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Recent Applications of Model Theory

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FINITENESS RESULTS IN HENSEL MINIMAL STRUCTURES

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1. EXTENDED ABSTRACT

We present work with Halupczok, Rideau-Kikuchi, Vermeulen [3], [4], [5] which is partially still work in progress and which provides non-archimedean analogues to o-minimality and to the general Pila-Wilkie counting theorem.

The original Pila-Wilkie counting theorem on rational points on definable sets in o-minimal structures states the following.

Theorem 1.1 ([8]). *Let $X \subset \mathbb{R}^n$ be definable in an o-minimal structure. Then for every $\varepsilon > 0$ there exists $c = c_\varepsilon$ such that for every $H \geq 1$ one has*

$$\#X^{\text{trans}}(\mathbb{Q}, H) < cH^\varepsilon.$$

Here, $X^{\text{trans}}(\mathbb{Q}, H)$ is the set of points $(x_1, \dots, x_n)$ in $\mathbb{Q}^n$ lying on X^{trans} and with each x_i of height at most H , and X^{trans} is the set $X \setminus X^{\text{alg}}$ where X^{alg} is the algebraic part of X . Recall that X^{alg} is the set of $x \in X$ for which there exists a semi-algebraic curve C lying in X which is of constant local dimension 1.

Pila-Wilkie's Theorem 1.1 plays an important role in many arithmetic applications, with a culmination quite recently in the solution of the André-Oort Conjecture in [7].

A precursor to Theorem 1.1 is the theory of o-minimal structures, where cell decomposition plays important roles. In the talk I recalled in detail the definitions of o-minimality and of Hensel minimality and I explained their similarity.

Let us right away state the non-archimedean analogue of Theorem 1.1 for Hensel minimal structures on p -adic fields instead of o-minimal structures, as follows.

Theorem 1.2 (Pila-Wilkie type bounds for 1-h-minimal structures, [5]). *Let K be a finite field extension of $\mathbb{Q}_p$ for some prime number p . Let $X \subset K^n$ be definable in a 1-h-minimal structure on K . Then for every $\varepsilon > 0$ there exists $c = c_\varepsilon$ such that for every $H \geq 1$ one has*

$$\#X^{\text{trans}}(\mathbb{Q}, H) < cH^\varepsilon.$$

The notion of Hensel minimality (and its instance of 1-h-minimality) is an analogue of o-minimality for the non-archimedean setting and has been recently developed in [3] and [4]. These notions of 1-h- and of o-minimality are built upon insights coming from cell decomposition results, and their definitions boil down to conditions on unary sets.

Theorem 1.1 comes in many variants, like a variant for definable families, and versions with so-called blocks that allow for example to bound the number of points of bounded degree over $\mathbb{Q}$ (of bounded height) instead of just rational points (see [6]).

Also Theorem 1.2 comes in many variants, including for definable families, a block version (allowing to bound algebraic points of bounded degree), and a version which works uniformly in the p -adic field K . Such uniformity in the p -adic field implies similar counting results in large positive characteristic, namely when counting tuples of rational functions $a(t)/b(t)$ in $\mathbb{F}_q(t)$ with a and b polynomials of bounded degree in t , lying on a definable subset of $\mathbb{F}_q((t))^n$ for some power q of a large prime p .

In order to show Theorem 1.2, we develop Taylor approximation results for definable functions in general dimension and up to any finite degree, allowing us to provide general parametrization results for definable sets analogous to Yomdin-Gromov parametrizations. Previously only two special cases of these results were known: for subanalytic sets on the one hand [1] [2], and for dimension one in Hensel minimal structures on the other hand [4]. For curves, a generalization of Theorem 1.2 has recently been obtained in [9].

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DEFINABILITY OF HENSELIAN VALUATIONS VIA PROPERTIES OF ORDERED ABELIAN GROUPS

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Keywords: Augmentable ordered abelian groups, spines, definable henselian valuations.

This talk is based on recent joint work with B. Boissonneau, F. Jahnke, and P. Touchard [3], where we study model-theoretic properties of ordered abelian groups and their connections with the definability of henselian valuations.

Ordered abelian groups form a classical subject of model theory since the foundational work of Robinson and Zakon [8]. A central insight, going back to Schmitt [9, 10] and by Cluckers and Halupczok [4], is that many first-order properties of ordered abelian groups can be studied via their spines, i.e., chains of uniformly definable convex subgroups. They prove relative quantifier elimination for the theory of ordered abelian groups in an enriched language with auxiliary sorts for the spines, providing a robust framework that has since been applied to questions of classification, elimination of imaginaries, and dividing lines in ordered abelian groups.

The study of ordered abelian groups is closely tied to that of valued fields: by the Ax–Kochen/Ershov principle [2, 6], understanding value groups is a key step in understanding valued fields, and conversely, value groups often serve as test cases for definability problems concerning henselian valuations.

In this talk, we introduce and develop the notion of augmentability for ordered abelian groups. Given ordered abelian groups G, H , let $G \oplus H$ denote their lexicographic sum. We give the following definitions.

Definition 0.1. *Let G be an ordered abelian group. Then*

- *G is Augmentable by Infinites if there exists a non-trivial ordered abelian group H such that $G \preceq H \oplus G$;*
- *G is augmentable by infinitesimals if there exists a non-trivial ordered abelian group H such that $G \preceq G \oplus H$.*

Our main result shows that:

Theorem 0.2. *Every non-trivial ordered abelian group is Augmentable by Infinites.*

The proof proceeds via spine reduction: augmentability can be reformulated as a problem about colored multi-orders (multi-sorted generalizations of linear orders encoding spines). Using quantifier elimination and embedding arguments for these structures, we show that Augmentability by Infinites is always guaranteed. In particular, we prove that given an elementary pair $A \preceq B$ of coloured multi-orders, the convex hull of A , as well as the left and right closures of A , are elementary substructures of B , giving a multi-order analog of the Delon-Lucas result for linear orders [5].

As an application of our results, we study definable henselian valuations. Indeed, Augmentability by Infinites ensures the existence of controlled elementary extensions of the residue field. Namely, we show that if k is a field of characteristic 0 that is $\mathcal{L}_{\text{ring}}$ -elementarily equivalent to a field admitting a non-trivial henselian valuation (that is, if k is t -henselian), then $k \preceq k((\Gamma))$ for some non-trivial ordered abelian group Γ . We apply this to deduce the following:

Theorem 0.3. *Let k be a field of characteristic 0. The following are equivalent:*

- (1) k is not t -henselian;
- (2) for every henselian valued field (K, v) with residue field k , the valuation ring $\mathcal{O}_v$ is $\mathcal{L}_{\text{ring}}$ -definable (possibly using parameters),
- (3) for every henselian valued field (K, v) with residue field k , the valuation ring $\mathcal{O}_v$ is $\emptyset$ - $\mathcal{L}_{\text{ring}}$ -definable,
- (4) All henselian valuation rings with residue field elementarily equivalent to k are uniformly $\emptyset$ - $\mathcal{L}_{\text{ring}}$ -definable in $\mathcal{L}_{\text{ring}}$.

This gives a characterization of the class of fields of characteristic 0 such that there exists a uniform $\emptyset$ - $\mathcal{L}_{\text{ring}}$ -definition for all henselian valuations with residue field in that class. This complements earlier work by Anscombe–Fehm [1] on existential and universal definability, and provides a partial answer to a question of Krapp–Kuhlmann–Link [7].

To conclude the talk, we give some insights into augmentability by infinitesimals. Here, the situation is subtler: not every ordered abelian group admits such an embedding, and we present partial results isolating the spine-theoretic obstructions. In particular, we show that the class of ordered abelian groups augmentable by infinitesimals via a divisible augment is first-order axiomatizable. Finally, we show the following result, which relates augmentability by infinitesimals to the definability of a given valuation.

Theorem 0.4. *Let (K, v) be a henselian valued field with value group G and residue field k of characteristic 0. Then v is not definable in $\mathcal{L}_{\text{ring}}$ if and only if there exists a non-trivial ordered abelian group Γ such that $G \preceq G \oplus \Gamma$ and $k \preceq k((\Gamma))$.*

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IMAGES OF ALGEBRAIC SETS UNDER LATTICE QUOTIENTS

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This talk is based on joint work with Hunter Spink [1]. Suppose we are given a lattice $\Lambda \subset \mathbb{C}^n$, and let $\pi: \mathbb{C}^n \rightarrow \mathbb{C}^n/\Lambda$ be the projection map; for example, π could be the covering map of an abelian variety, or the complex exponential function. Then, given an affine algebraic set $X \subset \mathbb{C}^n$, it is natural to ask about the image set $\pi(X)$. In particular, what can be said about its closure? Ullmo-Yafaev [3] observed that the closure $\overline{\pi(X)}$ is the union of two components, namely $\pi(X)$ itself and the "flow" $\text{Fl}(X)$, which is the set of limit points of unbounded sequences in X .

In the case where the quotient $\mathbb{C}^n/\Lambda$ is compact (a complex torus), several results about $\text{Fl}(X)$ are available. For X an algebraic curve, Ullmo-Yafaev [3] showed that $\text{Fl}(X)$ is a finite union of cosets for closed subgroups (that is, *real* subtori) of $\mathbb{C}^n/\Lambda$. Peterzil-Starchenko [2] extended this result to arbitrary algebraic subsets $X \subset \mathbb{C}^n$ after appropriate modifications: They show that $\text{Fl}(X)$ is always a finite union of algebraic *families* of translated real subtori. Our work gives a similar result for many lattices Λ which are not cocompact:

Theorem 0.1 (D-Spink). *Let $X \subset \mathbb{C}^n$ be algebraic, and let $\Lambda \subset \mathbb{C}^n$ be a lattice whose real and complex spans coincide. Then we have*

$$\text{Fl}(X) = \bigcup_{i=1}^n (\pi(C_i) + \mathbb{T}_i),$$

where the C_i are algebraic sets with $\dim C_i < \dim X$, and the $\mathbb{T}_i$ are positive-dimensional closed subgroups of $\mathbb{C}^n/\Lambda$.

One could ask an analogous question over $\mathbb{R}$, phrased in terms of o-minimal sets: Suppose that $\Lambda \subset \mathbb{R}^n$ is a lattice, and that $X \subset \mathbb{R}^n$ is definable in some o-minimal structure? Then can we characterize $\text{Fl}(X)$? When Λ is cocompact, Peterzil-Starchenko [2] show that $\text{Fl}(X)$ is a finite union of definable families of real subtori, in precise analogy with their algebraic result. We show the same result for arbitrary Λ , completely eliminating the compactness assumption:

Theorem 0.2 (D-Spink). *Let $X \subset \mathbb{R}^n$ be definable in an o-minimal structure, and let $\Lambda \subset \mathbb{R}^n$ be a lattice. Then we have*

$$\text{Fl}(X) = \bigcup_{i=1}^n (\pi(C_i) + \mathbb{T}_i),$$

where the C_i are definable sets with $\dim C_i < \dim X$, and where the $\mathbb{T}_i$ are positive-dimensional closed subgroups of $\mathbb{R}^n/\Lambda$.

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VAPNIK-CHERVONENKIS DIMENSION OF APPROXIMATE SUBGROUPS

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1. THE DEFINITION OF APPROXIMATE SUBGROUPS

Approximate subgroups are fundamental objects of study in additive combinatorics aiming to capture the group-like behaviour in many combinatorial phenomena. Although the definition of approximate subgroups is now well established and widely accepted as the correct formal approach to capturing group-like combinatorial phenomena, it was actually introduced in 2008 by Terence Tao [19] after many decades of development involving numerous mathematical areas.

Originally, the study of approximate subgroups was strongly motivated by number theory and harmonic analysis, focussing essentially only on the abelian framework, and more specifically on $\mathbb{Z}$ (e.g. [7, 16]). Initially, sum-set phenomena were usually studied in a crude way by comparing the size of a set with that of its product sets. This method of capturing group-like structure is commonly known as *small doubling*. It is particularly prevalent in combinatorics, where the size of a finite set is computed by its cardinality. It is also well represented in the Minkowski's classical theory of convex geometry.

In [8], alongside the Plunnecke-Ruzsa inequalities, it was observed that, in the abelian setting, the quantitative property of small doubling is essentially equivalent to a more algebraic property. In [19], while generalising several fundamental facts of additive combinatorics to the non-abelian setting, this algebraic property was formalised as a definition.

Definition 1.1 (Approximate subgroups). *A k -approximate subgroup is a symmetric set containing the identity such that its set of pairwise products is contained in k many translates of itself.*

In other words, the definition of approximate subgroups was derived from an extraordinary fact. Several criteria for identifying subgroups can be weakened to arrive at a notion of “approximate subgroup”. Nevertheless, it turns out that the various notions of “approximate subgroup” obtained through this process are essentially equivalent [19, Theorem 5.4].

