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EXT-GROUPS OF REPRESENTATIONS OF REDUCTIVE p-ADIC GROUPS

KEI YUEN CHAN

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The classical Langlands program focuses on the classification of irreducible smooth representations over $\mathbb{C}$ in terms of arithmetic data. A modern perspective on this subject is the categorical Langlands program, aiming at certain 'Langlands duals' to describe the whole category. On the other hand, the Ext-group is one basic invariant in describing a non-semisimple category. The first part of the talk investigates how Ext-groups can play a role in the categorical Langlands program. The second part of the talk discusses the possibilities on putting the reflection group of H_4 in the framework of Langlands program, emphasizing on homological viewpoint.

As an example of the interplay between Ext-groups and the categorical Langlands, we consider the classical cohomological duality: Let π be a smooth representation lying in certain Bernstein component. One defines a contravariant functor [4]

$$\pi \mapsto \mathrm{Ext}^d_G(\pi, C^\infty(G)),$$

where π lies in certain Bernstein component, d is the cohomological degree of the Bernstein component and C^∞ is the space of smooth functions of G. This functor is extensively studied in the literature. In particular, it is shown to send irreducible representations to irreducible representations [1], and its Ext-duality theory is explored in [29], [8] and [24]. It is expected to correspond to variants of Grothendieck-Serre duals in the spectral side [15, 16].

A first important result on Ext-groups for representation theory of p-adic groups is the higher Ext-vanishing results between discrete series. A reformulation in the language of derived category of the complex smooth representation category $\mathrm{Rep}(G)$ of G is as follows:

Theorem 0.1. *(*[23, 25, 8]*) Let $\mathrm{Rep}_{ds}(G)$ be the full Serre subcategory of $\mathrm{Rep}(G)$ precisely consisting of representations of finite length whose simple composition factors are discrete series. Then $\mathrm{Rep}_{ds}(G)$ is a semisimple subcategory of $\mathrm{Rep}(G)$. Moreover, the natural embedding $D^b(\mathrm{Rep}_{ds}(G)) \hookrightarrow D^b(\mathrm{Rep}(G))$ is fully-faithful.*

We briefly explain the methods of proving the theorem. For [23, 25], the approach is to consider the Schwartz algebra for p-adic groups or affine Hecke algebras. One shows that discrete series and tempered modules can be equiped with an action of Schwartz algebra, and discrete series are projective objects in the category of representations of Schwartz algebra. Now, [23, 25] show a comparison theorem to transfer the vanishing theorem from Schwartz algebras to affine Hecke algebras. For the algebraic approach

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in [7, 8], the proof is inductive in nature utilizing the tools of parabolic induction and Jacquet functor, and in particular, there is an inductive description of discrete series in [7].

Beyond discrete series, one considers extensions of tempered representations. Classically, tempered representations can be parametrized by R-groups of some unitarily parabolically induced modules. Connections to Euler-Poincaré pairing is hinted in [3, 27], and the work [26] establishes a spectral description of Ext-groups between tempered modules in terms of R-groups. The study on extensions of standard representations can be partly reduced to extensions of tempered representations [9].

We now turn to applications of Ext-groups. We investigate the possibility of developing a Langlands theory for the Coxeter group W of type H_4. Although there is no algebraic group of H_4, several characterizations in the Langlands correspondence can be purely done in the data of root systems and reflections groups. For instance, the role of distinguished nilpotent orbits can be replaced by the Heckman-Opdam distinguished points [17] (a generalization of the Bala-Carter theory).

On the other hand, Kazhdan-Lusztig [20] (also see [14]) establishes the Deligne-Langlands conjecture for the Iwahori block by using affine Hecke algebras. Their graded version, called graded Hecke algebras, is well-defined for type H_4 and this provides a ground to look for a Langlands theory for type H_4.

A first step towards the goal is a classification of 'discrete series'. Here discrete series and tempered modules for graded Hecke algebras

can be defined from an algebraic criteria due to Casselman. The homological property of discrete series allows one to establish an upper bound on the number of discrete series [25, 8], which is the key for the exhaustion part of the classification theorem. For the construction part, in addition to homological properties in [8], we also use structure of calibrated modules developed in [12] and [21].

Theorem 0.2. [10] *Let $\mathbb{H}$ be the graded Hecke algebra of type H_4. There are precisely 20 isomorphism classes of discrete series of $\mathbb{H}$.*

The next step is a classification of tempered modules:

Theorem 0.3. [10] *Let $\mathbb{H}$ be the graded Hecke algebra of type H_4. Then all tempered modules are precisely parabolically induced from a discrete series. Moreover, there are 14 isomorphism classes of tempered modules with real central characters.*

It is well-known that a tempered module is a direct summand in a parabolically induced module, and so one has to prove such module is irreducible. Our approach is to use the orthogonality of discrete series in the homological elliptic pairing.

We finally discuss a Springer theory for type H_4. Its importance comes from that the structure information for geometrically constructed modules is controlled by Springer theory, see [22, 14, 6]. Although the reflection group of type H_4 does not possess underlying (known) geometry governing such theory, other perspectives [2, 21] suggest possible generalizations using representation theoretic invariants such as fake degrees.

From the perspective of graded Hecke algebras, the W-structures of tempered modules are crucial to obtain a Springer theory for type H_4 in [10, 11]. The main ingredients in obtaining the W-structure of discrete series include:

(1) the orthogonality of discrete series in the elliptic space;

(2) dimensions of calibrated discrete series; and
(3) a bit W-structure of some parabolically induced modules.

In view of the work in [5], one may need some more from the Springer correspondence for the categorical Langlands program. On the other hand, the derived Springer correspondence [28] suggests one may look at the skew group ring $\mathbb{A}_W$-structure rather than W-structure. (Precisely, $\mathbb{A}_W = \mathrm{Sym}(V) \rtimes \mathbb{C}[W]$ for the polynomial ring $\mathrm{Sym}(V)$ of the reflection representation V of W.) In order to define $\mathbb{A}_W$-structure on tempered modules X, one picks a certain irreducible W-subspace σ in X naturally corresponding to a Springer representation. Then one defines

$$X_i := \mathrm{Sym}^i(V).\sigma,$$

where $\mathrm{Sym}^i(V)$ is the subspace in $\mathbb{H}$ with all polynomials on V with degree less than or equal to i. Define

$$\overline{X} := \oplus_{i \in \mathbb{Z}_{\geq 0}} X_i / X_{i-1},$$

with a natural $\mathbb{A}_W$-action descend from $\mathbb{H}$-action on X. In the case of discrete series, we have the following result (details in progress):

Theorem 0.4. [11] *Let $\mathbb{H}$ be the graded Hecke algebra of type H_4. Let X be a discrete series of $\mathbb{H}$ and let $\overline{X}$ be defined as above. Then the multiplicity of $\tau \in \mathrm{Irr}(W)$ of degree i-the piece comes from a coefficient of q^{2i}-degree of the Green polynomials.*

The remaining unexplained notation in the theorem is the Green polynomials, which can be defined and computed from some semi-orthogonality property of a graded elliptic pairing on $\mathbb{A}_W$-modules (see [19]). Such result also suits well in the theory of Dirac cohomology, see [13].

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THE UNIVERSITY OF HONG KONG
Email address: kychan1@hku.hk

DIMENSION FORMULA, FOLLOWING FRIEDBERG-GINZBURG, FOR PERIODS OF SMALL REPRESENTATIONS W.R. TO SMALL SUBGROUPS

DMITRY GOUREVITCH

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Let $\mathbf{G}$ be an algebraic reductive group defined over a number field $\mathbb{K}$, and let $\mathbf{H}\subset\mathbf{G}$ be an algebraic subgroup, also defined over $\mathbb{K}$. Let π be an irreducible representation of $\mathbf{G}(\mathbb{A}_{\mathbb{K}})$ in the space of automorphic forms. There are many known examples in which the period integral over $\mathbf{H}(\mathbb{A}_{\mathbb{K}})/\mathbf{H}(\mathbb{K})$, viewed as a functional on π, or on a family of representations constructed from π, has a number-theoretic importance, *e.g.* has an unfolding, is Eulerian, equals a special value of an L-function. In the classical examples, the subgroup $\mathbf{H} \subset \mathbf{G}$ is spherical. However, there are alos many well-known examples in which the subgroup is not spherical. Ginzburg suggested that it would be useful to have a simple geometric criterion that would enable to detect pairs $(\mathbf{H}, \pi)$ that have a potential to have a number-theoretic importance. He suggested such a criterion in [Gin14] and a more refined one, joint with Friedberg, in [FG21]. In our work in progress [GS] with Eitan Sayag we suggest a related criterion, in terms of the nilpotent orbits attached to π using Fourier coefficients.

Let $\mathfrak{h} \subset \mathfrak{g}$ denote the Lie algebras of $\mathbf{H}\subset\mathbf{G}$. Let $\mathfrak{g}^*$ denote the dual space to $\mathfrak{g}$, and let $\mathfrak{h}^{\perp}\subset\mathfrak{g}^*$ denote the space of functionals vanishing on $\mathfrak{h}$. Let $\mathcal{N}$ denote the nilpotent cone of $\mathfrak{g}^*$. It has finitely many orbits under the coadjoint action of $\mathbf{G}$.

Definition 0.1. *Let $\Xi\subset\mathcal{N}$ be a $\mathbf{G}$-invariant subset. We say that $\mathfrak{h}$ is* Ξ-spherical *if for any $\mathbf{G}$-orbit $\mathbf{O} \subset \Xi$, we have*

(0.1) $$\dim \mathbf{O} \cap \mathfrak{h}^{\perp} \leq \dim \mathbf{O}/2$$

We say that $\mathfrak{h}$ is adapted *to Ξ if it is Ξ-spherical and satisfies*

(0.2) $$\dim \Xi \cap \mathfrak{h}^{\perp} = \dim \Xi/2$$

Let $\mathrm{WO}(\pi) \subset \mathfrak{g}^*(\mathbb{K})$ denote the union of nilpotent orbits O such that the Fourier coefficients $\mathcal{F}_O$ do not vanish identically on π. Let $\mathcal{V}(\pi) \subset \mathfrak{g}^*$ denote the Zariski closure of $\mathrm{WO}(\pi)$. Our criterion for the period integral on $\mathbf{H}(\mathbb{A}_{\mathbb{K}})/\mathbf{H}(\mathbb{K})$ to have a number-theoretic meaning on π is:

(0.3) $$\mathfrak{h} \text{ is adapted to } \mathcal{V}(\pi)$$

We define the Gelfand-Kirillov dimension of π by $\dim_{\mathrm{GK}}(\pi) := \dim \mathcal{V}(\pi)/2$. As we show below, if $\mathfrak{h}$ is adapted to $\mathcal{V}(\pi)$ then $\dim \mathfrak{h} \geq \dim_{\mathrm{GK}}(\pi)$. This relates us to the criterion in [Gin14] which is $\dim \mathfrak{h} = \dim_{\mathrm{GK}}(\pi)$. In [FG21] this criterion is refined, to adress cases in which $\dim \mathfrak{h} > \dim_{\mathrm{GK}}(\pi)$, but the period integral still is Eulerian. They say that among all subgroups that define periods that are equivalent by unfolding, one should consider the one with of minimal dimension.

Conjecture 0.2. *If $\mathfrak{h}$ is $\mathcal{V}(\pi)$-spherical but not $\mathcal{V}(\pi)$-adapted then the period integral over $\mathbf{H}$ vanishes identically on π.*

Our criterion, as well as those of [Gin14, FG21], does not consider convergence issues. Neither does it take into account the continuous invariants of π. It is motivated by the recent results [GS21, AG24] on multiplicities of representations of real reductive groups. Possibly one should refine our criterion using the symplectic structure of nilpotent orbits.

In the next section we give some geometric statements that allow to grasp the notion of Ξ-spherical.

