ERROR BOUNDS FOR DISCRETE MINIMIZERS OF THE GINZBURG–LANDAU ENERGY IN THE HIGH-κ REGIME[*]

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Abstract. In this work, we study discrete minimizers of the Ginzburg–Landau energy in finite element spaces. Special focus is given to the influence of the Ginzburg–Landau parameter κ. This parameter is of physical interest as large values can trigger the appearance of vortex lattices. Since the vortices have to be resolved on sufficiently fine computational meshes, it is important to translate the size of κ into a mesh resolution condition, which can be done through error estimates that are explicit with respect to κ and the spatial mesh width h. For that, we first work in an abstract framework for a general class of discrete spaces, where we present convergence results in a problem-adapted κ-weighted norm. Afterward we apply our findings to Lagrangian finite elements and a particular generalized finite element construction. In numerical experiments we confirm that our derived L^2- and H^1-error estimates are indeed optimal in κ and h.

1. Introduction. Superconductors are materials that allow electricity to be conducted without any electrical resistance. Letting $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, denote a bounded polyhedral Lipschitz domain occupied by a superconducting material, the superconductivity in Ω can be modeled by a complex-valued wave function $u : \Omega \to \mathbb{C}$ which is called the order parameter. The physical quantity of interest is $|u|^2$, which denotes the density of the superconducting electron pairs, where in the appropriate scaling, it holds that $0 \leq |u|^2 \leq 1$. This means that the material is not superconducting (in the normal state) in $x \in \Omega$ if $|u(x)|^2 = 0$ and behaves like a perfect superconductor if $|u(x)|^2 = 1$. In between, different degrees of superconductivity are possible. Of particular interest are mixed normal-superconducting states where both phases coexist in a lattice of quantized vortices [1].

Mathematically, the order parameter can be characterized as a minimizer of the Ginzburg–Landau energy (or Gibbs free energy) given by

$$(1.1) \qquad E(u) = \frac{1}{2} \int_\Omega |\nabla u + \mathrm{i}\kappa A u|^2 + \frac{\kappa^2}{2}\left(1 - |u|^2\right)^2 dx,$$

where $A : \Omega \to \mathbb{R}^d$ is a real-valued magnetic potential and κ is the so-called Ginzburg–Landau parameter, a material parameter that correlates with the temperature and

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determines the type of superconductor. By the necessary condition for local extrema, any minimizer $u \in H^1(\Omega)$ must fulfill the condition $E'(u) = 0$, which is known as the Ginzburg–Landau equation (GLE) and reads written out (cf. [19])

$$(1.2) \quad \mathrm{Re} \int_\Omega \left(\nabla u + \mathrm{i}\kappa A u\right) \cdot \left(\nabla \varphi + \mathrm{i}\kappa A \varphi\right)^* + \kappa^2\big(|u|^2 - 1\big)u\varphi^* \, dx = 0 \ \text{ for } \varphi \in H^1(\Omega).$$

The real-valued magnetic potential $A : \Omega \to \mathbb{R}^d$ in the GLE is typically unknown and can be inferred from an external magnetic field H through the condition $H = \mathrm{curl}\, A$, which is then added as a penalty term to the energy. In this work we consider the simplifying case that A is given, where the focus of our analysis is rather the influence of κ on the accuracy of numerical approximations. In fact, the size of the parameter κ is crucial for the appearance of vortices [44, 45, 46, 47]. On the one hand, if κ is too small, no vortices will appear. On the other hand, the larger the value of κ, the more vortices appear in the lattice and the more point-like they become [2, 44]. The so-called high-κ regime is hence the physically most interesting regime, but numerically it is also the most challenging one because it requires fine meshes to resolve all lattice structures. This raises an important practical question: how fine do we have to select the mesh size relative to the size of κ so that the numerical approximations capture the correct vortex pattern?

Motivated by this question, the main goal of this work is to derive rigorous error bounds for the discrete minimizers with constants that are explicit and optimal in the spatial parameter h and the Ginzburg–Landau parameter κ.

The first work where the approximation properties of discrete solutions to the stationary GLE were analyzed is the seminal *SIAM Review* article by Du, Gunzburger, and Peterson [19] (see also [20] for periodic boundary conditions). The paper considers H^1-error estimates in finite element spaces for both the order parameter u and the magnetic potential A. The proof technique considers fixed (compact) intervals of κ-values and does not trace all κ-dependencies that enter through the size of these intervals and through uniform bounds for certain operator norms (that are linked to the chosen interval). The proof also considers a modified setting where $E''(u)$ is assumed to have a trivial kernel. However, the solutions to the GLE (1.2) can be only locally unique up to gauge transformations [19]. In our case, these transformations are of the form $u \mapsto \exp(-\mathrm{i}\theta)u$ for any $\theta \in \mathbb{R}$. In fact, it is easily seen that $E(u) = E(\exp(-\mathrm{i}\theta)u)$ for all such θ, which hence leads to a cluster of (qualitatively equivalent) solutions $\exp(-\mathrm{i}\theta)u$. In turn, we have $\langle E''(u)\exp(-\mathrm{i}\frac{\pi}{2})u, \cdot \rangle = 0$, which shows that $E''(u)$ can become singular. Hence, it makes sense to revisit the results [19] with new proof techniques that allow us to follow all κ-dependencies and which allow us to avoid an assumption of local uniqueness. To the best of our knowledge, there are only two other works that address the convergence of discrete solutions to the stationary GLE: In [22] a spatial discretization based on a covolume method is suggested, and in [21] a finite volume discretization is used to solve the GLE on the sphere. In both papers the convergence of a subsequence of discrete solutions to a continuous minimizer is established, however without rates in h and κ.

With the goal to close this gap in the literature for finite element discretizations, our error analysis is performed in a general framework of finite element methods, where we state our results under natural assumptions on the discrete spaces. We first establish bounds on the discrete minimizers which are explicit in κ and independent of h. This enables us to provide an abstract convergence result which identifies a suitable, continuous minimizer of (1.1). This a priori information is crucial in the derivation of the error bounds. In order to exploit the structure of the problem, we

have to study the properties of the second Fréchet derivative of the energy E. In particular, we carry over the inf-sup stability to our discrete setting under a smallness condition related to the product κh. Let us emphasize that this is not a technical issue, but is indeed observed in our numerical experiments. We employ a problem adapted scalar product and its Ritz projection, which captures the one-dimensional kernel of $E''(u)$, to extract optimal error bounds not only for the H^1-norm, but also new error bounds for the L^2-norm and the energy. Our numerical experiments confirm that the predicted scaling of the error in κ and h is asymptotically sharp.

It is worth mentioning that, aside from stationary GLEs, there has been a lot of work on the numerical analysis of the time-dependent problem that describes the dynamics of superconductors, where we exemplarily refer to [11, 12, 15, 16, 17, 18, 23, 25, 26, 34, 35, 36] and the references therein. For works with a particular emphasis on tracing the influence of κ in the estimates, we refer to [7, 8, 9] for the case of vanishing vector potentials A. Due to the different nature of the time-dependent problem, we will not discuss the equation any further here.

The rest of the paper is organized as follows. In section 2, we introduce the analytical framework and present some results on continuous minimizers of (1.1). In particular, we discuss the assumptions concerning uniqueness of minimizers. For an abstract finite element space discretization, we present in section 3 our main results on the existence, boundedness, and approximation of discrete minimizers. An application to linear Lagrange finite elements is also given. Numerical experiments which illustrate our theoretical findings and confirm the convergence rates as well as the κ-dependency of our bounds are shown in section 4. The proofs of our main results are given in section 5. Finally, in section 6 we present a nonstandard application of the abstract result to spaces based on the localized orthogonal decomposition.

Notation. For a complex number $z \in \mathbb{C}$, we use z^* for the complex conjugate of z. In the whole paper we further denote by $L^2 := L^2(\Omega, \mathbb{C})$ the Hilbert space of L^2-integratable complex functions, but equipped with the *real* scalar product $m(u, v) :=$ Re $\int_\Omega v\, w^*\, dx$ for $v, w \in L^2$. Hence, we interpret the space as a *real* Hilbert space. Analogously, we equip the space $H^1 := H^1(\Omega, \mathbb{C})$ with the scalar product $m(v, w) + m(\nabla v, \nabla w)$. This interpretation is crucial so that the Fréchet derivatives of E are meaningful and exist on H^1. For any space X, we denote its dual space by X'. Note that this implies that the elements of the dual space of H^1 consist of real-linear functionals, which are not necessarily complex-linear. For example, if $F(v) := m(f, v)$ for some $f \in L^2$, then it holds that $F(\alpha v) = \alpha F(v)$ if $\alpha \in \mathbb{R}$, but in general *not* if $\alpha \in \mathbb{C}$.

In the following C will denote a generic constant which is independent of κ and the spatial mesh parameter h, but might depend on numerical constants as well as Ω and A. In particular, we will write $\alpha \lesssim \beta$ if there is a constant C independent of κ and h such that $\alpha \leq C\beta$.

2. Analytical framework. In this section, we present several results concerning the continuous minimizers of (1.1).

From now on, we assume that the magnetic potential A satisfies

$$(2.1) \qquad A \in L^\infty(\Omega, \mathbb{R}^d), \qquad \operatorname{div} A = 0 \text{ in } \Omega, \qquad A \cdot \nu = 0 \text{ on } \partial\Omega.$$

The above assumption can be rigorously justified on convex and on smooth domains Ω; cf. [19]. However, we also note that the L^∞-regularity of A might not be available on general complex geometries with re-entrant corners.

Further, we introduce the dual pairing $\langle u, \varphi \rangle := \langle u, \varphi \rangle_{(H^1)', H^1}$, and the bilinear forms given by

$$(2.2) \quad m(u, \varphi) := \operatorname{Re} \int_\Omega u\varphi^* \, dx, \quad a_A(u, \varphi) := \operatorname{Re} \int_\Omega \big(\nabla u + \mathrm{i}\kappa A u\big) \cdot \big(\nabla \varphi + \mathrm{i}\kappa A \varphi\big)^* \, dx,$$

as well as the norm $||\varphi||_{H^1}^2 := ||\nabla \varphi||_{L^2}^2 + ||\varphi||_{L^2}^2$, the scaled norms

$$(2.3) \qquad ||\varphi||_{H^1_\kappa}^2 := ||\nabla \varphi||_{L^2}^2 + \kappa^2 ||\varphi||_{L^2}^2, \qquad ||\varphi||_{H^2_\kappa}^2 := ||\varphi||_{H^2}^2 + \kappa^2 ||\varphi||_{H^1_\kappa}^2,$$

and the induced norm $||f||_{(H^1_\kappa)'} = \sup_{\varphi \in H^1} \frac{f(\varphi)}{||\varphi||_{H^1_\kappa}}$. We abbreviate $A_\infty = ||A||_{L^\infty}$ and define the stabilized inner product on $H^1 = H^1(\Omega)$ for $u, \varphi \in H^1$ by

$$(2.4) \qquad \hat{a}_\kappa(u, \varphi) := a_A(u, \varphi) + \beta^2 m(u, \varphi)_{L^2} \quad \text{with } \beta^2 = \kappa^2(A_\infty^2 + 1).$$

We call it stabilized since this modification enables us to show boundedness and coercicvity of $\hat{a}_\kappa(\cdot, \cdot)$ with respect to the H^1_κ-norm defined in (2.3).

LEMMA 2.1. *There are κ-independent constants $C_{bnd}, C_{coe} > 0$ such that for all $v, \varphi \in H^1$*

$$\hat{a}_\kappa(v, \varphi) \leq C_{bnd} ||v||_{H^1_\kappa} ||\varphi||_{H^1_\kappa} \qquad and \qquad \hat{a}_\kappa(\varphi, \varphi) \geq C_{coe} ||\varphi||_{H^1_\kappa}^2.$$

Proof. The boundedness is a straightforward application of the Cauchy–Schwarz inequality. For the coercivity, we note that by Young's inequality it holds that

$$|\nabla \varphi + \mathrm{i}\kappa A \varphi|^2 \geq |\nabla \varphi|^2 - 2|\nabla \varphi||\kappa A \varphi| + |\kappa A \varphi|^2 \geq \frac{1}{2}|\nabla \varphi|^2 - \kappa^2 A_\infty^2 |\varphi|^2.$$

By the choice of β, we conclude the lower bound. □

A straightforward calculation shows that the energy is (real-)Fréchet differentiable and satisfies for all $\varphi \in H^1$

$$(2.5) \qquad \langle E'(u), \varphi \rangle = \operatorname{Re} \int_\Omega \big(\nabla u + \mathrm{i}\kappa A u\big) \cdot \big(\nabla \varphi + \mathrm{i}\kappa A \varphi\big)^* + \kappa^2\big(|u|^2 - 1\big)u\varphi^* \, dx.$$

In particular any minimizer $u \in H^1$ satisfies $E'(u) = 0$. The natural boundary condition is given by $\big(\nabla u + \mathrm{i}\kappa A u\big) \cdot \nu = 0$ on $\partial\Omega$. Since A has a vanishing trace by assumption, the natural boundary condition collapses to the standard homogeneous Neumann condition $\nabla u \cdot \nu = 0$ on $\partial\Omega$. Our first result collects the existence of a minimizer u and its properties.

THEOREM 2.2. *For every $\kappa \geq 0$ there exists a minimizer $u \in H^1$ of (1.1). Further, any minimizer fulfills*

$$|u(x)| \leq 1 \text{ for all } x \in \Omega, \qquad ||u||_{H^1_\kappa} \lesssim \kappa,$$

and if Ω is convex, then $u \in H^2$ and

$$||\nabla u||_{L^4} \lesssim \kappa, \qquad ||u||_{H^2_\kappa} \lesssim \kappa^2,$$

where the hidden constants in the above estimates are independent of κ and u.