Given that there are several possible definitions which are essentially equivalent, it is only natural to wonder why Tao's definition has gained general acceptance as the standard one. For most of the community, the main argument in favour of Tao's definition is probably that, being algebraic in nature, it works well with group homomorphisms. While this is a very useful technical feature, there are also other important reasons for this consensus.

Significantly influenced by the results of Helfgott [10] (and further motivated by the work of Bourgain and Gamburd [1] in the context of expander graphs), the main aim of [19] was to extend the study of approximate subgroups from the abelian case to the non-abelian setting. While small doubling captures group-like behaviour well in the abelian context, it actually needs to be replaced by small tripling in the non-commutative framework. In contrast, Tao's definition is equally applicable to both contexts, which is likely what motivated [19] to use it in the first place.

Another key advantage of Tao's definition is that it only involves the group operation, making it feasible to study infinite sets. Infinite approximate subgroups have been shown to capture another relevant group-like phenomenon: the work of Meyer on quasicrystals [15], which is now being studied in the more general context of approximate lattices [3, 14].

In light of all this evidence, we are tempted to propose the following general thesis:

Thesis. *Group-like phenomena are correctly captured by the current definition of approximate subgroups.*

This thesis should be interpreted as a means of making two types of conjectures:

- If there is a group-like structure, there must be an associated approximate subgroup.
- Any generalisation of approximate subgroups is either too wild (so it does not truly capture group-like behaviour) or too subtle (so it can be reduced to approximate subgroups in some way).

2. VAPNIK-CHERVONENKIS DIMENSION

In a group G , it is natural to identify subsets with relations via the group operation. Given a subset A of G , we can define the binary relation invariant under left translations given by $\sim^A (x, y) \Leftrightarrow y^{-1}x \in A$. Conversely, for a binary relation invariant under left translations $x \sim y$, we get the subset $A_\sim = \{y^{-1}x : x \sim y\}$. Through this duality, we can transfer terminology about relations to sets and *vice versa*. One fundamental notion in mathematics concerning binary relations is the Vapnik-Chervonenkis dimension.

The VC dimension measures the complexity of the binary relation. It is named after Vapnik and Chervonenkis, who introduced the concept working in statistics in [21] in 1968. It is well known that this notion was developed independently in several areas at the same time. In extremal combinatorics, it was considered without giving it a specific name [17]. In model theory, it was introduced by Shelah as the negation of the *independence property* [18] from which the acronym NIP (no independence property) has become ubiquitous in the area. In computational learning theory, it is also called *probably approximately correct* [22].

Definition 2.1. *Let X be a subset of a group G . The VC dimension of X is the VC dimension of its associated binary relation, that is the VC dimension of the family $\{aX : a \in G\}$ of translates of X . We say that X is VC if it has finite VC dimension.*

In [5], it was suggested that the VC dimension partially captures some group-like behaviour. In particular, the following fact was noted:

Fact 2.2. *Let $X \subset G$ be a subset. Then, its associated binary relation omits the induced subgraph $(\{0, 1\}, \{0, 1\}, \leq)$ if and only if X is the coset of a proper subgroup of G .*

The paper [5] is dedicated to the study of finite approximate subgroups with bounded VC dimension. For these approximate subgroups, the authors present some improvements to several known results about finite approximate subgroups. One might naturally wonder which approximate subgroups have finite VC dimension. It is easy to note that geometric progressions (the most basic example of an approximate subgroup) have finite VC dimension. At a conference in Cambridge in 2022, Conant asked whether nilprogressions (the natural non-abelian generalisation of geometric progressions) have finite VC dimension [23, §14.3].

Question 2.3 (Conant). *Is there a constant $C(d, s)$ such that every nilprogression of dimension d and nilpotent length s has VC dimension smaller than $C(d, s)$?*

After this conference, in a private conversation at Oberwolfach, Tran¹ suggested to me that having finite VC dimension should essentially be similar to being an approximate subgroup. To make this precise, we have the following conjecture:

Conjecture 2.4 (Tran-Jing). *Every approximate subgroup is commensurable to a VC approximate subgroup.*

According to the observation that VC dimension partially captures group-like behaviour, Tran-Jing conjecture may be interpreted in relation to our general thesis.

3. TRAN-JING CONJECTURE

The VC dimension of a subset $A \subseteq G$ in a group G seems to depend on the ambient group G . However, it is easy to see that this is actually a local property. Write $\text{VC}_X(A/Y) = \text{VC}(\sim_{|X \times Y}^A) = \text{VC}(\{yA \cap X : y \in Y\})$.

Lemma 3.1. *Let $X \subseteq G$. Then, $\text{VC}_X(X/XX^{-1}) \leq \text{VC}(X) \leq \text{VC}_X(X/XX^{-1}) + 1$.*

Corollary 3.2. *Let G be a Lie group. Then, every neighbourhood of the identity contains a compact neighbourhood of the identity which is a VC approximate subgroup.*

Proof sketch. Lie groups are analytic and $\mathbb{R}_{\text{an}}$ is o -minimal [6]. □

Consequently, we can partially solve Tran-Jing conjecture in the *laminar case*, i.e. when there is a Lie model (see [11]).

Proposition 3.3. *Let X be a definable approximate subgroup on a definable group G and suppose X has a Lie model $\pi: H \rightarrow L$ with $\ker(\pi) \subseteq X^m$. Then, X is commensurable to a type-definable VC approximate subgroup.*

Unfortunately, this is an incomplete solution. To solve Tran-Jing conjecture in the laminar case, we need to make V definable in place of type-definable. Nevertheless, working with type-definable sets does provide a continuous-like weak solution.

Definition 3.4. *Let $R(x, y)$ and $Q(x, y)$ be two disjoint binary relations on $X \times Y$. A subset $S \subseteq X$ is shattered by $R(x, y)$ against $Q(x, y)$ if, for every $D \subseteq S$, there is $y_D \in Y$ such that*

$$\begin{cases} x \in D & \text{Rightarrow} & R(x, y_D) \\ x \notin D & \text{Rightarrow} & Q(x, y_D) \end{cases}$$

¹Jing indicated to me that this conjecture was motivated by some previous discussions in relation to [13].

Corollary 3.5. *For every $k \in \mathbb{N}$ there is $d(k)$ such that every finite k -approximate subgroup X is $d(k)$ -commensurable to an approximate subgroup $Y \subseteq X^4$ such that no subset of size $d(k)$ is shattered by Y against $G \setminus X^4$.*

According to Hrushovski's Lie quasi-model theorem [12], solving Tran-Jing conjecture in the non-laminar case requires a better understanding of the approximate subgroups obtained from quasimorphisms.

Let G and H be groups and $E \subseteq H$ a symmetric subset containing the identity. A quasimorphism $f: G \rightarrow H$ from G to H of error set E is a function from G to H such that $f(1_G) = 1_H$ and $f(x)f(y)f(xy)^{-1} \in E$ for all $x, y \in G$.

Example 3.6 (Brooks' quasimorphisms). *Consider a free group $F = \langle X \rangle$ and v a reduced word of F . Let $f: F \rightarrow \mathbb{Z}$ be the map counting (with sign) the number of occurrences of v in reduced forms:*

$$f(w) = (\# \text{occurrences of } v \text{ in } w) - (\# \text{occurrences of } v^{-1} \text{ in } w).$$

Then, $f: F \rightarrow \mathbb{R} : [-3, 3]$ is a quasimorphism with error set $[-l, l]$, where l only depends on the length of v . Hence, $f^{-1}([-l, l])$ is an approximate subgroup [12, Proposition 5.12]. These quasimorphisms were first described by Brooks in [4, §3(a)].

During the conference, I asked whether the approximate subgroups given by Brooks' counting quasimorphisms have finite VC dimension as a starting point to attack the Tran-Jing conjecture in the non-laminar case. I am pleased to announce that Will Johnson has privately communicated to me a positive solution for this particular case.

Proposition 3.7 (Will Johnson). *Let $F = \langle X \rangle$ be a free group and $v_1, \dots, v_n$ be reduced words of F . For each i , let $f_i: F \rightarrow \mathbb{Z}$ be Brooks' quasimorphism counting v_i . Consider the quasimorphism $h = \sum_{i=1}^n c_i f_i$ with $c_1, \dots, c_n \in \mathbb{R}$. Let Γ be an approximate subgroup of F of the form $\Gamma = h^{-1}[-m, m]$ for some $m \in \mathbb{R}$. Then, the VC dimension of Γ is finite and bounded by a function on $v_1, \dots, v_n$ (independently of $c_1, \dots, c_n, m$).*

4. CONANT'S QUESTION

The first known examples of approximate subgroups were geometric progressions².

Definition 4.1 (Geometric Progressions). *Let $u_1, \dots, u_d$ be elements generating an abelian group. Set $N_1, \dots, N_d \in \mathbb{N}$. The geometric progression on $\bar{u}$ of formal length $\bar{N}$ is the set of words on $\bar{u}$ with at most N_i occurrences of u_i, u_i^{-1} for each i . In other words:*

$$P(\bar{u}; \bar{N}) = \{u_1^{k_1} \cdots u_d^{k_d} : |k_i| \leq N_i \text{ for each } i\}.$$

In the process of studying approximate subgroups in the non-abelian context, nilprogressions has been introduced as a generalisation of geometric progression.

Definition 4.2 (Nilprogressions). *Let $u_1, \dots, u_d$ be elements generating a nilpotent group of class s . Set $N_1, \dots, N_d \in \mathbb{N}$. The nilprogression (or generalised nilpotent progression) $P_g(\bar{u}; \bar{N})$ on $\bar{u}$ of formal length $\bar{N}$ is the set of words on $\bar{u}$ with at most N_i occurrences of u_i, u_i^{-1} for each i .*

²In the abelian context, when using additive notation, geometric progressions are called *arithmetic progressions*.

However, we should note that the definition of nilprogressions is actually quite problematic. While the elements of geometric progressions can be explicitly written in a common form, the elements of nilprogressions are words of arbitrary form. In other words, nilprogressions might seem to be the natural generalisation of geometric progressions, but they are far from behaving as well as geometric progressions. For instance, contrary to popular belief, nilprogressions are not generally approximate subgroups. Only nilprogressions of long enough formal length are approximate subgroups. The proof of this key fact is based on Hall's collecting process [9], a computational method commonly used in group theory for studying nilpotent groups.

Let G be a group and $x_1, \dots, x_n \in G$. A *commutator* on $\bar{x}$ is a term u on $\bar{x}$ in the language that only has a binary operation for group commutation. Its *weight* $\chi(u)$ is the total number of instances of $x_1, \dots, x_n$ it contains (counting repetitions). A *weight preserving order* $<$ of the commutators is a total order on the commutators such that, if $\chi(u) < \chi(v)$, then $u < v$. Fixed a weight preserving order of the commutators, a commutator α of weight k is *basic* with respect to this order when $k = 1$, or $\alpha = [w, v]$ with $w > v$ when $k = 2$, or $\alpha = [[w, u], v]$ with $[w, u] > v > u$.