1. Geometry

Theorem 1.1 ([AG24, Theorem B]). *Let $\mathbf{P}\subset\mathbf{G}$ be a parabolic subgroup. Let $\mathbf{O_P}\subset\mathcal{N}$ denote the Richardson orbit of $\mathbf{P}$, and $\overline{\mathbf{O_P}}$ denote the (Zariski) closure of $\mathbf{O_P}$. Then the following are equivalent.*

(i) *$\mathbf{P}$ has finitely many orbits on $\mathbf{G}/\mathbf{H}$.*
(ii) *$\mathfrak{h}$ is an $\overline{\mathbf{O_P}}$-spherical subalgebra of $\mathfrak{g}$.*

Corollary 1.1. *If $\mathfrak{h}$ is a spherical subalgebra of $\mathfrak{g}$ then it is Ξ-spherical for every $\mathbf{G}$-invariant subvariety $\Xi\subset\mathfrak{g}^*$.*

Corollary 1.2 ([AG24, Corollary J]). *Let $\mathbf{P}\subset\mathbf{G}$ be a parabolic subgroup defined over $\mathbb{R}$, and let P be the corresponding parabolic subgroup of G. Suppose that for all but finitely many orbits of $\mathbf{H}$ on $\mathbf{G}/\mathbf{P}$, the set of real points is non-empty and orientable. Then the following are equivalent.*

(i) *$\mathbf{H}$ is $\overline{\mathbf{O_P}}$-spherical, where $\mathbf{O_P}$ denotes the Richardson orbit of $\mathbf{P}$.*
(ii) *Every $\pi \in \mathcal{M}_{\overline{\mathbf{O_P}}}(G)$ has finite multiplicities.*
(iii) *H has finitely many orbits on G/P.*
(iv) *$\mathbf{H}$ has finitely many orbits on $\mathbf{G}/\mathbf{P}$.*

Proposition 1.3 ([AG24, Proposition 2.2.9]). *Let $\mathbf{O} \subset \mathcal{N}$ be a Richardson nilpotent orbit. Then $\mathbf{O} \cap \mathfrak{h}^{\perp}$ is either empty, or has dimension at least $\dim \mathbf{O}/2$.*

Let us also record the following straightforward lemma.

Lemma 1.4. *Let $\Xi\subset\mathcal{N}$ be a closed $\mathbf{G}$-invariant subvariety. Then*

(i) *$\mathfrak{h}$ is Ξ-spherical if and only if it is $\overline{\mathbf{O}}$-spherical for every orbit $\mathbf{O}\subset\Xi$ of maximal dimension.*
(ii) *Suppose $\mathfrak{h}$ is Ξ-spherical. Then $\mathfrak{h}$ is Ξ-adapted if and only if $\dim \mathbf{O} \cap \mathfrak{h}^{\perp} = \dim \mathbf{O}/2$ for* ***some*** *orbit $\mathbf{O}\subset\Xi$ of maximal dimension.*

1.1. **Diagonal subgroups.** Assume that $\mathbf{H}$ is reductive, and consider $\Delta\mathbf{H}$ as a subgroup of $\mathbf{G} \times \mathbf{H}$. This allows to consider integrals of the form

$$\int_{[\mathbf{H}]} \varphi(h) f(h) dh,$$

where φ is an automorphic form on $\mathbf{G}(\mathbb{A})$ and f is an automorphic form on $\mathbf{H}(\mathbb{A})$. Clearly this is the same as the period integral of the automorphic for $\varphi \boxtimes f$ on $\mathbf{G} \times \mathbf{H}$ over $\Delta\mathbf{H}$.

Let $\mathbf{B}\subset\mathbf{G}$ and $\mathbf{B_H}\subset\mathbf{H}$ be Borel subgroups. Let $\mathcal{N}\subset\mathfrak{g}^*$ and $\mathcal{N}_{\mathfrak{h}}\subset\mathfrak{h}^*$ denote the nilpotent cones. From Theorem 1.1 we obtain the following corollary

Corollary 1.5. *Let $\mathbf{P}\subset\mathbf{G}$ and $\mathbf{Q}\subset\mathbf{H}$ be parabolic subgroups, and let $\mathbf{O_P}\subset\mathfrak{g}^*$ and $\mathbf{O_Q}\subset\mathfrak{h}^*$ be the corresponding nilpotent orbits. Then*

(i) $\Delta\mathfrak{h}$ *is* $\overline{\mathbf{O_P}}\times\overline{\mathbf{O_Q}}$*-spherical if and only if the set of double cosets* $\mathbf{P}\backslash\mathbf{G}/\mathbf{Q}$ *is finite*
(ii) $\Delta\mathfrak{h}$ *is* $\mathcal{N}\times\overline{\mathbf{O_Q}}$*-spherical if and only if* $\mathbf{G}/\mathbf{Q}$ *is a spherical* $\mathbf{G}$*-space*
(iii) $\Delta\mathfrak{h}$ *is* $\overline{\mathbf{O_P}}\times\mathcal{N}_{\mathfrak{h}}$*-spherical if and only if* $\mathbf{P}\backslash\mathbf{G}$ *is a spherical* $\mathbf{H}$*-space*

For the case when the commutant $[\mathfrak{g},\mathfrak{g}]$ is simple, all pairs $(\mathbf{H},\mathbf{P})$ such that $\mathbf{H}$ is a symmetric subgroup of $\mathbf{G}$, and $\mathbf{G}/\mathbf{P}$ is a spherical $\mathbf{H}$-space are classified in [HNOO13, §5, Table 2]. More generally, a strategy for classifying all pairs $(\mathbf{H},\mathbf{P})$ such that $\mathbf{H}$ is reductive, $\mathbf{P}$ is a parabolic subgroup, and $\mathbf{P}\backslash\mathbf{G}$ is a spherical $\mathbf{H}$-space is given in [AP14]. This strategy is also implemented in *loc. cit.* for $\mathbf{G}=\mathrm{SL}_n$, and in [AP21] for all the other classical groups.

The pairs of subgroups $\mathbf{Q}\subset\mathbf{H}\subset\mathbf{G}$ such that $\mathbf{H}$ is a symmetric subgroup of $\mathbf{G}$, and $\mathbf{Q}$ is a parabolic subgroup of $\mathbf{H}$ that is also a spherical subgroup of $\mathbf{G}$ are classified in [HNOO13, §6, Table 3]. The main example of non-simple $[\mathfrak{g},\mathfrak{g}]$ is the diagonal symmetric pair: $\mathbf{G}=\mathbf{H}\times\mathbf{H}$, with $\mathbf{H}$ embedded diagonally. The classification all pair of parabolic subgroups of $\mathbf{R_1},\mathbf{R_2}\subset\mathbf{H}$ such that $\mathbf{H}/\mathbf{R_1}\times\mathbf{H}/\mathbf{R_2}$ is $\Delta\mathbf{H}$-spherical is given in [Ste03].

2. Comparison to [Gin14, FG21]

- Our (wave-front sphericity) condition automatically holds for spherical subgroups, while the condition of [FG21] not always holds for them (e.g. Shalika or linear periods).
- Our condition does not hold for non-Borel constant term ($P=LU$, Eisenstein series), while the condition of [FG21] does.
- Our condition requires (complicated) dimension computations, while the condition of [FG21] requires unfolding.

The following lemma gives some relation between our conditions hold as well.

Lemma 2.1. *Let $\Xi\subset\mathcal{N}$ be a closed $\mathbf{G}$-invariant subset such that $\mathfrak{h}$ is Ξ-spherical. Then*

(i) $\dim\mathfrak{h}\geq\dim\Xi/2$
(ii) If $\dim\mathfrak{h}=\dim\Xi/2$ *then the projection* $\mathfrak{g}^*\twoheadrightarrow\mathfrak{h}^*$ *maps* Ξ *onto* $\mathfrak{h}$*, and* $\mathfrak{h}$ *is* Ξ*-adapted.*

In all the examples given in [Gin14, FG21] our conditions hold.

3. Evidence

3.1. **Archimedean evidence.** For a representation τ of a real reductive group, let $\mathcal{V}(\tau)$ denote the associated variety of the annihilator of τ in $\mathcal{U}(\mathfrak{g})$.

Theorem 3.1 ([AG24, GS21]). *Suppose that $\mathfrak{h}$ is $\mathcal{V}(\tau)$-spherical subalgebra of $\mathfrak{g}$. Then*

(i) $m_H(\tau)<\infty$
(ii) If $m_H(\tau)>0$ *then* $\mathfrak{h}$ *is adapted to* $\mathcal{V}(\tau)$*.*

Call an automorphic representation π "real-wavefront" if $\mathcal{V}(\pi)=\mathcal{V}(\pi_\nu)$ for some Archimedean place ν.

Corollary 3.2. *For any real-wavefront π, Conjecture 0.2 holds.*

3.2. GL_n. In [MW89], Moeglin and Waldspurger showed that the discrete spectrum of $\mathrm{GL}_n(\mathbb{A})$ consists of residues of Eisenstein series $L(\rho, m) := E_{P_{m,n/m}}(\rho^{\otimes n/m})$, where m is a divisor of n, $P_{m,n/m}$ is the standard parabolic subgroup of GL_n with Levi subgroup $(\mathrm{GL}_m)^{n/m}$, and ρ is a cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A})$.

Theorem 3.1 ([Gin06, Proposition 5.3 and its proof], cf. [JL13]). *We have*

$$\mathcal{V}(L(\rho, m)) = \overline{\mathcal{O}_{m^{n/m}}} = \mathcal{V}(L(\rho, m)_\nu)$$

for every non-archimedean place ν at which $L(\rho, m)$ is unramified.

Theorem 3.2 ([GS]). *We have $\mathcal{V}(L(\rho, m)) = \overline{\mathcal{O}_{m^{n/m}}} = \mathcal{V}(L(\rho, m)_\nu)$ for every place ν.*

In particular, $L(\rho, m)$ is real-wavefront, and satisfies Conjecture 0.2. The proof uses restriction to the mirabolic subgroup $P_n(\mathbb{A})$.

4. Twisted version

For a character ψ of the unipotent part of $\mathfrak{h}$, we use $\mathcal{O} \cap p_{\mathfrak{h}}^{-1}(\psi)$ in place of $\mathcal{O} \cap \mathfrak{h}^\perp$, where $p_{\mathfrak{h}} : \mathfrak{g}^* \to \mathfrak{h}^*$.

Examples: Shalika model, Klyachko models.

$$G = \mathrm{GL}_n \ supset \ H_k = \begin{pmatrix} \mathrm{Sp}_{2k} & * \\ 0 & N_{n-2k} \end{pmatrix},$$

where N_{n-2k} is the unipotent radical of the Bore subgroup of GL_{n-2k}, and ψ is a generic character of N_{n-2k}.

Theorem 4.1 ([OS08]). *Every π in the discrete spectrum has a non-vanishing Klyachko period for exactly one k, and this peirod is Eulerian.*

They describe the k explicitly, and this is compatible with the twisted version of Conjecture 0.2.

5. Integral kernels

Let $H = H_1 \times H_2 \subset G$ be reductive. Let Θ be an automorphic function on $\mathbf{G}(\mathbb{A})$. Define transfer of representations of $\mathbf{H}_1(\mathbb{A})$ to representations of $\mathbf{H}_2(\mathbb{A})$ by sending τ into the representation σ spanned by all functions of the form

$$f(h_2) = \int_{[\mathbf{H}_1]} \varphi(h_1)\Theta(h_1 h_2) dh_1, \quad \varphi \in \tau \tag{5.1}$$

Friedberg and Ginzburg suggest the following criterion for this correspondence to be useful and meaningful:

$$\dim \sigma = \dim \tau + \dim \Theta - \dim \mathbf{H}_1 \tag{5.2}$$

Assume that $\mathcal{V}(\Theta)$ is the closure of a single orbit $\mathbf{O}_\Theta$. Let $p_i : \overline{\mathbf{O}_\Theta} \twoheadrightarrow \mathfrak{h}_i^*$ denote the restrictions to $\overline{\mathbf{O}_\Theta}$ of the natural projections $\mathfrak{g} \twoheadrightarrow \mathfrak{h}_i$.

We suggest to require that 3 conditions hold:

(a) $p_1(\overline{\mathbf{O}_\Theta}) = \mathfrak{h}_1^*$
(b) $\dim p_1^{-1}(\mathcal{V}(\tau)) = \dim p_1^{-1}(\mathcal{V}(\tau)) \cap p_2^{-1}(\mathcal{V}(\sigma))$
(c) $\Delta\mathfrak{h} \subset \mathfrak{h} \times \mathfrak{g}$ is $\mathcal{V}(\tau) \times \mathcal{V}(\sigma) \times \mathcal{V}(\Theta)$-adapted.

Examples: Θ-correspondence, doubling.

