sufficiently large finite set forces ρ to be close to a genuine unitary representation.

Examples include finite groups, $\mathbb{Z}$ and free groups, but Voiculescu's example shows $\mathbb{Z}^2$ is not matricially stable. More precisely, for each $n > 1$, consider $n \times n$ unitary matrices

$$u_n = \begin{pmatrix} 1 & & & & \\ & e^{\frac{1}{n}2\pi i} & & & \\ & & e^{\frac{2}{n}2\pi i} & & \\ & & & \ddots & \\ & & & & e^{\frac{n-1}{n}2\pi i} \end{pmatrix} \text{ and } v_n = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{pmatrix}$$

Since they approximately commute when n is sufficiently large, they induce quasi-representations ρ_n of $\mathbb{Z}^2$ by sending the two canonical generators to u_n and v_n, respectively, but the fact that these unitary matrices cannot be perturbed to commuting ones (a particularly slick proof of this fact is given in [3]) amounts to saying that for large n, the quasi-representations ρ_n's are not close to genuine representations. A bit more generally, for finitely generated abelian groups, matricial stability holds precisely when the group is virtually $\mathbb{Z}$.

Almost-flat Bundles and a Character Map. Given a compact manifold M with $\pi_1(M) = G$, previous work [1, 5] shows that quasi-representations correspond asymptotically to *almost-flat* vector bundles [2], i.e., Hermitian vector bundles equipped with connections whose curvature can be made arbitrarily small. Given a quasi-representation ρ, such a bundle then determines a class

$$[E_\rho] \in K^0(M),$$

which gives a topological invariant that can be used to obstruct the perturbation of quasi-representations into genuine ones.

Since this invariant is K-theoretic in nature, it only distinguishes quasi-representations up to *stable equivalence*—here one can show that two natural equivalence relations for quasi-representations, namely stable homotopy and stable approximate unitary equivalence, coincide. Hence we form the Grothendieck group $\mathrm{QR}(G)$ consisting of formal differences of stable equivalence classes of quasi-representations. More precisely, we actually form the group $\mathrm{QR}(F,\varepsilon)$ for each fixed pair (F,ε) and then take an inverse limit. With this Grothendieck construction, we see that the above prescription $\rho \mapsto [E_\rho] \in K^0(M)$ produces a group homomorphism

$$\chi : \mathrm{QR}(G) \to K^0(M)$$

that we call the *character map*.

Combined with the Chern character, this reproduces classical obstructions: for example, a flat bundle has vanishing higher Chern classes, so if ρ were close to a genuine representation, $\mathrm{ch}(\chi(\rho))$ would lie in $H^0(M)$.

However, this map into $K^0(M)$ is generally far from injective or surjective. Even if we take a limit over M and obtain a more canonical map

$$\chi : \mathrm{QR}(G) \to K^0(BG)$$

replacing M with the classifying space BG (if no finite model for BG exists, we need to take an inverse limit to define $K^0(BG)$), we still cannot expect χ to be a bijection, particularly when torsion is present, even in simple examples such as $G = \mathbb{Z}/2$. This motivates a refinement of the character map χ.

A Refined Target: Equivariant K-theory of $\underline{E}G$. The universal free G-space EG yields the classifying space BG. However, taking a lesson from the formulation of the Baum-Connes conjecture, we argue that topological information relevant to quasi-representations naturally lives in the *universal proper G-space $\underline{E}G$*, characterized by being H-equivariantly contractible for every finite subgroup $H \leq G$.

We consider the equivariant K-theory

$$K^0_G(\underline{E}G) \;=\; \varinjlim_{Y} K^0_G(Y),$$

where Y ranges over proper cocompact G-spaces, and where $K^0_G(Y) \cong K_0(C_0(Y) \rtimes G)$ via the Green–Julg theorem. There is a canonical map $K^0_G(\underline{E}G) \to K^0(BG)$; this is an isomorphism when G is torsion-free, but only a rational surjection in general.

For each proper cocompact G-space Y and sufficiently good (F, ε), we construct a homomorphism $\chi : \mathrm{QR}(F, \varepsilon) \to K_G^0(Y)$. Parallel to what was done above, we take an inverse limit of the right-hand side over Y and obtain a homomorphism

$$\chi : \mathrm{QR}(G) \longrightarrow K_G^0(\underline{E}G).$$

There is a relatively simple construction of this refined character map χ, with the help of Fell's absorption principle. Composing this χ with the canonical map $K_G^0(\underline{E}G) \to K^0(BG)$ recovers the classical character map discussed above, but our adapted character map χ remembers more information: we show that it completely classifies quasi-representations up to stable equivalence.

Theorem 0.2 (Weinberger–Wu–Yu). *If G is amenable, then the character map*

$$\chi : \mathrm{QR}(G) \longrightarrow K_G^0(\underline{E}G)$$

is an isomorphism.