Theorem 4.3 (Hall's collecting process). *Let $x_1, \dots, x_r$ be the generators of a nilpotent group G of class s . Pick a weight preserving order of the commutators. Let $u_1, \dots, u_t$ be the basic commutators on $x_1, \dots, x_r$ listed in order. Every element g in G is of the form $u_1^{n_1} \dots u_t^{n_t}$ for some $n_1, \dots, n_t$.*

Using Hall's collecting process, we obtain a better behaved notion of nilpotent progression [20, Definition 5.6.2]:

Definition 4.4 (Nilpotent progressions). *Let $x_1, \dots, x_r$ be elements generating a nilpotent group of class s . Pick a weight preserving order of the commutators and let $u_1, \dots, u_t$ be the ordered list of basic commutators in $x_1, \dots, x_r$ with respect to this order. Let $N_1, \dots, N_r \in \mathbb{N}$. The nilpotent progression in $\bar{x}$ of formal length $\bar{N}$ is the set*

$$P(\bar{x}; \bar{N}) = \{u_1^{l_1} \dots u_t^{l_t} : |l_i| \leq L_i^{\chi(u_i)}\}.$$

Fact 4.5. [20, §5.6] *Every nilpotent progression is a $k(r, s)$ -approximate subgroup, where $k(r, s)$ only depends on the rank r and the class s . If $\bar{N}$ is large enough, the nilprogression $P_g(\bar{x}; \bar{N})$ is $C(r, s)$ -commensurable to the nilpotent progression $P(\bar{x}, \bar{N})$ and is therefore a $k'(r, s)$ -approximate subgroup too, where $C(r, s)$ and $k'(r, s)$ only depend on the rank r and the class s .*

Thus, Conant's question should first be asked in the case of nilpotent progressions. We have the following partial answer:

Theorem 4.6. *Let $P(\bar{x}, \bar{N})$ be a pseudofinite (i.e. $\bar{N}$ no standard) nilpotent progression of rank r and class s . Then $P(\bar{x}, \bar{N} + o(\bar{N}))$ has finite VC dimension.*

Proof sketch. The idea is to see that Breuillard-Green-Tao theorem [2] adds no new information when we start with a nilpotent progression. After that, we simply use that Lie groups are piecewise definable in $\mathbb{R}_{\text{an}}$ and this is o -minimal. $\square$

Corollary 4.7. *For every r, s, ε there is $d(r, s, \varepsilon)$ such that for every nilpotent progression $P(\bar{x}, \bar{N})$ of rank r and class s there is no set S of size $|S| > d$ shattered by $P(\bar{x}, \bar{N})$ against $P(\bar{x}, (1 + \varepsilon)\bar{N})$.*

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DEFINABLE $\mathcal{C}^r$ STRUCTURES ON DEFINABLE TOPOLOGICAL GROUPS: D-MINIMAL CASE

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Pillay proved that any group G definable in an o-minimal structure can be equipped with a definable topology which makes G an abstract definable manifold whose group operation and inverse induced from G are continuous [6]. Let r be a positive integer. We can easily extend Pillay's result to the definable $\mathcal{C}^r$ category when the o-minimal structure is an expansion of an ordered field. In [7], Wencel gave a purely topological proof for Pillay's results. His method is applicable to all first-order topological structures which have dimension functions satisfying van den Dries's requirements [1, Definition] and the continuity property. Examples of such structures are definably complete locally o-minimal structures [4] and weakly o-minimal structures in which the dimension functions satisfy the addition property [8].

D-minimality is a weaker concept than definably complete local o-minimality. D-minimal structures were first introduced by Miller [5] and generalized by Fornasiero [2]. The structure $\mathcal{F} = (F, <, \dots)$ is *d-minimal* if it is definably complete, and every definable subset X of F is the union of an open set and finitely many discrete sets, where the number of discrete sets does not depend on the parameters of definition of X . The presenter tried to generalize Pillay's result to d-minimal cases, and he partially succeeded in [3]. He proved that a definable $\mathcal{C}^r$ variant of Pillay's result holds if the structure is a d-minimal expansion of an ordered field and G is a definable topological group. In this talk, the presenter introduces the results in [3]. A good news is that d-minimal expansions of ordered groups have dimension functions satisfying van den Dries's requirements. An obstacle to mimic Wencel's proof in the d-minimal cases is that they do not necessarily enjoy the continuity property.

In [3], the function called partition degree is proposed. It works as a supplement of dimension function. First we give the definition of partition degree. For a given definable set X of dimension d , there exists a definable $\mathcal{C}^r$ submanifold of dimension d which is contained and open in X . The largest definable $\mathcal{C}^r$ submanifold of dimension d satisfying the above condition is denoted by $\text{Reg}_r(X)$. The difference $X \setminus \text{Reg}_r(X)$ may be still of dimension d , but it contains a definable $\mathcal{C}^r$ submanifold of dimension d which is open in it. We obtain a definable set of dimension smaller than d after repeatedly removing definable $\mathcal{C}^r$ submanifolds of dimension d from X finitely many times. The partition degree $\text{p.deg}(X)$ of X is the number of removed definable $\mathcal{C}^r$ submanifolds through the above process.

Basic properties of $\text{p.deg}(X)$ were investigated in [3]. The first significant property is that

the partition degree is preserved under definable homeomorphisms

and the second is the following variant of continuity property:

Proposition 0.1 (A variant of continuity property). *Let X be a definable set and $f_1, \dots, f_k : X \rightarrow F$ be definable functions. There exists a definable open subset U of X such that*

- U is a definable C^r submanifold;
- the function f_i restricted to U is of class C^r for each $1 \leq i \leq k$;
- at least one of the inequalities

$$\dim X \setminus U < \dim X \text{ or } \text{p. deg } X \setminus U < \text{p. deg } X$$

holds.

Using the above proposition instead of the continuity property of dimension and the following the strategy employed in [7], we prove the following theorem by induction on the pair $(\dim X, \text{p. deg } X)$ under the lexicographic order.

Theorem 0.2. *Let G be a topological group definable in a d -minimal expansion of an ordered field. There exists a definable open subset V and finitely many elements $g_1, \dots, g_m$ of G such that V is a definable C^r submanifold and $(\varphi_i : U_i := g_i V \ni g_i \cdot g \mapsto g \in V)_{1 \leq i \leq m}$ is a definable C^r structure on G .*

The definition of definable C^r structures is not repeated here because it is a technical and straightforward modification of the definable topology given in Pillay's result [6] to the definable C^r case.

Unfortunately, $\text{p. deg}(X)$ is not necessarily preserved under definable bijections. This makes it impossible to apply the same strategy to the case where G is a definable (not necessarily topological) group in the d -minimal setting.

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LIKELY INTERSECTIONS IN POWERS OF THE MULTIPLICATIVE GROUP

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1. INTRODUCTION

This talk was based on the paper [1], which was written in collaboration with Gabriel Dill of the University of Neuchâtel.

Let W be an algebraic subvariety of the complex multiplicative group $(\mathbb{C}^\times)^n$, and let H be a connected algebraic subgroup of $\mathbb{C}^n$. If $\dim W + \dim H \geq n$, then W is “expected”, in a sense, to intersect H ; since all cosets of H have the same dimension, the same expectation holds for every coset of H .

Of course the dimension condition does not guarantee the existence of an intersection; if W is defined by the equation $z_1 + z_2 + 1 = 0$ and H by the equation $z_1 z_2^{-1} = 1$, it is not hard to single out a coset of H which does not intersect W .

The aim of this work is to give a sufficient condition which grants the existence of these intersections. Consider first the following notion.

Definition 1.1. *Let W be an irreducible algebraic subvariety of $(\mathbb{C}^\times)^n$. We say W is geometrically non-degenerate if for every connected algebraic subgroup H of $(\mathbb{C}^\times)^n$ we have*

$$\dim \pi_H(W) = \min\{\dim W, n - \dim H\}$$

where $\pi_H : (\mathbb{C}^\times)^n \rightarrow (\mathbb{C}^\times)^n/H$ denotes the natural projection.

The following is the main result:

Theorem 1.2. *Let W be a geometrically non-degenerate algebraic subvariety of $(\mathbb{C}^\times)^n$. There is a finite set $\mathcal{G}$ of connected algebraic subgroups of $(\mathbb{C}^\times)^n$ such that for every algebraic subgroup H of $(\mathbb{C}^\times)^n$ with $\dim W + \dim H \geq n$, one of the following holds:*

- (1) *For all $z \in (\mathbb{C}^\times)^n$, we have $W \cap (z \cdot H) \neq \emptyset$.*
- (2) *There is $G \in \mathcal{G}$ such that $H \subseteq G$.*

We also prove a more precise version in which we focus only on translates of H by torsion points. In that case, we prove that there is a finite list $\mathcal{G}'$ of subgroups such that for every torsion point ζ and every algebraic subgroup of sufficiently large dimension of $(\mathbb{C}^\times)^n$, if $W \cap (\zeta \cdot H) = \emptyset$ then $(\zeta \cdot H) \subseteq G$ for some $G \in \mathcal{H}'$. In this extended abstract I will only focus on Theorem 1.2.

2. MOTIVATION

The motivation for this work comes from two active areas of research, *unlikely intersection problems* and the *Exponential-Algebraic Closedness conjecture*.

Unlikely intersection problems are a family of problems in arithmetic geometry. They are concerned with the interactions between the arithmetic and the geometric structure in arithmetic varieties; the general “unlikely intersections philosophy” is that these should not interact any more than they are supposed to. A famous example is the *Manin-Mumford conjecture*, now a theorem. In the following, by a *torsion coset* I mean a coset of an algebraic subgroup of $(\mathbb{C}^\times)^n$ by a torsion point.

Theorem 2.1 (Manin-Mumford conjecture). *Let $W \subseteq (\mathbb{C}^\times)^n$ be an algebraic subvariety which is not a torsion coset. The union of all torsion cosets contained in W is not Zariski-dense in W .*

For example, if W is a curve, this says that W contains only finitely many torsion points (unless it is a torsion coset). See for example [5] for one (of many) proofs of Theorem 2.1; in fact, we also give a seemingly new proof of this in [1, Section 5] as an application of our results.

In unlikely intersection problems one shows that intersections which are not supposed to exist do not exist, unless there is a reason for it. Our work aims to “dualize” this idea, in a sense. Theorem 1.2 can be read as saying that intersections which are supposed to exist do exist, once we forget about a “small” set of exceptions.

The Exponential-Algebraic Closedness conjecture is due to Boris Zilber, stemming from his work on the model theory of the complex exponential function [6]. The conjecture predicts sufficient conditions for systems of equations involving algebraic operations and the exponential function (so-called *exponential-polynomial equations*) to have solutions in the complex numbers. The statement has a geometric form: it gives conditions, called *freeness* and *rotundity*, for an algebraic subvariety of $\mathbb{C}^n \times (\mathbb{C}^\times)^n$ to contain a point of the form $(z_1, \dots, z_n, e^{z_1}, \dots, e^{z_n})$. The definition of rotundity is particularly relevant for this work.

Definition 2.2. *Let $V \subseteq \mathbb{C}^n \times (\mathbb{C}^\times)^n$ be an irreducible algebraic subvariety. We say V is rotund if for every connected algebraic subgroup H of $(\mathbb{C}^\times)^n$, we have*

$$\dim \pi_{TH}(V) \geq n - \dim H$$

where TH is the tangent bundle of H , identified with an algebraic subgroup of $\mathbb{C}^n \times (\mathbb{C}^\times)^n$, and $\pi_{TH} : \mathbb{C}^n \times (\mathbb{C}^\times)^n \twoheadrightarrow (\mathbb{C}^n \times (\mathbb{C}^\times)^n)/TH$ is the projection map.

Remark 2.3. *It takes an easy calculation to see that if W is a geometrically non-degenerate algebraic subvariety of $(\mathbb{C}^\times)^n$, and $L \leq \mathbb{C}^n$ is a linear subspace with $\dim L + \dim W \geq n$, then $L \times W$ is rotund.*

Working with subvarieties $L \times W$ of $\mathbb{C}^n \times (\mathbb{C}^\times)^n$, with L linear, allows us to consider in particular the algebraic subgroups of $(\mathbb{C}^\times)^n$, which we can recover as the sets $\exp(L)$ when L is defined over $\mathbb{Q}$. This has the advantage that we can make L vary among linear subspaces defined over $\mathbb{R}$ of fixed dimension and make use of the better topological properties of the reals.

3. PROOF STRATEGY

While the definition of rotundity is algebraic (the tangent bundle used in Definition 2.2 can easily be described algebraically), in the complex setting we can give the following analytic characterization, essentially due to Kirby.

Proposition 3.1 ([3, proof of Proposition 6.2 and Remark 6.3]). *Let $V \subseteq \mathbb{C}^n \times (\mathbb{C}^\times)^n$ be an irreducible algebraic subvariety. V is rotund if and only if there is a Zariski-open dense subset V° of V such that the map $\delta : V^\circ \rightarrow (\mathbb{C}^\times)^n$ defined by*

$$\delta(v_1, \dots, v_{2n}) := \left(\frac{v_1}{e^{v_{n+1}}}, \frac{v_2}{e^{v_{n+2}}}, \dots, \frac{v_n}{e^{v_{2n}}} \right)$$

is open in the complex topology.

This proposition is proved by combining Remmert’s open mapping theorem from complex analysis, the fibre dimension theorem from algebraic geometry, and the uniform version of the Ax–Schanuel theorem.

The second key ingredient is the following previous result of myself.

Theorem 3.2 ([2, Lemma 6.15]). *Let $L \leq \mathbb{C}^n$ be a linear subspace defined over $\mathbb{R}$, $W \subseteq (\mathbb{C}^\times)^n$ an algebraic subvariety, and assume $L \times W$ is a rotund subvariety of $\mathbb{C}^n \times (\mathbb{C}^\times)^n$.*

Then there is $(\ell_1, \dots, \ell_n, w_1, \dots, w_n) \in L \times W$ such that

$$\left(\frac{w_1}{\exp(\ell_1)}, \dots, \frac{w_n}{\exp(\ell_n)} \right) \in \mathbb{S}_1^n$$

where $\mathbb{S}_1$ denotes the unit circle $\{s \in \mathbb{C} \mid |s| = 1\}$.

