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SCIENTIFIC REPORTS

Intertwining between Probability, Analysis and Statistical Physics

05 Aug 2024–15 Aug 2024

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COUPLING OF BROWNIAN MOTIONS WITH SET VALUED DUAL PROCESSES ON RIEMANNIAN MANIFOLDS

MARC ARNAUDON

Classification AMS 2020: 60J60, 60J65, 60H10, 58J65, 53E10, 60J55, 35K93

Keywords: Brownian motion on Riemannian manifolds, intertwining relations, set-valued dual processes, couplings of primal and dual processes, stochastic mean curvature evolutions, boundary and skeleton local times, generalized Pitman theorem

The purpose of this talk is to construct a Brownian motion $(X_t)_{t\geq 0}$ taking values in a Riemannian manifold M, coupled (intertwined) with a compact set-valued process $(D_t)_{t\geq 0}$ such that, at least for small enough (D_t)-stopping time $\tau > 0$ and conditioned by $D_{[0,\tau]}$, the law of X_τ is the normalized Lebesgue measure on D_τ. This intertwining result is a generalization of Pitman's $2M - X$ theorem in the real line, where the set-valued process is $D_t = [-R_t, R_t]$ with $R_t = 2M_t - X_t$, $M_t = \max_{s\leq t} X_s$, (X_t) is a Brownian motion started at 0.

Such couplings are very important for the construction of strong stationary times, as explained by Diaconis and fill [5] in a discrete time and finite setting. A strong stationary time τ for $(X_t)_{t\geq 0}$ is a finite stopping time for $(X_t)_{t\geq 0}$ (and some independent randomness) such that τ and X_τ are independent and X_τ is distributed according to the invariant probability π of $(X_t)_{t\geq 0}$ (which is in our situation, assuming M is compact, the renormalized volume measure in M). Strong stationary times are important for two reasons (cf. [5]):

- they enable to sample exactly the invariant probability π, contrary to the usual approximations provided by Monte Carlo techniques.
- They provide a probabilistic alternative to functional analysis approaches for the quantitative investigation of convergence to equilibrium. More precisely, for any strong stationary time τ, we have

$$\forall\, t \geq 0, s(\mathcal{L}(X_t), \pi) \leq \mathbf{P}[\tau > t],$$

where the separation discrepancy $s(\mu, \pi)$ between two probability measures μ and π is defined by

$$s(\mu, \pi) := \mathrm{esssup}_\pi \left(1 - \frac{d\mu}{d\pi}\right)$$

(where $d\mu/d\pi$ is the Radon-Nikodym density). The separation discrepancy dominates the total variation norm. In the context of convergence to equilibrium, it is very difficult to estimate the discrepancy $s(\mathcal{L}(X_t), \pi)$ via functional inequalities.

Date: September 3rd, 2024.

In our context, the first time τ (if it exists) such that $D_\tau = M$ is a strong stationary time.

We first construct regular intertwined processes related to Stokes' theorem. Then using several limiting procedures we construct synchronous intertwined, free intertwined processes. The local times of the Brownian motion on the (morphological) skeleton or the boundary of each D_t play an important role.

More precisely, the synchronous intertwined process $(D_t)_{t\geq 0}$ has motion at $y \in \partial D_t$:

$$\begin{aligned} d\partial D_t(y) &= N^{D_t}(y)\left(\langle dX_t, N^{D_t}(X_t)\rangle + \frac{1}{2}h^{D_t}(y)dt + \left(-h^{D_t}(X_t)dt - 2\sin\theta^{S_t}(X_t)dL^{S_\cdot}(X)\right)\right) \\ &= N^{D_t}(y)\left(\langle dX_t, \nabla\rho_{\partial D_t}(X_t)\rangle + \frac{1}{2}h^{D_t}(y)dt + \text{"}\Delta\rho_{\partial D_t}(X_t)dt\text{"}\right) \end{aligned}$$

with N^{D_t} the inward unit normal vector (resp. h^{D_t} the mean curvature) of level sets of distance to boundary $\rho_{\partial D_t}$, $L^{S_\cdot}(X)$ the local time of X_t on the morphological skeleton S_t of D_t, $\theta^{S_t}(X_t)$ the angle between the regular skeleton and any of the two minimal geodesics from ∂D_t to X_t (when $X_t \in S_t$).