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Dmitry Gourevitch, Faculty of Mathematics and Computer Science, Weizmann Institute of Science, POB 26, Rehovot 76100, Israel

Email address: `dimagur@weizmann.ac.il`

ON A BRAIDED MONOIDAL HALL 2-CATEGORY

QUOC P. HO

Classification AMS 2020: Primary 20C08, 18N25, 18N65. Secondary 57K18.

Keywords: Hall algebras, braided monoidal categories, Hecke categories, categorification, 2-Segal objects.

The lecture reported on a joint research program in progress with Jonte Gödicke, Yang Hu, and Walker Stern, whose aim is to provide a novel mechanism to construct braided monoidal $(\infty, 2)$-categories, refining the classical Hall algebra construction.

1. 2-SEGAL OBJECTS, WALDHAUSEN $S_\bullet$ CONSTRUCTION, AND HALL ALGEBRAS

Let $\mathcal{C}$ be a category with finite limits. It has been understood by Dyckerhoff and Kapranov in [1] (and subsequently refined by Stern and Jonte in [2, 3]) that 2-Segal objects in a category $\mathcal{C}$, which are simplicial objects satisfying certain Segal-like conditions, are the same as algebra objects in the category of correspondences in $\mathcal{C}$. These algebra objects can then be turned into more conventional algebra objects via a certain linearization procedure.

An abundant source of 2-Segal objects is given by the Waldhausen $S_\bullet$ construction applied to suitable input data, such as abelian categories and stable ∞-categories. Upon linearization, the resulting algebra objects are usually referred to as Hall algebras, which have been studied extensively in representation theory.

The key observation of our work is that the Waldhausen $S_\bullet$ construction can be iterated, resulting in n-fold 2-Segal objects. Focusing on the case where $n = 2$, we show that the resulting double 2-Segal objects give rise to lax braided algebra objects (or more precisely, lax E_2-algebra objects) in the category of higher correspondences. Very roughly speaking, the term lax refers to the fact that the braiding is not required to be invertible.

As part of our work, we also construct a linearization procedure which is a lax symmetric monoidal $(\infty, 2)$ functor from the $(\infty, 2)$-category of higher correspondences in stacks to the category of $(\infty, 2)$-categories, leveraging [4], for example. This allows us to obtain a new class of lax braided monoidal $(\infty, 2)$-categories.

2. SOME APPLICATIONS

We were motivated by a question posed by Elias and Tolmachov in [5, 6] regarding the existence of a monoidal functor from the affine Hecke category to the finite Hecke category for GL_n. The main idea behind our approach comes from factorization homology, necessitating the construction of a certain $(\infty, 2)$-category built out of finite Hecke categories for GL_n, for all n together. Affine Hecke categories would then appear from taking the factorization homology of this braided monoidal category over an annulus. The conjectural functor would then be induced by the inclusion of the annulus back to the 2-dimensional disk.

From the perspective of Soergel bimodules, such a braided monoidal $(\infty, 2)$-category was already constructed by hand in [7], using obstruction theoretic methods. Nonetheless, computing factorization homology is known to be much easier once we have a geometric model, as already seen in my previous work [8, 9], ultimately inspired by Ben-Zvi–Nadler's work in the Betti Geometric Langlands program [10, 11, 12]. As a direct consequence for the general paradigm discussed above, we obtain a geometric construction of this braided monoidal $(\infty, 2)$-category.

Our work also has applications to the study of categorified representations of categorified quantum groups, which will appear in a forthcoming work.

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Department of Mathematics, The Hong Kong University of Science and Technology, Clear Water Bay, Hong Kong

Email address: `quoc.ho@ust.hk`

DESIDERATA AND UNIQUENESS OF LOCAL LANGLANDS CORRESPONDENCE

CHI-HENG LO

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Keywords: Local Langlands correspondence

The local Langlands conjecture predicts a ”canonical” bijection between the set of smooth irreducible representations of a p-adic reductive algebraic group G and the set of enhanced L-parameters of G, known as the local Langlands correspondence (LLC). There are several constructions of LLC in the literature, either on specific types of groups or on special classes of representations. We refer to [Tai25, §6.6] for a non-exhaustive summary of known cases. The comparison between different constructions of LLC is a non-trivial problem. In this talk, I reported on joint work with Tasho Kaletha and Cheng-Chiang Tsai, where we collect a list of desiderata of LLC for general quasi-split p-adic group G from the literature. Our main theorem shows that, when the residue characteristic of the p-adic field F is sufficiently large, any LLC satisfying these desiderata is unique (if it exists). This provides a unified framework for comparing different LLC.

We now introduce the notation needed to formulate the desiderata and the main theorem precisely. Let F be a finite extension of $\mathbb{Q}_p$, and let G be a connected reductive algebraic group defined and quasi-split over F. For simplicity, we do not distinguish between G and its group of F-points. Denote by $\widehat{G}$ the complex dual group of G, and by ${}^LG$ its Langlands dual group. Let $\Pi(G)$ be the set of irreducible smooth representations of G, and $\Phi(G)$ the set of L-parameters for G, i.e. the set of $\widehat{G}$-conjugacy classes of admissible homomorphisms

$$\phi : W_F \times \mathrm{SL}_2 \longrightarrow {}^LG.$$

To each L-parameter ϕ, we attach an infinitesimal parameter (L-parameter trivial on SL_2)

$$\lambda_\phi(w) := \phi\left(w, \begin{pmatrix} |w|^{1/2} & \\ & |w|^{-1/2} \end{pmatrix}\right), \qquad w \in W_F,$$

and define the component group

$$\mathcal{S}_\phi := \pi_0(Z_{\widehat{G}}(\phi)/Z(\widehat{G})^\Gamma).$$

An *enhanced* L-parameter of G is a pair (ϕ, ϵ) consisting of $\phi \in \Phi(G)$ and an irreducible representation ϵ of $\mathcal{S}_\phi$. We denote the set of all such pairs by $\Phi^{\mathfrak{e}}(G)$.

A local Langlands correspondence (resp. enhanced local Langlands correspondence) is a surjective (resp. bijective) map

$$\mathrm{LLC}_G : \Pi(G) \longrightarrow \Phi(G) \quad (\text{resp. } \mathrm{LLC}_G^{\mathfrak{e}} : \Pi(G) \longrightarrow \Phi^{\mathfrak{e}}(G)).$$

For a parameter ϕ, we write $\Pi_\phi := \mathrm{LLC}_G^{-1}(\phi)$ for the corresponding L-packet. Finally, denote by $\mathcal{L}(G)$ the set of standard Levi subgroups of G. We shall consider a compatible system of local Langlands correspondences $\{\mathrm{LLC}_M\}_{M \in \mathcal{L}(G)}$.

For $\pi \in \Pi(G)$ and $\phi \in \Phi(G)$, we write $\pi \leftrightarrow (P_\pi, \pi_t, \nu_\pi)$ and $\phi \leftrightarrow (P_\phi, \phi_t, \nu_\phi)$ for the Langlands classification for representations (see [Kon03]) and for L-parameters (see [SZ18]). We let $M(\pi) = \mathrm{Ind}_{P_\pi}^G \pi_t \otimes \nu_\pi$ (normalized induction) denote the standard module of π.

Now we state our list of desiderata for LLC. We fix a choice of Whittaker datum $\mathfrak{w}$ of G for simplicity.

- (LC) Suppose that $\mathrm{LLC}_G(\pi) = \phi$, $\pi \leftrightarrow (P_\pi, \pi_t, \nu_\pi)$, and $\phi \leftrightarrow (P_\phi, \phi_t, \nu_\phi)$. Then $P_\pi = P_\phi$, $\nu_\pi = \nu_\phi$ and $\mathrm{LLC}_M(\pi_t) = \phi_t$. (See [SZ18, §7.2])
- (Inf) Let $\pi_1, \pi_2 \in \Pi(G)$. If the standard module $M(\pi_1)$ is $\mathfrak{w}$-generic and π_2 is the unique $\mathfrak{w}$-generic subquotient of $M(\pi_1)$, then $\lambda_{\mathrm{LLC}_G(\pi_1)} = \lambda_{\mathrm{LLC}_G(\pi_2)}$. (See [Hai14, Conjecture 5.2.2])
- (IT) If π is a subquotient of $\mathrm{Ind}_P^G \sigma$ for some tempered representation σ of M, then $\mathrm{LLC}_G(\pi) = ({}^L M \hookrightarrow {}^L G) \circ \mathrm{LLC}_M(\sigma)$.
- (Ka) If π is generic F-non-singular supercuspidal, then $\mathrm{LLC}_G(\pi)$ coincides with the one constructed in [Kal19].
- (Sh) For each tempered L-parameter ϕ, the L-packet Π_ϕ contains a unique $\mathfrak{w}$-generic member π_{gen}. (See [Sha90, Conjecture 9.4])
- (Disc) Let $\Pi_{disc}(G)$ denote the set of discrete series representations of G and let $\Phi_{disc}(G)$ denote the set of discrete L-parameters of G. Then $\Pi_{disc}(G) = \bigsqcup_{\phi \in \Phi_{disc}(G)} \Pi_\phi$.
- (St) For each discrete L-parameter ϕ, there exists a finite sum

  $$\eta_\phi = \sum_{\pi \in \Pi_\phi} m_\pi \pi,$$

  which gives a stable distribution. Moreover, $m_{\pi_{gen}} = 1$.
- (SAS) Suppose $S = \{\pi_i\}_{i=1}^s$ is a finite set of discrete series representations and $\{a_i\}_{i=1}^s$ is a set of nonzero complex numbers such that $\Theta := \sum_{i=1}^s a_i \pi_i$ is a stable distribution. Then, S is a disjoint union of discrete L-packets

  $$S = \bigsqcup_{j=1}^r \Pi_{\phi_j}, \quad \text{and } \Theta = \sum_{j=1}^r b_j \eta_{\phi_j}.$$

  (See [BY23, §4])

We prove that when p is sufficiently large (with respect to G), every F-non-singular supercuspidal representation is generic with respect to some Whittaker datum. This is the main ingredient of the proof of the following theorem on the uniqueness of the local Langlands correspondence.

Theorem 0.1. *Assume that $p \gg 0$. Then the local Langlands correspondence $\{LLC_M\}_{M \in \mathcal{L}(G)}$ satisfying the above desiderata is unique if it exists.*

Next, we consider the uniqueness of the enhanced local Langlands correspondence. We shall consider the set of endoscopic groups of G, which we denote by $\mathcal{E}(G)$. The enhanced local Langlands correspondence depends on a choice of Whittaker datum $\mathfrak{w}$ of G. Thus, we write

$$\begin{aligned} \mathrm{LLC}^{\mathfrak{e}}_{G,\mathfrak{w}} : \Pi(G) &\xrightarrow{\text{bij.}} \Phi^{\mathfrak{e}}(G), \\ \pi_{\mathfrak{w}}(\phi, \varepsilon) &\longmapsto (\phi, \varepsilon). \end{aligned}$$

We need one more desideratum on endoscopic character relation for tempered L-packets. Suppose that ϕ is a tempered L-parameter of G. For each $s \in \mathcal{S}_\phi$, take any preimage $x \in \widehat{G}$ and form an endoscopic group (G', x). The L-parameter ϕ factors through ${}^L G'$ as an L-parameter ϕ' of G'. We define $\eta_{\phi,x} = \mathrm{Trans}^G_{(G',x)} \eta_{\phi'}$, the Langlands-Shelstad transfer from G' to G of the stable distribution $\eta_{\phi'}$. Here is the last desideratum we need.

(ECR) In the above setting, the distribution $\eta_{\phi,x}$ depends only on s. Hence, we write $\eta_{\phi,s} := \eta_{\phi,x}$. Moreover, we have

$$\pi_{\mathfrak{w}}(\phi, \varepsilon) = \frac{1}{|\mathcal{S}_\phi|} \sum_{s \in \mathcal{S}_\phi} \overline{\mathrm{trace}(\varepsilon(s))} \eta_{\phi,s}.$$

Here is our main result on the uniqueness of the enhanced local Langlands correspondence.