Proof. First note that the energy E is continuous in $H^1(\Omega)$, and further weakly lower semicontinuous; see, e.g., [48, Thm. 1.6]. In addition, a simple calculation shows

$$E(u) = \frac{1}{2}\hat{a}_\kappa(u,u) + \frac{\kappa^2}{4}\int_\Omega \left(1 + \frac{\beta^2}{\kappa^2} - |u|^2\right)^2 + 1 - \left(1 + \frac{\beta^2}{\kappa^2}\right)^2 dx,$$

and hence $E(u) \to \infty$ as $||u||_{H^1_\kappa} \to \infty$. The standard arguments then imply the existence of a minimizer; see, e.g., [48, Thm. 1.2]. For the pointwise bound, we refer to [19, Prop. 3.11], which implies a bound in L^2 independent of κ. We further have

$$||\nabla u||_{L^2} \leq ||\nabla u + \mathrm{i}\kappa Au||_{L^2} + \kappa A_\infty ||u||_{L^2} \lesssim E(0)^{1/2} + \kappa \lesssim \kappa.$$

Since $E'(u) = 0$, we rearrange to

$$a_A(u,\varphi) = -\kappa^2 \operatorname{Re}\int_\Omega \left(|u|^2 - 1\right)u\varphi^* \, dx = m(f,\varphi)$$

with $||f||_{L^2} \lesssim \kappa^2$ and obtain with (2.1)

$$\operatorname{Re}\int_\Omega \nabla u \cdot \nabla\varphi^* \, dx + \operatorname{Re}\int_\Omega u\,\varphi^* \, dx = m(f,\varphi) - \operatorname{Re}\int_\Omega \left(-2\mathrm{i}\kappa A \cdot \nabla u + \kappa^2|A|^2 u + u\right)\varphi^* \, dx.$$

If Ω is convex, standard elliptic regularity theory for the Poisson problem with homogeneous Neumann boundary condition (cf. [28, Theorem 3.2.1.3]) gives us $u \in H^2$ with

$$||u||_{H^2} \lesssim ||f||_{L^2} + (1 + \kappa^2)||u||_{L^2} + \kappa||\nabla u||_{L^2} \lesssim \kappa^2,$$

where we used the L^2- and H^1-bounds for u in the last step. Finally, we have

$$||\nabla u||_{L^4}^4 = \int_\Omega \nabla u \cdot \nabla u^* |\nabla u|^2 \, dx = -\int_\Omega u \operatorname{div}\left(\nabla u^* |\nabla u|^2\right) dx \lesssim ||u||_{L^\infty} ||u||_{H^2} ||\nabla u||_{L^4}^2,$$

which yields the last estimate. □

Since u is a global minimizer of the energy E, it must not only hold that $\langle E'(u), \varphi\rangle = 0$ but also $\langle E''(u)\varphi, \varphi\rangle \geq 0$ for all $\varphi \in H^1$. Later we will make use of these conditions. For that we require a corresponding representation of the second Fréchet derivative of E. This and its properties are summarized in the following lemma.

LEMMA 2.3. (a) *The energy is twice (real-)Fréchet differentiable and satisfies for* $\varphi, v \in H^1$

$$\langle E''(u)v, \varphi\rangle = \operatorname{Re}\int_\Omega \left(\nabla v + \mathrm{i}\kappa Av\right) \cdot \left(\nabla\varphi + \mathrm{i}\kappa A\varphi\right)^* \\ + \kappa^2\left((|u|^2 - 1)v\varphi^* + u^2 v^*\varphi^* + |u|^2 v\varphi^*\right) dx.$$

(b) *For* $\varphi, v \in H^1$ *it holds that*

$$\langle E''(u)v, \varphi\rangle = \langle E''(u)\varphi, v\rangle \quad \textit{and} \quad |\langle E''(u)v, \varphi\rangle| \lesssim ||v||_{H^1_\kappa} ||\varphi||_{H^1_\kappa}.$$

Proof. The Fréchet derivative is computed in a straightforward manner, and the symmetry follows from the representation by noting the real part in front of the integral. For the bound, we employ Lemma 2.1 as well as $|u| \leq 1$. □

As explained in the introduction, a minimizer of (1.1) cannot be unique due to the invariance of the energy under complex rotations, i.e., if u is a minimizer, then

$\exp(-\mathrm{i}\phi)u$ is also a minimizer for any $\phi \in \mathbb{R}$. This property is known as gauge invariance and the mapping $u \mapsto \exp(-\mathrm{i}\phi)u$ is called a gauge transformation. On polygonal domains, minimizers are believed to be locally unique, which means that in a sufficiently small environment of u, the only other minimizers are exactly the ones obtained by gauge transformations. However, a general proof for this local uniqueness property is one of the great challenges of the field and has not yet been established. For partial results in various important settings, we refer to [42, 43, 44, 49] and the references therein.

In the following, we hence make the local uniqueness an (reasonable) assumption for our analysis. Furthermore, we later describe how to check the validity of the assumption numerically for any given setting so that we explicitly know if it is fulfilled.

In the definition below we express the local uniqueness by curves passing through a minimizer. To be precise, we look at the energy level $\ell : t \mapsto E(\gamma(t))$ for a smooth curve $\gamma : \mathbb{R} \to H^1$ with $\gamma(0) = u$. If u is a minimizer of E, then $\ell(t)$ has a local minimum at $t = 0$ (i.e., $\ell'(0) = 0$ and $\ell''(0) \geq 0$). Furthermore, u is locally unique up to gauge if $\ell''(0) > 0$ for all directions $\gamma'(0)$ that are not parallel to $\mathrm{i}u$. Note that the direction $\mathrm{i}u$ is the tangent in u on the circle line $t \mapsto \exp(-\mathrm{i}t)u$, i.e., the direction in which the value of the energy does not change; cf. Figure 2.1. Since the energy is constant on the circle line, we naturally have $\ell''(0) = 0$ in this direction. The following definition formalizes this type of local uniqueness.

DEFINITION 2.4 (local uniqueness up to gauge transformation). *Let u be a minimizer of* (1.1). *We call u a* locally unique minimizer up to gauge transformation *if, for all smooth curves $\gamma : [-\pi, \pi) \to H^1$ with $\gamma(0) = u$, it holds that*

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} E(\gamma(t))|_{t=0} = 0 \iff \gamma'(0) \in \mathrm{span}\{\mathrm{i}u\} := \{\alpha\,\mathrm{i}u \,|\, \alpha \in \mathbb{R}\}.$$

Note that u being a minimizer always implies that $\frac{\mathrm{d}}{\mathrm{d}t} E(\gamma(t))|_{t=0} = 0$ as well as $\frac{\mathrm{d}^2}{\mathrm{d}t^2} E(\gamma(t))|_{t=0} \geq 0$, independent of whether it is locally unique or not.
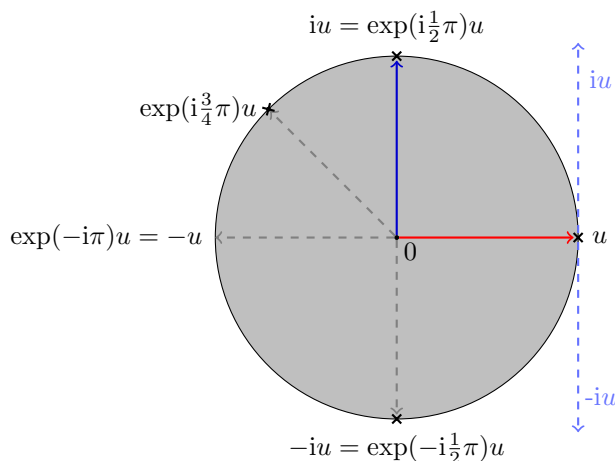

FIG. 2.1. *For a given minimizer u, the figure illustrates the circle line parametrized by the 2π-periodic curve $\gamma : t \mapsto \exp(-\mathrm{i}t)u$ for $t \in [-\pi, \pi)$. The tangent direction in u is given by $\gamma'(0) = \mathrm{i}u$ and the energy E is constant and minimal on the whole circle line, i.e., $\frac{\mathrm{d}^2}{\mathrm{d}t^2} E(\gamma(t)) = 0$.*

From now on, we assume local uniqueness for our error analysis.

Assumption 2.5. The minimizers u of the Ginzburg–Landau energy (1.1) are locally unique up to gauge transformation in the sense of Definition 2.4.

For any minimizer u, the above assumption implies inf-sup stability of $E''(u)$ on the $m(\cdot,\cdot)$-orthogonal complement of span$\{iu\}$ in H^1. To formalize and prove this result, we define the corresponding space by

$$H^1_{iu} := H^1 \cap (iu)^\perp := \{\varphi \in H^1 \,|\, m(iu,\varphi) = 0\}.$$

Note that H^1_{iu} is a closed subspace of H^1 and that our error analysis will be naturally restricted to it. The claimed inf-sup stability is specified in the following proposition.

PROPOSITION 2.6. *Let u be a minimizer of* (1.1) *and let Assumption* 2.5 *be fulfilled, i.e., u is locally unique up to gauge transformation. Then, there is a constant $C_{\mathrm{sol}}(u,\kappa) \gtrsim 1$ such that*

$$(2.6) \qquad C_{\mathrm{sol}}^{-1}(u,\kappa) \leq \inf_{v \in H^1_{iu}} \sup_{\varphi \in H^1_{iu}} \frac{\langle E''(u)v,\varphi\rangle}{\|v\|_{H^1_\kappa}\|\varphi\|_{H^1_\kappa}}.$$

Furthermore, $E''(u)$ is singular in the direction iu, i.e.,

$$\langle E''(u)\,iu,\varphi\rangle = 0 \qquad \text{for all } \varphi \in H^1.$$

Since Assumption 2.5 implies a positive spectrum of $E''(u)$ on H^1_{iu}, the inf-sup stability follows from the Gårding inequality. For completeness we elaborate the short proof below, also with the purpose to emphasize how to check the local uniqueness numerically.

Proof. Let $v \in H^1_{iu} \setminus \{0\}$ be arbitrary and $\gamma : [-\pi,\pi) \to H^1$ a smooth curve with $\gamma(0) = u$ and $\gamma'(0) = v$. Then Assumption 2.5 implies

$$(2.7) \;\; 0 < \frac{\mathrm{d}^2}{\mathrm{d}t^2}E(\gamma(t))|_{t=0} = \langle E'(\gamma(0)),\gamma''(0)\rangle + \langle E''(\gamma(0))\,\gamma'(0),\gamma'(0)\rangle = \langle E''(u)v,v\rangle.$$

Hence, the eigenvalue problem seeking $(w_\ell,\lambda_\ell) \in H^1 \times \mathbb{R}$ with

$$(2.8) \qquad \langle E''(u)w_\ell,v\rangle = \lambda_\ell\, m(w_\ell,v) \quad \text{for all } v \in H^1$$

has a nonnegative spectrum. A direct calculation shows that the smallest eigenvalue is $\lambda_1 = 0$ with eigenfunction $w_1 = iu$. Furthermore, since $\langle E''(u)\cdot,\cdot\rangle$ is a symmetric bilinear form on H^1, the second smallest eigenvalue λ_2 is positive due to the Courant–Fischer theorem which yields

$$\lambda_2 = \inf_{\substack{v \in H^1_{iu} \\ v \neq 0}} \frac{\langle E''(u)v,v\rangle}{m(v,v)} \overset{(2.7)}{>} 0.$$

Observe that the strict positivity of the second eigenvalue of $E''(u)$ is in fact equivalent to the local uniqueness of u in the sense of Definition 2.4. Hence, by computing λ_2 we can practically verify if a computed minimizer u is locally unique.

Thanks to $\lambda_2 > 0$, we obtain that $E''(u)$ is injective on the $m(\cdot,\cdot)$-orthogonal complement of the first eigenspace, i.e., on H^1_{iu}. Furthermore, it is easily seen that the following Gårding inequality holds for all $v \in H^1$ (and in particular $v \in H^1_{iu}$):

$$(2.9) \qquad \langle E''(u)v,v\rangle \geq \frac{1}{2}\|v\|^2_{H^1} - (1+\kappa^2)(1+A_\infty^2)\|v\|^2_{L^2}.$$

It is well known (cf. [27, 51]) that the injectivity on $H^1_{\mathrm{i}u}$ and the Gårding inequality (in combination with the Lax–Milgram theorem) imply the Fredholm alternative for $E''(u) \colon H^1_{\mathrm{i}u} \to (H^1_{\mathrm{i}u})'$, i.e., $E''(u)$ has a bounded inverse on the orthogonal complement of $\mathrm{i}u$. With this insight, let $v \in H^1_{\mathrm{i}u}$ be arbitrary and let $z \in H^1_{\mathrm{i}u}$ be the unique solution to

$$\langle E''(u)z, w\rangle = \mu\, m(v, w) \qquad \text{for all } w \in H^1_{\mathrm{i}u}$$

for some constant $\mu > (1 + \kappa^2)(1 + A_\infty^2)$. Since $E''(u)$ has a bounded inverse it also holds

$$(2.10) \qquad \qquad \|z\|_{H^1} \le C\,\mu\,\|v\|_{H^1}$$

for some constant $C = C(u, \kappa)$. In conclusion we obtain

$$\langle E''(u)(v + z), v\rangle = \langle E''(u)v, v\rangle + \mu\, m(v, v) \overset{(2.9)}{\ge} \frac{1}{2}\|v\|^2_{H^1} \overset{(2.10)}{\ge} C\,\|v\|_{H^1}\|v + z\|_{H^1}$$

with $C = C(u, \kappa, \mu) > 0$. Hence,

$$\sup_{\phi \in H^1_{\mathrm{i}u}} \frac{\langle E''(u)\phi, v\rangle}{\|\phi\|_{H^1}\|v\|_{H^1}} \ge \frac{\langle E''(u)(v + z), v\rangle}{\|v + z\|_{H^1}\|v\|_{H^1}} \ge C.$$

Since $v \in H^1_{\mathrm{i}u}$ was arbitrary, the desired inf-sup stability follows from the symmetry of $\langle E''(u)\cdot, \cdot\rangle$ and the equivalence of the H^1-norm and the H^1_κ-norm (up to κ-constants). $\qquad \Box$

Remark 2.7 (verifying the local uniqueness numerically). The proof of Proposition 2.6 shows that Assumption 2.5 is fulfilled for a minimizer u if and only if the second smallest eigenvalue, λ_2, of

$$\langle E''(u)w_\ell, v\rangle = \lambda_\ell\, m(w_\ell, v) \quad \text{for all } v \in H^1$$

is positive. Given a computed discrete minimizer u_h that approximates u, we can check this condition numerically. Since $\|u_h - u\|_{H^1} \le \varepsilon(h) \to 0$ independent of local uniqueness (cf. Proposition 5.1 below), the second smallest eigenvalue of $E''(u_h)$ converges to the second smallest eigenvalue of $E''(u)$. Hence, if λ_2 is clearly bounded away from zero, u must be locally unique in the sense of Definition 2.4. The practical assembly of the bilinear form $\langle E''(u_h)\cdot, \cdot\rangle$ is discussed in Appendix A. In our numerical experiments we present the corresponding values for λ_2 and we observed that the local uniqueness up to gauge transformations was always fulfilled.

In the next step, we will derive stability and regularity estimates for solutions to variational problems on $H^1_{\mathrm{i}u}$. The variational problems will later play a crucial role in our error analysis and involve the stabilized bilinear form $\hat{a}_\kappa(\cdot, \cdot)$, as well as the inf-sup stable bilinear form $\langle E'(u)\cdot, \cdot\rangle$.

LEMMA 2.8. *For any $f \in L^2(\Omega)$, there is $z \in H^1_{\mathrm{i}u} \subset H^1(\Omega)$ such that*

$$\hat{a}_\kappa(z, \varphi) = m(f, \varphi) \quad \text{for all } \varphi \in H^1_{\mathrm{i}u},$$

and there hold the bounds

$$\|z\|_{H^1_\kappa} \lesssim \|f\|_{(H^1_\kappa)'} \lesssim \frac{1}{\kappa}\|f\|_{L^2} \text{ and, if } \Omega \text{ is convex, then } z \in H^2 \text{ and } \|z\|_{H^2_\kappa} \lesssim \|f\|_{L^2},$$

where the (hidden) constants in the bounds are independent of κ.