The free intertwined process has motion at $y \in \partial D_t$:

$$d\partial D_t(y) = N^{D_t}(y)\left(dW_t + \left(\frac{1}{2}h^{D_t}(y)dt - dL_t^{\partial D_\cdot}(X)\right)\right)$$

where (W_t) is a real-valued Brownian motion independent of (X_t) and $L_t^{\partial D_\cdot}(X)$ is the local time of $X_\cdot$ on $\partial D_\cdot$.

For both couplings the initial conditions are D_0, and X_0 uniformly distributed in D_0.

Under these conditions, we prove that (D_t) is Markovian and has same law as the moving set $(\tilde{D}_t)$ started at $\tilde{D}_0 = D_0$ and satisfying the equation at $y \in \partial\tilde{D}_t$:

$$d\partial\tilde{D}_t(y) = N^{\tilde{D}_t}(y)\left(d\tilde{W}_t + \left(\frac{1}{2}h^{\tilde{D}_t}(y) - \frac{\underline{\mu}(\partial\tilde{D}_t)}{\mu(\tilde{D}_t)}\right)dt\right)$$

with $\underline{\mu}(\partial\tilde{D}_t)$ and $\mu(\tilde{D}_t)$ the respective volumes of $\partial\tilde{D}_t$ and $\tilde{D}_t$, $(\tilde{W}_t)$ a real-valued Brownian motion. The process $(\tilde{D}_t)$ is the so-called renormalized stochastic mean curvature flow, playing the role of a Bessel(3) process: notice that the process $(\mu(\tilde{D}_t))$ is a time-changed Bessel(3) process with speed $\underline{\mu}(\partial\tilde{D}_t)$ ([4]).

The talk is mainly based on the reference [1]. Infinite lifetime for some set-valued processes in the Euclidean plane is proven in [2]. Strong stationary times are exhibited in [3].

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INTERACTING PARTICLE SYSTEMS, CONDITIONED RANDOM WALKS AND THE AZTEC DIAMOND

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Classification AMS 2020: 82C22, 82C23, 82C41, 60K35.

Keywords: Interacting particle systems, non-colliding random walks, intertwinings, Aztec diamond.

In this talk I introduced a class of so-called integrable (or exactly-solvable models) in **inhomogeneous-space**. These are interacting particle systems in interlacing arrays, namely configurations of points $(x_i^{(n)})_{1\leq i\leq n, n=1,\dots,N}$ satisfying the interlacing inequalities:

$$x_i^{(n+1)} \leq x_i^{(n)} < x_{i+1}^{(n+1)}. \tag{0.1}$$

Particles move independently except when interlacing is about to be break if a move goes through in which case they interact via so-called "push-block" interactions, see [2]. Lower level particles push and block higher level particles in order for the interlacing to remain, see [2]. The main novelty here is that the transition mechanism of individual particles depends on the particle space location. Namely, particles can move in continuous-time with general pure-birth chain jumps and in discrete-time with inhomogeneous-space Bernoulli jumps or inhomogeneous-space geometric jumps, see [2, 1]. These models generalise homogeneous models studied in seminal papers by Borodin-Ferrari [4], Warren-Windridge [11] and Dieker-Warren [6]. It is also possible to introduce inhomogeneities in time as well while retaining the exact-solvability of the model, see [2].