Theorem 0.2. *Assume that $p \gg 0$. Then the enhanced local Langlands correspondence $\{LLC^{\mathfrak{e}}_{G',\mathfrak{w}'}\}_{G' \in \mathcal{E}(G)}$ satisfying the desiderata* (*LC*), (*Inf*), (*IT*), (*Ka*), (*Sh*), (*Disc*), (*St*), (*SAS*), (*ECR*) *is unique if it exists.*

We also show that our list of desiderata automatically implies several other desiderata in the literature. These include the recipe for computing the central character [GR10, §8.2], the Adams–Vogan conjecture on the contragredient of L-packets [AV16], the decomposition formula for parabolic induction of tempered representations, and the recipe describing the change of Whittaker datum.

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Department of Mathematics, National University of Singapore, 119076, Singapore

Email address: ch_lo@nus.edu.sg

EXTENSIONS OF HARISH-CHANDRA MODULES AND A-PACKETS

ARVIND NAIR

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Let G be a connected reductive real algebraic group, $G(\mathbb{R})$ its group of real points, $\mathfrak{g}$ the complexified Lie algebra of $G(\mathbb{R})$, and $K \subset G(\mathbb{R})$ a maximal compact subgroup with associated Cartan involution θ. Let G^u be the compact form of $G(\mathbb{R})$ containing K.

Let $\mathscr{M}(\mathfrak{g}, K)$ be the category of $(\mathfrak{g}, K)$-modules and let $\mathscr{H}(\mathfrak{g}, K)$ be the full subcategory of Harish-Chandra modules (i.e. finite-length objects in $\mathscr{M}(\mathfrak{g}, K)$). We discuss some results showing that the groups $Ext^i_{(\mathfrak{g},K)}(V, W) := Ext^i_{\mathscr{M}(\mathfrak{g},K)}(V, W)$ behave well for those modules which are (expected to be) the Archimedean components of automorphic forms, i.e. for Harish-Chandra modules belonging to Arthur packets.

Unitary cohomological representations. An irreducible $(\mathfrak{g}, K)$-module V is *cohomological* if $Ext^*_{(\mathfrak{g},K)}(E, V) \neq \{0\}$ for some finite-dimensional irreducible algebraic representation of $G(\mathbb{C})$, in which case the infinitesimal character of V agrees with that of E. We will assume $E = \mathbb{C}$ here; the case of general E reduces to this one by means of translation functors. The irreducible unitary cohomological representations (i.e. irreducible unitarizable $(\mathfrak{g}, K)$-modules) were classified by Vogan and Zuckerman [10]: They are cohomologically induced modules $\mathscr{R}^S_{\mathfrak{q}}(\mathbb{C})$ as $\mathfrak{q}$ runs over the Lie algebras of θ-stable parabolic subgroups for G, i.e. parabolics $Q \subset G(\mathbb{C})$ for which $\theta(Q) = Q$ and $L = Q \cap \bar{Q}$ is a Levi in Q (and in $\bar{Q}$). Associated with a θ-stable parabolic Q are the Levi $L = Q \cap \bar{Q}$, the compact form $L^u := G^u \cap L(\mathbb{C})$, the compact group $K_L := K \cap L(\mathbb{R})$, and a $K_{\mathbb{C}}$-orbit O in the flag variety X of $G(\mathbb{C})$. The orbit O associated with Q is defined as follows: Let X_Q be the generalized flag variety of parabolics of $G(\mathbb{C})$ conjugate to Q, and let $\pi_Q : X \to X_Q$ be the canonical map taking a Borel B to the unique B-standard conjugate of Q. Then $Q \in X_Q$ has a closed $K_{\mathbb{C}}$-orbit, and its preimage in X is the closure of a single $K_{\mathbb{C}}$-orbit, which is O. We will call orbits arising from θ-stable Q parabolics *special orbits*.

Theorem 0.1. *Let Q_1 and Q_2 be θ-stable parabolics and let $\mathfrak{q}_i, L_i, L_i^u, K_{L_i}$ be as above for $i = 1, 2$. Let $O_i \subset X$ be the special $K_{\mathbb{C}}$-orbit associated with Q_i and let $\bar{O}_i$ be its closure.*

If $\bar{O}_1 \cap \bar{O}_2 = \emptyset$ or, equivalently, if $Q_1 \cap kQ_2k^{-1}$ is not parabolic for any $k \in K$ then $Ext^*_{(\mathfrak{g},K)}(\mathscr{R}^{S_1}_{\mathfrak{q}_1}(\mathbb{C}), \mathscr{R}^{S_2}_{\mathfrak{q}_2}(\mathbb{C})) = \{0\}$.

If $\bar{O}_1 \cap \bar{O}_2 \neq \emptyset$ then, replacing Q_2 by a K-conjugate, we may assume that $Q := Q_1 \cap Q_2$ is parabolic, and then

$$Ext^*_{(\mathfrak{g},K)}(\mathscr{R}^{S_1}_{\mathfrak{q}_1}(\mathbb{C}), \mathscr{R}^{S_2}_{\mathfrak{q}_2}(\mathbb{C})) = H^{*-d_{Q_1,Q_2}}(L^u/K_L) \tag{0.1}$$

where $d_{Q_1,Q_2} = \dim_{\mathbb{C}} O_1 + \dim_{\mathbb{C}} O_2 - 2\dim_{\mathbb{C}} \bar{O}_1 \cap \bar{O}_2 = \frac{1}{2}\dim_{\mathbb{R}} L_1^u/K_{L_1} + \frac{1}{2}\dim_{\mathbb{R}} L_2^u/K_{L_2} - \dim_{\mathbb{R}} L^u/K_L$.

In particular, if $Q_1 = Q_2 = Q$ we have $d_{Q,Q} = 0$ and (0.1) is an isomorphism of rings. In general, (0.1) is an isomorphism of bimodules for $(H^*(L_1^u/K_{L_1}), H^*(L_2^u/K_{L_2}))$ via the restrictions $H^*(L_i^u/K_{L_i}) \to H^*(L^u/K_L)$ for $i = 1, 2$.

The theorem can be upgraded to an equivalence of categories. The full triangulated subcategory generated by unitary cohomological representations

$$\mathcal{D}_{coh} := \left\langle\, \mathscr{R}^S_{\mathfrak{q}}(\mathbb{C}) \;\middle|\; \theta\text{-stable } Q \,\right\rangle \quad \subset \quad D^b(\mathscr{H}(\mathfrak{g}, K))$$

is equivalent to a certain category of sheaves on $K\backslash G^u$. Let $\mathscr{O}$ be the collection of special $K_{\mathbb{C}}$-orbits in X. (This includes the closed orbits, which come from θ-stable Borels). Fix $x \in X$ belonging to a closed $K_{\mathbb{C}}$-orbit on X and let $\alpha_x : G^u \to X$ be the orbit map $g_u \mapsto g_u \cdot x$. Then $\left\{K\backslash\alpha_x^{-1}\left(\bar{O}\right)\right\}_{O\in\mathscr{O}}$ is a collection of smooth connected closed submanifolds of $K\backslash G^u$ which is closed under taking nonempty intersections. They generate a stratification $\mathscr{S}$ of $K\backslash G^u$. Define $\mathcal{D}_{\mathscr{S}}$ to be the following full subcategory of the bounded derived category of $\mathscr{S}$-constructible complexes of $\mathbb{C}$-sheaves:

$$\mathcal{D}_{\mathscr{S}} := \{F \in D^b_{\mathscr{S}}(\mathbb{C}_{K\backslash G^u}) : H^i(F)|_S \text{ is constant for } S \in \mathscr{S}\} \quad \subset \quad D^b_{\mathscr{S}}(\mathbb{C}_{K\backslash G^u})$$

The dimensions of strata in $\mathscr{S}$ have the same parity as the discrete series defect δ (:= rank of the $\mathbb{R}$-split part of a maximally $\mathbb{R}$-anisotropic maximal torus) of G. So if $\delta = 0$ there is a well-defined middle perverse t-structure on $\mathcal{D}_{\mathscr{S}}$.

Theorem 0.2. *There is a natural equivalence of triangulated categories $\mathcal{D}_{coh} \simeq \mathcal{D}_{\mathscr{S}}$. If $\delta = 0$ it relates the standard t-structure on $\mathcal{D}_{coh}$ and the perverse t-structure on $\mathcal{D}_{\mathscr{S}}$.*

The proofs of Theorems 0.1 and 0.2 use Beilinson-Bernstein localization for derived categories of $(\mathfrak{g}, K)$-modules – well-known results from [4, 5, 7] – and the properties of special $K_{\mathbb{C}}$-orbits in the flag variety. The key point is the smoothness of the closures of special $K_{\mathbb{C}}$-orbits, and of the intersections of these closures, which ensures that the subtleties of Kazhdan-Lusztig-Vogan theory do not intervene.

Cohomological A-packets. For Harish-Chandra modules V, W let $\dim Ext^*_{(\mathfrak{g},K)}(V, W) := \sum_{i\geq 0} \dim Ext^i_{(\mathfrak{g},K)}(V, W)$. For a finite nonempty set Π of inequivalent $(\mathfrak{g}, K)$-modules, set $\pi(\Pi) := \bigoplus_{V\in\Pi} V$. The special case $\Pi_1 = \{\mathbb{C}\}$ of the following was proved in [8]:

Theorem 0.3. *For cohomological A-packets Π_1, Π_2 with the same infinitesimal character,*

$$\dim Ext^*_{(\mathfrak{g},K)}\left(\pi(\Pi_1), \pi(\Pi_2)\right) = 2^{\dim A}\left|\frac{W(G, T^c)^\theta}{W(G(\mathbb{R}), T^c(\mathbb{R}))}\right| \tag{0.2}$$

where $T^c = TA$ is a θ-stable fundamental (i.e. maximally anisotropic) torus in G with anisotropic part T and split part A, $W(G, T^c)$ is the Weyl group of T^c in G, $W(G, T^c)^\theta = \{w \in W(G, T^c) : \theta w = w\theta\}$, and $W(G(\mathbb{R}), T^c(\mathbb{R})) = N_{G(\mathbb{R})}(T^c(\mathbb{R}))/T^c(\mathbb{R})$.

Here by cohomological A-packets we mean the packets of unitary cohomological representations defined by Adams and Johnson [2] (cf. [8] for a review). For the trivial A-packet $\Pi_1 = \Pi_2 = \{\mathbb{C}\}$, (0.2) asserts that

$$\dim Ext^*_{(\mathfrak{g},K)}(\mathbb{C}, \mathbb{C}) = \dim H^*(G^u/K) = 2^{\dim A}\left|\frac{W(G, T^c)^\theta}{W(G(\mathbb{R}), T^c(\mathbb{R}))}\right|. \tag{0.3}$$

This is Theorem 14 of [8]. The proof of Theorem 0.3 bootstraps this identity using the computation (0.1) in Theorem 0.1.

The choice of Π_1, Π_2 as in Theorem 0.3 determines cohomological A-packets $\Pi_1(G')$ and $\Pi_2(G')$ on the real points of any inner form G' of G. Summing over the Vogan packet, i.e. over pure inner forms of G gives the following (which is [8, Theorem 12] when $\Pi_1 = \{\mathbb{C}\}$ and follows from Theorem 0.3 by the same arguments made there)

Theorem 0.4. *With notation as in Theorem 0.3,*

$$\sum_{G' \in H^1(\mathbb{R},G)} \dim Ext^*_{(\mathfrak{g}',K')}(\pi(\Pi_1(G')), \pi(\Pi_2(G'))) = |T^c(\mathbb{R})_2| \tag{0.4}$$

where $T^c(\mathbb{R})_2 = \{t \in T^c(\mathbb{R}) : t^2 = e\}$.

Tempered representations. Tempered L-packets are A-packets, and for these we have:

Theorem 0.5. *For tempered L-packets* Π_1, Π_2 *,* $Ext^*_{(\mathfrak{g},K)}(\pi(\Pi_1), \pi(\Pi_2)) \neq \{0\} \iff \Pi_1 = \Pi_2$.

The proof of this theorem is similar to the computation in Theorem 0.3. The key point is that the orbits in the flag variety corresponding to tempered representations are special and the corresponding D-modules come from local systems on these special orbits which are clean (i.e. their !-extension, !$*$-extension, and $*$-extension to the orbit closure all coincide). The theorem then follows by the same methods as earlier.

Parameters and the dual group. The numerical or nonvanishing results (Theorems 0.3, 0.4, 0.5) above can be reformulated nicely in terms of parameters, which also points the way to generalizations. (It seems unlikely that explicit computations like Theorem 0.1 are possible in general.) Let $W_\mathbb{R}$ be the Weil group of $\mathbb{R}$ and ${}^LG$ the L-group.