Proof. Since $\hat{a}_\kappa(\cdot,\cdot)$ is still coercive on $H^1_{\mathrm{i}u}$, we immediately obtain the unique solution, and also the bounds in H^1_κ. Furthermore, we have for any $f \in L^2$ that

$$(2.11) \qquad ||f||_{(H^1_\kappa)'} = \sup_{||\varphi||_{H^1_\kappa}=1} m(f,\varphi) \le \sup_{||\varphi||_{H^1_\kappa}=1} \frac{1}{\kappa}||f||_{L^2}\kappa||\varphi||_{L^2} \le \frac{1}{\kappa}||f||_{L^2},$$

which yields the second inequality. For the bound in the H^2_κ-norm for convex domains, let $\varphi \in H^1$ and decompose as $\varphi = \widehat{\varphi} + \alpha(\mathrm{i}u)$ with $\widehat{\varphi} \in H^1_{\mathrm{i}u}$ and $\alpha = m(\varphi,\mathrm{i}u)||u||_{L^2}^{-2}$. Then,

$$\begin{aligned}\hat{a}_\kappa(z,\varphi) &= \hat{a}_\kappa(z,\widehat{\varphi}) + \alpha\,\hat{a}_\kappa(z,\mathrm{i}u) = m(f,\widehat{\varphi}) + \alpha\,\hat{a}_\kappa(z,\mathrm{i}u) \\ &= m(f,\varphi) - \alpha\,m(f,\mathrm{i}u) + \alpha\,a_A(z,\mathrm{i}u),\end{aligned}$$

where we used (2.4) in the last step. We first note

$$|m(f,\varphi) - \alpha\,m(f,\mathrm{i}u)| \le 2||f||_{L^2}||\varphi||_{L^2},$$

and then employ $E'(\mathrm{i}u) = 0$ to obtain

$$|a_A(z,\mathrm{i}u)| = |\langle E'(\mathrm{i}u), z\rangle - \kappa^2\,\mathrm{Re}\int_\Omega \big(|u|^2 - 1\big)\mathrm{i}uz^*\,dx| \le \kappa^2||u||_{L^2}||z||_{L^2} \lesssim ||f||_{L^2},$$

where we exploited $\kappa^2||z||_{L^2} \lesssim ||f||_{L^2}$ in the last line. Altogether we have shown that there exists some $f_z \in L^2$ such that it holds for all $\varphi \in H^1$

$$(2.12) \qquad \hat{a}_\kappa(z,\varphi) = m(f_z,\varphi), \quad ||f_z||_{L^2} \lesssim ||f||_{L^2}.$$

We conclude as in Theorem 2.2: We write

$$(2.13) \quad \hat{a}_\kappa(z,\varphi) = \mathrm{Re}\int_\Omega \nabla z \cdot \nabla\varphi^* + \beta^2 z\,\varphi^*\,dx + \mathrm{Re}\int_\Omega \big(-2\mathrm{i}\kappa A \cdot \nabla z + \kappa^2|A|^2 z\big)\varphi^*\,dx,$$

and since the second term is in L^2, we have, again by regularity theory for the homogeneous Neumann problem, $z \in H^2$ and

$$||z||_{H^2} \lesssim ||f_z||_{L^2} + \kappa^2||z||_{L^2} + \kappa||\nabla z||_{L^2} \lesssim ||f||_{L^2},$$

where we used the L^2- and H^1-bounds for z in the last step. $\qquad\square$

Using the inf-sup stability established in Proposition 2.6, we can formulate an analogous result for variational problems based on $E''(u)$ in $H^1_{\mathrm{i}u}$. Note here that the inclusion $H^1_{\mathrm{i}u} \subset H^1$ implies $(H^1)' \subset (H^1_{\mathrm{i}u})'$ for the dual spaces.

COROLLARY 2.9. *Let Assumption* 2.5 *hold.*
(a) *For any* $f \in (H^1_{\mathrm{i}u})'$, *there is a unique* $z \in H^1_{\mathrm{i}u}$ *such that*

$$(2.14) \qquad \langle E''(u)z, \varphi\rangle = \langle f, \varphi\rangle \quad \text{for all} \quad \varphi \in H^1_{\mathrm{i}u},$$

which satisfies the estimate

$$||z||_{H^1_\kappa} \le C_{\mathrm{sol}}(u,\kappa)||f||_{(H^1_\kappa)'}.$$

(b) *Let* $z \in H^1_{\mathrm{i}u}$ *be the solution of* (2.14) *with* $f \in L^2$. *Then, it further holds that*

$$||z||_{H^1_\kappa} \le \frac{C_{\mathrm{sol}}(u,\kappa)}{\kappa}||f||_{L^2}$$

and, if Ω *is convex, then* $z \in H^2$ *and*

$$||z||_{H^2_\kappa} \lesssim C_{\mathrm{sol}}(u,\kappa)||f||_{L^2}.$$

Proof. By standard theory for indefinite differential equations (cf. [5]), the inf-sup stability in Proposition 2.6 directly gives the well-posedness of (2.14) together with the stability estimate $||z||_{H_\kappa^1} \leq C_{\mathrm{sol}}(u,\kappa)||f||_{(H_\kappa^1)'}$, hence proving (a). The first estimate in (b) is obtained from (2.11). Using this observation, we conclude that $z \in H_{\mathrm{i}u}^1$ solves

$$\hat{a}_\kappa(z,\varphi) = m(\widetilde{f},\varphi) \quad \text{for all } \varphi \in H_{\mathrm{i}u}^1,$$

for some $\widetilde{f} \in L^2$ with $||\widetilde{f}||_{L^2} \lesssim C_{\mathrm{sol}}(u,\kappa)||f||_{L^2}$, and thus Lemma 2.8 gives the claim. $\square$

3. Space discretization and main results. Let us consider some finite dimensional finite element space V_h which is a subspace of $H^1(\Omega)$ and where we recall that we assume Ω to be a polygonal (or polyhedral) Lipschitz domain. By h we denote a spatial parameter which tends to zero for a finer spatial resolution.

In order to derive a priori estimates for the error between a discrete minimizer u_h and a continuous minimizer u, we introduce the closed subspace $V_h^\perp$ of V_h by $V_h^\perp = V_h \cap (\mathrm{i}u)^\perp \subset H_{\mathrm{i}u}^1$, where the orthogonality is with respect to the inner product $m(\cdot,\cdot)$. Before we proceed, let us stress that $V_h^\perp$ is just an auxiliary space that is not required for practical computations but is only used as a tool in the analysis. For the moment, $u \in H^1$ can denote any minimizer of the energy E; however, later we will link every discrete minimizer $u_h \in V_h$ to a specific minimizer u in order to ensure that the error becomes small.

As another theoretical tool, we require the orthogonal projection $\mathrm{R}_{\kappa,h}^\perp : H_{\mathrm{i}u}^1 \to V_h^\perp$ to satisfy

$$\hat{a}_\kappa(\mathrm{R}_{\kappa,h}^\perp w, \varphi_h) = \hat{a}_\kappa(w, \varphi_h) \quad \text{for all } \varphi_h \in V_h^\perp.$$

In the following assumption, we introduce some abstract conditions which are sufficient to carry out our error analysis and which are later verified for our examples.

Assumption 3.1. The family of (nonempty) finite element spaces V_h has the following properties:
   (a) The family of spaces V_h is dense in $H^1(\Omega)$ in the sense that for each $\varphi \in H^1$ we have

$$\lim_{h \to 0} \inf_{\varphi_h \in V_h} ||\varphi - \varphi_h||_{H^1} = 0.$$

   (b) Let $w \in H_{\mathrm{i}u}^1$ and $f \in L^2$ such that $\hat{a}_\kappa(w,\varphi) = m(f,\varphi)$ for all $\varphi \in H_{\mathrm{i}u}^1$. Then, it holds that

$$(3.1) \qquad ||w - \mathrm{R}_{\kappa,h}^\perp w||_{H_\kappa^1} \lesssim h||f||_{L^2},$$

   where the constant is independent of h and κ.

The most prominent example that fulfills Assumption 3.1 is linear Lagrange finite element spaces which are discussed at the end of this section. Property (a) is obvious and to verify property (b), one first replaces $||f||_{L^2}$ by $||w||_{H_\kappa^2}$ and uses (for a convex domain Ω) H^2-regularity. We give the details below. Another, nontrivial example of generalized finite elements spaces is presented in section 6.

Recall that we want to minimize the functional E from (1.1) over V_h, i.e.,

$$(3.2) \quad E(u_h) = \inf_{\varphi_h \in V_h} E(\varphi_h), \qquad E(\varphi_h) = \frac{1}{2}\int_\Omega |\nabla\varphi_h + \mathrm{i}\kappa A\varphi_h|^2 + \frac{\kappa^2}{2}\big(1 - |\varphi_h|^2\big)^2 \, dx.$$

Note that since V_h is finite dimensional, the existence of a minimizer u_h is always guaranteed. Our first result shows bounds on the discrete minimizer u_h in different norms and the corresponding energy, which are independent of h and behave in the parameter κ the same way as the exact minimizer u studied in Theorem 2.2. The proof is postponed to section 5.

LEMMA 3.2. *For all $h > 0$ let u_h be a minimizer of* (3.2). *Then there hold the bounds*

$$E(u_h) \lesssim \kappa^2 \qquad ||u_h||_{L^2} \lesssim 1, \qquad ||\nabla u_h||_{L^2} \lesssim \kappa, \qquad ||u_h||_{H_\kappa^1} \lesssim \kappa,$$

where the hidden constants are independent of h and κ.

Our main findings are collected in the following theorem. We provide error bounds for the discrete minimizers which are explicit in the parameter κ and the mesh width h. In addition, we show that the error behaves as the quasi-best approximation of $H_{\mathrm{i}u}^1$ in $V_h^\perp$.

THEOREM 3.3. *Let Assumptions 2.5 and 3.1 hold, and let $h \leq h_0$ be sufficiently small such that in particular $\kappa C_{\mathrm{sol}}(u, \kappa) h$ is small. Then, there is a neighborhood $U \subset H^1(\Omega)$ of each discrete minimizer u_h of* (3.2) *such that there is a unique minimizer $u \in U$ of* (1.1) *with*

$$m(u_h, \mathrm{i}u) = 0,$$

and we have the error bounds

$$||u - u_h||_{H_\kappa^1} \lesssim (1 + \kappa C_{\mathrm{sol}}(u, \kappa) h) \inf_{\varphi_h \in V_h^\perp} ||u - \varphi_h||_{H_\kappa^1},$$

$$||u - u_h||_{L^2} \lesssim h \, C_{\mathrm{sol}}(u, \kappa) \, (1 + \kappa C_{\mathrm{sol}}(u, \kappa) h) \inf_{\varphi_h \in V_h^\perp} ||u - \varphi_h||_{H_\kappa^1},$$

as well as the following estimate on the error in the energy:

$$0 \leq E(u_h) - E(u) \lesssim ||u - u_h||_{H_\kappa^1}^2 \left(1 + \kappa^{1/2} ||u - u_h||_{H_\kappa^1} + \kappa ||u - u_h||_{H_\kappa^1}^2\right),$$

where the hidden constants are independent of κ, $C_{\mathrm{sol}}(u, \kappa)$, and h.

The proof is divided into several steps which are outlined in detail in section 5. The first application of the results is Lagrangian finite elements. In the following, we denote by $\mathcal{T}_h$ a conforming family of partitions of the domain Ω consisting of simplical elements K. For the space $\mathcal{P}_1(K)$ of complex-valued polynomials of degree less than or equal to 1 on K, we consider the finite element space

(3.3) $$V_h := \{\varphi_h \in H^1(\Omega) \,|\, \varphi_h|_K \in \mathcal{P}_1(K) \text{ for all } K \in \mathcal{T}_h\}.$$

We assume that the partition $\mathcal{T}_h$ is shape-regular and the L^2-projection, defined via $m(\pi_h v, \varphi_h) = m(v, \varphi_h)$ for all $\varphi_h \in V_h$, is H^1-stable, i.e.,

(3.4) $$||\pi_h \varphi||_{H^1} \lesssim ||\varphi||_{H^1},$$

with a constant independent of h. This condition is always fulfilled for quasi-uniform triangulations but is also valid for certain adaptively refined meshes. For a detailed discussion on criteria when (3.4) holds, we refer to [6, 13]. In this setting, we obtain convergence rates which are explicit in the parameter κ and the mesh width h.

Corollary 3.4. *Let the conditions of Theorem 3.3 hold, and assume in addition that Ω is convex. For Lagrangian finite elements which satisfy (3.4), the error bounds in Theorem 3.3 can be further estimated by*

$$||u - u_h||_{H^1_\kappa} \lesssim \kappa^2 h \qquad and \qquad ||u - u_h||_{L^2(\Omega)} \lesssim C_{\mathrm{sol}}(u,\kappa)\kappa^2 h^2,$$

where the hidden constants are independent of κ, $C_{\mathrm{sol}}(u,\kappa)$, and h.

The proof is presented in section 5.

4. Numerical experiments. Before we present the proof of our main result, we illustrate our theoretical findings with some numerical examples confirming the rates and the κ-dependence in our error bounds. In the following we will hence compute discrete minimizers

$$u_h = \arg\min_{\varphi_h \in V_h} E(\varphi_h)$$

in $\mathcal{P}_1$-spaces V_h for different mesh sizes h and different material parameters κ. Note that, just like analytical minimizers, discrete minimizers can be at most unique up to gauge transformations, i.e., if $u_h \in V_h$ is a minimizer, so is $\exp(-i\theta)u_h$ for any $\theta \in [-\pi, \pi)$.