If we denote by $(\mathsf{x}_i^{(n)}(t))_{i,n}$ the process in the interlacing array informally described above then by construction the projections $(\mathsf{x}_1^{(k)}(t))_{k=1}^N$ and $(\mathsf{x}_k^{(k)}(t))_{k=1}^N$ on the left and right edge of the array respectively are autonomous and evolve as a TASEP (totally asymmetric exclusion processs)-like and PushTASEP systems in inhomogeneous space. The first result I presented in the talk is that under certain "Gibbs-type" initial conditions for $(\mathsf{x}_i^{(n)}(t))_{i,n}$ the projection on any fixed level k of the array $(\mathsf{x}_i^{(k)}(t))_{i=1}^k$ is Markovian with explicit transition probabilities. These k-th level dynamics $(\mathsf{x}_i^{(k)}(t))_{i=1}^k$ can be understood as the discrete inhomogeneous-space generalisation of Dyson Brownian motion [7] or inhomogeneous walks conditioned to not intersect, see [8] for work on the homogeneous case. In particular, the process $(\mathsf{x}_i^{(n)}(t))_{i,n}$ provides a highly non-obvious coupling between TASEP-like and PushTASEP systems in inhomogeneous space and discrete inhomogeneous analogues of Dyson Brownian motion. From a symmetric function point of view this result can also be thought of as the factorial Schur process generalisation of seminal work of Borodin on dynamics for the Schur processes [3].

The second result I presented was that the aforementioned dynamics viewed as a dynamical point process have determinantal correlations functions along certain space-time directions. This means that the correlations of the model (along these

space-time directions) can be written as determinants of a function called the correlation kernel $\mathfrak{K}$. All probabilistic information about the model is then encoded in this kernel $\mathfrak{K}$. This allows to study the model asymptotically. A short-time asymptotic for the dynamics was presented. The explicit formula for $\mathfrak{K}$ was essential to establish this result.

A key ingredient in the proof of the results above are certain new intertwining relations satisfied by the transition probabilities of the projections $(\mathsf{x}_i^{(k)}(t))_{i=1}^k$ on single-levels of the array. The main tool I used to prove this and perform computations more generally is a natural generalisation of an infinite Toeplitz matrix T_f associated to a "symbol" function f. For certain choices of the symbol f one obtains the transition probabilities of an individual free (non-interacting) particle in $\mathbb{Z}$. The way the single-level intertwined semigroups are coupled comes from a simple coalescing random walk model. It is interesting to note that a general recipe for coupling intertwined semigroups exists, proposed by Diaconis and Fill [5] and later developed further by Borodin and Ferrari [3, 4]. It gives the same coupling as the one in [2] for continuous-time dynamics and discrete-time Bernoulli jumps. However, in the case of discrete-time geometric jumps the coupling given by the general recipe of Diaconics-Fill is different from the one in [2] and has only a single Markovian projection on the right edge particles.

I then explained how such dynamics and in fact generalisations thereof are connected to tilings of the Aztec diamond with general inhomogeneous weights. This statistical mechanics model can equivalently be viewed as random dimer covers (perfect matchings) of a certain graph called the Aztec diamond graph. One associates general weights to each edge in this graph and attached to this data there is a canonical Boltzmann probability measure which is the object of interest. The way dynamics are introduced in the first place is via the so-called shuffling algorithm that was introduced in seminal work by Propp [10]. This gives an exact sampling method using only local moves for picking random tilings of the Aztec diamond with arbitrary weights according to the aforementioned Boltzmann measure. The homogeneous case (when all edges receive equal weight) of the Aztec diamond tiling model is very well-understood but the inhomogeneous case is much less so. However, there has been intense activity and spectacular progress in the past 10 years or so on inhomogeneous Aztec diamond tiling models using a variety of techniques. Many of these advances are focused on the multi-periodic case. The contribution of this part of [2] is to connect these models to walks conditioned to never intersect which could be analysed further asymptotically using probabilistic techniques. This generalises the work of Nordenstam [9] for the homogeneous case.

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LAST PASSAGE PERCOLATION IN A STRIP

GUILLAUME BARRAQUAND

We will present a method for computing the stationary measures of integrable probabilistic systems with boundaries. Focusing on the example of a model called last passage percolation, we will emphasize the key role played by certain commutation relations involving the transition kernel of a geometric random walk. These relations can also be interpreted, in terms of symmetric functions, as variants of the Cauchy and Littlewood summation identities for Schur symmetric functions. The method works as well for other models and their associated families of symmetric functions, such as Whittaker functions or Hall-Littlewood polynomials. Finally, we will also discuss connections to the traditional approach for computing stationary measures of interacting particle systems between boundary reservoirs: the matrix product ansatz.