The proof, inspired by the Baum-Connes conjecture, uses the Dirac–dual-Dirac method [4], viewing χ as a type of coassembly map. Quasidiagonality of $C^*(G)$ also plays an important role by allowing finite-dimensional approximation of Fredholm quasi-representations.

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DEGENERATIONS OF ALGEBRAIC MANIFOLDS AND ANALYTIC TORSIONS OF NAKANO SEMI-POSITIVE VECTOR BUNDLES

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1. Analytic torsion

Let M be a compact Kähler manifold. Let F be a holomorphic Hermitian vector bundle on M. Let $A^{0,q}(M,F)$ be the vector space of F-valued smooth $(0,q)$-forms on M, which is endowed with the L^2 metric with respect to the metrics on M and F. Let $\square_F^{0,q}$ be the Hodge-Kodaira Laplacian acting on $A^{0,q}(M,F)$. We denote by $\sigma(\square_F^{0,q})$ the eigenvalues of $\square_F^{0,q}$. Let $E(\lambda;\square_F^{0,q})$ be the eigenspace of $\square_F^{0,q}$ with eigenvalue $\lambda\in\sigma(\square_F^{0,q})$. Define

$$Z(s) := \sum_{q\geq 0}(-1)^q q \sum_{\lambda\in\sigma(\square_F^{0,q})\setminus\{0\}} \lambda^{-s}\,\dim E(\lambda;\square_F^{0,q}),\qquad s\in\mathbf{C}.$$

It is classical that $Z(s)$ converges when $\operatorname{Re} s>\dim M$, extends to a meromorphic function on $\mathbf{C}$ and is holomorphic at $s=0$.

Definition 1.1. *([12], [3]) The analytic torsion of (M,F) is the real number defined as*

$$\tau(M,F) := \exp(-Z'(0)).$$

For the basic properties of analytic torsion, we refer to [3], [4]. Analytic torsion plays a crucial role in several areas in mathematics, such as Arakelov geometry [13], mirror symmetry at genus one [2], [5], [7], [19], [17], [11]. In this note, we report a recent progress on the boundary behavior of analytic torsion.

2. Singularity of analytic torsion

Let X be a connected complex manifold of dimension $n+1$ and let $S\subset\mathbf{C}$ be the unit disc. Let $\pi\colon X\to S$ be a surjective holomorphic map with connected fibers. Let Σ be the critical locus of π. Assume that $\pi(\Sigma)=\{0\}$, that there is an ample line bundle on X, and that X is an open subset of a projective manifold. We set $S^o=S\setminus\{0\}$. Then $\pi\colon\pi^{-1}(S^o)\to S^o$ is a family of projective algebraic manifolds.

Let h_X be a Kähler metric on X. Let $K_{X/S}$ be the relative canonical bundle of π. Let $\xi\to X$ be a holomorphic vector bundle on X endowed with a Hermitian metric h_ξ. We assume that ξ extends to a holomorphic vector bundle on the projective manifold containing X as an open subset. We define $K_{X/S}(\xi)=K_{X/S}\otimes\xi$. For $s\in S$, we set $X_s:=\pi^{-1}(s)$ and $\xi_s=\xi|_{X_s}$. Throughout this note, we make the following:

Assumption $(\xi,h_\xi)|_X$ is Nakano semi-positive. Namely, if R_ξ denotes the curvature form of (ξ,h_ξ), then $\sqrt{-1}R_\xi$ induces a semi-positive Hermitian form on $TX\otimes\xi$.

Under this assumption, $\dim H^q(X_s, K_{X_s}(\xi_s))$ is independent of $s \in S$ ([14], [9]). We set $h^q := \dim H^q(X_s, K_{X_s}(\xi_s))$. Then $R^q\pi_* K_{X/S}(\xi)$ is a locally free sheaf on S of rank h^q for all $q \geq 0$. By [8], the family $\pi\colon X \to S$ admits a semi-stable reduction, whose base space is a ramified covering of S with ramification index $d \in \mathbf{N}$ (cf. Section 3).