4.1. Implementation. For the discretization in space with linear Lagrange finite elements, we use the open source Python tool FEniCS [3, version 2019.2.0]. To compute a discrete minimizer, we applied a steepest descent approach using an implicit Euler method for the L^2 gradient flow. A direct application yields the following nonlinear iteration:

$$m(u_h^{n+1}, \varphi_h) = m(u_h^n, \varphi_h) - \tau \langle E'(u_h^{n+1}), \varphi_h \rangle,$$

where $\tau > 0$ is some parameter. To avoid the solution of nonlinear systems several times, we replace $E'(u^{n+1})$ by the linearization

$$\langle E'(u_h^{n+1}), \varphi_h \rangle \to a_A(u_h^{n+1}, \varphi_h) + \kappa^2 \mathrm{Re} \int_\Omega (|u_h^n|^2 - 1)u_h^{n+1}\varphi_h^* \, dx,$$

and thus we have to solve the following linear system for $u_h^{n+1} \in V_h$:

$$m(u_h^{n+1}, \varphi_h) + \tau\, a_A(u_h^{n+1}, \varphi_h) + \tau\, \kappa^2 \mathrm{Re} \int_\Omega (|u_h^n|^2 - 1)u_h^{n+1}\varphi_h^* \, dx = m(u_h^n, \varphi_h)$$

for all $\varphi_h \in V_h$. In our experiments, we set $\Omega = [0,1] \times [0,1] \subset \mathbb{R}^2$ and use on the finest grid the initial value $u_0 = 0.8 + 0.6\mathrm{i}$. For the coarser grids, we project this reference solution and use this as a starting value. The magnetic potential is chosen as

$$A(x,y) := \sqrt{2}\begin{pmatrix} \sin(\pi x)\cos(\pi y) \\ -\cos(\pi x)\sin(\pi y) \end{pmatrix}$$

and satisfies the assumptions in (2.1). Further, we set $\tau = \kappa^{-2}$ and used the stopping criterion $\kappa^{-2}|E(u_h^{n+1}) - E(u_h^n)| < \delta$ for a tolerance $\delta = 10^{-9}$. Below this tolerance, we use a Newton method for the equation $E'(u_h) = 0$ with the previous approximation u_h^n as starting value and the same stopping criterion, but $\delta = 10^{-12}$. The code to reproduce the results presented in this paper is available at https://doi.org/10.35097/1924.

Let us note that the steepest descent approach as described above can potentially also find local minimizers of E in V_h, depending on the choice of the initial value.

However, we also note that global minimizers are typically found more easily and that local minimizers (that are not global minimizers) can be identified and discarded by comparing the energy levels.

Finally, let us also mention that the phase of the minimizer (i.e., which $\exp(-i\theta)u_h$ we find for some $\theta \in [-\pi, \pi)$) depends only on the choice of the initial value. Hence, it is important to start the steepest descent method always with the same initial value in order to ensure a reasonable comparison between minimizers on different meshes (since phase differences might otherwise dominate the error).

4.2. Numerical results. We first illustrate the convergence in the spatial parameter h for different values of κ. To this end, we computed a reference solution on a finer grid using $h_{\max} \sim 2.5 \cdot 10^{-3}$. In order to numerically verify Assumption 2.5 on the local uniqueness of the computed minimizers, we proceeded as in Remark 2.7 and computed the five smallest eigenvalues (in absolute value) of $E''(u_h)$ and collected them in Table 4.1. We observe that the first eigenvalue is essentially zero up to numerical precision and we verified that it indeed belongs to iu_h. The second smallest eigenvalue is clearly bounded away from zero, and the computed minimum is hence indeed locally unique in the sense of Definition 2.4.

In order to compare the results for different values of κ, we divide the error in the H^1_κ- and L^2-norm by κ^2 and the energy by κ^4; see Figure 4.1. Here we recall that according to Corollary 3.4 we expect the H^1_κ-error to convergence with the rate $\kappa^2 h$, the L^2-error with the rate $\kappa^2 h^2$, and the energy-error with the rate $\kappa^4 h^2$. Indeed, we observe the predicted asymptotic convergence in h and, in particular, the numerical experiments confirm the κ-scaling in our error estimates. The plot further indicates that the constants in front of the normalized errors are independent of κ. Further, we observe in Figure 4.2 the uniform boundedness of $\|u_h\|_{H^1_\kappa}$ and $E(u_h)$ as predicted in Lemma 3.2.

Let us also note that for larger values of κ, we observe a preasymptotic behavior in h. We expect that this is related to the smallness condition for $\kappa C_{\mathrm{sol}}(u, \kappa)h$ stated in the theorem, which is required below in Lemma 5.4 for the discrete inf-sup stability. Since beyond the (numerically observed) threshold $\kappa h < 1$, the errors coincide for all values of κ, this is still in alignment with our theory.

To further investigate the preasymptotic effect, we added a comparison of our errors with the best-approximation $R_{\kappa,h}(u)$ in V_h with respect to $\hat{a}_\kappa(\cdot, \cdot)$. To be precise, for our reference solution u, we computed the orthogonal projection $R_{\kappa,h}(u) \in V_h$ such that

$$(4.1) \qquad \hat{a}_\kappa(u - R_{\kappa,h}(u), \phi_h) = 0 \quad \text{for all} \quad \phi_h \in V_h.$$

By comparing our errors $\|u - u_h\|_{H^1_\kappa}$ with $\|u - R_{\kappa,h}(u)\|_{H^1_\kappa}$, we can identify possible numerical pollution effects related to κ. The corresponding results are depicted in

TABLE 4.1

The numerically computed five smallest eigenvalues of $E''(u_h)$ with reference solution u_h and $\kappa = 8, 10, 17, 24$.

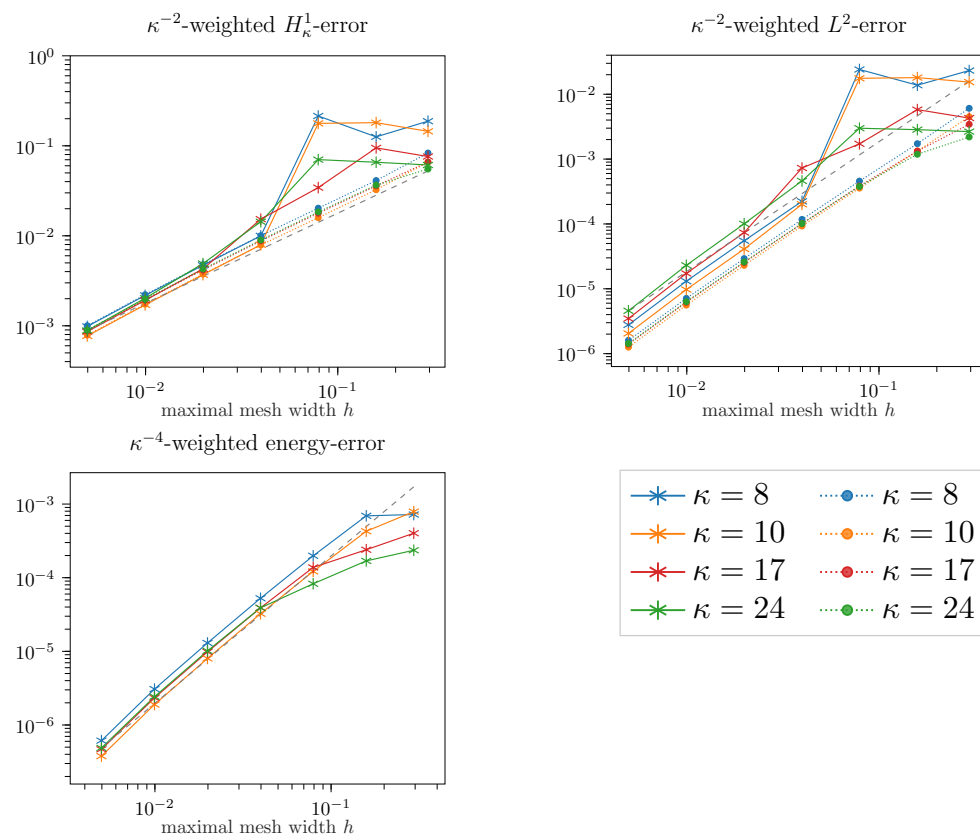| κ	λ_1	λ_2	λ_3	λ_4	λ_5
8	$\sim 10^{-12}$	2.65	2.65	7.54	7.71
10	$\sim 10^{-12}$	3.37	3.37	3.78	10.70
17	$\sim 10^{-8}$	0.46	0.57	2.63	2.75
24	$\sim 10^{-9}$	0.51	2.03	2.03	5.52

FIG. 4.1. *Convergence in the mesh size h for κ-weighted errors in the H^1_κ- and L^2-norm and for the energy, for $\kappa = 8, 10, 17, 24$. The errors between u and u_h in L^2 and H^1_κ are scaled by κ^{-2} and the error in energy by κ^{-4}. The dotted lines indicate the corresponding errors (in L^2 and H^1_κ, respectively) between u and its best-approximation $\mathrm{R}_{\kappa,h}(u)$ in V_h with respect to $\hat{a}_\kappa(\cdot, \cdot)$; cf. (4.1). The dashed lines indicate order $\mathcal{O}(h)$ in the upper left figure and order $\mathcal{O}(h^2)$ in the upper right and bottom left figures.*

Figure 4.1 and reveal a very interesting phenomenon. Contrary to u_h itself, the best-approximation does not show any visible preasymptotic convergence. On the contrary, in the asymptotic regime both approximations have a very similar behavior and follow the expected rates closely. Since both errors are expected to behave asymptotically as

$$||u - u_h||_{H^1_\kappa} + ||u - \mathrm{R}_{\kappa,h}(u)||_{H^1_\kappa} \lesssim \kappa \min\{1, h\kappa\}$$

(due to the energy bounds $||u||_{H^1_\kappa} + ||u_h||_{H^1_\kappa} \lesssim \kappa$), we suspect that the necessary resolution condition $h\kappa \lesssim 1$ will only become visible for larger values of κ, whereas the preasymptotic convergence of u_h must be related to the smallness condition $\kappa C_{\mathrm{sol}}(u, \kappa)h$. This is further supported by the right-hand side of Figure 4.2, where we compare the energy $E(\mathrm{R}_{\kappa,h}(u))$ of the best-approximation with the energy $E(u_h)$ of the actual minimizer in V_h. The energies in Figure 4.2 on the finest level are given by

$$\kappa^{-2}E(u_h) = 1.29 \cdot 10^{-1} \text{ for } \kappa = 8, \qquad \kappa^{-2}E(u_h) = 8.24 \cdot 10^{-2} \text{ for } \kappa = 17,$$
$$\kappa^{-2}E(u_h) = 1.05 \cdot 10^{-1} \text{ for } \kappa = 10, \qquad \kappa^{-2}E(u_h) = 6.84 \cdot 10^{-2} \text{ for } \kappa = 24.$$
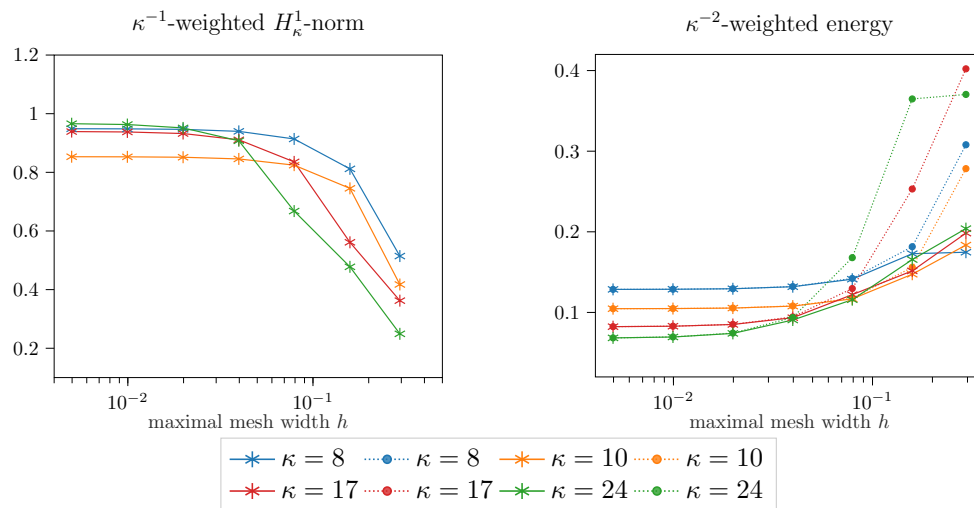
FIG. 4.2. *Boundedness of the scaled H^1_κ-norm and energy E with respect to h for $\kappa = 8, 10, 17, 24$. The dotted lines show the energy of the best-approximation $\mathrm{R}_{\kappa,h}(u)$ in V_h with respect to $\hat{a}_\kappa(\cdot, \cdot)$.*
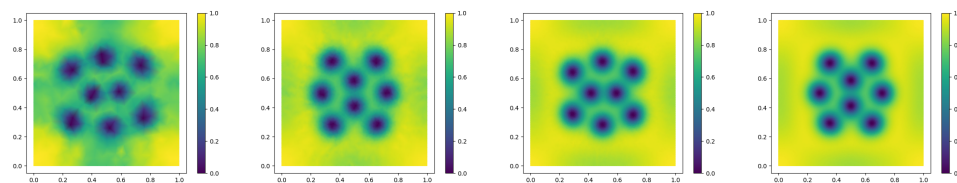

FIG. 4.3. *Minimizers for the Ginzburg–Landau parameter $\kappa = 20$ and different mesh widths $h \approx 8 \cdot 10^{-2}, 4 \cdot 10^{-2}, 2 \cdot 10^{-2}, 1 \cdot 10^{-2}$ (from left to right).*

We can see that, in the preasymptotic regime, the energy of $\mathrm{R}_{\kappa,h}(u)$ is indeed significantly larger than the one of u_h. This explains why good approximations are not found on coarse meshes and indicates that one cannot capture the correct vortex pattern on meshes which do not satisfy some further resolution condition on h with respect to κ.

In our second experiment, we first computed for $\kappa = 20$ the discrete minimizers for different values of $h \approx 8 \cdot 10^{-2}, 4 \cdot 10^{-2}, 2 \cdot 10^{-2}, 1 \cdot 10^{-2}$; see Figure 4.3. We observe that the number of vortices remains constant on the different discretization levels, but the minimizer is rotated by $\frac{\pi}{2}$. A simple calculation shows that by our choice of A this rotation of the coordinate system leaves the energy invariant. In particular, this illustrates that the density $|u|^2$ of minimizers is not necessarily unique and that convergence of discrete minimizers can only be expected up to a subsequence, even for a fixed gauge condition. On the other hand, we plotted the minimizers for the values $\kappa = 8, 10, 17, 24$; see Figure 4.4. We observe that the number of vortices increases with larger values of κ, which is in agreement with analytical results [2, 44].

5. Proof of the main result. In this section, we provide the proof of our main results, Theorem 3.3 and Corollary 3.4. We first show an abstract convergence result in order to identify possible limits of a sequence of discrete minimizers. Those are then used to establish convergence with rates, if we are sufficiently close to a continuous minimizer. Throughout this section, we let Assumptions 2.5 and 3.1 hold.
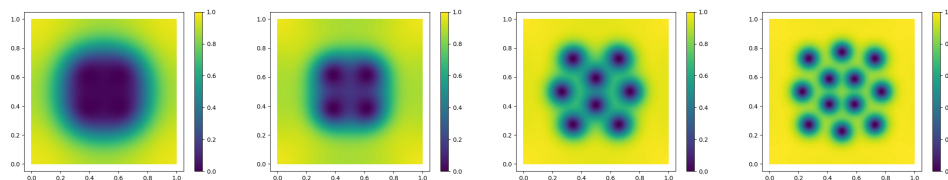
FIG. 4.4. *Different minimizers corresponding to the Ginzburg–Landau parameters* $\kappa = 8, 10, 17, 24$ *(from left to right) for* $h \approx 2.5 \cdot 10^{-3}$.