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CANONICAL LIFTING OF INTERTWINING TO HIGHER DIMENSIONS AND APPLICATIONS

ANDREW CHEE

We present a theory of canonical lifting and related transformations from one-dimension to higher dimension and related spaces, e.g. the Weyl chamber, that preserve intertwining. Such constructions coincide with rich classes of intertwined complex models (1) whose connections have not previously been identified or explored and (2) have not traditionally been studied in terms of their simpler one-dimensional counterparts. These include the well studied non-intersecting Markov processes, e.g. Dyson Brownian Motion and Dyson Laguerre processes, their discrete counterparts and their associated stationary ensembles. We leverage this theory to extend results on the universality of certain scaling limits to even broader classes of models.

This work is joint with Pierre Patie (Cornell).

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A ONE-PARAMETER FAMILY OF INTERTWININGS USING CURVATURE, AND PITMAN'S CELEBRATED 2M-X THEOREM

REDA CHHAIBI

Classification AMS 2020: Primary 46L53; Secondary 58B32, 60B99.

Keywords: Orbit method, Jimbo-Drinfeld's quantum groups, Non-commutative (=quantum) probability, Quantum random walks, Brownian motion on $\mathbb{H}^3 = SL_2(\mathbb{C})/SU_2$, Infinite curvature.

This talk is based on a joint work with François Chapon [1].

The classical theorem by Pitman (1975) states that a Brownian motion minus twice its running infimum enjoys the Markov property. It has the same law as the norm of a 3-dimensional Brownian motion.

We start by recalling the long history of this theorem. First and foremost, it is for for this example that the technology of intertwining has been developed by Rogers and Pitman in their seminal 1981 paper. Then, it was further understood that this theorem, in its discrete form, is heavily connected to combinatorial representation theory. Even in the context of solvable models in mathematical physics, this theorem bridges random matrix theory and directed percolation.

After recalling this history, we present a problem which remained and which is to explain this theorem through a relation between the representation theory and the geometry of SL_2. Let us explain the problem in more details. In passing, we find a one-parameter family of intertwinings that interpolate between the Rogers-Pitman interwining and the classical Dynkin-type interwining between a 3-dimensional Brownian motion and its norm. It is this interpolation which I want to advertise and present in this conference on interwinings.

The problem: On the one hand, Biane understood that Pitman's theorem is intimately related to the representation theory of the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$, in the so-called crystal regime $q \to 0$. On the other hand, Bougerol and Jeulin showed the appearance of exactly the same Pitman transform in the infinite curvature limit $r \to \infty$ of a Brownian motion on the hyperbolic space $\mathbb{H}^3 = SL_2(\mathbb{C})/SU_2$. Our work aims at understanding this phenomenon by giving a unifying point of view.

The result of [1]**:** We exhibit a presentation $\mathcal{U}_q^{\overline{h}}(\mathfrak{sl}_2)$ of the Jimbo-Drinfeld quantum group which isolates the role of curvature r and that of the Planck constant $\overline{h}$. The simple relationship between parameters is $q = e^{-r}$. The semi-classical limits $\overline{h} \to 0$ are the Poisson-Lie groups dual to $SL_2(\mathbb{C})$ with varying curvatures $r \in \mathbb{R}^+$. We also construct classical and quantum random walks, drawing a full picture which includes Biane's quantum walks and the construction of Bougerol-Jeulin. Taking the curvature

Date: 14 August 2024.

parameter r to infinity leads indeed to the crystal regime at the level of representation theory ($\overline{h} > 0$) and to the Bougerol-Jeulin construction in the classical world ($\overline{h} = 0$).

All these results are neatly in accordance with the philosophy of Kirillov's orbit method.