In the rest of the proof we show that if W is geometrically non-degenerate then, using the uniformity granted by Remark 2.3 we can obtain a strong version of Theorem 3.2 which is “uniform in L ” where L is an affine subspace of $\mathbb{C}^n$. More precisely, we show that, given a geometrically non-degenerate W , there is a positive real ϵ such that for every affine subspace L of $\mathbb{C}^n$ of dimension at least $n - \dim W$ which is parallel to a linear subspace defined over $\mathbb{R}$, the image of $L \times W$ under the map δ defined in Proposition 3.1 contains a ball of radius ϵ centred at a point in $\mathbb{S}_1^n$.

The main tools for this come from *tropical geometry*. We embed the complex numbers into a certain larger algebraically closed field $\mathfrak{C}$, equipped with the Archimedean valuation $v : \mathfrak{C} \rightarrow \Gamma$ (where Γ is an appropriate non-trivial ordered abelian group; we assume the valuation is surjective). To any algebraic variety W defined over $\mathbb{C}$ of a power of the multiplicative group we can then attach a semilinear subset $\text{Trop}(W)$ of Γ^n , which is the image of $W(\mathfrak{C})$ under the valuation map but which can also be computed through the *initial forms* of the ideal defining W . The set $\text{Trop}(W)$ contains information on the behaviour of the points of $W(\mathbb{C})$ whose coordinates are “very large” or “very small”; in particular, it provides local approximations of W by algebraic subvarieties of $(\mathbb{C}^\times)^n$ of a simpler form (i.e., invariant under translation by a positive-dimensional algebraic subgroup of $(\mathbb{C}^\times)^n$, which is convenient for certain arguments by induction on the dimension). See [4] for a comprehensive introduction to tropical geometry; the results we need are recalled (and a few of them proved) in [1, Section 2]. If W is geometrically non-degenerate then we manage using these approximations to prove the desired uniform version of Theorem 3.2.

With this at hand, we conclude by equidistribution methods: for fixed $\epsilon > 0$, if L is defined over $\mathbb{Q}$ and it cannot be defined by equations that are “too simple” (say, with integer coefficients of low absolute value) then every ball of radius ϵ centred at a point of $\mathbb{S}_1^n$ contains points of $\exp(L)$ (more precisely a point $\exp(\ell)$ where $\ell \in L \cap (i\mathbb{R}^n)$). This is sufficient to conclude that W intersects every translate of $\exp(L)$ as soon as the equations defining L are sufficiently complicated, thus showing a finite set as in our statements exists.

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AROUND DEFINABLE TYPES IN VALUED FIELDS AND OTHER STRUCTURES

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Keywords: model theory, definable types, valued fields, Ax-Kochen-Ershov principle, pro-definability, belles paires, beautiful pairs, amalgamation property.

1. SPACES OF DEFINABLE TYPES IN HENSELIAN VALUED FIELDS

In their celebrated work [3], Hrushovski and Loeser obtained strong topological tameness properties for Berkovich analytifications of algebraic varieties, using a novel, model-theoretic, approach to non-archimedean geometry. They introduced the so-called *stable completion* of an algebraic variety V , the space of all definable types concentrating on V , in the theory ACVF of algebraically closed non-trivially valued fields, which are orthogonal to the value group. A key property of the stable completion is its strict pro-definability, i.e., the fact that it may be given the structure of a pro-definable set, where the transition functions in the projective system are surjective. This is a much stronger property than mere pro-definability, as it allows for a use of logical compactness very much like in ordinary definable sets.

Motivated by the pro-definability of the stable completion, in joint work with Cubides and Ye [1], we study other spaces of definable types in ACVF, in particular the space of all definable types (corresponding to the Zariski-Riemann space), the space of all bounded definable types, i.e. types orthogonal to the type at $+\infty$ in the value group Γ (corresponding to the Huber analytification). We obtain as one of our main results:

Theorem 1.1 ([1]). *If V is an algebraic variety defined over an algebraically closed non-trivially valued field K , then for all natural classes $\mathcal{C}$ of definable types, the space of types from $\mathcal{C}$ concentrating on V is strict pro-definable. In particular, this holds for the space of all definable types and for the space of all bounded definable types concentrating on V .*

In particular, this theorem gives an entirely new proof of the strict pro-definability of the stable completion. For many other henselian valued fields, we obtain analogous results, e.g., for the theory RCVF of real closed fields with a proper convex valuation, for p -adically closed fields and for other valued fields like, e.g., $\mathbb{C}((t))$ and $\mathbb{R}((t))$.

The key idea of our approach to strict pro-definability is an adaptation of Poizat's theory of belles paires [4] to the unstable context. Poizat characterizes the stable theories in which the space of definable types is strict pro-definable as the ones without the finite cover property, and also as those stable theories T in which the theory of *belles paires*, i.e., elementary pairs $(M, P(M))$ of models of T with $P(M)$ being $|T|^+$ -saturated and M being $|T|^+$ -saturated over $P(M)$, admits a $|T|^+$ -saturated model which is a belle paire.

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In [1], we introduce the notion of a *beautiful pair* for a not necessarily stable theory T , which in the stable case is equivalent to a belle paire. Beautiful pairs exist precisely when the class of all (global) definable types has the amalgamation property (AP). Assuming AP, we show in [1] that any two beautiful pairs are elementarily equivalent and that they are elementary pairs precisely when the class of definable types has the extension property (EP). In case there is a beautiful pair which is $|T|^+$ -saturated, we say that the theory has *beauty transfer*. It is then routine to show that if the definable types in T have AP and beauty transfer holds, then the space of all definable types is strict pro-definable. Moreover, the notion of a beautiful pair may be relativized to natural classes of definable types. It is in this way that we show Theorem 1.1. The variants for other henselian valued fields are obtained through the following Ax-Kochen-Ershov principle.

Theorem 1.2 ([1]). *Let (K, v) be a henselian valued field of residue characteristic 0. Let C_{res} and C_{val} be natural classes of definable types in the theory of the residue field and value group of (K, v) , respectively, and assume that both C_{res} and C_{val} have AP, EP and beauty transfer.*

Then, in the theory of (K, v) , the class of definable types induced by C_{res} and C_{val} has AP, EP and beauty transfer.

2. THE AMALGAMATION PROPERTY FOR DEFINABLE TYPES

In [1], we show that in the theory of the binary branching levelled meet tree with level set $(\omega, <)$, the class of definable types does not have EP. Nevertheless, it has AP and beauty transfer. Starting from this example, using a combinatorial construction involving generic surjections level-by-level, in joint work with Mennuni [2], we construct a theory in which the class of definable types does not have AP. Although, this theory is wild from a classification theoretic point of view, as it has TP_2 (the tree property of the second kind).

Using a modification of the example from [2], using, among other things, the $\mathbb{F}_2$ -vector space structure in the standard binary branching levelled tree and a copy of the tree which, level-by-level, has the structure of a principal homogenous space, we obtain the following result.

Theorem 2.1 (Hils-Mennuni, unpublished). *There is a dp -minimal (so in particular NIP) theory T in which the class of definable types does not have the amalgamation property.*

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ACCESSIBLE INDISCERNIBLE SEQUENCE OF ARITHMETIC AND ITS APPLICATIONS

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Classification AMS 2020: 03H15, 05D10, 05C55

Keywords: iterated star-maps, indiscernible sequence, Ramsey type theorems

By iterating the star-map internally or externally from a standard model of arithmetic, one can construct a sequence of hyperfinite integers, which are accessible but first-order indiscernible, in a nonstandard model. The sequence offers a structural framework for applying model theoretical techniques to some problems in combinatorial number theory. The results presented here will be in a sequel of [1].

Fix a non-principal ultrafilter $\mathcal{F}$ on the set $\mathbb{N}$ of all natural numbers. After taking an ultrapower of the model $\mathcal{N}_0 = (\mathbb{N}_0; \mathcal{A})$ where $\mathbb{N}_0 := \mathbb{N}$ and $\mathcal{A}$ is the set of all functions and relations on $\mathbb{N}_0$ modulo $\mathcal{F}$ one can obtain a nonstandard model $\mathcal{N}_1$ with a hyperfinite integer a . If one take repeatedly the ultrapowers of the previously obtained models starting with $\mathcal{N}_1$ modulo $\mathcal{F}$, the number a then generates an indiscernible sequence $a = a_1 < a_2 < \dots$.

Let $\bar{a}$ be the n -tuple $\{a_1 < a_2 < \dots < a_n\}$. We have the following theorem:

Let $\varphi(\bar{x})$ be a first-order formula with parameters from $\mathcal{N}_0$. If $\varphi(\bar{a})$ is true, then there exists an infinite set W of the numbers in $\mathbb{N}_0$ such that $\varphi(\bar{w})$ is true for every n -tuple $\bar{w}$ of elements from W in increasing order.

Note that Ramsey theorem for n -tuples is an easy consequence. Just let $\varphi(\bar{a})$ say that $\bar{a}$ has color c_0 . One can also obtain easily an infinite version of Folkman's theorem. By assuming that $\mathcal{F}$ is an idempotent ultrafilter, one can then obtain a theorem combining Hindman's theorem and an infinite set version of Folkman's theorem.

The arguments mentioned above assume only the existence of one non-principal ultrafilter $\mathcal{F}$ which is a weak version of the axiom of choice.

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LARGE FIELDS, HENSELIAN RINGS, AND TAME TOPOLOGY

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Classification AMS 2020: 03C60, 12L12.

Keywords: t-minimal theories, henselian rings, NIP theories, large fields.

1. T-MINIMALITY AND HENSELIANITY

“Rings” are commutative unital; “field topologies” are Hausdorff non-discrete.

Definition 1.1. Let D be a definable set in a structure M . A topology τ on D is definable if some (uniformly) definable family $\{U_a\}_{a \in Y}$ is a basis for τ .

Definition 1.2. A complete theory T is t-minimal (in the sense of Mathews [24]) if there is a Hausdorff definable topology τ on models M such that for definable $D \subseteq M^1$, D has non-empty interior iff it is infinite.

For example, dense o-minimal structures and algebraically closed valued fields (ACVF) are t-minimal. T-minimal theories have a dimension theory for definable sets, and under additional technical assumptions they have theorems about generic continuity and cell decomposition [28, 3, 14].

Theorem 1.3 ([15]). If M is t-minimal and K is an infinite definable (not just interpretable) field, then there’s a “canonical” definable field topology τ_K on K .

Definition 1.4 ([2, 15, 17]). A field topology τ on K is gold t-henselian or gt-henselian if it satisfies the following equivalent conditions:

- (1) If $X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$ has a simple root b , then there is a neighborhood (U, V) of $(\bar{a}, b)$ and a continuous function $f : U \rightarrow V$ such that $f(\bar{c})$ is a simple root of $X^n + c_{n-1}X^{n-1} + \cdots + c_0$ for $\bar{c} \in U$.
- (2) If $V \rightarrow W$ is an étale morphism of K -varieties, then $V(K) \rightarrow W(K)$ is a local homeomorphism with respect to K .

τ is silver t-henselian or st-henselian if it satisfies the two equivalent weaker conditions:

- (1'): Condition (1) holds when $X^n + a_{n-1}X^{n-1} + \cdots + a_0$ is separable.
- (2'): Condition (2) holds for finite étale morphisms.

Fact 1.5 ([2]). A field topology τ is t-henselian [26] iff it is a gt-henselian V -topology.

Definition 1.6. Let R be a local ring with maximal ideal $\mathfrak{m}$. Then R is henselian if for any $c_0, c_1, \dots, c_n \in \mathfrak{m}$, the polynomial $X^{n+2} + X^{n+1} + c_nX^n + \cdots + c_1X + c_0$ has a root in $-1 + \mathfrak{m}$.

Fact 1.7 ([2]). (1) Let R be a henselian local domain that isn’t a field, and let $K = \text{Frac}(R)$. Then the family $\{aR + b : a, b \in K, a \neq 0\}$ is a basis for a gt-henselian field topology on K , called the R -adic topology.

- (2) If (K, τ) is a gt-henselian topological field, then $K \equiv \text{Frac}(R)$ for some henselian local domain R that isn’t a field.