Theorem 2.1. ([20]) *For $s \in S^o$, let $\tau(X_s, K_{X_s}(\xi_s))$ be the analytic torsion of $(X_s, K_{X_s}(\xi_s))$ with respect to $h_X|_{X_s}$ and $h_\xi|_{X_s}$, where K_{X_s} is the canonical bundle of X_s. Then*

$$\log \tau(X_s, K_{X_s}(\xi_s)) = \kappa \log |s|^2 + c + \sum_{0\leq m\leq n} \sum_{i\in I} |s|^{2r_i} (\log |s|^{-2})^m \phi_{i,m}(s)$$
$$- \sum_{q\geq 0} (-1)^q \log \left(\sum_{0\leq k\leq nh^q} \{c_k^q + \sum_{1\leq j\leq d} |s|^{\frac{2j}{d}} \psi^q_{j,k}(s)\} (\log |s|^{-2})^k \right),$$

where $\kappa \in \mathbf{Q}$, $\{r_i\}_{i\in I} \subset \mathbf{Q} \cap (0,1]$ is a finite set of positive rational numbers, $c \in \mathbf{R}$, $(c_0^q, \ldots, c^q_{nr^q}) \neq (0, \ldots, 0)$ is a non-zero real vector, and $\phi_{i,m}(s)$, $\psi^q_{j,k}(s)$ are smooth functions on S. In particular, by setting $\varrho := \sum_{q\geq 0}(-1)^q \varrho^q \in \mathbf{Z}$ with $\varrho^q := \max\{0 \leq k \leq nh^q;\ c_k^q \neq 0\}$, there exists a constant $\gamma \in \mathbf{R}$ such that as $s \to 0$

$$\log \tau(X_s, K_{X_s}(\xi_s)) = \kappa \log |s|^2 - \varrho \log \log(|s|^{-2}) + \gamma + O\left(1/\log |s|^{-1}\right).$$

In this theorem, κ is given by an integral of certain characteristic classes associated to the semi-stable reduction of $\pi\colon X \to S$. Since the formula for κ is complicated, we omit the detail here. See [20, Sects. 6, 8]. However, when X_0 has only isolated singularities, there is a simple formula for κ in terms of Milnor number and spectral genus. Let us recall these invariants to give an explicit formula for κ when $\dim \operatorname{Sing} X_0 = 0$.

We identify an isolated hypersurface singularity germ $(X_0, 0) \subset (\mathbf{C}^{n+1}, 0)$ with its defining equation $f(z) = 0$, where $f(z) \in \mathcal{O}_{\mathbf{C}^{n+1},0}$ has an isolated critical point at the origin. The *Milnor number* of f, denoted by $\mu(f)$, is defined as

$$\mu(f) := \dim \mathcal{O}_{\mathbf{C}^{n+1},0} / (\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}) \mathcal{O}_{\mathbf{C}^{n+1},0}.$$

We need another invariant of f called the *spectral genus*, introduced recently by Eriksson-Freixas i Montplet [6]. Let Mil_f be the Milnor fiber of f. Then $H^n(\operatorname{Mil}_f)$ carries a mixed Hodge structure. Let $F^\bullet H^n(\operatorname{Mil}_f)$ be the Hodge filtration on $H^n(\operatorname{Mil}_f)$. Let $M_s \in \operatorname{GL}(H^n(\operatorname{Mil}_f))$ be the semi-simple part of the monodromy acting on $H^n(\operatorname{Mil}_f)$. Let $\log z$ be the branch of the logarithm with imaginary part lying in $[0, 2\pi)$. Let $\log M_s$ be the corresponding logarithm of M_s. Since M_s preserves $F^\bullet H^n(\operatorname{Mil}_f)$, $\log M_s$ acts on $\operatorname{Gr}^n_F H^n(\operatorname{Mil}_f)$. By [6], the spectral genus of f is the rational number defined as

$$\widetilde{p}_g(f) := \frac{1}{2\pi i} \operatorname{Tr}\left[\log M_s|_{\operatorname{Gr}^n_F H^n(\operatorname{Mil}_f)}\right].$$

Theorem 2.2. ([20]) *Suppose that $\operatorname{Sing} X_0$ consists of isolated points. Let $r(\xi)$ be the rank of ξ. Then*

$$\kappa = -r(\xi) \sum_{x\in \operatorname{Sing} X_0} \left(\frac{\mu(x)}{(n+2)!} - \widetilde{p}_g(x) \right).$$

In [6], it is conjectured that $\frac{\mu(x)}{(n+2)!} - \widetilde{p}_g(x) > 0$ for any isolated hypersurface singularity of dimension n.

3. Singularity of the L^2-metric

Since the analytic torsion is the ratio of the Quillen metric and the L^2-metric on the determinant of the cohomology [3], Theorem 2.1 is reduced to the behavior of the Quillen and L^2 metrics as $s \to 0$. For the Quillen metrics, this was determined in [18]. For the L^2-metrics, it will be determined in [20], which we explain briefly here.