For an A-parameter $\psi : W_\mathbb{R} \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^LG$, the *tempered companion* $T(\psi)$ of ψ is the composition $W_\mathbb{R} \xrightarrow{\sigma_1} W_\mathbb{R} \times \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\psi} {}^LG$ where $\sigma_1 : W_\mathbb{R} \to {}^LPGL(2,\mathbb{R}) = W_\mathbb{R} \times \mathrm{SL}_2(\mathbb{C})$ is the L-parameter of the lowest discrete series representation of $PGL(2,\mathbb{R})$ (cf. [8, Definition 2]). The p-adic analogue of $\psi \mapsto T(\psi)$ is restriction of an A-parameter $\psi : W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG$ to $W_F \times \Delta\mathrm{SL}_2(\mathbb{C})$. The following conjecture is then a uniform statement for all local fields, although we only consider the Archimedean case:

Conjecture 0.6 (K. Y. Chan – D. Prasad)**.** *For G quasisplit and A-packets Π_1, Π_2 with A-parameters ψ_1, ψ_2, $Ext^*_{(\mathfrak{g},K)}(\pi(\Pi_1), \pi(\Pi_2)) \neq \{0\}$ if and only if $T(\psi_1) \cong T(\psi_2)$.*

Corollary 0.7. *The conjecture holds for (1) cohomological representations, or, more generally, A-packets of representations with regular infinitesimal character and for (2) tempered representations.*

Here (1) is reduced to the cohomological case, where Theorem 0.3 and [8, Corollary 1] (which identifies the tempered companions of cohomological A-parameters) do the job. (2) follows from Theorem 0.5.

Here is a consequence of Theorem 0.4 reformulated in terms of parameters; it refines a special case of the conjecture and is suggestive:

Corollary 0.8. *If G is a pure inner form of a compact group, then for cohomological A-packets Π_1 and Π_2 with parameters ψ_1 and ψ_2 with $T(\psi_1) \cong T(\psi_2) = \phi$, we have*

$$\sum_{G' \in H^1(\mathbb{R},G)} \dim Ext^*_{(\mathfrak{g}',K')}(\pi(\Pi_1(G')), \pi(\Pi_2(G'))) = |\pi_0(C(\phi)/Z(\hat{G}))| \tag{0.5}$$

where $C(\phi) = \{g \in \hat{G} : Ad(g) \circ \phi \cong \phi\}$ is the centralizer of the L-parameter ϕ.

Twisted elliptic pairings and dual group geometry. Corollary 0.7 seems to be as far as we can go using localization. To treat general A-packets in Conjecture 0.6, by which we mean here the A-packets defined by Adams-Barbasch-Vogan [1] using the microlocal geometry of the space of Langlands parameters, a different approach is needed. In the talk we outlined such an approach, which is work in progress, based on twisted elliptic pairings of characters and Vogan's Kazhdan-Lusztig duality. The basic observation, first made in the very special case (0.3) in [8], is that the θ acts on $Ext^i(\mathscr{R}^{S_1}_{\mathfrak{q}_1}(\mathbb{C}), \mathscr{R}^{S_2}_{\mathfrak{q}_2}(\mathbb{C}))$ is by $(-1)^{i+c_1-c_2}$ where $c_i = \operatorname{codim}_X O_i$ (this follows from Theorem 0.1), so that $\dim Ext^*(\mathscr{R}^{S_1}_{\mathfrak{q}_1}(\mathbb{C}), \mathscr{R}^{S_2}_{\mathfrak{q}_2}(\mathbb{C}))$ is $\pm$ the Lefschetz number of θ. Remarkably, this holds in some generality: The recent work of Davis and Vilonen [6] shows that the unitarity of (Hermitian) representations is completely characterized by the action of θ on the graded pieces of the Hodge filtration: it is by a sign depending only on parity. This allows one to compute θ on Ext groups and relate $\dim Ext^*(\pi(\Pi_1), \pi(\Pi_2))$ to elliptic pairings using [3]. (Indeed, all the numerical results above e.g. Theorem 0.3 etc. are statements about twisted elliptic pairings.) The natural thing to compute is the sum over pure inner forms, i.e. the twisted elliptic pairings of Vogan A-packets, using the refined Langlands parametrization in [1]. Vogan's KL duality (cf. [1, §16]) can then be used to transfer the computation to the dual side, i.e. to a geometric pairing of perverse sheaves on the space of Langlands parameters defined in [1]. This being done, proving the criterion for nonvanishing becomes a geometric problem. More naturally, as in Corollary 0.8, there is an expression for $\dim Ext^*$ between Vogan packets in terms of the symmetry group of the tempered companion parameter. At the moment we have checked that this approach works in a special case, namely *Conjecture 0.6 holds if one of* Π_1 *or* Π_2 *is tempered.* We expect that the general case can be treated, but this is work in progress.

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School of Mathematics, Tata Institute of Fundamental Research, Mumbai 400005, India
E-mail address: arvind@math.tifr.res.in

CONVEX ELEMENTS AND COHOMOLOGY OF DEEP LEVEL DELIGNE-LUSZTIG VARIETIES

SIAN NIE
JOINT WITH IVANOV, PANJUN TAN AND QINGCHAO YU

Classification AMS 2020: 11S37, 20G25, 11G25.

Keywords: Convex elements, deep level Deligne-Lusztig varieties, Fargues-Scholze paremeters.

Deligne-Lusztig varieties/representations have been a cornerstone in the representation theory of finite reductive groups since their introduction in [2]. Lusztig's extension of these constructions to the p-adic setting, known as deep level Deligne-Lusztig varieties/representations, has since become a major research focus. The significance of these deep level analogs stems from their dual appeal: the varieties themselves exhibit exceptional cohomological and arithmetic properties, while their associated representations serve as key tools for realizing irreducible supercuspidal representations of p-adic groups, particularly in advancing the local Langlands correspondence. We refer to

In [1], Boyarchenko–Weinstein started a program toward a complete description of the cohomology of deep level Deligne-Lusztig varieties. various related/partial results in this direction were obtained in [3, 6, 4, 5, 8] on deep level Deligne–Lusztig varieties of Coxeter type. To handle the general case, we introduce the notion of Convex elements in Weyl groups, and then complete the program of Boyarchenko–Weinstein for deep level Deligne–Lusztig varieties associated to convex elements.

We sketch the main results as follows.

First we introduce the definition of convex element

Definition 0.1 (Ivanov-Nie)**.** *Let Φ be root system and let W be its Weyl group. We fix a set Φ^+ of positive roots. Let σ be an automorphism of Φ. Let $x \in W\sigma$ be an elliptic element. For $\gamma \in \Phi^+$ we define*

$$n_x(\gamma) := \min\{i \in \mathbb{Z}_{\geq 1}; x^i(\gamma) \in -\Phi^+\}.$$

We say x is quasi-convex *if*

$$n_x(\alpha+\beta) \leq \max\{n_x(\alpha), n_x(\beta)\}$$

for all $\alpha, \beta \in \Phi^{\pm}$ such that $\alpha+\beta \in \Phi$. Moreover, we say x is convex *if both x and x_1 are quasi-convex.*

It is proved in [10] that convex elements afford very nice properties.

Theorem 0.2. *[Nie-Tan-Yu] Let notation be as in Definition 0.1. We have*

(1) Each elliptic W-conjugacy of $W\sigma$ has a Convex representative;

(2) Each convex element satisfies the Steinberg cross-section theorem.

Let G be a reductive group over a p-adic field k, and let T be an unramified elliptic torus of G. Let W denote the Weyl group of T in G. Fix another prime number $\ell \neq p$ and let $\Lambda = \{\overline{\mathbb{Q}}_\ell, \overline{\mathbb{F}}_\ell\}$.

It is prove in [9] that there is an explicit description of the cohomology of deep level Deligne-Lusztig varieties associated to convex elements.

Theorem 0.3 (Ivanov-Nie). *Let X be a deep level Deligne-Lusztig variety associated to a convex element in W. Let $\phi : T(k) \to \Lambda^\times$ be a smooth character. Then we have*

$$R\Gamma_c(X, \overline{\mathbb{Q}}_\ell)[\phi] \cong \pm\kappa_\phi \otimes R\Gamma_c(\bar{X}, \Lambda)[\phi_{-1}],$$

where $\pm\kappa_\phi$ is the complex of the geometric Weil-Heisenberg representation associated to ϕ, and $R\Gamma_c(\bar{X}, \Lambda)[\phi_{-1}]$ is the complex of certain classical Deligne-Lusztig representation.

As a consequence, if ϕ_{-1} is a non-singular character, then the above complex concentrates at a single cohomological degree $N_\phi \in \mathbb{Z}_{\geq 0}$.

Combining the above results with [7, Corollary 10.4.2], the above result gives description of the Fargues–Scholze parameters of modular supercuspidal representations.

Corollary 0.4. *Let T, X, ϕ be as in Theorem 0.2. Assume $\Lambda = \overline{\mathbb{F}}_\ell$ and ϕ is a toral character, Then*

$$\pi_{T,\phi} := \text{c-ind}_{Z(k)\mathcal{G}_{\mathbf{x}}(\mathcal{O}_k)}^{G(k)} H_c^{N_\phi}(X_r, \lambda)[\phi],$$

is an irreducible supercuspidal representation of $G(k)$. Here Z is the center of G and $\mathcal{G}_{\mathbf{x}}$ is the parahoric group scheme, over the integer ring $\mathcal{O}_k$ of k, associated to T.

Moreover, the Fargues–Scholze parameter of $\pi_{T,\phi}$ is

$$W_k \xrightarrow{{}^L\phi} {}^LT(\lambda) \xrightarrow{{}^L\phi} {}^LG(\lambda),$$

where ${}^L\phi$ is the L-parameter given by class field theory and ${}^Lj$ is the canonical L-embedding (notation as in [7, Theorem 10.4.1]*).*

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Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China

School of Mathematical Sciences, University of Chinese Academy of Sciences, Chinese Academy of Sciences, Beijing 100049, China
Email address: niesian@amss.ac.cn

CHARACTER THEORY AT A TORSION ELEMENT

DIPENDRA PRASAD

Classification AMS 2020: Primary 11F70; Secondary 22E55

Keywords: Branching laws, principal $\mathrm{SL}(2)$, Coxeter element, character theory, Weyl character formula

The lecture reported on a joint work done with Santosh Nadimpalli and Santosha Pattanayak, [NPP]. This work calculates the character of an irreducible representation of a complex reductive algebraic group G at any element of its principal $\mathrm{SL}_2(\mathbb{C})$, in particular, at *principal elements* of a maximal torus which operates on all simple roots by the same scalar, giving us another proof of a theorem of Kostant on the character values at the Coxeter conjugacy class, [Kos76], and more generally for any power of the Coxeter element.

We prove that these principal elements of order m in the adjoint group have smallest dimensional centralizer among elements of order m in the adjoint group. Our main theorem on character values becomes sharper when these principal elements of order m in the adjoint group are the only elements up to conjugacy having the smallest dimensional centralizer among elements of order m in the adjoint group. This turns out to be true for most groups if $m|h$, where h is the Coxeter number of G, though for certain pairs (G, m) it fails.

We define a group $\mathrm{G}(m)$ which is the dual group of the (connected component of identity of the) centralizer of the principal element of order m in the adjoint group of $\widehat{\mathrm{G}}$, which plays an important role in this work. Our main theorem on character values depends on identifying a particular irreducible representation of the simply connected cover of $\mathrm{G}(m)^{\mathrm{der}}$, the derived subgroup of $\mathrm{G}(m)$, actually only its dimension, of highest weight $\rho/m - \rho_m$ (restricted to the maximal torus of $\mathrm{G}(m)^{\mathrm{der}}$ where it is integral), involving half-sum of positive roots of G and $\mathrm{G}(m)$, which in turn depends on the precise heights of the simple roots in the centralizer of the powers of the Coxeter element. A preliminary analysis due to us is completed by Patrick Polo in [Polo].