Theorem 1.8 ([15]). *If K is an infinite definable field in a t -minimal theory, then the canonical topology τ_K on K is st-henselian.*

Conjecture 1.9. *If K is an infinite definable field in a t -minimal theory, then the canonical topology τ_K on K is gt-henselian.*

Example 1.10 ([7, 22, 19]). *Let T be the expansion of $\text{ACVF}_{0,0}$ by a generic derivation ∂ . If $(K, +, \cdot, v, \partial) \models T$, let $R = \{x \in K : v(x) \geq 0 \text{ and } v(\partial x) \geq 0\}$.*

- (1) *R is a henselian local ring. The R -adic topology on K is gt-henselian but not a V -topology, so not t -henselian.*
- (2) *The reduct $(K, +, \cdot, R)$ is t -minimal with respect to the R -adic topology.*

Question 1.11. *Is every st-henselian field topology gt-henselian?*

2. LARGENESS AND HENSELIANITY

Definition 2.1 ([25]). *A field K is large if the following equivalent conditions hold:*

- (1) *For any smooth algebraic curve C , if there is a K -point on C , then there are infinitely many K -points.*
- (2) *If $f(X, Y) \in K[X, Y]$ satisfies $f(0, 0) = 0 \neq \frac{\partial f}{\partial Y}(0, 0)$, then there are infinitely many $(a, b) \in K^2$ such that $f(a, b) = 0$.*

Number fields and function fields are non-large. Most known fields with model-theoretically nice properties (model completeness in a natural language, classification-theoretic properties, or decidability) are large or finite.

Theorem 2.2 ([25, 2, 15]). *If τ is an st-henselian topology on K , then K is large.*

Corollary 2.3. *If K is a definable field in a t -minimal theory, then K is finite or large. $\mathbb{Q}$ is not definable in a t -minimal theory.*

What else do Theorem 1.8 and Conjecture 1.9 say about definable fields? Not much:

Theorem 2.4 ([20]). *Let K be a sufficiently saturated and resplendent large field. Then $K = \text{Frac}(R)$ for some henselian local domain $R \subsetneq K$. The R -adic topology on K is gt-henselian.*

Theorem 2.5 ([16]). *Any countable large field admits a gt-henselian field topology.*

Question 2.6. *Does some t -minimal structure define a pseudofinite field?*

3. NIP AND T-MINIMALITY

Definition 3.1. *A theory T has the independence property (IP) if it interprets a bipartite graph (V_1, V_2, E) into which all finite bipartite graphs embed (as induced subgraphs). Otherwise, T is NIP.*

The class of NIP theories and structures includes the stable and o-minimal theories and structures, ordered abelian groups (e.g., Presburger arithmetic), local fields of characteristic 0 (e.g., $\mathbb{Q}_p$), and algebraically closed valued fields (ACVF) [27].

Definition 3.2 ([29, 23, 4]). *A highly saturated model $\mathbb{M}$ has dp-rank at least κ if there is a singleton $b \in \mathbb{M}^1$ and κ -many mutually indiscernible sequences $\{I_\alpha\}_{\alpha < \kappa}$ none of which are b -indiscernible. $\mathbb{M}$ is dp-minimal if it has dp-rank 1, and dp-finite if it has finite bounded dp-rank.*

Fact 3.3 ([4]). *Strongly, o -, P -, C -, and weakly o - minimal theories are dp -minimal.*

Theorem 3.4 ([6, 28, 11, 8]). (1) *If $(K, +, \cdot, \dots)$ is dp -finite but not ω -stable, then there is a canonical definable field topology τ_K on K .*

(2) *If $(K, +, \cdot, \dots)$ is dp -minimal but not ω -stable, then K is t -minimal with respect to τ_K . Moreover, generic continuity and cell decomposition theorems hold.*

4. NIP AND HENSELIANITY

Conjecture 4.1 (Shelah conjecture). *If K is an NIP field, then K is finite or real closed or algebraically closed or K admits a non-trivial henselian valuation.*

Conjecture 4.2 (Henselianity conjecture). *NIP valuation rings are henselian.*

Fact 4.3 ([5]). *The SC implies the HC.*

Fact 4.4 ([1]). *The SC implies a full classification of NIP fields.*

Theorem 4.5 ([10, 9]). (1) *The HC holds in positive characteristic.*

(2) *The SC and HC hold in the dp -finite case.*

(3) *Dp -finite fields have such-and-such a classification.*

Conjecture 4.6 (Generalized henselianity conjecture). *The following five equivalent [18] statements hold:*

(1) *Every NIP local domain is henselian.*

(2) *Every NIP integral domain is a local ring.*

(3) *Every NIP ring is a finite product of henselian local rings.*

(4) *If $(K, +, \cdot, \dots)$ is NIP, then every definable field topology on K is gt -henselian.*

(5) *If $(K, +, \cdot, \dots)$ is NIP, then there cannot be two independent definable field topologies τ_1, τ_2 on K .*

Theorem 4.7 ([12, 13]). *The GHC holds in positive characteristic, and in finite dp -rank.*

5. NIP AND LARGENESS

Conjecture 5.1 (Stable fields conjecture). *Stable fields are finite or separably closed.*

The Shelah conjecture implies NIP fields are large or finite.

Theorem 5.2 ([21]). *If K is stable and large, then K is separably closed.*

Theorem 5.3 ([17]). *If K is NIP and large, then K is separably closed or there's a unique coarsest st -henselian field topology τ_K on K .*

Question 5.4. *Is τ_K gt -henselian? Is it definable?*

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RESIDUAL DOMINATION FOR HENSELIAN FIELDS

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Classification AMS 2020: 12J10, 12L12 Secondary: 03C10, 03C60.

Keywords: valued fields, henselian, orthogonality, domination.

One of the most celebrated results in recent model theory of valued fields is Hrushovski-Loeser's characterization of the homotopy type of the Berkovich analytification of quasi-projective varieties over complete (non-trivially valued) non-archimedean valued fields [9, Theorem 11.1.1]. Inherent to their approach, they introduced a model theoretic analogue of Berkovich's analytification of an algebraic variety X as the space of types concentrating on X which are orthogonal to the value group. One of the key features that renders so useful this approach is the following result proven in [11, 8]: in ACVF (the theory of algebraically closed non-trivially valued fields) the following conditions are equivalent for a global A -invariant type p :

- p is orthogonal to the value group,
- p is stably dominated,
- p is generically stable.

The aim of the present work is to extend such a theorem to arbitrary henselian fields of equicharacteristic zero. Part of the difficulty lies in understanding what should play the role of stability in such a generality. For example, note that, in the case of real closed valued fields, a generically stable type over a model must be realized, and hence the previous equivalence is clearly false. It turns out that, independently of what the stable part of the structure is or of what generically stable types correspond to, the types which are orthogonal to the value group coincide with those which are “controlled” by the residue field, meaning that they are *residually dominated*. The notion we introduce here encompasses previous notions of domination present in [4], [6] and [14], where such a behaviour was already noticed but results were only obtained over models for pure henselian valued fields. Our contribution here is two-folded: first, we extend such results over algebraically closed sets of imaginary elements and, second, we show that they remain valid in various expansions of henselian valued fields. In addition, we show that our notion of residual domination can be evaluated by looking at the corresponding type in an algebraically closed extension. Here is an informal version of our main contribution:

Theorem. *Let M be an RV-expansion of a henselian valued field of equicharacteristic zero such that the value group Γ_M is either:*

- *dense with property D or*
- *a pure discrete ordered abelian group of bounded regular rank.*

Assume that the residue field and the value group are stably embedded and orthogonal to each other. Then, the following are equivalent for a definable type p of fields points (over an algebraically closed base of imaginary elements):

- (1) p is orthogonal to Γ ;
- (2) the quantifier free part of p , seen as a type in ACVF, is orthogonal to Γ ;
- (3) p is residually dominated.

Along the way, we obtain a change of base statement for residual domination in arbitrary henselian valued fields of equicharacteristic zero (possibly with more structure). This result follows from descent results of [13] for generically stable types in arbitrary theories.

We also extend the theorem above to certain valued fields with operators. This results applies in particular to valued difference fields as in [5], equicharacteristic zero ∂ -henselian fields with a monotone derivation [12], as well as models of $\text{Hen}_{(0,0)}$ with generic derivations as in [3] and [7].

We then apply those results to show that residually dominated types inherit tame behavior from the residue field. In particular, when the residue field is stable we show that an invariant type of field points is orthogonal to the value group if and only if it is generically stable. Similarly, in an NTP_2 context, if the residue field is simple and algebraically closed sets of imaginary elements are extension basis, then a type of field points invariant over some (imaginary) base A , is orthogonal to the value group if and only if, for all set of imaginary parameters $B \supseteq A$, the type $p|_B$ is generically simple.

These results apply in particular to multiplicative difference valued fields and ultraproducts of the p -adics. On particularly striking example is the limit theory VFA_0 of an algebraically closed valued field of characteristic p with the Frobenius automorphism (when p tends to infinity), giving an answer to questions of Chernikov and Hils in [1].

The main motivation for this work is the question of whether Hrushovski-Loeser's ideas can be adapted to characterize the homotopy type of topological spaces arising from types orthogonal to the value group. When the underlying theory is the theory of real closed valued fields, one can view such spaces of types as the Berkovich analytification of a semi-algebraic set as defined in [10]. To characterize the homotopy type of such spaces remains an open question. It is also related to the wildly open question about the homotopy type of spaces of real places. Our hope is to use the results of this paper together with structural properties of such type spaces (for example strict pro-definability, which was shown in [2]) to attack this question.

This is a joint work with Silvain Rideau-Kikuchi and Mariana Vicaria.

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APPROXIMATE RINGS

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Keywords: Approximate ring, locally compact model, finite dimensional algebra, escape norm.

The new results presented in this talk belong to my joined project with Simon Machado.

An *approximate subgroup* is a symmetric subset X of a group such that $X \cdot X$ is covered by finitely many left translates of X ; it is a K -*approximate subgroup* if K translates are enough. This notion was introduced by Tao in [8] and has become one of the central objects in additive combinatorics. It originates in fundamental considerations in additive combinatorics on sets of small doubling, tripling, etc. For example: each finite symmetric subset X of a group with small tripling (meaning that $|X^3| \leq K|X|$) has small n -pling for every $n \geq 3$ (namely $|X^n| \leq K^{2n-5}|X|$); if $X^5 \leq K|X|$, then X^2 is a K -approximate subgroup. The advantage of the notion of approximate subgroup is that it is more algebraic and makes perfect sense also for infinite subsets. Another origin of approximate subgroups are approximate lattices whose theory goes back to the seminal monograph of Meyer [7]. Approximate lattices are approximate subgroups of locally compact groups which are uniformly discrete and with finite co-volume.

Passing from groups to rings, the role of sets of small doubling, tripling, etc., is played by finite subsets X of a ring satisfying the various sum-product conditions, e.g. $\max(|X+X|, |X \cdot X|) \leq K|X|$, $|X + X \cdot X| \leq K|X|$, $|X \cdot X - X \cdot X| \leq K|X|$, or $|X + X + X \cdot X| \leq K|X|$. And, very roughly speaking, sum-product phenomena (which also lie in the center of additive combinatorics and have many applications) assert that this kind of conditions imply that X is “close” to a subring. Similarly to the context of approximate subgroups, it is then natural to study a more algebraic condition of an approximate subring which makes sense also for infinite sets. In [5], I defined an *approximate subring* of a ring as an additively symmetric subset X such that both $X+X$ and $X \cdot X$ are covered by finitely many additive translates of X ; it is a K -*approximate subring* if K translates are enough. For example, one can show that the sum-product condition $|X + X + X \cdot X| \leq K|X|$ implies that $X - X$ is a $(K^5 + K^{19})$ -approximate subring. For commutative unital rings more statements of this form can be found for example in the lecture notes of Emmanuel Kowalski entitled “Introduction to additive combinatorics”. Similarly to approximate subgroups, also some special approximate subrings were studied already in Meyer’s monograph [7], where uniform approximate lattices in $(\mathbb{R}, +)$ closed under multiplication (so a particular case of approximate subrings in our sense) were classified in arithmetic terms of Pisot-Salem numbers.

A breakthrough in the study of approximate subgroups was done by Hrushovski in [3]. He introduced *locally compact models* of approximate subgroups (i.e.

homomorphisms from the group generated by the approximate subgroup in question to a locally compact group which have some good properties), and, using some model theory methods related to the theory of stable groups, he proved the existence of locally compact models for wide classes of approximate subgroups, in particular for pseudofinite ones. This paved the road for Breuillard, Green, and Tao to classify finite approximate subgroups in the celebrated paper [1]. However, in general, locally compact models need not exist. In another breakthrough paper [4], Hrushovski weakened the notion of a locally compact model by allowing quasi-homomorphisms with compact error sets and constructed such models via a new theory of *definability patterns structures* in the context of newly introduced *local logics*. In [6], we gave a much shorter construction of such generalized locally compact models, developing and using topological dynamics of suitable locally compact flows.