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A RATE-DISTORTION FRAMEWORK FOR MCMC ALGORITHMS: GEOMETRY AND FACTORIZATION OF MULTIVARIATE MARKOV CHAINS

MICHAEL CHOI

We introduce a framework rooted in a rate distortion problem for Markov chains, and show how a suite of commonly used Markov Chain Monte Carlo (MCMC) algorithms are specific instances within it, where the target stationary distribution is controlled by the distortion function. Our approach offers a unified variational view on the optimality of algorithms such as Metropolis-Hastings, Glauber dynamics, the swapping algorithm and Feynman-Kac path models. Along the way, we analyze factorizability and geometry of multivariate Markov chains. Specifically, we demonstrate that induced chains on factors of a product space can be regarded as information projections with respect to a particular divergence. This perspective yields Han–Shearer type inequalities for Markov chains as well as applications in the context of large deviations and mixing time comparison.

This is based on joint work with Youjia Wang and Geoffrey Wolfer.

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ON THE SEPARATION CUT-OFF PHENOMENON FOR BROWNIAN MOTIONS ON HIGH DIMENSIONAL ROTATIONALLY SYMMETRIC COMPACT MANIFOLDS

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Classification AMS 2020: primary: 58J65, secondary: 37A25 58J35 60J60 35K08.

Keywords: Rotationally symmetric Brownian motions, strong stationary times, separation discrepancy, hitting times.

Given a family of rotationally symmetric compact manifolds indexed by the dimension and a weight function, the goal is to investigate the cut-off phenomenon for the Brownian motions on this family. We provide a class of compact manifolds with non-negative Ricci curvatures for which the cut-off in separation occurs, in high dimension, with different explicit mixing times. We also produce counter-examples, still with non-negative Ricci curvatures, where there are no cut-off in separation. Our proof is based on a previous construction of a sharp strong stationary times by the authors, and some quantitative estimates on the two first moments of the covering time. The concentration of measure phenomenon for the above family of manifolds appears to be relevant for the study of the corresponding cut-off phenomenon. This is a joint work with Marc Arnaudon and Laurent Miclo.

0.1. **rotationally symmetric compact manifolds.** For $n \geq 2$, let M_f^n be the product manifold $[0,L] \times \mathbb{S}^{n-1}/\sim$, where $(r_1,\theta_1) \sim (r_2,\theta_2)$ if $(r_1,\theta_1) = (r_2,\theta_2)$ or $r_1 = r_2 = 0$ or $r_1 = r_2 = L$, endowed with the warping product metric

$$ds^2 = dr \otimes dr + f^2(r) d\theta \otimes d\theta,$$

where $\mathbb{S}^{n-1}$ is the usual sphere of dimension $n-1$ and radius 1, $d\theta \otimes d\theta$ is the standard metric on the sphere and f is a regular real function that satisfies the following assumption:

$$\left\{ \begin{array}{rcl} f : [0,L] & \to & \mathbb{R}_+, \\ f(s) \sim_0 s & , & f(L-s) \sim_0 s \\ f^{(2k)}(0) & = & f^{(2k)}(L) = 0, k \in \mathbb{Z}_+ \end{array} \right. \tag{0.1}$$

We will call such function a **weight** function, we will assume all along the paper that f is a weight function. Later, further conditions will be required to ensure the regularity of the metric at $\widetilde{0} \sim (0,.)$ and $\widetilde{L} \sim (L,.)$. The volume of the geodesic ball $B(\widetilde{0},r)$ in M_f^n centered at $\widetilde{0}$ of radius $r \in [0,L]$ is given by $\mathrm{Vol}_n(B(\widetilde{0},r)) = c_n \int_0^r f^{n-1}(s)ds$, where $c_n = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$ is the volume of $\mathbb{S}^{n-1}$. The area of the geodesic sphere $\partial B(\widetilde{0},r)$ is $c_n f^{n-1}(r)$ and the mean curvature of any point in $\partial B(\widetilde{0},r)$ is given by $(n-1)\frac{f'(r)}{f(r)}$. We have $\mathrm{Ric}(v) = \left((n-2)\frac{1-f'(r)^2}{f^2(r)} - \frac{f''(r)}{f(r)}\right) v$ if $v \in T\mathbb{S}^{n-1}$ and $\mathrm{Ric}(\partial_r) = \left(-(n-1)\frac{f''(r)}{f(r)}\right)\partial_r$,

Date: 03 Sep. 2024.

where Ric denote the Ricci tensor. For a good introduction to warped products, see Chapter 3 in Petersen [9].