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D.P.: Indian Institute of Technology Bombay, Powai, Mumbai-400076
Email address: prasad.dipendra@gmail.com

GLUING CLUSTER STRUCTURES

GUS SCHRADER

Fock and Goncharov [FG09a, FG09b, FG10] have constructed a quantization of higher rank Teichmüller theory based on the cluster Poisson structure on the moduli space of decorated $G = PGL_n(\mathbb{C})$-local systems on a marked surface S introduced in their earlier work [FG06]. Such a cluster Poisson variety admits a 'chartwise' non-commutative deformation, in which the combinatorics of the cluster atlas is used to construct something like a non-commutative scheme where each chart in the atlas is assigned a non-commutative 'quantum torus algebra' $\mathcal{T}^q$ defined over $\mathbb{Z}[q^{\pm 1}]$, whose $q = 1$ specialization recovers the coordinate ring of the chart in question. For a pair of charts related by a single mutation, it was explained in [FG09a] that there exists a non-commutative analog of the classical gluing map, defined using the quantum dilogarithm function. Taking 'global sections' of the resulting object yields a single non-commutative algebra $\mathbb{L}$ known as the *quantum universal Laurent ring* associated to the cluster Poisson variety, which consists of elements that are skew Laurent polynomials in every quantum torus in the cluster atlas. Moreover, any symmetry of the classical variety which can be realized via cluster transformations automatically extends to an automorphism of $\mathbb{L}$.

In this way, given a pair (G, S) as above Fock and Goncharov construct in [FG09a] a non-commutative algebra $\mathbb{L}_{G,S}$ carrying an action of the mapping class group Γ_S by quantum cluster transformations. In the same paper, Fock and Goncharov conjectured that for fixed G, the assignment $S \rightsquigarrow \mathbb{L}_{G,S}$ should behave like an algebraic analog of a modular functor in 2D conformal field theory in the following sense.

Each interior 'puncture' point p on S gives rise to a central (and thus mutation-invariant) subalgebra $R_T(p) \subset \mathbb{L}_{G,S}$ isomorphic to the representation ring of the maximal torus T of G. Moreover, whenever p is not the only special point on its connected component there is also an action of the Weyl group W_G on $\mathbb{L}_{G,S}$ by cluster transformations, which restricts to the reflection representation on the central subalgebra $R_T(p)$. Hence the subalgebra of invariants $R_G(p) = R_T(p)^{W_p}$ gives a canonical copy of the representation ring of G associated to each such puncture.

Now suppose that c is a simple closed curve on S, and S' the surface obtained by cutting S open along c, and S° the surface obtained by shrinking the corresponding boundary circles $c_\pm$ on S' to punctures $p_\pm$. Then there is a central ideal $\mathcal{I}_c \subset \mathbb{L}_{G,S'}$ generated by the relations $\chi_{p_+} = \chi^*_{p_-}$ identifying the central element in $R_T(p_+)$ corresponding to a character χ with the dual character in $R_T(p_-)$. Assuming that neither of $p_\pm$ is the only special point on its connected component in S', we have an action of the product of Weyl groups $W_{p_+} \times W_{p_-}$, and the ideal $\mathcal{I}_c$ is preserved by the diagonal subgroup $W(c)$. The centralizer $\Gamma_{S;c}$ of the Dehn twist along c in the mapping class group Γ_S can be identified with the quotient of the mapping class group $\Gamma_{S'}$ by the central subgroup generated by the product $\tau_{c_+}\tau_{c_-}$ of Dehn twists along $c_\pm$.

Conjecture 0.1. *[Algebraic modular functor conjecture for $G = \mathrm{PGL}_n$; see* [FG09a] *Section 6.2,* [GS19] *Conjectures 2.27 and 2.29.] For any essential simple closed curve c as above,* *there is a canonical subalgebra $R_G(c) \subset \mathbb{L}_{G,S}$, and an isomorphism of algebras*

$$\eta_c \colon \mathbb{L}_{G,S}^{R_G(c)} \simeq (\mathbb{L}_{G,S'}/\mathcal{I}_c)^{W(c)} \tag{0.1}$$

where $\mathbb{L}_{G,S}^{R_G(c)}$ denotes the centralizer of $R_G(c)$ in $\mathbb{L}_{G,S}$. The map (0.1) *is equivariant with respect to the action of $\Gamma_{S;c}$ by cluster transformations on both sides, and restricts to an isomorphism $R_G(c) \simeq R_G(p)$.*

In this talk we describe the proof of this conjecture obtained in the joint work [SS25] with A. Shapiro. Our proof requires introducing an enhanced version of the moduli space of decorated local systems incorporating extra data associated to the boundary components $c_\pm$ created when cutting S along c. When $G = PGL_2(\mathbb{C})$, the extra data in the new moduli space can be roughly thought of as incorporating 'twist' coordinates canonically conjugate to the 'length coordinates' coming from the copies of $R_T(c_\pm)$ associated to the $c_\pm$ as in [Mir07, AM24]. In order to construct the quantization of the enhanced moduli space we need to go beyond the standard theory of cluster algebras and work with a new object we call the *residue universal Laurent ring*. It is obtained by localizing $\mathbb{L}_{G,S'}$ at a canonical mutation-invariant collection of divisors $\emptyset(c_\pm)$, taking invariants for $W(c_\pm)$, and then passing to the subalgebra consisting of elements having only simple poles at each such divisor with residues satisfying a certain symmetry condition related to the affine Weyl group similar to the ones introduced in [GKV97] in the study of affine Hecke algebras. With these definitions, we in fact prove the following stronger result allowing us to completely reconstruct $\mathbb{L}_{G,S}$ in terms of data associated to the cut surface S':

Main Theorem. *Let (G, S, c) be as in Conjecture 0.1. Then there is a $\Gamma_{S;c}$ equivariant algebra isomorphism*

$$\eta_c \colon \mathbb{L}_{G,S} \simeq \mathbb{L}_{G,S';\phi}. \tag{0.2}$$

The isomorphism η_c is constructed using a version of the *Whittaker transform* for the $U_q(sl_n)$ open Toda integrable system, and the mapping class group equivariance is deduced as a consequence of identities satisfied by the $U_q(sl_n)$ Whittaker functions, generalizing the (generating function of all) Pieri rules. We also discuss an interesting special case of the Theorem is when S is a punctured disk with two boundary marked points (cf [SS19]), so that the corresponding universal Laurent ring is a variant of the quantum group $U_q(sl_n)$. As we will explain in the talk, the isomorphism (0.2) can then be understood as an explicit 'universal Clebsch-Gordan intertwiner' for the quantum group.

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NORTHWESTERN UNIVERSITY, USA
Email address: gus.schrader@northwestern.edu

BRAID GROUP SYMMETRIES ON POISSON ALGEBRAS ARISING FROM QUANTUM SYMMETRIC PAIRS

JINFENG SONG

For a complex semisimple Lie algebra $\mathfrak{g}$, the associated Drinfeld-Jimbo quantum group $\mathrm{U} = U_q(\mathfrak{g})$ is a central object in modern mathematics. Let $\mathcal{A} = \mathbb{Z}[q^{\pm 1/2}]$. The quantum group U admits two well-known integral $\mathcal{A}$-forms: one, introduced by Lusztig, specializes to the universal enveloping algebra $U(\mathfrak{g})$ at $q = 1$; the other arises from the work of De Concini–Kac in the study of the quantum groups at roots of unity. It was proved by De Concini–Kac–Procesi [DCKP92] that after rescaling, the De Concini–Kac form specializes at $q = 1$ to the coordinate algebra $\mathbb{C}[G^*]$ for the dual Poisson-Lie group G^*. This builds a direct connection between the rescaled De Concini–Kac form and Poisson geometry. We refer to the rescaled De Concini–Kac form as the DCKP-integral form, and denote it by $\mathrm{U}_{\mathcal{A}}$.

Let θ be an algebra involution on $\mathfrak{g}$ and $\mathfrak{g}^\theta \subset \mathfrak{g}$ be the fixed-point subalgebra. Associated with a symmetric pair $(\mathfrak{g}, \mathfrak{g}^\theta)$, the *quantum symmetric pair* $(\mathrm{U}, \mathrm{U}^\imath)$, introduced by Letzter, consists of the quantum group U and a coideal subalgebra $\mathrm{U}^\imath \subset \mathrm{U}$, called an *$\imath$quantum group*. Proposed by Bao–Wang, $\imath$quantum groups can be viewed as vast generalizations of quantum groups, and the $\imath$program aims at generalizing fundamental constructions on quantum groups to $\imath$quantum groups.

The Lusztig-type integral form on modified $\imath$quantum groups was introduced by Bao and Wang [BW18] in the study of the $\imath$canonical basis, and it has been extensively studied over the last decade in representation theory, categorification, and geometry.

On the other hand, to relate $\imath$quantum groups with Poisson geometry, one is forced to consider the DCKP-type integral form $\mathrm{U}^\imath_{\mathcal{A}}$, which was introduced recently [So24], defined by $\mathrm{U}^\imath_{\mathcal{A}} = \mathrm{U}_{\mathcal{A}} \cap \mathrm{U}^\imath$. This integral form $\mathrm{U}^\imath_{\mathcal{A}}$ specializes to the coordinate algebra of the Poisson homogeneous space $K^\perp \backslash G^*$, and it has revealed exciting connections with cluster algebras. Unlike the Lusztig-type integral form, the DCKP-type integral form $\mathrm{U}^\imath_{\mathcal{A}}$ has not been deeply studied, and its basic algebraic properties remain unclear.

In this talk, we study the DCKP-type integral form for $\mathrm{U}^\imath$, as well as its semi-classical limit. We establish relative braid group symmetries and PBW bases on this integral form. By taking the semi-classical limit, we obtain relative braid group symmetries and a set of polynomial generators on the coordinate Poisson algebra $\mathbb{C}[K^\perp \backslash G^*]$. This allows us to describe the Poisson bracket on $K^\perp \backslash G^*$ explicitly.

It is well-known that $\mathrm{U}_{\mathcal{A}}$ is invariant under Lusztig's braid group symmetries and that there exists a rescaled PBW basis of $\mathrm{U}_{\mathcal{A}}$ established using braid group symmetries. As a generalization of Lusztig braid group symmetries on quantum groups, relative braid group symmetries $\mathbf{T}_i$ on $\mathrm{U}^\imath$ were systematically constructed for arbitrary finite type by Wang–Zhang [WZ23]. Using the relative braid group symmetries, a PBW basis was constructed on $\mathrm{U}^\imath$ in [LYZ25].

Therefore, it is natural to expect that the DCKP-type integral form $\mathrm{U}^\imath_{\mathcal{A}}$ is invariant under the relative braid group symmetries and admits a (rescaled) PBW basis in general. In this talk we settles this problem for $\mathrm{U}^\imath_{\mathcal{A}'}$ where $\mathcal{A}'$ is a certain localization of $\mathcal{A}$.

Theorem 0.1. *Let $(\mathrm{U}, \mathrm{U}^\imath)$ be a quantum symmetric pair of arbitrary type.*

(1) *The relative braid group symmetries $\mathbf{T}_i$ on $\mathrm{U}^\imath$ preserve the integral form $\mathrm{U}^\imath_{\mathcal{A}'}$.*
(2) *There exists a rescaled PBW basis for $\mathrm{U}^\imath_{\mathcal{A}'}$. In particular, the root vectors arising from the rescaled PBW basis form a finite generating set of $\mathrm{U}^\imath_{\mathcal{A}'}$ as an $\mathcal{A}'$-algebra.*

Let us briefly explain the ideas in the proof. The difficulty is that $\mathrm{U}^\imath_{\mathcal{A}'}$ is not automatically equipped with a finite generating set and it is hard to directly establish a generating set without the relative braid group symmetries. However, in the $q = 1$ case, if one allows the use of Poisson brackets, then a finite set of Poisson generators for $\mathbb{C}[K^\perp \backslash G^*]$ has been obtained in [So24]. Motivated by this result, we introduce the *rescaled q-commutator* as a q-analog of the Poisson bracket. Then we show that there is a finite set $\mathcal{G}$ such that any element of $\mathrm{U}^\imath_{\mathcal{A}'}$ can be obtained by taking algebraic operations and rescaled q-commutators of elements in $\mathcal{G}$. Therefore, to show the integrability of $\mathbf{T}_i$, it suffices to check formulas of $\mathbf{T}_i(x)$ for $x \in \mathcal{G}$ and this can be done directly.