Let T be another unit disc in $\mathbf{C}$. Set $T^o := T \setminus \{0\}$. By the semi-stable reduction theorem [8], there is a commutative diagram

$$\begin{array}{ccc} (Y, Y_0) & \xrightarrow{F} & (X, X_0) \\ {\scriptstyle f}\downarrow & & {\scriptstyle \pi}\downarrow \\ (T, 0) & \xrightarrow{\mu} & (S, 0). \end{array}$$

Here Y is a complex manifold of dimension $n+1$, $\mu\colon (T,0) \to (S,0)$ is given by $\mu(t) = t^d$, $Y_t := f^{-1}(t)$ is isomorphic to $X_{\mu(t)}$ for $t \neq 0$, and $Y_0 = f^{-1}(0)$ is a *reduced* normal crossing divisor of Y. Since $(F^*\xi, F^*h_\xi)$ is Nakano semi-positive, $R^q f_* K_Y(F^*\xi)$ is a locally free sheaf on T of rank h^q.

Theorem 3.1. ([20]) *Let $\mathfrak{m}_0^\infty(T)$ be the smooth functions on T vanishing at $t = 0$. By choosing suitable bases $\{\theta_1, \ldots, \theta_{h^q}\}$ of $R^q\pi_* K_{X/S}(\xi)$ and $\{\widetilde{\theta}_1, \ldots, \widetilde{\theta}_{h^q}\}$ of $R^q f_* K_{Y/T}(F^*\xi)$ respectively, there exist integers $e_1^q, \ldots, e_{h^q}^q \geq 0$ with the following properties:*

(1) *The $h^q \times h^q$-Hermitian matrix $H(s) := (H_{\alpha\beta}(s))$, $H_{\alpha\beta}(s) := (\theta_\alpha|_{X_s}, \theta_\beta|_{X_s})_{L^2}$ is expressed as follows:*

$$H(\mu(t)) = D(t) \cdot \widetilde{H}(t) \cdot \overline{D(t)}, \qquad D(t) = \mathrm{diag}(t^{-e_1^q}, \ldots, t^{-e_{h^q}^q}),$$

where $\widetilde{H}(t) = (\widetilde{H}_{\alpha\bar{\beta}}(t))$, $\widetilde{H}_{\alpha\beta}(s) := (\widetilde{\theta}_\alpha|_{Y_t}, \widetilde{\theta}_\beta|_{Y_t})_{L^2}$, admits an expression

$$\widetilde{H}(t) \equiv \sum_{0 \leq m \leq n} (\log|t|^{-2})^m A_m \mod \bigoplus_{0 \leq k \leq n} (\log|t|^{-2})^k \mathfrak{m}_0^\infty(T) \otimes M_{h^q}(\mathbf{C})$$

with some constant Hermitian $h^q \times h^q$-matrices A_m $(1 \leq m \leq n)$. In particular, there exist $c_m^q \in \mathbf{R}$ $(1 \leq m \leq nh^q)$ such that

$$\det \widetilde{H}(t) = \sum_{0 \leq m \leq nh^q} c_m^q (\log|t|^{-2})^m \mod \bigoplus_{0 \leq k \leq nh^q} (\log|t|^{-2})^k \mathfrak{m}_0^\infty(T).$$

(2) *There exists a constant $C > 0$ such that $\widetilde{H}(t) \geq C\, I_{h^q}$ for all $t \in T^o$ as positive definite Hermitian matrices. In particular, $c_m^q \neq 0$ for some $1 \leq m \leq nh^q$.*

(3) *Set $\delta^q = \sum_{1 \leq \alpha \leq h^q} e_\alpha^q/d$. Then there exist real-valued smooth functions $\psi_{j,k}^q(s)$ on S such that*

$$\|\theta_1 \wedge \ldots \wedge \theta_{h^q}(s)\|_{L^2}^2 = |s|^{-2\delta^q} \sum_{0 \leq m \leq nh^q} \{c_m^q + \sum_{1 \leq j \leq d} |s|^{\frac{2j}{d}} \psi_{j,m}^q(s)\} (\log|s|^{-2})^m.$$

In particular, setting $\varrho^q := \max\{0 \leq m \leq nh^q;\ c_m^q \neq 0\}$, as $s \to 0$, one has

$$\log\|\theta_1 \wedge \ldots \wedge \theta_{h^q}(s)\|_{L^2}^2 = -\delta^q \log|s|^2 + \varrho^q \log\log(|s|^{-2}) + c^q + O\left(1/\log|s|^{-1}\right).$$

Moreover, if $\varrho^q = 0$, then $c^q \neq 0$.

(4) *The rational numbers e_α^q/d $(1 \leq \alpha \leq h^q)$ are independent of the choice of semi-stable reduction of $\pi\colon X \to S$.*

The proof of Theorem 3.1 relies on the theory of harmonic integrals for Nakano semi-positive vector bundles on open Kähler manifolds [14], the existence of an asymptotic expansion of the fiber integral of a differential form [1], [15], [16] and the non-degeneracy of the L^2-metric on the higher direct image sheaves of $K_{Y/T}(F^*\xi)$ [10].

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