In [5], I introduced *locally compact models for approximate rings*, and, using rather basic model theory tools related to model-theoretic connected components of groups and rings, I proved the existence of locally compact models for *arbitrary* approximate subrings. This had some quick structural consequences, e.g. each approximate subring of a ring of *positive characteristic* is commensurable to a subring contained in $4X + X \cdot 2X$ (where $nX := X + \cdots + X$ is the n -fold sum).

In the current joint work in progress with Simon Machado, we give deeper applications of the existence of locally compact models for approximate subrings. First of all, we obtain a ring-theoretic counterpart of the structural result of Breuillard, Green and Tao from [1], i.e. we prove structural results on finite approximate subrings, which in turn yield a very general qualitative sum-product phenomenon as well as a ring-theoretic counterpart of Gromov's theorem on groups of polynomial growth. Secondly, we obtain generalizations to some (possibly noncommutative) real algebras of the aforementioned Meyer's result on approximate lattices in $(\mathbb{R}, +)$ closed under multiplication (yielding in particular a new proof of Meyer's result). In this talk, I focus on the first part, i.e. on the structure of finite approximate subrings.

Main results

Let X be an approximate subring. We recursively define: $X_0 := X$ and $X_{n+1} := X_n X_n + (X_n + X_n)$. Then the ring $\langle X \rangle$ generated by X coincides with $\bigcup_n X_n$.

An approximate subring X is *definable* in a structure M if all X_n 's are definable in M and the restrictions of $+$ and $\cdot$ to any X_n are definable in M . Note that any approximate subring X is trivially definable in the structure M being the ring $\langle X \rangle$ expanded by predicates for all subsets of all finite Cartesian powers.

Definition 0.1. A locally compact model of an approximate subring X is a ring homomorphism $f: \langle X \rangle \rightarrow S$ to some locally compact ring S such that:

- (1) $f[X]$ is relatively compact in S ,
- (2) $f^{-1}[U] \subseteq X_m$ for some $m < \omega$ and $U \subseteq S$ a neighborhood of 0.

In the definable context, we additionally require definability of f :

- (3) for any $C \subseteq U \subseteq S$ where C is compact and U is open, there exists a definable Y such that $f^{-1}[C] \subseteq Y \subseteq f^{-1}[U]$.

The next fact is the main result of [5].¹

Theorem 0.2. *Each definable approximate subring X has a definable locally compact model $f: \langle X \rangle \rightarrow S$ with a neighborhood U of 0 in S such that $f^{-1}[U] \subseteq 4X + X \cdot 2X \subseteq X_2$.*

Recall the following classical notion.

Definition 0.3. *We say that a ring R is nilpotent of class at most n if $R^{n+1} = \{0\}$. (In particular, R being nilpotent of class at most 0 means that $R = \{0\}$, and being nilpotent of class at most 1 means that $R^2 = \{0\}$.)*

Example 0.4. *The ring of upper triangular $(n \times n)$ -matrices with zeros on the diagonal is nilpotent of class at most $n - 1$.*

We are ready to state two versions of our main structural result on finite approximate subrings.

Theorem 0.5. *For any $K \in \mathbb{N}$ there exist $N(K) \in \mathbb{N}$ such that for every finite K -approximate subring X there exists an $N(K)$ -approximate subring $Y \subseteq 4X + X \cdot 2X$ which is $N(K)$ -commensurable to X for which there exists an ideal $I \triangleleft \langle Y \rangle$ contained in Y such that $\langle Y \rangle / I$ is nilpotent of class at most $N(K)$.*

Theorem 0.6. *For any $K \in \mathbb{N}$ there exists $N(K) \in \mathbb{N}$ such that for every finite K -approximate subring X there exists an $N(K)$ -approximate subring $Y \subseteq 4X + X \cdot 2X$ which is $N(K)$ -commensurable to X for which there exists an ideal $I \triangleleft \langle Y \rangle$ contained in $Y_{N(K)}$ such that $\langle Y \rangle / I$ is nilpotent of class at most $\lfloor 4 \log_2(K) \rfloor$.*

A general strategy of the proof is as follows.

- (1) Improving the target space: Use Theorem 0.2 to obtain a commensurable definable approximate subring $Y \subseteq 4X + X \cdot 2X$ and a definable locally compact model $f: \langle Y \rangle \rightarrow \mathcal{A}$ where $\mathcal{A}$ is a finite dimensional real algebra. This is good enough for Theorem 0.6. In the case of Theorem 0.5, we find a commensurable $Y \subseteq 4X + X \cdot 2X$ and a definable locally compact model $f: \langle Y \rangle \rightarrow S$ with the target ring S whose additive group can be written as $\mathbb{R}^n \times C$ for a connected compact Lie group C and such that $f^{-1}[U] \subseteq Y$ for some neighborhood U of 0.
- (2) Using the first step in the context of a pseudofinite approximate subring X , we further modify the obtained Y to make sure that the associated escape norm has several good properties (in the spirit of subadditivity and submultiplicativity).
- (3) Having the above steps at our disposal, we adapt the general strategy of the proof of the theorem of Breuillard, Green and Tao as outlined by van den Dries in [2]. The pseudofinite context is adequate, since assuming that the theorem in question fails, we get a pseudofinite “counter-example”.

As a by-product of the first step above, we get the following surprising corollary.

Corollary 0.7. *Any definable approximate subring X is commensurable with a definable approximate subring $Y \subseteq 4X + X \cdot 2X$ which is closed under multiplication (i.e., $Y \cdot Y \subseteq Y$).*

Applications

¹In [5], there was expression $4X + X \cdot 4X$ which was later improved by Mateusz Rzepecki to $4X + X \cdot 2X$.

Sum-product phenomena usually give meaningful structural information when there are “few” 0-divisors, e.g. see [9]. In some particular rings, also the case of many 0-divisor was successfully studied by more advanced techniques, such as Bourgain’s multiscale analysis. Using Theorems 0.5 and 0.6, we get very general qualitative sum-product phenomena which are meaningful also in the case of many 0-divisors. Here is one of them.

Corollary 0.8. *Let $\epsilon > 0$. There is a non-decreasing unbounded function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that the following holds. Let R be a ring and $X \subseteq R$ be a finite subset. Then:*

- *either, $|X + X + X \cdot X| \geq f(|X|)|X|$, or*
- *there is a subring $R' \subseteq R$ and an ideal $I \subseteq R' \cap (4(X - X) + (X - X) \cdot 2(X - X)) \subseteq R' \cap X'_3$ such that R'/I is nilpotent and $|X'_3 \cap R'| \geq |X|^{1-\epsilon}$, where $X'_3 := (X - X)_2$.*

Another application of Theorem 0.6 is a counterpart of Gromov’s theorem in the context of torsion-free rings.

For a subset X of a ring R by $X^{\leq n}$ we will denote the subset of R consisting of the elements obtained from X using $+$ and $\cdot$ so that the elements of X are used at most n times (counting repetitions).

Definition 0.9. *A ring R generated by a finite set X has polynomial growth if there exists $d \in \mathbb{N}$ and constant C such that $|X^{\leq n}| \leq Cn^d$ for all positive integers n .*

Proposition 0.10. *A finitely generated virtually nilpotent ring has polynomial growth.*

Theorem 0.11. *(A ring-theoretic counterpart of Gromov’s theorem) A finitely generated torsion-free ring of polynomial growth is virtually nilpotent. More precisely, given $d > 0$, if R is a torsion-free ring generated by a finite symmetric set X for which $|X^{\leq n}| \leq n^d |X|$ for arbitrarily large n , then R has an ideal of index at most $O_d(1)$ which is nilpotent of class at most $4(57d + 1)$ (in particular, R is virtually nilpotent).*

In fact, in the last theorem, the assumption about R being torsion-free can be weakened to the requirement that R contains no non-zero finite ideals. It remains open whether this assumption could be removed.

Other structural results

Besides Theorems 0.5 and 0.6, we also obtain several much quicker structural consequences of Theorem 0.2. Here we state only two of them.

The first one is a sum-product phenomenon in the style of [9]. *Thickness* is a combinatorial notion of largeness whose definition is skipped in this abstract.

Theorem 0.12. *For every $K \in \mathbb{N}$ there exists $N(K) \in \mathbb{N}$ such that for every finite K -approximate subring X of a ring either there is an $N(K)$ -thick (in particular, of cardinality at least $\frac{|Y|}{N(K)-1}$) subset of $Y := 4X + X \cdot 2X$ consisting of zero divisors or Y is a subring which is (additively) K^7 -commensurable with X .*

The next result goes beyond the finitary context. The class of NSOP (i.e. *non strict order property*) theories is rich. It contains all stable, and, more generally, all simple theories. Thus, among many interesting examples, this class includes the theories of algebraically closed fields, separably closed fields, differentially closed fields, or bounded PAC fields. In particular, Theorem 0.13 applies to all definable in M approximate subrings of $M_n(K)$,

where $M := (K, +, \cdot)$ [$M := (K, +, \cdot, D)$ when K is a differentially closed field] is any of the above fields.

Theorem 0.13. *Let X be a definable (in a structure M) approximate subring. If $\text{Th}(M)$ (i.e. the theory of M) has NSOP, then either there is a definable thick subset D of $Y := 4X + X \cdot 2X$ consisting of zero divisors or Y is a subring K^7 -commensurable with X .*

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ERDŐS-HAJNAL AND VC-DIMENSION

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In what follows, graphs are finite and simple. A graph H is an *induced subgraph* of a graph G if H can be obtained from G by removing vertices. A class $\mathcal{C}$ of graphs is *hereditary* if it is closed under taking induced subgraphs and under isomorphism; and a hereditary class $\mathcal{C}$ is *proper* if it is not the class of all graphs. We say that $\mathcal{C}$ has the *Erdős-Hajnal property* if there exists $c > 0$ such that every graph $G \in \mathcal{C}$ has a clique or independent set of size at least $|G|^c$, where $|G|$ denotes the number of vertices of G . A conjecture of Erdős and Hajnal [9, 10] (see [7, 16] for surveys and [3, 8, 25, 26] for some recent partial results) asserts that:

Conjecture 0.1. *Every proper hereditary class of graphs has the Erdős-Hajnal property.*

For a set $\mathcal{F}$ of subsets of a set V , a subset S of V is *shattered* by $\mathcal{F}$ if for every $A \subset S$ there exists $B \in \mathcal{F}$ with $B \cap S = A$. The *VC-dimension* of $\mathcal{F}$ (introduced by Vapnik and Chervonenkis in [33]) is the largest cardinality of a subset of V that is shattered by $\mathcal{F}$. Since its introduction in 1971, the notion of VC-dimension has proved to be relevant in a number of areas of pure and applied mathematics. The *VC-dimension* of a graph G is the VC-dimension of the set $\{N_G(v) : v \in V(G)\}$ of subsets of $V(G)$, where $N_G(v)$ denotes the set of all neighbours of v (not including v itself). It is not hard to see that for every $d \geq 1$, the class of graphs of VC-dimension at most d is a proper hereditary class. The aim of this paper is to confirm a conjecture of Fox, Pach, and Suk [11] that every hereditary class of graphs of bounded VC-dimension has the Erdős-Hajnal property; in their paper, they came close to settling this by proving a bound of $2^{(\log n)^{1-o(1)}}$ where the constant depending on the VC-dimension is hidden in the $o(1)$ term. (Here $\log$ denotes the binary logarithm.) Our result is:

Theorem 0.2. *For every $d \geq 1$, there exists $c > 0$ such that every graph G of VC-dimension at most d contains a clique or independent set of size at least $|G|^c$.*

As a result, this paper can be viewed as part of a growing body of results [11, 12, 13, 17] that, when VC-dimension is bounded, completely settle or significantly improve bounds for well-known open problems in extremal combinatorics.