Let us mention some future directions. The DCKP-type integral forms are closely related to the quantum groups at roots of unity. In [DCP93], the authors showed that the Poisson algebra $\mathbb{C}[G^*]$ can be identified with a central subalgebra of U_ϵ, the quantum group at roots of unity. This central subalgebra is crucial in the study of representations of U_ϵ. In a future work, we will see the relation between the Poisson algebra $\mathcal{P}$ and the $\imath$quantum group at roots of unity. The rescaled PBW bases are crucial for the cluster realization of quantum groups and $\imath$quantum groups. We expect that our results provide fundamental ingredients for the cluster realization of $\imath$quantum groups of general types.

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Keywords: Quantum symmetric pair, Poisson algebra, Braid group action, PBW basis

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Hong Kong University of Science and Technology, Clearwater Bay, Kowloon, Hong Kong SAR

Email address: `jfsong@ust.hk`

WAVE-FRONT SETS FOR A REDUCTIVE GROUP OVER A LOCAL FIELD

CHENG-CHIANG TSAI

Classification AMS 2020: 22E50

Keywords: wave-front sets, asymptotic cones

Let F be a local field, for simplicity assuming $\mathrm{char}(F) = 0$. Let $\mathbb{G}$ be a connected reductive group over F. When F is p-adic we furthermore assume that $p \gg \mathrm{rank}_{\bar{F}}\, \mathbb{G}$. We are interested in irreducible admissible complex representations of $G := \mathbb{G}(F)$. Let π be such a representation. Its character Θ_π is an invariant distribution on G, i.e. a G-(conjugation-)invariant linear functional on the space of test functions on G.

Let $\mathfrak{g} := (\mathrm{Lie}\, \mathbb{G})(F)$ be the Lie algebra and $\mathfrak{g}^* := (\mathrm{Lie}\, \mathbb{G})^*(F)$ the (linear) dual. Let $(\mathfrak{g}^*)^{\mathrm{nil}} \subset \mathfrak{g}^*$ be the nilpotent cone. There is a notion of $\mathrm{WF}(\pi)$, the **wave-front set** of π (see e.g. [BV80] for F archimedean and [BM97] for F non-archimedean). It is an $\mathrm{Ad}^*(G)$-invariant closed subset of $(\mathfrak{g}^*)^{\mathrm{nil}}$ that roughly speaking measures the (asymptotic) support of the Fourier transform of Θ_π.

When F is archimedean, $\mathrm{WF}(\pi)$ is related to associated varieties and are known to be an important invariant of π. When F is non-archimedean, it is known by [MW87] that $\mathrm{WF}(\pi)$ almost describes the existence of degenerate Whittaker models. Besides the local character and the degenerate Whittaker models (including Whittaker models as a special case), there are many examples in which $\mathrm{WF}(\pi)$ for certain collection of π is related to the enhanced Langlands parameter or other arithmetic properties. Such kind of result for general π is not available yet to the best of our knowledge.

Let $T \subset G$ be a maximal torus and $\theta : T \to \mathbb{C}^\times$ a "general position" unitary character. With appropriate setup we can consider the functorial lift from T to G. This will be an irreducible representation $\pi_{(T,\theta)}$ of G; for example when T is contained in a Borel subgroup it is given by the parabolic induction, when T is elliptic and $F = \mathbb{R}$ it is given by cohomological induction, and when T is elliptic and F is p-adic this is the regular supercuspidal representations of Kaletha [Kal19]. What is in common in all these cases is that $\mathrm{WF}(\pi_{T,\theta})$ has the following description: There exists an element $X_\theta \in \mathfrak{t}^* \subset \mathfrak{g}^*$ (the dual Lie algebra of T) such that

$$\mathrm{WF}(\pi_{(T,\theta)}) = \mathrm{AC}(X_\theta) := (\mathfrak{g}^*)^{\mathrm{nil}} \cap \overline{((F^\times)^2 \cdot \mathrm{Ad}^*(G)X_\theta)}. \tag{0.1}$$

Here $\mathrm{AC}(X_\theta)$ is called the **asymptotic cone** of the orbit $\mathrm{Ad}^*(G)X_\theta \subset \mathfrak{g}^*$. It is very difficult to compute in general. When $F = \mathbb{R}$, there is an algorithm to compute it, which makes uses of the Kostant-Sekiguchi correspondence and also the formalism of Cartan decomposition. In particular, it is crucial that a real Lie group has a unique conjugacy class of maximal compact subgroups.

In [Tsa23], we give a badly complicated algorithm to compute $\mathrm{AC}(X)$[1] for any $X \in \mathfrak{g}^*$. One feature of the algorithm is that it makes use of Bruhat-Tits theory, and the fact that G can have more than one conjugacy classes of maximal compact subgroups is reflected. In the talk we explain this feature.

In particular, in [Tsa23, Example 6.2] we consider the following: Denote by $\varpi_F \in F$ be a fixed uniformizer. Fix an integer $n \geq 6$. Let $\mathbb{G} = U_n$ be a unitary group over F that splits over the quadratic unramified extension E/F, given by the following hermitian form:

$$\langle x, y\rangle = x_1\bar{y}_1 + \ldots + x_{n-1}\bar{y}_{n-1} + \varpi_F x_n \bar{y}_n. \tag{0.2}$$

Note that there is an additional ϖ_F for the n-th coordinate. Fix $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathcal{O}_E^\times$ with distinct non-zero residues such that $\mathrm{Tr}_{E/F}\,\lambda_i = 0$. Consider the following diagonal elements

$$X_{-1} := \varpi_F^{-1} \cdot \begin{bmatrix} 0 & & & & & \\ & 0 & & & & \\ & & \lambda_3 & & & \\ & & & \lambda_4 & & \\ & & & & \ldots & \\ & & & & & \lambda_n \end{bmatrix}$$

$$X_0 := \begin{bmatrix} \lambda_1 & & & & & \\ & \lambda_2 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & \ldots & \\ & & & & & 0 \end{bmatrix}, \ X := X_{-1} + X_0.$$

Then, in the calculation of the asymptotic cone of $\mathrm{Ad}^*(G)(X)$, two different maximal compact subgroups of $G = U_n(E/F)$ will contribute, eventually making $\mathrm{AC}(X)$ the union of the closure of a $(n-3,3)$-orbit and the closure of another $(n-2,1,1)$-orbit. This is the first known example for which p is arbitrarily large and the wave-front set consists of two orbits whose geometric orbits are incomparable.

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Institute of Mathematics, 6F, Astronomy-Mathematics Building, No. 1, Sec. 4, Roosevelt Road, Taipei 10617, Taiwan

Email address: chengchiangtsai@gmail.com

LEVEL-RANK DUALITIES FROM D-HARISH-CHANDRA SERIES

TING XUE

Classification AMS 2020: 20G99, 14L99

Keywords: Affine Springer fibers, character sheaves, rational Cherednik algebras, cyclotomic Hecke algebras

In this talk we explain a conjectural generalisation of Uglov's level-rank (Koszul) duality that arises from d-Harish-Chandra series introduced by Broué-Malle-Michel. We discuss connections with character sheaves for graded Lie algebras and Oblomkov-Yun's construction of rational Cherednik algebra modules using affine Springer fibres. This is based on joint work with Minh-Tam Trinh [5].

1. d-Harish-Chandra series

Let $\mathbf{G}$ be a connected reductive group defined over an algebraically closed field $\overline{\mathbb{F}}_q$ of characteristic p. For simplicity, we will assume that $\mathbf{G}$ is split over $\mathbb{F}_q$. Let $F : \mathbf{G} \to \mathbf{G}$ be the Frobenius morphism and $G = \mathbf{G}^F$. By the work of Deligne and Lusztig, there are induction and restriction maps

$$R_{\mathbf{L}}^{\mathbf{G}} : \mathbb{Z}\operatorname{Irr}(L) \to \mathbb{Z}\operatorname{Irr}(G), \ \ {}^*R_{\mathbf{L}}^{\mathbf{G}} : \mathbb{Z}\operatorname{Irr}(G) \to \mathbb{Z}\operatorname{Irr}(L)$$

where $\mathbf{L}$ is an F-stable Levi subgroup of $\mathbf{G}$ and $L = \mathbf{L}^F$. If $\mathbf{L}$ is contained in a F-stable parabolic subgroup of $\mathbf{G}$, then this is the usual Harish-Chandra induction and restriction.

Consider the set of unipotent irreducible representations of G

$$\operatorname{Uch}(G) = \{\rho \in \operatorname{Irr}(G) \mid (\rho : R_{\mathbf{T}}^{\mathbf{G}}(1)) \neq 0 \text{ for some } F\text{-stable maximal torus } \mathbf{T} \subset \mathbf{G}\}$$

where 1 denotes the trivial representation of $T = \mathbf{T}^F$. Lusztig has shown that the set $\operatorname{Uch}(G)$ can be parametrized in a way that is independent of q, and that only depends on the Weyl group W of $\mathbf{G}$.

Motivated by ℓ-modular representation theory of G at primes $\ell \neq p$, Broué–Malle–Michel [2] show that there exists a decomposition of the set $\operatorname{Uch}(G)$ into d-Harish-Chandra series for each positive integer d, with the case $d = 1$ being the usual Harish-Chandra series.

Theorem 1.1 (Broué-Malle-Michel'93 [2])**.** *For each positive integer d, there exists a partition*

$$\operatorname{Uch}(G) = \coprod_{(\mathbf{L},\lambda) \text{ } d\text{-cuspidal pairs}/\sim} \operatorname{Uch}_{(\mathbf{L},\lambda)_d}$$

where $\operatorname{Uch}_{(\mathbf{L},\lambda)_d} = \{\rho \mid (\rho : R_{\mathbf{L}}^{\mathbf{G}}\lambda) \neq 0\}$. Moreover, there exist bijections

$$\varphi_{(\mathbf{L},\lambda)_d} : \operatorname{Uch}_{(\mathbf{L},\lambda)_d} \xrightarrow{\sim} \operatorname{Irr}(W_{(\mathbf{L},\lambda)_d})$$

and signs

$$\varepsilon_{(\mathbf{L},\lambda)_d} : \operatorname{Uch}_{(\mathbf{L},\lambda)_d} \to \{\pm 1\}$$

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that are compatible with induction and restriction.

Here $W_{(\mathbf{L},\lambda)_d} = Z_{W_{\mathbf{L}}^{\mathbf{G}}}(\lambda)$ are the relative Weyl groups, where $W_{\mathbf{L}}^{\mathbf{G}} = N_G(L)/L$. They are always complex reflection groups as shown by Broué-Malle-Michel. A d-cuspidal pair consists of a d-split Levi $\mathbf{L}$ and $\lambda \in \mathrm{Uch}(L)$ such that ${}^*R_{\mathbf{M}}^{\mathbf{L}}\lambda = 0$ for any d-split Levi subgroup $\mathbf{M} \subset \mathbf{L}$. A d-split Levi is an F-stable Levi subgroup of the form $\mathbf{L} = Z_{\mathbf{G}}(\mathbf{T})^\circ$ for some d-torus $\mathbf{T}$ and a d-torus is an F-stable torus $\mathbf{T} \subset \mathbf{G}$ such that the corresponding $|T|$ is a power of $\Phi_d(q)$.

Remark 1.2. *d-HC series are non-trivial only if d is a singular number, that is, if the d-th cyclotomic polynomial $\Phi_d(q)$ divides $|G|$.*

Let ζ_d denote a primitive d-th root of unity.

Conjecture 1.3 (Broué-Malle'93 [1]). *For each $W_{(\mathbf{L},\lambda)_d}$ there is a generic cyclotomic Hecke algebra $H_{(\mathbf{L},\lambda)_d}(x)$ over $\bar{\mathbb{Q}}[x^{\pm 1/\infty}]$ such that*

$$\mathbb{C} \otimes_{\bar{\mathbb{Q}}[x^{\pm 1/\infty}]} H_{(\mathbf{L},\lambda)_d}(x)|_{x\mapsto\zeta_d} \cong \mathbb{C}W_{(\mathbf{L},\lambda)_d}$$

and such that $\bar{\mathbb{Q}}_\ell \otimes_{\bar{\mathbb{Q}}[x^{\pm 1/\infty}]} H_{(\mathbf{L},\lambda)_d}(x)|_{x\mapsto q}$ is the endomorphism algebra of a Deligne-Lusztig representation associated to (L,λ).