5.1. Abstract convergence result. In order to deduce convergence, we first establish bounds on minimizers in the discrete space V_h which are independent of the spatial parameter h as formulated in Lemma 3.2.

Proof of Lemma 3.2. First note that for all $h > 0$ we have $0 \in V_h$, and thus by the minimizing property, we conclude the bound on the energy

$$E(u_h) \leq E(0) \leq \frac{\kappa^2}{2} \operatorname{vol}(\Omega).$$

This gives on the one hand

$$||\nabla u_h + \mathrm{i}\kappa A u_h||_{L^2} \leq E(u_h)^{1/2} \leq \kappa \operatorname{vol}(\Omega)^{1/2},$$

and on the other hand we estimate

$$\frac{\kappa^2}{2} ||1 - |u_h|||_{L^2}^2 \leq \frac{\kappa^2}{2} \int_\Omega \left(1 - |u_h|\right)^2 \left(1 + |u_h|\right)^2 dx \leq E(0) = \frac{\kappa^2}{2} \operatorname{vol}(\Omega)^{1/2},$$

and thus conclude

$$||u_h||_{L^2} \leq ||1 - |u_h|||_{L^2} + \operatorname{vol}(\Omega)^{1/2} \leq 2 \operatorname{vol}(\Omega)^{1/2}.$$

Combining the estimates above, the bound on $||\nabla u_h||_{L^2}$ directly follows. □

With the uniform estimates on the discrete minimizers, following the approach in [10], we employ the Banach–Alaoglu theorem to obtain some limit which is an exact minimizer and by Assumption 2.5 locally unique up to complex rotation.

PROPOSITION 5.1. *Denote by* $(u_h)_{h>0}$ *a family of minimizers of* (3.2). *Then, there exists a minimizer* u_0 *of* (1.1) *such that there is a monotonically decreasing sequence* $(h_n)_{n\in\mathbb{N}}$ *with*

$$\lim_{n\to\infty} ||u_0 - u_{h_n}||_{H_\kappa^1} = 0.$$

In particular, we can define the twisted approximations

$$\widetilde{u}_{h_n} := e^{\mathrm{i}\phi_n} u_{h_n} \qquad \textit{where } \phi_n \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \textit{ is chosen such that } m(\widetilde{u}_{h_n}, \mathrm{i}u_0) = 0,$$

which also converge in H^1, *i.e.,*

$$\lim_{n\to\infty} ||\widetilde{u}_{h_n} - u_0||_{H_\kappa^1} = 0.$$

Conversely, for any n, the minimizer u_{h_n} *is an approximation to* $e^{-\mathrm{i}\phi_n} u_0$.

Remark 5.2. The assertion of Proposition 5.1 can be interpreted as follows. Assume that there exists a (sub-)sequence of discrete minimizers that keeps a positive distance to all exact minimizers; then this would be a contradiction to Proposition 5.1. Hence, for h sufficiently small, one always arrives at a neighborhood of some minimizer u_0, which is precisely the claim in Theorem 3.3.

Proof of Proposition 5.1. The proof of convergence of a subsequence is along the lines of [10] if one takes into account the bounds provided in Lemma 3.2 together with the weak lower semicontinuity of E; see Theorem 2.2 and Assumption 3.1.

For the twisted approximations, we note that we can find some $\phi_n \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ such that real part of the inner product with iu_0 vanishes if n is large enough. Thus, we obtain by the choice of ϕ_n

$$\sin \phi_n \ m(u_{h_n}, u_{h_n}) = m(e^{i\phi_n} u_{h_n}, iu_{h_n}) = m(e^{i\phi_n} u_{h_n}, iu_{h_n} - iu_0).$$

Since the right-hand side tends to zero, either $u_0 = 0$ or $\phi_n \to 0$ holds. In any case, we have

$$||\widetilde{u}_{h_n} - u_0||_{H_\kappa^1} \le ||u_{h_n} - u_0||_{H_\kappa^1} + |1 - e^{i\phi_n}| \, ||u_{h_n}||_{H_\kappa^1} \to 0,$$

which yields the assertion. $\qquad \square$

5.2. Discrete inf-sup stability. In order to derive the error estimates, we first establish a discrete version of the inf-sup condition in (2.6). In the proof, we need the following consequence of Assumption 3.1.

COROLLARY 5.3. *Let Assumption* 3.1 *hold, and let* $z \in H_{iu}^1$ *be the solution of*

$$\langle E''(u)z, \varphi \rangle = \langle f, \varphi \rangle \quad \text{for all } \varphi \in H_{iu}^1.$$

Then, there holds the estimate

(5.1) $$||z - R_{\kappa,h}^\perp z||_{H_\kappa^1} \lesssim C_{\text{sol}}(u, \kappa) h ||f||_{L^2}.$$

Proof. According to the representation of E'' in Lemma 2.3, we bring the terms depending on u to the right-hand side and for $f_z = \kappa^2 \big((|u|^2 - 1)z + u^2 z^* + |u|^2 z \big) - \beta z$ we obtain

$$\hat{a}_\kappa(z, \varphi) = m(f, \varphi) - m(f_z, \varphi) \quad \text{for all } \varphi \in H_{iu}^1.$$

Here, f_z satisfies

$$||f_z||_{L^2} \lesssim \kappa^2 ||z||_{L^2} \lesssim C_{\text{sol}}(u, \kappa) ||f||_{L^2},$$

where we used part (b) in Corollary 2.9, and the approximation (3.1) in Assumption 3.1 gives the claim. $\qquad \square$

The proof of the next lemma, which states the discrete inf-sup stability, is inspired by the thesis [41, Prop. 8.2.7], where this was done for the Helmholtz equation.

LEMMA 5.4. *Let Assumption* 2.5 *be fulfilled.*
(a) *If* $\kappa C_{\text{sol}}(u, \kappa) h$ *is sufficiently small, it holds for all* $w_h \in V_h^\perp$ *that*

$$||w_h||_{H_\kappa^1} \lesssim C_{\text{sol}}(u, \kappa) \sup_{\varphi_h \in V_h^\perp} \frac{\langle E''(u)w_h, \varphi_h \rangle}{||\varphi_h||_{H_\kappa^1}},$$

where the constant is independent of h *and* κ.

(b) *For any $f \in (H^1_{\mathrm{iu}})'$, there is a unique $w_h \in V^\perp_h$ such that*

$$\langle E''(u)w_h, \varphi_h \rangle = \langle f, \varphi_h \rangle \quad \textit{for all } \varphi_h \in V^\perp_h$$

and it holds that

$$\|w_h\|_{H^1_\kappa} \lesssim C_{\mathrm{sol}}(u, \kappa)\|f\|_{(H^1_\kappa)'}.$$

Proof. Part (b) is a classical stability bound for inf-sup stable problems; cf. [5, Thm. 2.1]. Hence, claim (b) directly follows once we have shown (a). To do so, we fix $w_h \in V^\perp_h$ and observe for arbitrary $z \in H^1_{\mathrm{iu}}$

$$
(5.2) \quad
\begin{aligned}
&\langle E''(u)w_h, w_h + z \rangle \\
&= a_A(w_h, w_h) + \kappa^2 \operatorname{Re} \int_\Omega \big(2|u|^2 - 1\big)w_h w^*_h + u^2 w^*_h w^*_h \, dx + \langle E''(u)z, w_h \rangle.
\end{aligned}
$$

Let $z \in H^1_{\mathrm{iu}}$ be the unique solution to

$$\langle E''(u)z, \varphi \rangle = m(f, \varphi) \quad \text{for all } \varphi \in H^1_{\mathrm{iu}} \quad \text{with} \quad f = (\beta^2 + 2\kappa^2)w_h,$$

which exists by Proposition 2.6 and Corollary 2.9, and insert it into (5.2). Then, we obtain from (2.4) together with Lemmas 2.1 and 2.3 that

$$
\begin{aligned}
\|w_h\|^2_{H^1_\kappa} &\lesssim \langle E''(u)w_h, w_h + z \rangle \lesssim \langle E''(u)w_h, w_h + \mathrm{R}^\perp_{\kappa,h}z \rangle + \|w_h\|_{H^1_\kappa}\|\mathrm{R}^\perp_{\kappa,h}z - z\|_{H^1_\kappa} \\
&\lesssim \sup_{\varphi_h \in V^\perp_h} \frac{\langle E''(u)w_h, \varphi_h \rangle}{\|\varphi_h\|_{H^1_\kappa}}\|w_h + \mathrm{R}^\perp_{\kappa,h}z\|_{H^1_\kappa} + \|w_h\|_{H^1_\kappa}\|\mathrm{R}^\perp_{\kappa,h}z - z\|_{H^1_\kappa}.
\end{aligned}
$$

It remains to study the terms with z. Here, we establish with Corollary 2.9 and (5.1) the bound

$$\kappa\|z\|_{H^1_\kappa} + h^{-1}\|z - \mathrm{R}^\perp_{\kappa,h}z\|_{H^1_\kappa} \lesssim C_{\mathrm{sol}}(u, \kappa)\|f\|_{L^2} \lesssim \kappa C_{\mathrm{sol}}(u, \kappa)\|w_h\|_{H^1_\kappa}.$$

From this, we finally conclude

$$\|w_h\|^2_{H^1_\kappa} \lesssim \sup_{\varphi_h \in V^\perp_h} \frac{\langle E''(u)w_h, \varphi_h \rangle}{\|\varphi_h\|_{H^1_\kappa}} C_{\mathrm{sol}}(u, \kappa)\|w_h\|_{H^1_\kappa} + \kappa C_{\mathrm{sol}}(u, \kappa)h\|w_h\|^2_{H^1_\kappa},$$

and obtain the assertion (a) if $\kappa C_{\mathrm{sol}}(u, \kappa)h$ is sufficiently small by absorption. □

5.3. Convergence with rates. After these preparations, we can derive the error equation employing the second Fréchet derivative E''. To this end, we strive for a representation of the form

$$(5.3) \qquad\qquad \langle E''(u)(\mathrm{R}^\perp_{\kappa,h}u - u_h), \varphi_h \rangle = \varepsilon_h(\varphi_h)$$

for $\varphi_h \in V^\perp_h$ and employ Lemma 5.4 to conclude a bound for $\mathrm{R}^\perp_{\kappa,h}u - u_h$. The right-hand side ε_h is studied in the following lemma.

LEMMA 5.5. *Let u and u_h be minimizers of* (1.1) *and* (3.2), *respectively.*
(a) *For $\varphi_h \in V^\perp_h$ there holds the representation* (5.3) *where $\varepsilon_h = \varepsilon^{lin}_h + \varepsilon^{nonlin}_h$ and*

$$
\begin{aligned}
\varepsilon^{lin}_h(\varphi_h) &= \kappa^2 \operatorname{Re} \int_\Omega \Big((2|u|^2 - 1)(\mathrm{R}^\perp_{\kappa,h}u - u) + u^2(\mathrm{R}^\perp_{\kappa,h}u - u)^*\Big)\varphi^*_h \, dx \\
&\quad - \beta^2 \operatorname{Re} \int_\Omega (\mathrm{R}^\perp_{\kappa,h}u - u)\varphi^*_h \, dx, \\
\varepsilon^{nonlin}_h(\varphi_h) &= 2\kappa^2 \operatorname{Re} \int_\Omega |u|^2 u\varphi^*_h \, dx + \kappa^2 \operatorname{Re} \int_\Omega |u_h|^2 u_h \varphi^*_h \, dx \\
&\quad - \kappa^2 \operatorname{Re} \int_\Omega \big(2|u|^2 u_h + u^2 u^*_h\big)\varphi^*_h \, dx.
\end{aligned}
$$

(b) *The error terms are bounded by*

$$||\varepsilon_h^{lin}||_{(H_\kappa^1)'} \lesssim \kappa ||u - \mathrm{R}_{\kappa,h}^\perp u||_{L^2},$$
$$||\varepsilon_h^{nonlin}||_{(H_\kappa^1)'} \lesssim \kappa \big(||u - u_h||_{L^4}^2 + ||u - u_h||_{L^6}^3 \big),$$

where the constants are independent of h and κ.

Proof. Inserting the exact solution u, we decompose ε_h as

$$\varepsilon_h^{lin}(\varphi_h) = \langle E''(u)(\mathrm{R}_{\kappa,h}^\perp u - u), \varphi_h \rangle, \qquad \varepsilon_h^{nonlin}(\varphi_h) = \langle E''(u)(u - u_h), \varphi_h \rangle,$$

and treat the two terms separately. We begin with the linear part and use Lemma 2.3, the definition of $\hat{a}_\kappa(\cdot, \cdot)$ in (2.4), and the orthogonality condition of $\mathrm{R}_{\kappa,h}^\perp$ to obtain

$$
\begin{aligned}
&\langle E''(u)(\mathrm{R}_{\kappa,h}^\perp u - u), \varphi_h \rangle \\
&= a_A(\mathrm{R}_{\kappa,h}^\perp u - u, \varphi_h) \\
&\quad + \kappa^2 \operatorname{Re} \int_\Omega \Big((|u|^2 - 1)(\mathrm{R}_{\kappa,h}^\perp u - u) + u^2 (\mathrm{R}_{\kappa,h}^\perp u - u)^* + |u|^2 (\mathrm{R}_{\kappa,h}^\perp u - u) \Big) \varphi_h^* \, dx \\
&= -\beta^2 \operatorname{Re} \int_\Omega (\mathrm{R}_{\kappa,h}^\perp u - u) \varphi_h^* \, dx \\
&\quad + \kappa^2 \operatorname{Re} \int_\Omega \Big((|u|^2 - 1)(\mathrm{R}_{\kappa,h}^\perp u - u) + u^2 (\mathrm{R}_{\kappa,h}^\perp u - u)^* + |u|^2 (\mathrm{R}_{\kappa,h}^\perp u - u) \Big) \varphi_h^* \, dx.
\end{aligned}
$$

Using that $\kappa ||\varphi_h||_{L^2} \leq ||\varphi_h||_{H_\kappa^1}$ gives the first estimate in part (b).