The story behind Theorem 0.2, perhaps, began in geometric graph theory with the result of Larman, Matoušek, Pach, and Törőcsik [18] that the class of intersection graphs of line segments in general position in the plane has the Erdős-Hajnal property (in fact, they proved more, for intersection graphs of convex compact sets in the plane). Alon, Pach, Pinchasi, Radoičić, and Sharir [1] generalized this result to classes of semi-algebraic graphs of bounded description complexity. Fox, Pach, and Tóth [14] provided another extension of [18] by verifying the Erdős-Hajnal property of the class

of string graphs where every two curves cross a bounded number of times (and a recent result by Tomon [32] shows that the condition “every two curves cross a bounded number of times” can be dropped). In yet another direction, Sudakov and Tomon [31] recently proved Conjecture 0.1 for the classes of algebraic graphs of bounded description complexity, which is an analogue of the result of [1]. All of these hereditary classes (except intersection graphs of convex compact sets, and more generally, string graphs) turn out to have bounded VC-dimension. Indeed,

- for the classes of semi-algebraic graphs of bounded description complexity, this is true by the classical Milnor–Thom theorem in real algebraic geometry (see [23]);
- for the classes of algebraic graphs of bounded description complexity, this is a consequence of a theorem of Rónyai, Babai, and Ganapathy [28] on the number of zero-patterns of polynomials; and
- for the classes of string graphs where every two curves intersect at a bounded number of points, this follows from a result of Pach and Tóth [27, Lemma 4.2] together with the standard fact that a hereditary class has bounded VC-dimension if and only if it does not contain a bipartite graph, the complement of a bipartite graph, and a split graph.

In a model-theoretic setting, Theorem 0.2 states that every class of graphs edge-definable in NIP (non-independence property) structures has the Erdős-Hajnal property (see [29] for a general reference on NIP theories), which was formally stated as a conjecture by Chernikov, Starchenko, and Thomas [6]. Two notable special cases of NIP graphs include distal graphs and stable graphs. Malliaris and Shelah [20, 21] implicitly proved Conjecture 0.1 for stable graphs (which contains the result of [31]) by developing regularity lemmas for these graphs (see [4] for a short proof using pseudo-finite model theory). In the case of distal graphs, Basu [2] proved the Erdős-Hajnal property for graphs definable by o-minimal structures (which in fact extends [1]), before Chernikov and Starchenko [5] made use of the theory of Keisler measures in NIP to formulate regularity lemmas for distal graphs and settle the general problem in this direction (see also [30] for a short and pure model-theoretic proof). Recently, Fu [15] also combined tools from model theory and a result from [8] to prove the Erdős-Hajnal property of the class of graphs of VC-dimension at most two.

Our proof of Theorem 0.2 uses the ultra-strong regularity lemma for graphs of bounded VC-dimension proved by Lovász and Szegedy [19], and builds on the method of iterative sparsification introduced in earlier papers of the series “Induced subgraph density” [24, 25]. The method involves passing through a sequence of induced subgraphs that are successively more ‘restricted’.

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SHARPLY O-MINIMAL STRUCTURES

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Keywords: Sharp o-minimality, Hilbert 16th problem, Pfaffian functions, Pila-Wilkie theorem, Wilkie conjecture.

A *structure* on $\mathbb{R}$ is a family of nonempty collections of subsets of $\mathbb{R}^n$ ($n \in \mathbb{Z}_{\geq 0}$), called *definable sets*, which is closed under Boolean operations, coordinate projections, and Cartesian product with $\mathbb{R}$. The structure is *o-minimal* if every definable subset of $\mathbb{R}$ is a finite union of points and open intervals. Classical examples include the structure of semialgebraic sets (by Tarski–Seidenberg) and the structure $\mathbb{R}_{an}$ generated by restricted analytic functions. The smallest expansion of $\mathbb{R}$ containing the graph of the exponential map, denoted $\mathbb{R}_{exp}$, was the first nontrivial o-minimal example beyond the semialgebraic world [11]. More generally, inspired by Khovanskii’s fewnomial theory [5], Pfaffian expansions of o-minimal structures retain o-minimality [8].

One of the driving motivations for developing o-minimality (see the exposition in [10]) comes from Hilbert’s 16th problem, which asks for a uniform upper bound on the number of limit cycles of a real planar polynomial vector field. This problem encapsulates a broader principle: solutions of algebraic differential equations should inherit quantitative finiteness properties from their coefficients.

A more accessible avatar is the *Infinitesimal Hilbert 16th problem* (IH16), which concerns uniform bounds for the number of zeros of *Abelian integrals*, i.e. periods of restrictions of polynomial forms to the level curves of a bivariate polynomial (called Hamiltonian) considered as (multivalued) functions of the form and the Hamiltonian. Varchenko and Khovanskii established uniform finiteness for (complex) zeros of Abelian integrals associated with polynomial Hamiltonians, essentially by showing that such integrals are definable in $\mathbb{R}_{an,exp}$ [9, 6]. Much later, a *constructive* uniform bound - depending only on the degrees of the form and the Hamiltonian - was obtained in [1]. This represented a shift from existential finiteness to quantitative, complexity-sensitive estimates for solutions of algebraic differential equations.

These developments, together with quantitative advances for Pfaffian geometry such as [2], led to the introduction of the notion of *sharply o-minimal* structures in [3]. Fix an o-minimal structure $\mathcal{S}$ equipped with a two-parameter filtration by *format* $\mathcal{F}$ and *degree* $\mathcal{D}$ which roughly measures the complexity of definitions and is compatible with Boolean operations, projections, and products. We say that $\mathcal{S}$ is *sharply o-minimal* if the topological complexity of any definable set is controlled polynomially in $\mathcal{D}$ with coefficients depending only on $\mathcal{F}$: specifically, the number of connected components is $\leq \text{poly}_{\mathcal{F}}(\mathcal{D})$. While $\mathbb{R}$ is sharply o-minimal, $\mathbb{R}_{an}$ is not; by contrast, the restricted sub-Pfaffian structure $\mathbb{R}_{rPfaff}$ is sharply o-minimal, essentially by the effective cell-decomposition and Betti bounds of [2] and the general sharp framework of [4]. Many fundamental theorems of o-minimality admit sharp analogues in this setting [4].

It is natural – guided by [1] – to conjecture that the o-minimal structure generated by Abelian integrals (and, more broadly, by periods) is also sharp.

The Pila–Wilkie theorem [7] revealed a striking bridge to Diophantine geometry: outside the *algebraic part* of a definable set A , the number of rational points of height H grows subpolynomially in H . This bound is sharp for $\mathbb{R}_{an}$. Wilkie conjectured that in sufficiently well-behaved o-minimal structures (e.g. $\mathbb{R}_{exp}$) this bound should improve to a polynomial in $\log H$. Sharp o-minimality formalizes this ”well-behaved” assumption, and indeed Wilkie’s conjecture has now been proved [3] for sharply o-minimal structures with *sharp derivatives*, in particular for $\mathbb{R}_{exp}$.

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ARITHMETIC PROGRESSIONS OF LENGTH 3 IN FINITE FIELDS

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Keywords: Model Theory, Supersimplicity, Additive Combinatorics

This talk is based on joint work with Amador Martin-Pizarro [4]. The goal is to present connections between the existence of arithmetic progressions of length 3 with well-known tools and theorems for definable groups in supersimple theories of rank 1.

The class of supersimple theories of rank 1 include important examples of first-order theories, such as the theory of non-principal ultraproducts of finite fields and the theory of the integers, among others. Furthermore, a few years ago, Kaplan and Shelah [3] showed that the group of integers expanded by a unary predicate interpreted as the set Pr of prime integers is supersimple of rank 1, yet assuming Dickson's conjecture. Similarly, Bhardwaj and Tran [1] proved that this is also the case whenever the predicate is interpreted as the set of square-free integers.

Applying a group version of the independence theorem, due to Pillay, Scanlon and Wagner [5], we prove the existence of many 3-term arithmetic progressions for definable subsets of supersimple groups of rank 1. Our main purely model-theoretic result is the following statement:

Theorem 0.1. *Consider a definable subset X of an abelian group G without involutions, all definable in a supersimple group theory of rank 1. For every definable subgroup H of G of finite index, the subset*

$$\{x \in X : \text{there are only finitely many } g \text{ in } H \text{ with } x, x + g \text{ and } x + 2g \text{ in } X\}$$

is finite.

This result, together with the Lang-Weil-type estimates for uniformly definable sets over finite fields [2], allows us to bound the number of points starting few arithmetic progressions in finite fields. More precisely, we prove:

Theorem 0.2. *Given a complexity $C > 0$ and natural numbers r and s , there is a constant $t = t(C, r, s)$ in $\mathbb{N}$ such that for every finite field k , every definable additive subgroup H of k of index at most s and every definable subset X of k , if both X and H have complexity at most C , then the set*

$$\{x \in X : \exists^{\leq r} g \text{ in } H \text{ with } x, x + g \text{ and } x + 2g \text{ in } X\}$$

has size at most t . Moreover, there are two constants $\eta = \eta(C, s) > 0$ in $\mathbb{N}$ and $\delta = \delta(C) > 0$ such that whenever $|X| \geq \eta$, the subset

$$\{x \in X : \exists^{\geq \delta|k|} g \text{ in } H \text{ with } x, x + g \text{ and } x + 2g \text{ in } X\}$$

has cardinality at least $\delta|k|$.

Here, by the complexity of a definable set we mean the length of a formula defining it.

On the other hand, Theorem 0.1 applied to the expansion of the integers $(\mathbb{Z}, 0, +, \text{Pr})$ gives:

Theorem 0.3. *Given a complexity $C > 0$ and an integer $k \geq 1$, there are constants M and N in $\mathbb{N}$ such that for every definable subset $X(x; \bar{y})$ in the structure $(\mathbb{Z}, 0, +, \text{Pr})$ of complexity at most C and every choice $\bar{b}$ of parameters, the set*

$$\{a \in X(\mathbb{Z}, \bar{b}) : \exists^{\leq M} g \text{ in } k\mathbb{Z} \text{ with } a, a + g \text{ and } a + 2g \text{ in } X(\mathbb{Z}, \bar{b})\}$$

is finite of size at most N . Moreover, if there are at least $M + 1$ many g 's in $k\mathbb{Z}$ for the element a in $X(\mathbb{Z}, \bar{b})$, then there are infinitely many such g 's.

The same statement holds for the expansion of integers with a distinguished predicate for all square-free integers, without assuming Dickson's conjecture.

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GLOBALLY VALUED FIELDS: FOUNDATIONS AND PERSPECTIVES

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Keywords: heights, Arakelov geometry, continuous model theory

The goal of the talk was to introduce to the audience *globally valued fields* and to present results from works [3, 9, 15] that appeared in the author's PhD thesis.

1. FOUNDATIONS

Let F be a field. A *global height* on F is a collection of functions $h : \mathbb{P}^n(F) \rightarrow \mathbb{R}_{\geq 0}$, for every natural n , satisfying the following axioms.

Height of one:		$h(1 : 1) = 0$
Invariance:	$\forall x \in \mathbb{P}^n(F), \forall \sigma \in \text{Sym}_{n+1},$	$h(\sigma x) = h(x)$
Additivity:	$\forall x \in \mathbb{P}^n(F), \forall y \in \mathbb{P}^m(F),$	$h(x \otimes y) = h(x) + h(y)$
Monotonicity:	$\forall x \in \mathbb{P}^n(F), \forall y \in \mathbb{P}^m(F),$	$h(x) \leq h(x : y)$
Triangle inequality:	$\forall x, y \in F^n, x + y \neq 0,$	$h(x + y) \leq h(x : y) + e$

for some real number $e \geq 0$ which is called the archimedean error, and $(x_i)_i \otimes (y_j)_j = (x_i y_j)_{i,j}$ is the Segre embedding. A *globally valued field* is a field equipped with a global height. The axioms above define an unbounded continuous logic theory GVF_e . Number fields with the Weil projective height are models of this theory if $e \geq \log 2$. Function fields of curves provide another examples, with $e = 0$. In fact Weil projective heights are the unique global height on $\overline{\mathbb{Q}}$ with $h(2 : 1) = \log 2$.

Another class of examples is induced by proper adelic curves, due to Chen and Moriawaki [7]. A *proper adelic curve* structure on F is a measure space (Ω, μ) together with a map $\phi : \Omega \rightarrow M_F$ (denoted $\omega \mapsto |\cdot|_\omega$), where M_F is the space of absolute values on F , such that for all $x \in F^\times$ the function

$$\omega \mapsto \log |x|_\omega$$

is measurable and integrable, with the integral equal to zero. In that case, functions

$$(1.1) \quad h(x_0 : \dots : x_n) = \int_{\Omega} \max_i \log |x_i|_\omega \, d\mu(\omega)$$

define a global height on F .

In a joint work with Itai Ben Yaacov, Pablo Destic and Ehud Hrushovski [3], we prove that a converse is true for countable fields.

Theorem 1.1. *Assume that F is countable. Then for every global height on F , there is a proper adelic curve structure that induces it via the formula (1.1).*

In fact, this theorem is a part of a dictionary of equivalent structures defining globally valued fields. Our main result is the following.