2. Level-rank (Koszul) duality from d-Harish-Chandra series

Fix positive integers $l \neq m$. Consider the following diagram

$$\begin{tikzcd}
 & & \mathrm{Uch}_{(\mathbf{L},\lambda)_l} \cap \mathrm{Uch}_{(\mathbf{M},\mu)_m} \arrow[dll, "\varphi_{(\mathbf{L},\lambda)_l}"'] \arrow[drr, "\varphi_{(\mathbf{M},\mu)_m}"] \arrow[ddddll, "\Psi_l"] \arrow[ddddrr, "\Psi_m"'] & & \\
\mathrm{Irr}(W_{(\mathbf{L},\lambda)_l})_{(\mathbf{M},\mu)_m} \arrow[rrrr, leftrightarrow, "\upsilon_{l,m}", "\sim"'] \arrow[d, "\text{Tits deformation}"] & & & & \mathrm{Irr}(W_{(\mathbf{M},\mu)_m})_{(\mathbf{L},\lambda)_l} \arrow[d, "\text{Tits deformation}"'] \\
K_0(H_{(\mathbf{L},\lambda)_l}(x))_{(\mathbf{M},\mu)_m} \arrow[dd, "d_{\zeta_m}"', "x\mapsto\zeta_m"] & & & & K_0(H_{(\mathbf{M},\mu)_m}(x))_{(\mathbf{L},\lambda)_l} \arrow[dd, "x\mapsto\zeta_l"', "d_{\zeta_l}"] \\
 & \mathcal{O}^{\mathrm{rat}}_{(\mathbf{M},\mu)_m}(W_{(\mathbf{L},\lambda)_l}) \arrow[rr, leftrightarrow, "\text{Koszul}"] \arrow[dl, "\text{KZ}"] & & \mathcal{O}^{\mathrm{rat}}_{(\mathbf{L},\lambda)_l}(W_{(\mathbf{M},\mu)_m}) \arrow[dr, "\text{KZ}"'] & \\
K_0(H_{(\mathbf{L},\lambda)_l}(\zeta_m))_{(\mathbf{M},\mu)_m} & & & & K_0(H_{(\mathbf{M},\mu)_m}(\zeta_l))_{(\mathbf{L},\lambda)_l}
\end{tikzcd}$$

where d_{ζ_m} and d_{ζ_l} are decomposition maps, $\mathcal{O}^{\mathrm{rat}}_{(\mathbf{M},\mu)_m}(W_{(\mathbf{L},\lambda)_l})$ denotes a block (specified by $(\mathbf{M},\mu)_m$) of the category $\mathcal{O}$ of a rational Cherednik algebra $H^{\mathrm{rat}}(W_{(\mathbf{L},\lambda)_l})$, similarly for $\mathcal{O}^{\mathrm{rat}}_{(\mathbf{L},\lambda)_l}(W_{(\mathbf{M},\mu)_m})$.

Conjecture 2.1 (Trinh-X.[5]).
1. *The image of Ψ_l, resp. Ψ_m, is a union of blocks of the specialised Hecke algebra $H_{(\mathbf{L},\lambda)_d}(\zeta_m)$, resp. $H_{(\mathbf{M},\mu)_m}(\zeta_l)$.*
2. *The bijection $\upsilon_{l,m}$ categorifies to a derived (Koszul) equivalence between highest weight covers of respective blocks of $\mathrm{Rep}\, H_{(\mathbf{L},\lambda)_d}(\zeta_m)$ and $\mathrm{Rep}\, H_{(\mathbf{M},\mu)_m}(\zeta_l)$, that is, an equivalence between $D^b(\mathcal{O}^{rat}_{(\mathbf{M},\mu)_m}(W_{(\mathbf{L},\lambda)_l}))$ and $D^b(\mathcal{O}^{rat}_{(\mathbf{L},\lambda)_l}(W_{(\mathbf{M},\mu)_m}))$.*

Theorem 2.2 (Trinh-X.[5]). *When l and m is coprime, the above conjecture holds for GL_n (and GU_n). In particular, the image of Ψ_l (or Ψ_m) is a single block.*

The complex reflection groups involved in Theorem 2.2 are of the form $S_N \ltimes (\mathbb{Z}/d)^N$. To prove the theorem, we use Lyle-Mathas' classification of blocks of cyclotomic Hecke algebras attached to $S_N \ltimes (\mathbb{Z}/d)^N$ at roots of unity. We show that the bijections $\upsilon_{l,m}$ are Uglov's level-rank duality bijections, and make use of the level-rank duality conjectured by Chuang-Miyachi and proved by Rouquier-Shan-Varagnolo-Vasserot, Shan-Varagnolo-Vasserot, Losev and Webster.

Remark 2.3. *Our conjecture generalises level-rank duality from complex reflection groups $S_N \ltimes (\mathbb{Z}/d)^N$ to complex reflection groups of exceptional types.*

Remark 2.4. *Recently Chlouveraki-Malle have verified that part (1) of the conjecture holds for exceptional groups. They also extend the result to Suzuki-Ree groups.*

3. Connections with character sheaves on graded Lie algebras and affine Springer fibers

In this section let G be a simply connected almost simple algebraic group over $\mathbb{C}$. For each regular elliptic number m of G, there exists a unique (up to conjugacy) order m automorphism $\theta : G \to G$ that is GIT-stable. Let $\mathfrak{g} = \operatorname{Lie} G$ and $\mathfrak{g} = \oplus_{i\in\mathbb{Z}/m}\mathfrak{g}_i$ the grading induced by θ. GIT-stable means that the action of $G_0 := G^\theta$ on $\mathfrak{g}_1$ has stable vectors. Vinberg shows that (for general $\mathbb{Z}/m$-graded Lie algebras) there exists Cartan subspace $\mathfrak{a} \subset \mathfrak{g}_1$ such that

$$\mathfrak{g}_1/\!/G_0 \cong \mathfrak{a}/C$$

where $C = N_{G_0}(\mathfrak{a})/Z_{G_0}(\mathfrak{a})$ is a complex reflection group.

Define character sheaves on $\mathfrak{g}_1$ to be the Fourier transform of simple G_0-equivariant perverse sheaves on the nilpotent cone $\mathfrak{g}_{-1}^{\mathrm{nil}}$, where the Fourier transform is the functor $\mathrm{Four} : \mathrm{Perv}_{G_0}(\mathfrak{g}_{-1}^{\mathrm{nil}}) \to \mathrm{Perv}_{G_0}(\mathfrak{g}_1)$ (we identify $\mathfrak{g}_{-1}^*$ with $\mathfrak{g}_1$). In our study of character sheaves on $\mathfrak{g}_1$, we consider the nearby cycle sheaf $P \in \mathrm{Perv}_{G_0}(\mathfrak{g}_{-1}^{\mathrm{nil}})$ associated to the adjoint quotient map $f : \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}/\!/G_0$.

Theorem 3.1 (Grinberg-Vilonen-X.[3], Vilonen-X.[6])**.**

$$\mathrm{Four}(P) \cong \mathrm{IC}(\mathfrak{g}_1^{rs}, \mathcal{H}_C)$$

where $\mathcal{H}_C$ is a Hecke algebra associated to the complex reflection group C with explicit Hecke relations.

Here we have $\pi_1^{G_0}(\mathfrak{g}_1^{rs}) \cong Br_C \ltimes I$, where I is a finite abelian group and Br_C is the braid group associated to C.

Combining the above theorem with theorems of Lusztig-Yun, W. Liu, and Etingof, we obtain (restricting attention to cuspidal character sheaves in the principal block where I acts trivially)

$$\begin{tikzcd}
\mathrm{Char}_{G_0}^{\mathrm{cusp}}(\mathfrak{g}_1)_{\mathrm{p,st}} \arrow[r, leftrightarrow, "\mathrm{Four}", "\sim"'] \arrow[d, "\sim"'] & \mathrm{SPerv}_{G_0}^{\mathrm{cusp}}(\mathfrak{g}_{-1}^{\mathrm{nil}})_{\mathrm{p,st}} \arrow[d, "\sim"'] \\
\mathrm{Irr}_{\mathrm{p}}(\mathcal{H}_C) \arrow[r, "\sim"'] & \mathrm{Irr}_{\mathrm{p}}^{\mathrm{f.d.}}(H_{\frac{1}{m}}^{\mathrm{rat}}(W))
\end{tikzcd} \tag{3.1}$$

Returning to d-HC series, let $\mathbf{A}$ be a 1-split maximal torus and $\mathbf{T}$ a m-split maximal torus. We observe that

$$\text{the relative Weyl groups } W_{(\mathbf{A},1)_1} = W \text{ and } W_{(\mathbf{T},1)_m} = C$$
$$\text{the block of category } \mathcal{O} \text{ of rational Cherednik algebras } \mathcal{O}^{\mathrm{rat}}_{(\mathbf{T},1)_m}(W) = \mathcal{O}_{\mathrm{p}}(H^{\mathrm{rat}}_{\frac{1}{m}}(W))$$
$$\text{and the specialised Hecke algebra and } H_{(\mathbf{T},1)_m}(1) \cong \mathcal{H}_C\,.$$

We conjecture that the bijection in the bottom row of (3.1) is induced by the $(1,m)$-duality conjecture discussed in §2.

Now we turn to affine Springer fibers. Let $\gamma \in \mathfrak{g}(\mathbb{C}(\!(t)\!))$ be an elliptic regular semisimple element, homogeneous of slope $\nu = 1/m$. Let $\mathrm{Sp}_\gamma \subset \mathcal{F}\ell$ be the affine Springer fiber at γ contained in the affine flag variety.

Theorem 3.2 (Oblomkov-Yun[4]). *The stable part $H^*_{\epsilon=1}(\mathrm{Sp}_\gamma)_{\mathrm{st}}$ of the specialised $\mathbb{G}_m$-equivariant cohomology of Sp_γ has a perverse filtration such that the associated graded $\mathrm{Gr}^p_* H^*_{\epsilon=1}(\mathrm{Sp}_\gamma)_{\mathrm{st}}$ has a commuting action of the rational Cherednik algebra $H^{\mathrm{rat}}_{\frac{1}{m}}(W)$ and the braid group Br_C.*

Let us write $\mathcal{E}_{\nu,\gamma} = t^{-\dim \mathrm{Sp}_\gamma} \sum_{i,j} (-1)^i t^j \,\mathrm{Gr}^p_j H^i_{\epsilon=1}(\mathrm{Sp}_\gamma)_{\mathrm{st}}$.

Conjecture 3.3 (Trinh-X.[5]). (i) *The action of the braid group Br_C on $\mathcal{E}_{\nu,\gamma}$ factors through $H_{(\mathbf{T},1)_m}(1) \cong \mathcal{H}_C$.*

(ii) *In the appropriate Grothendieck groups, we have*

$$[\mathcal{E}_{\nu,\gamma}] = \sum_{\rho \in \mathrm{Uch}_{(\mathbf{A},1)_1} \cap \mathrm{Uch}_{(\mathbf{T},1)_m}} \varepsilon_{(\mathbf{T},1)_m}(\rho) [\Delta_\nu(\varphi_{(\mathbf{A},1)_1}(\rho)) \otimes S(\varphi_{(\mathbf{T},1)_m}(\rho))]$$

where $\Delta_\nu(\chi)$ denotes the standard module of $H^{\mathrm{rat}}_{\frac{1}{m}}(W)$ corresponding to $\chi \in \mathrm{Irr}\, W$ and $S(E)$ denotes the Specht module of $H_{(\mathbf{T},1)_m}(1) \cong \mathcal{H}_C$ corresponding to $E \in \mathrm{Irr}\, C$.

We further expect that

$$[\mathcal{E}_{\nu,\gamma}] = \sum_{\tau \in \mathrm{Irr}^{\mathrm{f.d.}}_{\mathrm{p}}(H^{\mathrm{rat}}_{\frac{1}{m}}(W)),\, \sigma \in \mathrm{Irr}_{\mathrm{p}}(\mathcal{H}_C),\, \tau \leftrightarrow \sigma} \epsilon_{\tau,\sigma} [\tau \otimes \sigma]$$

where $\epsilon_{\tau,\sigma} \in \{\pm 1\}$ and the correspondence $\tau \leftrightarrow \sigma$ is given by the bottom arrow in (3.1).

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School of Mathematics and Statistics, University of Melbourne, VIC 3010, Australia, and Department of Mathematics and Statistics, University of Helsinki, Helsinki, 00014, Finland

Email address: ting.xue@unimelb.edu.au