For the nonlinear part, we note with Lemma 2.3 the identity for $v, \varphi \in H^1$

$$\langle E''(v)v, \varphi \rangle = \langle E'(v), \varphi \rangle + 2\kappa^2 \operatorname{Re} \int_\Omega |v|^2 v \varphi^* \, dx.$$

Since $\langle E'(u), \varphi_h \rangle = \langle E'(u_h), \varphi_h \rangle = 0$, we expand

$$
\begin{aligned}
\langle E''(u)(u - u_h), \varphi_h \rangle &= \langle E''(u)u, \varphi_h \rangle - \langle E''(u_h)u_h, \varphi_h \rangle + \langle \big(E''(u_h) - E''(u) \big) u_h, \varphi_h \rangle \\
&= 2\kappa^2 \operatorname{Re} \int_\Omega |u|^2 u \varphi_h^* \, dx - 2\kappa^2 \operatorname{Re} \int_\Omega |u_h|^2 u_h \varphi_h^* \, dx \\
&\quad + \kappa^2 \operatorname{Re} \int_\Omega 2\big(|u_h|^2 - |u|^2 \big) u_h \varphi_h^* + \big(u_h^2 - u^2 \big) u_h^* \varphi_h^* \, dx \\
&= 2\kappa^2 \operatorname{Re} \int_\Omega |u|^2 u \varphi_h^* \, dx + \kappa^2 \operatorname{Re} \int_\Omega |u_h|^2 u_h \varphi_h^* \, dx \\
&\quad - \kappa^2 \operatorname{Re} \int_\Omega \big(2|u|^2 u_h + u^2 u_h^* \big) \varphi_h^* \, dx,
\end{aligned}
$$

where we collected terms in the last step. For the estimate, we write $u_h = u - e_h$ and compute

$$(5.4) \qquad 2|u|^2 u + |u_h|^2 u_h - \big(2|u|^2 u_h + u^2 u_h^* \big) = 2u|e_h|^2 + e_h^2 u^* - |e_h|^2 e_h,$$

which together with $|u| \leq 1$ and the Hölder inequality gives the second bound. $\qquad \square$

Now we have everything together to prove the first part of Theorem 3.3, i.e., the H_κ^1-estimates for the discrete minimizers.

PROPOSITION 5.6. *Let u and u_h be minimizers of* (1.1) *and* (3.2), *respectively, and assume the orthogonality $m(u_h, iu) = 0$.*

(a) *We have for the fully discrete error*

$$||u - u_h||_{H^1_\kappa} \lesssim ||u - R^\perp_{\kappa,h} u||_{H^1_\kappa} + \kappa\, C_{\mathrm{sol}}(u, \kappa) ||u - R^\perp_{\kappa,h} u||_{L^2}$$
$$+ \kappa\, C_{\mathrm{sol}}(u, \kappa) \big(||u - u_h||^2_{L^4} + ||u - u_h||^3_{L^6} \big).$$

(b) *For h small enough and the (unique) minimizer u in Proposition* 5.1, *it holds that*

$$||u - u_h||_{H^1_\kappa} \lesssim ||u - R^\perp_{\kappa,h} u||_{H^1_\kappa} + \kappa\, C_{\mathrm{sol}}(u, \kappa) ||u - R^\perp_{\kappa,h} u||_{L^2}.$$

Let us point out that (a) holds for any minimizers u and u_h. But to ensure that the higher order terms are indeed negligible, we need the a priori information from the abstract convergence result in Proposition 5.1.

Proof of Proposition 5.6. (a) Using the triangle inequality, we obtain

$$||u - u_h||_{H^1_\kappa} \lesssim ||u - R^\perp_{\kappa,h} u||_{H^1_\kappa} + ||R^\perp_{\kappa,h} u - u_h||_{H^1_\kappa}$$

and are left to bound the second term. Lemmas 5.4 and 5.5 then give

$$||R^\perp_{\kappa,h} u - u_h||_{H^1_\kappa} \lesssim C_{\mathrm{sol}}(u, \kappa) ||\varepsilon_h||_{(H^1_\kappa)'}$$
$$\lesssim \kappa C_{\mathrm{sol}}(u, \kappa) \big(||u - R^\perp_{\kappa,h} u||_{L^2} + ||u - u_h||^2_{L^4} + ||u - u_h||^3_{L^6} \big),$$

and the bound is established.

(b) With the convergence shown in Proposition 5.1 for h sufficiently small, we can absorb the higher order terms, and obtain the claimed estimate for $h \leq h_0$. □

We can further show quadratic convergence in the L^2-norm for the discrete minimizers using an Aubin–Nitsche argument.

LEMMA 5.7. *Let u and u_h be a minimizers of* (1.1) *and* (3.2), *respectively, and assume the orthogonality $m(u_h, iu) = 0$. We have for the fully discrete error*

$$||u - u_h||_{L^2} \lesssim C_{\mathrm{sol}}(u, \kappa) h ||u - u_h||_{H^1_\kappa}$$
$$+ C_{\mathrm{sol}}(u, \kappa) \kappa ||u - u_h||_{L^2} \big(||u - u_h||_{L^3} + ||u - u_h||^2_{L^6} \big),$$

and hence for h sufficiently small, it holds for the (unique) minimizer u in Proposition 5.1

$$||u - u_h||_{L^2} \lesssim C_{\mathrm{sol}}(u, \kappa) h ||u - u_h||_{H^1_\kappa}.$$

Proof. Recall the abbreviation $e_h = u - u_h$, and let $z \in H^1_{iu}$ be the solution of

$$m(E''(u) z, \varphi) = m(e_h, \varphi),$$

and note that Corollary 2.9 and (5.1) give the estimate

(5.5)　　　　　$\kappa ||z||_{H^1_\kappa} + h^{-1} ||z - R^\perp_{\kappa,h} z||_{H^1_\kappa} \leq C_{\mathrm{sol}}(u, \kappa) ||e_h||_{L^2}.$

Using the symmetry of E'', we can decompose the error as

$$||e_h||^2_{L^2} = \langle E''(u)\, e_h, z - R^\perp_{\kappa,h} z \rangle + \langle E''(u)\, e_h, R^\perp_{\kappa,h} z \rangle = E^1 + \varepsilon^{\mathrm{nonlin}}_h (R^\perp_{\kappa,h} z),$$

where $\varepsilon_h^{\mathrm{nonlin}}$ is defined in Lemma 5.5. We estimate the first term with Lemma 2.3 and (5.5)

$$E^1 \lesssim ||e_h||_{H^1_\kappa}||z - \mathrm{R}^\perp_{\kappa,h}z||_{H^1_\kappa} \lesssim C_{\mathrm{sol}}(u,\kappa)h||e||_{H^1_\kappa}||e_h||_{L^2}.$$

For the second term, we use the representation of $\varepsilon_h^{\mathrm{nonlin}}$ in Lemma 5.5 and (5.4) together with (5.5) and the Hölder equation to obtain

$$|\varepsilon_h^{\mathrm{nonlin}}(\mathrm{R}^\perp_{\kappa,h}z)| \lesssim \kappa^2||u - u_h||_{L^2}\big(||u - u_h||_{L^3} + ||u - u_h||^2_{L^6}\big)||\mathrm{R}^\perp_{\kappa,h}z||_{H^1}$$
$$\lesssim \kappa C_{\mathrm{sol}}(u,\kappa)||u - u_h||_{L^2}\big(||u - u_h||_{L^3} + ||u - u_h||^2_{L^6}\big)||e_h||_{L^2},$$

where we used $\kappa||\mathrm{R}^\perp_{\kappa,h}z||_{H^1} \lesssim \kappa||z||_{H^1_\kappa} \lesssim C_{\mathrm{sol}}(u,\kappa)||e_h||_{L^2}$ in the last step. Combining the two bounds and dividing by $||e_h||_{L^2}$ gives the desired estimate. □

A similar trick gives the improved convergence of $\mathrm{R}^\perp_{\kappa,h}$ in the L^2-norm.

LEMMA 5.8. *For κh small enough, the following bound holds for all $w \in H^1_{\mathrm{i}u}$:*

$$||w - \mathrm{R}^\perp_{\kappa,h}(w)||_{L^2} \lesssim h||w - \mathrm{R}^\perp_{\kappa,h}(w)||_{H^1_\kappa},$$

where the constant is independent of h and κ.

Proof. We use an Aubin–Nitsche argument and let $z \in H^1_{\mathrm{i}u}$ be the solution of

$$\hat{a}_\kappa(z,\varphi) = m(w - \mathrm{R}^\perp_{\kappa,h}w, \varphi) \quad \text{for all } \varphi \in H^1_{\mathrm{i}u}.$$

Using orthogonality, we have by (3.1) that

$$||w - \mathrm{R}^\perp_{\kappa,h}w||^2_{L^2} = \hat{a}_\kappa(z - \mathrm{R}^\perp_{\kappa,h}z, w - \mathrm{R}^\perp_{\kappa,h}w) \lesssim h||w - \mathrm{R}^\perp_{\kappa,h}w||_{L^2}||w - \mathrm{R}^\perp_{\kappa,h}w||_{H^1_\kappa},$$

and the claim follows. □

Finally, we provide the error bounds for the energy which behaves in the lowest order as the square of the error in the H^1_κ-norm.

LEMMA 5.9. *Let u and u_h be minimizers of (1.1) and (3.2), respectively. The error in the energies is bounded by*

$$0 \le E(u_h) - E(u) \lesssim ||u - u_h||^2_{H^1_\kappa}\big(1 + \kappa^{1/2}||u - u_h||_{H^1_\kappa} + \kappa||u - u_h||^2_{H^1_\kappa}\big).$$

We note that the powers of κ can be improved in the case $d = 2$, but since the leading order term does not change, we will not give any details here.

Proof of Lemma 5.9. Since $V_h \subset H^1$, we have $E(u) \le E(u_h)$, and thus the lower bound. In the next step, we derive the representation

(5.6)
$$E(u_h) - E(u) = \frac{1}{2}a_A(u - u_h, u - u_h)$$
$$+ \frac{\kappa^2}{4}\mathrm{Re}\int_\Omega (1 - |u_h|^2)^2 - (1 - |u|^2)^2 + 4(|u|^2 - 1)u(u - u_h)^* \, dx.$$

Let us first note the identity

$$\frac{1}{2}a_A(u - u_h, u - u_h) = \frac{1}{2}a_A(u_h, u_h) - \frac{1}{2}a_A(u,u) + a_A(u, u - u_h),$$

and rewrite the energies as

$$
\begin{aligned}
E(u_h) &- E(u) \\
&= \frac{1}{2} a_A(u_h, u_h) - \frac{1}{2} a_A(u, u) + \frac{\kappa^2}{4} \operatorname{Re} \int_\Omega (1 - |u_h|^2)^2 - (1 - |u|^2)^2 \, dx \\
&= \frac{1}{2} a_A(u - u_h, u - u_h) + \frac{\kappa^2}{4} \operatorname{Re} \int_\Omega (1 - |u_h|^2)^2 - (1 - |u|^2)^2 \, dx - a_A(u, u - u_h).
\end{aligned}
$$

Since u is a minimizer, we have $\langle E'(u), u - u_h \rangle = 0$ and thus by (2.5)

$$
-a_A(u, u - u_h) = \kappa^2 \operatorname{Re} \int_\Omega (|u|^2 - 1) u (u - u_h)^* \, dx,
$$

and hence (5.6) holds. The first term of the representation gives the H_κ^1-norm in the estimate, and it remains to study the nonlinear part. We first investigate the difference of the squares. As before, we write $u_h = u - e_h$ and obtain

$$
\begin{aligned}
\left(1 - |u - e_h|^2\right)^2 &= \left(|u|^2 + |e_h|^2 - 1 - 2 \operatorname{Re}(u e_h^*)\right)^2 \\
&= |u|^4 + 1 - 2|u|^2 + 4(1 - |u|^2) \operatorname{Re}(u e_h^*) + \mathcal{O}(|e_h|^2 + |e_h|^3 + |e_h|^4),
\end{aligned}
$$

which gives

$$
(1 - |u_h|^2)^2 - (1 - |u|^2)^2 = 4(1 - |u|^2) \operatorname{Re}(u e_h^*) + \mathcal{O}(|e_h|^2 + |e_h|^3 + |e_h|^4).
$$

We now show that the part, which is linear in e_h, is canceled by the last term in (5.6). In fact, since it holds that

$$
4 \operatorname{Re}(|u|^2 - 1) u (u - u_h)^* = 4|u|^2 \operatorname{Re}(u e_h^*) - 4 \operatorname{Re}(u e_h^*),
$$

we conclude from (5.6), the fact that $|u| \leq 1$, and the Hölder inequality the bound

$$
E(u_h) - E(u) \lesssim \|u - u_h\|_{H_\kappa^1}^2 + \kappa^2 \left(\|u - u_h\|_{L^2}^2 + \|u - u_h\|_{L^3}^3 + \|u - u_h\|_{L^4}^4\right).
$$

To show the final estimate, we use interpolation theory (see, e.g., [37, Thm. 2.6]) with $\frac{1}{3} = \frac{\theta}{2} + \frac{1-\theta}{6}$ for $\theta = \frac{1}{2}$ to obtain for $w \in H^1$

$$
\kappa^2 \|w\|_{L^3}^3 \lesssim \kappa^{1/2} \left(\kappa \|w\|_{L^2}\right)^{3/2} \|w\|_{L^6}^{3/2} \lesssim \kappa^{1/2} \|w\|_{H_\kappa^1}^3,
$$

and similarly with $\frac{1}{4} = \frac{\theta}{2} + \frac{1-\theta}{6}$ for $\theta = \frac{1}{4}$

$$
\kappa^2 \|w\|_{L^4}^4 \lesssim \kappa \left(\kappa \|w\|_{L^2}\right) \|w\|_{L^6}^3 \lesssim \kappa \|w\|_{H_\kappa^1}^4,
$$

and the second claim is established. □

We can finally give the proof of our main result.

Proof of Theorem 3.3. We mainly collect the results shown in Proposition 5.6 and Lemma 5.7, together with the L^2-estimate in Lemma 5.8, and Lemma 5.9, and the claims are established. □

5.4. Application to Lagrange finite elements. In this section, we consider the linear Lagrange finite element space V_h as defined (3.3). In order to derive the corresponding error estimates through verifying the assumptions of Theorem 3.3, we require the L^2-orthogonal projection onto the ansatz space V_h as an auxiliary projection. We recall the L^2-projection for $v \in L^2$ as

$$
m(\pi_h v, \varphi_h) = m(v, \varphi_h) \quad \text{for all} \quad \varphi_h \in V_h.
$$

In the following lemma, we provide corresponding estimates in the H_κ^1-norm which are the first step toward verifying part (b) in Assumption 3.1.

LEMMA 5.10.
(a) *The L^2-projection π_h is stable in H^1_κ, i.e., it holds that*

$$||\pi_h\varphi||_{H^1_\kappa} \lesssim ||\varphi||_{H^1_\kappa}, \qquad \varphi \in H^1,$$

where the constant is independent of h and κ.
(b) *For all $z \in H^2$ it holds that*

$$||z - \pi_h z||_{H^1_\kappa} \lesssim h||z||_{H^2_\kappa},$$

where the constant is independent of h and κ.
(c) *If Ω is convex and $z \in H^1_{\mathrm{i}u}$ satisfies for $f \in L^2$ the equation $\hat{a}_\kappa(z,\varphi) = m(f,\varphi)$ for all $\varphi \in H^1_{\mathrm{i}u}$, then*

$$||z - \pi_h z||_{H^1_\kappa} \lesssim h||f||_{L^2}.$$

Proof. Due to (3.4), standard arguments lead to the bounds on the L^2-projection in parts (a) and (b). Part (c) is a direct consequence of part (b) and Lemma 2.8. $\quad\square$

In the next lemma, we relate the orthogonal projection, which takes into account the orthogonality to $\mathrm{i}u$ in $m(\cdot,\cdot)$, to the L^2-projection.