Theorem 1.2. *There is a bijective correspondence between the following structures on F :*

- (1) *Global heights,*
- (2) *Global functionals,*
- (3) *Renormalisation classes of global admissible measures,*
- (4) *Equivalence classes of global lattice valuations.*

What this roughly means is that globally valued fields can come from the following constructions.

(1) Weil heights	(2) Algebraic cycles	(3) Measure theory	(4) Banach lattices
Global heights as above	$\text{APic}(F) \rightarrow \mathbb{R}$ or $\text{NS}(F/k) \rightarrow \mathbb{R}$ positive on the effective cone	Proper adelic curves, M-fields satisfying the product formula	$v : F \rightarrow \Gamma \cup \{\infty\}$ 'valuation' valued in a lattice Γ , with $\Gamma \simeq L^1(\Omega, \mu)$

To illustrate (2), if $k \subset F = k(x, y)$, then a GVF structure on F trivial on k can be given by a nef b-divisor on $\mathbb{P}_k^2$. Similarly, (product with) a nef arithmetic line bundle defines a GVF structure on the function field of an arithmetic surface.

2. EXISTENTIAL CLOSEDNESS

A model theoretic analysis of a theory T often starts with identifying its model companion T^* and checking whether it exists. It would be desirable to know whether GVF_e has a model companion. As a natural step towards this question, some existentially closed models have been characterised in [2]. Ben Yaacov and Hrushovski proved there that for any field k , the unique GVF structure on $\bar{k}(t)$ that is trivial on k and satisfies $h(t : 1) = 1$ is existentially closed. Their proof crucially uses the differentiability of volume of divisors from [4]. The following is proved in [15] using analogous tools from Arakelov geometry [5, 14, 16].

Theorem 2.1. *The field $\bar{\mathbb{Q}}$ equipped with the Weil projective height is an existentially closed globally valued field.*

This means that if X is a variety over $\bar{\mathbb{Q}}$ and the function field of X is equipped with a global height h satisfying $h(2 : 1) = \log 2$, then there is a generic sequence of points $x_n \in X(\bar{\mathbb{Q}})$ such that for any tuple of rational functions $f_0, \dots, f_k$ on X , we have

$$\lim_n \text{ht}(f_0(x_n) : \dots : f_k(x_n)) = h(f_0 : \dots : f_k),$$

where on the left hand side ht is the Weil projective height on $\bar{\mathbb{Q}}$.

To understand existential closedness of more general globally valued fields, one needs to develop intersection theory over other GVFs. This is the motivation for the results presented in the following section.

3. DEFINABILITY OF ADELIC INTERSECTION

Let K be a GVF and X be a variety over K of dimension d . Pick closed embeddings $j_0 : X \rightarrow \mathbb{P}^{n_0}, \dots, j_d : X \rightarrow \mathbb{P}^{n_d}$. There exists a polynomial R called *resultant* such that:

$$R(\lambda_0, \dots, \lambda_d) = 0 \iff (\exists x \in X)(\forall i)(j_i(x) \subset \{\lambda_i = 0\}).$$

Here λ_i are linear forms on K^{n_i+1} defining hypersurfaces $\{\lambda_i = 0\}$ in $\mathbb{P}^{n_i}$ respectively. The resultant is unique up to scaling by an element of $K^\times$, provided that we specify its multi-degree by requiring

$$\deg_{\lambda_i} R = \deg(j_0^* \mathcal{O}(1) \cdot \dots \cdot j_{i-1}^* \mathcal{O}(1) \cdot j_{i+1}^* \mathcal{O}(1) \cdot \dots \cdot j_d^* \mathcal{O}(1)).$$

One can define the *adelic intersection pairing* over a GVF via a formula

$$j_0 \cdot \dots \cdot j_d := \lim_n \frac{\text{ht}(R_n)}{n^{d+1}}$$

where R_n is the resultant of the family of embeddings j_i composed with n 'th Veronese on each $\mathbb{P}^{n_i}$'s, and by ht we mean the height of its coefficients. Here one intersects closed embeddings $j : X \rightarrow \mathbb{P}^n$.

Over number fields this intersection product has a rich history starting with ideas of Arakelov [1], Deligne [8] and Philippon [13]. Over more general globally valued fields (or closely related structures) it has been studied by Gubler [10], Chen and Moriwaki [6], and by Yuan and Zhang [17].

For a variety S over K let us consider the space of quantifier-free types concentrated on S , over K . In this way, we get a locally compact topological space S_{GVF} whose points correspond to scheme theoretic points of S whose residue field is equipped with a GVF structure extending the one on K . In [9] together with Pablo Destic and Nuno Hultberg we prove that the adelic intersection pairing is definable in families.

Theorem 3.1. *Let $\mathcal{X} \rightarrow S$ be a flat projective morphism of finite type schemes over a globally valued field K , of relative dimension d . Let $j_i : \mathcal{X} \rightarrow \mathbb{P}_S^{n_i}$ be families of closed embeddings over S . Then, the map*

$$\begin{aligned} S_{\text{GVF}} &\rightarrow \mathbb{R} \\ s &\mapsto j_0(s) \cdot \dots \cdot j_d(s) \end{aligned}$$

is continuous, where by $j_i(s)$ we mean the restriction of j_i to the fiber over the scheme-theoretic point underlying s .

To prove this we use geometric results from [6] together with some analysis over archimedean places à la [13]. Furthermore, we use this theorem to prove a conjecture of Roberto Gualdi and Martin Sombra from [11, 12].

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REINTERPRETING STATEMENTS ABOUT SATURATED MODELS

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One of the core tools of model theory, especially in its applications to other parts of mathematics, is the passage to saturated models, where problems that may be unwieldy in the intended model suddenly become tractable. When this application involves bounds, however, the passage to saturated models appears to make proofs non-quantitative: precisely what saturated models do is hide elaborate calculations.

A number of papers [1, 2, 6] have given syntactic transformations which reinterpret statements about saturated models of various kinds as quantitative statements about the original model. The approach in [8] is specifically tailored to work in ultraproducts, which makes it especially suitable for applications of ultraproducts to other parts of mathematics.

One application is to the setting of “tame regularity lemmas”. The model example of such a lemma is the Malliaris–Shelah stable regularity lemma.

Definition 0.1. Let (V, E) be a finite graph. We say $A, B \subseteq V$ are ϵ -homogeneous if

$$\frac{|E \cap (A \times B)|}{|A \times B|} \in [0, \epsilon) \cup (1 - \epsilon, 1].$$

Lemma 0.2 ([5], see also [7]). For every k and every $\epsilon > 0$ there is an N so that whenever (V, E) is a k -stable finite graph, there is a partition $V = \bigcup_{i \leq n} V_i$ so that $n \leq N$, $\frac{|V_0|}{|V|} < \epsilon$, and every pair V_i, V_j with $0 < \min\{i, j\}$ is ϵ -homogeneous.

It is not so clear how to generalize this to hypergraphs. There are two natural generalizations of stability to ternary relations—*slicewise stability* (a symmetric relation $R \subseteq X^3$ is slicewise stable if every slice $R_x = \{(y, z) \mid (x, y, z) \in R\}$ is stable) and *partitionwise stability* (a symmetric relation $R \subseteq X^3$ is partitionwise stable if it is stable as a binary relation on $X \times X^3$). Neither of these notions are equivalent to any immediate generalization of the statement above to symmetric ternary relations [7, 3].

Chernikov and Towsner identified [3] a superficially stronger property which, in infinite saturated models, is equivalent to stability, and does generalize to a property equivalent to partitionwise stability.

Theorem 0.3. If $E \subseteq V^2$ is stable then there is a countable partition $V = \bigcup_i V_i$ so that every pair V_i, V_j is 0-homogeneous.

They then used a syntactic translation to turn this into a more complicated property which makes sense in finite graphs as well.

Theorem 0.4. E is partitionwise stable if and only if (uniformly under all measures), for each $\epsilon > 0$ and each F , there is a partition $V = \bigcup_{i \leq n} V_i$ so that $\mu(V_0) < \epsilon$ and each V_i with

$0 < i$ has the property that the set of x with $\mu(E_x \cap V_i) \in [F(n)\mu(V_i), (1 - F(n))\mu(V_i)]$ has measure $< F(n)$.

A second application of these syntactic translations is to quantifier elimination results. These ought to have quantitative content—given a formula, we should get a translation to an equivalent formula—but many proofs are given using “saturated embedding tests”, which obscure the explicit transformation on formulas.

Definition 0.5. A multiplicative subgroup G of $\mathbb{R}^{>0}$ has the Mann property if, for every linear equation

$$q_1x_1 + \cdots + q_nx_n = 1,$$

there are at most finitely many non-degenerate solutions $(g_1, \dots, g_n)$ in G^n .

Non-degenerate means that, for any non-empty $I \subseteq [1, n]$, $\sum_i q_i g_i \neq 0$.

Given any dense $\Gamma \subseteq \mathbb{R}^{>0}$ satisfying the Mann property, we extend the language of ordered rings by:

- a predicate U , and
- constants $\{c_\gamma\}_{\gamma \in \Gamma}$.

We can write down a theory $\text{RCF}(\Gamma)$ extending the theory of ordered fields by some facts about Γ and the constants.

We consider structures $\mathfrak{M} = (M, G)$ where:

- M is a real closed ordered field,
- $G = U^{\mathfrak{M}}$ is a distinguished subset.

Definition 0.6. A formula $\psi(\vec{x})$ is special if it has the form

$$\exists \vec{y} (U(\vec{y}) \wedge \theta_U(\vec{y}) \wedge \phi(\vec{x}, \vec{y}))$$

where θ_U has quantifiers restricted to U and ϕ is quantifier-free. (Plus some further technical restrictions on the language.)

Theorem 0.7 ([4]). In $\text{RCF}(\Gamma)$, every formula $\psi(x)$ is equivalent to a boolean combination of special formulas.

Again, a syntactic translation can be used to transform this proof into an explicit translation on formulas [9].

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TAMENESS, COMPLEXITY, AND THE SPACE OF QUANTUM FIELD THEORIES

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Keywords: O-minimality, complexity, quantum field theory, quantum gravity

In this talk, I give an overview of recent applications of model theory to quantum field theory. Quantum field theory is one of the central pillars of theoretical physics, providing a powerful framework for describing the quantum-mechanical interactions of particles. Recently, various foundational aspects of quantum field theory have been analyzed using o-minimality, a model-theoretic framework of tameness [1-9].

In the first part of the talk, I review sharp o-minimality; a refined notion of o-minimality, in which definable sets have an explicit measure of complexity. For sharply o-minimal structures, many of the interesting theorems in tame geometry become quantitative. A key example of such a structure is generated by the restricted Pfaffian functions, which arise as solutions of a particular type of system of differential equations. As discussed later in the talk, many functions arising in the formulation of physical theories belong to this class.

In the second part of the talk, I begin by briefly reviewing quantum field theory, focusing on the particular mathematical aspects which are central to our applications. I will explain that one of the key objects specifying a quantum field theory is the potential, a real-valued function on the space of fields. Given such a potential, the main objective of quantum field theory is to calculate scattering amplitudes, which are probability distributions for physical processes as real-valued functions of kinematic variables and parameters of the theory. Scattering amplitudes are usually calculated by perturbation theory, which results in a diagrammatical series expansion in terms of Feynman graphs. Due to this underlying combinatorial structure, scattering amplitudes possess a rich mathematical structure. It is shown in [2] that scattering amplitudes, at finite order in perturbation theory, are definable in an o-minimal structure – the first key result which shows that objects in quantum field theory have a tame mathematical structure. The proof of this statement relies on recent applications of o-minimality to Hodge theory. Continuing on this, various works attempt to use sharp o-minimality to assign a complexity to scattering amplitudes [4,5,7,8]. In addition, in reference [6] progress is made to understand the tameness of amplitudes beyond perturbation theory.

In the third part of the talk, I consider smaller class within the space of quantum field theories, namely those quantum field theories which can be consistently coupled to quantum gravity. Understanding how quantum field theory and general relativity (gravity) can be combined into a single theory of quantum gravity is one of the profound challenges of theoretical physics. A more modest question is to ask which

quantum field theories can be consistent with gravity at all, and it turns out that this requirement places strong constraints on quantum field theories. These strong constraints manifest themselves as conjectures, since the complete theory of quantum gravity is fully understood. In this part of the talk, I discuss recent works which show that some of these quantum gravity constraints come in the form of tameness [1,9]. It is conjectured that in a quantum field theory which is consistent with quantum gravity, the potential must be definable in an o-minimal structure, and that the geometry of the field space must admit a tame isometric embedding into Euclidean space. Ultimately, these works suggest that tameness may play a fundamental role as a physical principle.

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