LEMMA 5.11. *For κh small enough, the following bound holds:*

$$||\varphi - \mathrm{R}^\perp_{\kappa,h}(\varphi)||_{H^1_\kappa} \lesssim ||\varphi - \pi_h\varphi||_{H^1_\kappa}, \quad \varphi \in H^1_{\mathrm{i}u}.$$

Proof. For $\varphi_h \in V_h$ we let $P^\perp_{\mathrm{i}u} : V_h \to V^\perp_h$ be the mapping that adjusts the angle to $\mathrm{i}u$ via

$$P^\perp_{\mathrm{i}u}(\varphi_h) := \varphi_h - \frac{m(\varphi_h, \mathrm{i}u)}{m(\pi_h(\mathrm{i}u), \mathrm{i}u)}\pi_h(\mathrm{i}u).$$

We obtain for any $\varphi \in H^1_{\mathrm{i}u}$

$$||\varphi - \mathrm{R}^\perp_{\kappa,h}(\varphi)||_{H^1_\kappa}$$
$$\lesssim ||\varphi - (P^\perp_{\mathrm{i}u} \circ \pi_h)\varphi||_{H^1_\kappa} \le ||\varphi - \pi_h\varphi||_{H^1_\kappa} + \frac{m(\pi_h\varphi - \varphi, \mathrm{i}u)}{m(\pi_h(\mathrm{i}u), \mathrm{i}u)}||\pi_h(\mathrm{i}u)||_{H^1_\kappa}$$
$$\lesssim ||\varphi - \pi_h\varphi||_{H^1_\kappa} + ||\varphi - \pi_h\varphi||_{L^2}\frac{||u||_{L^2}}{m(\pi_h(\mathrm{i}u) - \mathrm{i}u, \mathrm{i}u) + ||u||^2_{L^2}}||u||_{H^1_\kappa}$$
$$\lesssim ||\varphi - \pi_h\varphi||_{H^1_\kappa} + \frac{\kappa}{1 - c\kappa h}||\varphi - \pi_h\varphi||_{L^2} \lesssim ||\varphi - \pi_h\varphi||_{H^1_\kappa},$$

where we used in the last step that $||\pi_h(\mathrm{i}u) - \mathrm{i}u||_{L^2} \lesssim h||u||_{H^1} \lesssim \kappa h$ holds. $\quad\square$

These preparations lead to the error bounds for our first application.

Proof of Corollary 3.4. From Lemmas 5.10 and 5.11, we obtain that Assumption 3.1 holds, and thus we can use the bounds in Theorem 3.3. In addition, we recall that Ω is assumed to be convex, and, hence, the approximation estimates due to Lemmas 5.10 and 5.11 yield

$$||u - \mathrm{R}^\perp_{\kappa,h}u||_{H^1_\kappa} \lesssim h||u||_{H^2_\kappa} \lesssim \kappa^2 h,$$

where we used Theorem 2.2 for the last step. This establishes the claims. $\quad\square$

6. Relaxed κ-dependencies in LOD spaces. In this final section we present a nonstandard application of the abstract approximation result in Theorem 3.3. For that we consider spaces based on the so-called localized orthogonal decomposition (LOD). LOD spaces were originally developed in the context of elliptic multiscale problems with rough coefficients to efficiently handle low regularity and unresolved scales [39]. An introduction to the methodology is given in the textbook by Målqvist and Peterseim [40] and the review article by Altmann, Henning, and Peterseim [4]. Recently, new applications of these spaces emerged in the field of quantum mechanics, where they were used to boost the performance of traditional discretizations [31, 33, 50]. As we will see, the GLE could be yet another promising application of LOD spaces in the context of quantum physics.

To define suitable LOD spaces for the GLE and to characterize its approximation properties in an abstract way, we start from a linear Lagrange finite element space V_h as defined in (3.3) and assume that the underlying triangulation $\mathcal{T}_h$ is shape-regular and quasi-uniform. The LOD space is now constructed from V_h by applying the inverse of a differential operator to the functions of V_h. In our case, we use the differential operator associated with the bilinear form $\hat{a}_\kappa(\cdot,\cdot,)$. The construction is made precise in the following definition.

DEFINITION 6.1 (LOD spaces). *Let $\hat{a}_\kappa(\cdot,\cdot)$ denote the symmetric, continuous, and coercive bilinear form on $H^1(\Omega,\mathbb{C})$ given by (2.2) and let $\hat{\mathcal{A}}_\kappa^{-1}$ denote the corresponding solution operator on L^2, i.e., for $f \in L^2(\Omega,\mathbb{C})$ the image $\hat{\mathcal{A}}_\kappa^{-1} f \in H^1$ is given by the solution to*

$$\hat{a}_\kappa(\hat{\mathcal{A}}_\kappa^{-1} f, \varphi) = m(f,\varphi) \qquad \text{for all } \varphi \in H^1.$$

With this definition, the LOD space based on $\hat{a}_\kappa(\cdot,\cdot)$ and V_h is given by

$$V_h^{\mathrm{LOD}} := \hat{\mathcal{A}}_\kappa^{-1} V_h.$$

We note that the above definition of LOD spaces formally differs from the construction given in the classical references [39, 32, 30]. However, the characterizations are indeed equivalent as can be extracted from, e.g., [29] and [4].

From a practical perspective it is also important to note that the space V_h^{LOD} admits a quasi-local basis, i.e., basis functions that are (super-)exponentially decaying in distances of the mesh size h. Details on the practical computation/approximation of such basis functions are given in [24] and recent superlocalization strategies are presented in [29]. Corresponding numerical errors that might arise from the approximation of basis functions are well understood [4] and will be for brevity disregarded in the following error analysis.

The approximation properties of the idealized space V_h^{LOD} are summarized in the following proposition.

PROPOSITION 6.2 (approximation properties of V_h^{LOD}). *Let V_h^{LOD} be the LOD space from Definition 6.1 and let $f \in L^2$ be given. If $u \in H^1$ denotes the solution to*

$$\hat{a}_\kappa(u,\varphi) = m(f,\varphi) \qquad \text{for all } \varphi \in H^1$$

and if $\mathrm{R}_{\kappa,h}^{\mathrm{LOD}} u \in V_h^{\mathrm{LOD}}$ denotes the corresponding $\hat{a}_\kappa(\cdot,\cdot)$-Ritz-projection of u in V_h^{LOD}, then it holds that

$$(6.1) \qquad \|u - \mathrm{R}_{\kappa,h}^{\mathrm{LOD}} u\|_{H_\kappa^1} \lesssim h \|f - \pi_h f\|_{L^2},$$

where we recall $\pi_h : L^2 \to V_h$ *as the L^2-projection on V_h. The hidden constant in* (6.1) *is generic and depends on the coercivity and continuity constants of $\hat{a}_\kappa(\cdot, \cdot)$, as well as the mesh regularity, but it does not depend on h and κ.*

Furthermore, for every $\phi \in H^1$ there exists a unique decomposition such that

(6.2)
$$\phi = \phi^{\mathrm{LOD}} + \phi_0, \quad where \ \ \phi^{\mathrm{LOD}} \in V_h^{\mathrm{LOD}}, \quad \pi_h \phi_0 = 0, \quad and \quad \hat{a}_\kappa(\phi^{\mathrm{LOD}}, \phi_0) = 0.$$

The result is standard and can be found, for instance, in [33] for homogeneous Dirichlet boundary conditions. For generalizations to higher order finite element spaces and to only piecewise smooth source terms f, we refer to [38].

To apply the general error estimates in Theorem 3.3, we need to verify Assumption 3.1 for the LOD space V_h^{LOD}. Analogously, to standard Lagrange finite elements, it is also possible to quantify the approximation properties of $\mathrm{R}_{\kappa,h}^{\mathrm{LOD}}$ for general smooth functions. This is done in the following lemma.

LEMMA 6.3. *Let V_h^{LOD} be the LOD space from Definition 6.1 and let the corresponding Ritz projection w.r.t. $\hat{a}_\kappa(\cdot, \cdot)$ be denoted by $\mathrm{R}_{\kappa,h}^{\mathrm{LOD}} : H^1 \to V_h^{\mathrm{LOD}}$. Then, for every $w \in H^2$ with $\nabla w \cdot \nu = 0$ on $\partial\Omega$, there is an $f_w \in L^2$ such that*

$$\hat{a}_\kappa(w, \varphi) = m(f_w, \varphi) \quad and \quad ||f_w||_{L^2} \lesssim ||w||_{H_\kappa^2}.$$

Consequently, for all $w \in H^2$ with $\nabla w \cdot \nu = 0$ on $\partial\Omega$ it holds that

$$||w - \mathrm{R}_{\kappa,h}^{\mathrm{LOD}} w||_{H_\kappa^1} \lesssim h ||w||_{H_\kappa^2}.$$

Proof. From (2.13), we obtain using integration by parts

$$\hat{a}_\kappa(w, \varphi) = \mathrm{Re} \int_\Omega \left(-\Delta w + \beta^2 w + 2\mathrm{i}\kappa A \nabla w + \kappa^2 |A|^2 w \right) \varphi^* \, dx =: m(f_w, \varphi)$$

and the bound for $||f_w||_{L^2}$ follows. Proposition 6.2 finishes the second part of the lemma. □

As the set of H^2-functions with a vanishing normal derivative on $\partial\Omega$ (for a polygonal Lipschitz domain) is dense in H^1 (cf. [14]), this lemma establishes property (a) in Assumption 3.1. For the second property, we need a variant of this result given in the next two lemmas. First, we give an estimate on the (standard) Ritz projection.

LEMMA 6.4. *Let V_h^{LOD} be the LOD space from Definition 6.1 with corresponding Ritz projection w.r.t. $\hat{a}_\kappa(\cdot, \cdot)$ given by $\mathrm{R}_{\kappa,h}^{\mathrm{LOD}} : H^1 \to V_h^{\mathrm{LOD}}$. For $f \in L^2$ let $w \in H_{\mathrm{i}u}^1$ be the solution of*

$$\hat{a}_\kappa(w, \varphi) = m(f, \varphi) \quad for \ all \quad \varphi \in H_{\mathrm{i}u}^1.$$

Then, it holds that

$$||w - \mathrm{R}_{\kappa,h}^{\mathrm{LOD}} w||_{H_\kappa^1} \lesssim h ||f||_{L^2}.$$

Proof. As in the proof of Lemma 2.8 in (2.12), we know that w solves the variational problem also tested against all $\varphi \in H^1$ for some modification of f which is bounded in L^2 by $||f||_{L^2}$. Hence, the assertion follows from Proposition 6.2. □

However, for property (b) in Assumption 3.1, we further need a bound on the Ritz projection which preserves the orthogonality with respect to $\mathrm{i}u$. The following lemma shows that one can reduce these error bounds to those established in Lemma 6.4.

LEMMA 6.5. *Let again* $\mathrm{R}_{\kappa,h}^{\mathrm{LOD}} : H^1 \to V_h^{\mathrm{LOD}}$ *denote the Ritz projection onto the LOD-space* V_h^{LOD} *and let*

$$\mathrm{R}_{\kappa,h}^{\perp,\mathrm{LOD}} : H_{\mathrm{i}u}^1 \to V_h^{\mathrm{LOD}} \cap (\mathrm{i}u)^\perp$$

denote the corresponding Ritz projection onto $V_h^{\mathrm{LOD}} \cap (\mathrm{i}u)^\perp$. *If* h *is small enough, in particular* $h \lesssim \kappa^{-1}$, *then it holds for all* $\varphi \in H_{\mathrm{i}u}^1$ *that*

$$||\varphi - \mathrm{R}_{\kappa,h}^{\perp,\mathrm{LOD}}\varphi||_{H_\kappa^1} \lesssim ||\varphi - \mathrm{R}_{\kappa,h}^{\mathrm{LOD}}\varphi||_{H_\kappa^1}.$$

Proof. To proceed as in the proof of Lemma 5.11, we note that by the LOD decomposition (6.2) we have $\pi_h\left(\mathrm{i}u - \mathrm{R}_{\kappa,h}^{\mathrm{LOD}}(\mathrm{i}u)\right) = 0$. Hence, with the approximation properties of π_h,

$$(6.3) \qquad ||\mathrm{R}_{\kappa,h}^{\mathrm{LOD}}(\mathrm{i}u) - \mathrm{i}u||_{L^2} \lesssim h\,||\mathrm{R}_{\kappa,h}^{\mathrm{LOD}}(\mathrm{i}u) - \mathrm{i}u||_{H_\kappa^1} \lesssim h\,||\mathrm{i}u||_{H_\kappa^1} \lesssim h\,\kappa.$$

This implies for all $\varphi \in H_{\mathrm{i}u}^1$

$$||\varphi - \mathrm{R}_{\kappa,h}^{\perp,\mathrm{LOD}}\varphi||_{H_\kappa^1}$$
$$\lesssim ||\varphi - \left(\mathrm{R}_{\kappa,h}^{\mathrm{LOD}}\varphi - \frac{m(\mathrm{R}_{\kappa,h}^{\mathrm{LOD}}\varphi, \mathrm{i}u)}{m(\mathrm{R}_{\kappa,h}^{\mathrm{LOD}}(\mathrm{i}u), \mathrm{i}u)}\mathrm{R}_{\kappa,h}^{\mathrm{LOD}}(\mathrm{i}u)\right)||_{H_\kappa^1}$$
$$\leq ||\varphi - \mathrm{R}_{\kappa,h}^{\mathrm{LOD}}\varphi||_{H_\kappa^1} + \frac{m(\mathrm{R}_{\kappa,h}^{\mathrm{LOD}}\varphi - \varphi, \mathrm{i}u)}{m(\mathrm{R}_{\kappa,h}^{\mathrm{LOD}}(\mathrm{i}u), \mathrm{i}u)}||\mathrm{R}_{\kappa,h}^{\mathrm{LOD}}(\mathrm{i}u)||_{H_\kappa^1}$$
$$\overset{(6.3)}{\lesssim} ||\varphi - \mathrm{R}_{\kappa,h}^{\mathrm{LOD}}\varphi||_{H_\kappa^1} + ||\varphi - \mathrm{R}_{\kappa,h}^{\mathrm{LOD}}\varphi||_{L^2} \frac{||u||_{L^2}}{m(\mathrm{R}_{\kappa,h}^{\mathrm{LOD}}(\mathrm{i}u) - \mathrm{i}u, \mathrm{i}u) + ||u||_{L^2}^2}||u||_{H_\kappa^1}$$
$$\lesssim ||\varphi - \mathrm{R}_{\kappa,h}^{\mathrm{LOD}}\varphi||_{H_\kappa^1} + \frac{\kappa}{1 - c\kappa h}||\varphi - \mathrm{R}_{\kappa,h}^{\mathrm{LOD}}\varphi||_{L^2} \lesssim ||\varphi - \mathrm{R}_{\kappa,h}^{\mathrm{LOD}}\varphi||_{H_\kappa^1}. \qquad \square$$

Lemma 6.5 together with Theorem 3.3 guarantees that the H_κ^1-error between an exact solution u and a corresponding approximation in the LOD space is bounded by the projection error $||u - \mathrm{R}_{\kappa,h}^{\mathrm{LOD}}u||_{H_\kappa^1}$. The next lemma quantifies this error.

LEMMA 6.6. *Let* u *be a minimizer of* (1.1) *and let* $\mathrm{R}_{\kappa,h}^{\mathrm{LOD}} : H^1 \to V_h^{\mathrm{LOD}}$ *be the Ritz projection onto* V_h^{LOD}. *Then it holds at least*

$$||u - \mathrm{R}_{\kappa,h}^{\mathrm{LOD}}u||_{H_\kappa^1} \lesssim \kappa^3\,h^2$$

and, if Ω *is convex, we have* $u \in H^2$ *and the estimate improves to*

$$||u - \mathrm{R}_{\kappa,h}^{\mathrm{LOD}}u||_{H_\kappa^1} \lesssim \kappa^4\,h^3.$$

Proof. We want to apply Proposition 6.2. By $E'(u) = 0$ we have for every $\phi \in H_\kappa^1$ that

$$\hat{a}_\kappa(u, \varphi) = \beta^2 \mathrm{Re} \int_\Omega u\varphi^*\,dx - \kappa^2 \mathrm{Re} \int_\Omega \left(|u|^2 - 1\right)u\varphi^*\,dx = m(\beta^2 u - \kappa^2\left(|u|^2 - 1\right)u, \varphi).$$

Since $\beta^2 u - \kappa^2\left(|u|^2 - 1\right)u$ is at least in H^1 and even in H^2 for convex domains, one easily verifies that for $s = 0, 1$

$$||\beta^2 u - \kappa^2\left(|u|^2 - 1\right)u||_{H^s} \lesssim \kappa^2||u||_{H^s} \leq \kappa^{s+2},$$

and for $s = 2$

$$||\beta^2 u - \kappa^2(|u|^2 - 1)u||_{H^2} \lesssim \kappa^2(||u||_{W^{1,4}}^2 + ||u||_{H^2}) \le \kappa^4,$$

where we used the bounds from Theorem 2.2 and in particular repeatedly $|u| \le 1$. The estimate now follows with Proposition 6.2 and standard estimates for the L^2-projection π_h on $P1$ finite element spaces. □

By collecting the previous results we obtain our final main result which shows the superapproximation properties of the LOD space, even on nonconvex domains.

THEOREM 6.7. *Let Assumption* 2.5 *hold and let h be sufficiently small in the sense of Theorem* 3.3. *If V_h^{LOD} denotes the LOD space from Definition* 6.1 *and if $u_h^{\mathrm{LOD}} \in V_h^{\mathrm{LOD}}$ is a corresponding minimizer of the Ginzburg–Landau energy with*

$$E(u_h^{\mathrm{LOD}}) = \inf_{\varphi \in V_h^{\mathrm{LOD}}} E(\varphi),$$

then there is neighborhood $U \subset H^1(\Omega)$ of u_h^{LOD} and a unique minimizer $u \in U$ of (1.1) *with $m(u_h^{\mathrm{LOD}}, \mathrm{i}u) = 0$ and such that*

$$C_{\mathrm{sol}}^{-1}(u, \kappa)\,||u - u_h^{\mathrm{LOD}}||_{L^2(\Omega)} + h\,||u - u_h^{\mathrm{LOD}}||_{H_\kappa^1} \lesssim \kappa^3 h^3,$$

and for convex domains Ω (and consequently H^2-solutions) it even holds that

$$C_{\mathrm{sol}}^{-1}(u, \kappa)\,||u - u_h^{\mathrm{LOD}}||_{L^2(\Omega)} + h\,||u - u_h^{\mathrm{LOD}}||_{H_\kappa^1} \lesssim \kappa^4 h^4.$$

Proof. Proposition 6.2 and Lemmas 6.3, 6.4, and 6.5 guarantee that Assumption 3.1 is fulfilled for V_h^{LOD}. Hence, we can apply Theorem 3.3 together with Lemmas 6.5 and 6.6 to conclude that for all sufficiently small h and for $u \in H^s$ with $s \in \{1, 2\}$ it holds that

$$||u - u_h^{\mathrm{LOD}}||_{H_\kappa^1} \lesssim ||u - \mathrm{R}_{\kappa,h}^{\perp,\mathrm{LOD}} u||_{H_\kappa^1} \lesssim ||u - \mathrm{R}_{\kappa,h}^{\mathrm{LOD}} u||_{H_\kappa^1} \lesssim \kappa^{s+2} h^{s+1}$$

and

$$||u - u_h^{\mathrm{LOD}}||_{L^2} \lesssim C_{\mathrm{sol}}(u, \kappa) h\, ||u - u_h^{\mathrm{LOD}}||_{H_\kappa^1} \lesssim C_{\mathrm{sol}}(u, \kappa)\, \kappa^{s+2} h^{s+2}. \qquad □$$

Remark 6.8. It is worth noting that, in LOD spaces, one can also improve the smallness condition on $\kappa C_{\mathrm{sol}}(u, \kappa)h$ required for the inf-sup condition in Lemma 5.4. In fact, a precise inspection of the proof leads to a smallness condition on $\kappa^2 C_{\mathrm{sol}}(u, \kappa)h^2$, which is in general weaker if $1 \lesssim C_{\mathrm{sol}}(u, \kappa)$. However, since the abstract result contains a term of the form $1 + \kappa C_{\mathrm{sol}}(u, \kappa)h$, one cannot exploit this any further in the error analysis, and we thus refrain from giving the proof here.

Appendix A. Matrix representation of $E''(u_h)$. In the following we briefly discuss the matrix representation of $E''(u_h)$ and the computation of the lowermost eigenvalues. For that, let $\mathcal{N}_h = \{z_1, \ldots, z_N\}$ denote the set of nodes of the mesh $\mathcal{T}_h$ and let $\phi_j \in V_h$ denote the *real* nodal shape function with the property

$$\phi_j(z_k) = \delta_{jk} \qquad \text{for all } 1 \le j, k \le N.$$

Consequently, a nodal basis of V_h is given by the set

$$\{\phi_1, \mathrm{i}\phi_1, \phi_2, \mathrm{i}\phi_2, \ldots, \phi_N, \mathrm{i}\phi_N\}.$$

We are seeking a matrix representation of the operator $\langle E''(u_h)v_h, w_h \rangle$ for arbitrary $v_h, w_h \in V_h$. By expanding v_h and w_h in terms of the nodal basis functions we obtain

$$v_h = \sum_{j=1}^N v_j^{\mathrm{Re}}\phi_j + \sum_{j=1}^N v_j^{\mathrm{Im}}\,\mathrm{i}\,\phi_j \qquad \text{and} \qquad w_h = \sum_{j=1}^N w_j^{\mathrm{Re}}\phi_j + \sum_{j=1}^N w_j^{\mathrm{Im}}\,\mathrm{i}\,\phi_j$$

for corresponding coefficient vectors $\mathbf{v}^{\mathrm{Re}}, \mathbf{v}^{\mathrm{Im}}, \mathbf{w}^{\mathrm{Re}}, \mathbf{w}^{\mathrm{Im}} \in \mathbb{R}^N$ with

$$\mathbf{v}^{\mathrm{Re}} = \begin{pmatrix} v_1^{\mathrm{Re}} \\ \vdots \\ v_N^{\mathrm{Re}} \end{pmatrix}, \quad \mathbf{v}^{\mathrm{Im}} = \begin{pmatrix} v_1^{\mathrm{Im}} \\ \vdots \\ v_N^{\mathrm{Im}} \end{pmatrix}, \quad \mathbf{w}^{\mathrm{Re}} = \begin{pmatrix} w_1^{\mathrm{Re}} \\ \vdots \\ w_N^{\mathrm{Re}} \end{pmatrix}, \quad \mathbf{w}^{\mathrm{Im}} = \begin{pmatrix} w_1^{\mathrm{Im}} \\ \vdots \\ w_N^{\mathrm{Im}} \end{pmatrix}.$$

Hence, we can write

$$\langle E''(u_h)v_h, w_h \rangle = \begin{pmatrix} \mathbf{w}^{\mathrm{Re}} \\ \mathbf{w}^{\mathrm{Im}} \end{pmatrix}^\top \begin{pmatrix} \mathbf{A}(u_h)^{\mathrm{RR}} & \mathbf{A}(u_h)^{\mathrm{IR}} \\ \mathbf{A}(u_h)^{\mathrm{RI}} & \mathbf{A}(u_h)^{\mathrm{II}} \end{pmatrix} \begin{pmatrix} \mathbf{v}^{\mathrm{Re}} \\ \mathbf{v}^{\mathrm{Im}} \end{pmatrix}$$

for a block matrix $\mathbf{A}(u_h) \in \mathbb{R}^{2N \times 2N}$. Recalling the representation of $E''(u_h)$ as

$$\begin{aligned} \langle E''(u_h)v_h, w_h \rangle = \mathrm{Re} \int_\Omega & \left(\nabla v_h + \mathrm{i}\kappa A w_h \right) \cdot \left(\nabla v_h + \mathrm{i}\kappa A w_h \right)^* \\ & + \kappa^2 \left((|u_h|^2 - 1)v_h w_h^* + u_h^2 v_h^* w_h^* + |u_h|^2 v_h w_h^* \right) dx, \end{aligned}$$

we compute the corresponding blocks of $\mathbf{A}(u_h)$ as follows. For the upper left block we obtain straightforwardly

$$\begin{aligned} \mathbf{A}(u_h)_{jk}^{\mathrm{RR}} &= \mathrm{Re} \int_\Omega \nabla\phi_k \cdot \nabla\phi_j + \kappa^2 \left(|A|^2 + 2|u_h|^2 - 1 + u_h^2 \right) \phi_k \phi_j \, dx \\ &= \int_\Omega \nabla\phi_k \cdot \nabla\phi_j + \kappa^2 \left(|A|^2 - 1 + 3\,\mathrm{Re}(u_h)^2 + \mathrm{Im}(u_h)^2 \right) \phi_k \phi_j \, dx. \end{aligned}$$

For the upper right block we have

$$\begin{aligned} & \mathbf{A}(u_h)_{jk}^{\mathrm{IR}} \\ &= \mathrm{Re} \int_\Omega \mathrm{i}\left(\nabla\phi_k + \kappa A \mathrm{i}\phi_k \right) \cdot \left(\nabla\phi_j + \mathrm{i}\kappa A \phi_j \right)^* + \mathrm{i}\kappa^2 \left((|u_h|^2 - 1) - u_h^2 + |u_h|^2 \right) \phi_k \phi_j \, dx \\ &= \int_\Omega \kappa\left(\phi_j A \cdot \nabla\phi_k - \phi_k A \cdot \nabla\phi_j \right) - \kappa^2\,\mathrm{Re}(\mathrm{i}u_h^2)\phi_k\phi_j \, dx \\ &= \int_\Omega \kappa\left(\phi_j A \cdot \nabla\phi_k - \phi_k A \cdot \nabla\phi_j \right) + 2\kappa^2\,\mathrm{Re}(u_h)\,\mathrm{Im}(u_h)\,\phi_k\phi_j \, dx. \end{aligned}$$

Analogously, we obtain for the upper left block

$$\mathbf{A}(u_h)_{jk}^{\mathrm{RI}} = \int_\Omega \kappa(\phi_k A \cdot \nabla\phi_j - \phi_j A \cdot \nabla\phi_k) + 2\kappa^2\,\mathrm{Re}(u_h)\,\mathrm{Im}(u_h)\,\phi_k\phi_j \, dx.$$

Finally, the lower left block is given by

$$\mathbf{A}(u_h)_{jk}^{\mathrm{II}} = \int_\Omega \nabla\phi_k \cdot \nabla\phi_j + \kappa^2 \left(|A|^2 - 1 + \mathrm{Re}(u_h)^2 + 3\,\mathrm{Im}(u_h)^2) \right) \phi_k\phi_j \, dx.$$

By assembling the blocks in a standard way, we obtain the matrix $\mathbf{A}(u_h)$. Since $\langle E''(u_h)\cdot,\cdot\rangle$ is symmetric, $\mathbf{A}(u_h)$ must be symmetric too. This is also easily seen by the observations that

$$\mathbf{A}(u_h)_{jk}^{\mathrm{RR}} = \mathbf{A}(u_h)_{kj}^{\mathrm{RR}}, \qquad \mathbf{A}(u_h)_{jk}^{\mathrm{II}} = \mathbf{A}(u_h)_{kj}^{\mathrm{II}}, \qquad \text{and} \quad \mathbf{A}(u_h)_{kj}^{\mathrm{IR}} = \mathbf{A}(u_h)_{jk}^{\mathrm{RI}}.$$

Note that a matrix representation of $m(v_h, w_h)$ is straightforwardly given by

$$m(v_h, w_h) = \begin{pmatrix} \mathbf{w}^{\mathrm{Re}} \\ \mathbf{w}^{\mathrm{Im}} \end{pmatrix}^\top \begin{pmatrix} \mathbf{M} & 0 \\ 0 & \mathbf{M} \end{pmatrix} \begin{pmatrix} \mathbf{v}^{\mathrm{Re}} \\ \mathbf{v}^{\mathrm{Im}} \end{pmatrix},$$

where $\mathbf{M} \in \mathbb{R}^{N \times N}$ is the conventional mass matrix with entries $\mathbf{M}_{jk} = \int_\Omega \phi_k \phi_j \, dx$. Hence, the eigenvalues of $E''(u_h)$ in the sense of definition (2.8) are given by the system

$$\begin{pmatrix} \mathbf{A}(u_h)^{\mathrm{RR}} & \mathbf{A}(u_h)^{\mathrm{IR}} \\ \mathbf{A}(u_h)^{\mathrm{RI}} & \mathbf{A}(u_h)^{\mathrm{II}} \end{pmatrix} \begin{pmatrix} \mathbf{v}_\ell^{\mathrm{Re}} \\ \mathbf{v}_\ell^{\mathrm{Im}} \end{pmatrix} = \lambda_\ell \begin{pmatrix} \mathbf{M} & 0 \\ 0 & \mathbf{M} \end{pmatrix} \begin{pmatrix} \mathbf{v}_\ell^{\mathrm{Re}} \\ \mathbf{v}_\ell^{\mathrm{Im}} \end{pmatrix}.$$

Due to the symmetry of the matrices, the smallest eigenvalues can be easily computed with a standard method such as the inverse power iteration.

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