Definition 0.1. *For any $n \in \mathbb{N} \setminus \{1\}$, $X_n := (X_n(t))_{t\geq 0}$ stands for the Brownian motion on M_f^n started at $\widetilde{0}$ and time-accelerated by a factor 2, i.e. the Δ-diffusion in M_f^n.*

0.2. **Intertwining dual process.** It was seen in [3] that X_n can be intertwined with a process $D := (D(t))_{t\geq 0}$ taking values in the closed balls of M_f^n centered at $\widetilde{0}$, starting at $\{\widetilde{0}\}$ and absorbed in finite time τ_n in the whole set M_f^n. In [2], several couplings of X_n and D were constructed (all with same marginal law), so that for any time $t \geq 0$, the conditional law of $X_n(t)$ knowing the trajectory $D([0,t]) := (D(s))_{s\in[0,t]}$ is the normalized uniform law over $D(t)$, which will be denoted $\Lambda(D(t), \cdot)$ in the sequel. Furthermore, D is progressively measurable with respect to X_n, in the sense that for any $t \geq 0$, $D([0,t])$ depends on X_n only through $X_n([0,t])$. Due to these couplings and to general arguments from Diaconis and Fill [5], τ_n is a strong stationary time for X_n, meaning that τ_n and $X_n(\tau_n)$ are independent and $X_n(\tau_n)$ is uniformly distributed over M_f^n. We deduce that

$$\forall\, t \geq 0, \qquad \mathfrak{s}(\mathcal{L}(X_n(t)), \mathcal{U}_n) \leq \mathbb{P}[\tau_n > t] \tag{0.2}$$

where the l.h.s. is the separation discrepancy between the law of $X_n(t)$ and the uniform distribution $\mathcal{U}_n$ over M_f^n. The separation discrepancy between two probability measures μ and ν defined on the same measurable space is given by

$$\mathfrak{s}(\mu, \nu) = \operatorname*{ess\,sup}_{\nu} 1 - \frac{d\mu}{d\nu}$$

where $d\mu/d\nu$ is the Radon-Nikodym density of μ with respect to ν.

Remarks 0.2. *Note that for any $t \in [0, \tau_n)$, the "opposite pole" $\widetilde{L}$ does not belong to the support of $\Lambda(D(t), \cdot)$. It follows from an extension of Remark 2.39 of Diaconis and Fill* [5] *that τ_n is even a sharp strong stationary time for X_n, meaning that*

$$\forall\, t \geq 0, \qquad \mathfrak{s}(\mathcal{L}(X_n(t)), \mathcal{U}_n) \;=\; \mathbb{P}[\tau_n > t]$$

Thus the understanding of the convergence in separation of X_n toward $\mathcal{U}_n$ amounts to understanding the distribution of τ_n. This time could be compute as follow :

Writing $B(\widetilde{0}, R(t)) := D(t)$ for $t \in [0, \tau_n]$, it has been seen in [3] *that $R := (R(t))_{t\in[0,\tau_n]}$ is solution to the stochastic differential equation*

$$\forall\, t \in (0, \tau_n), \qquad dR(t) \;=\; \sqrt{2}dB(t) + b_n(R(t))dt \tag{0.3}$$

$$\tau_n \;=\; \inf\{t \geq 0 \,:\, R(t) = L\} \tag{0.4}$$

where $(B(t))_{t\geq 0}$ is a standard Brownian motion in $\mathbb{R}$ and the mapping b_n is given by

$$\forall\, r \in (0, L), \qquad b_n(r) \;:=\; 2\frac{f^{n-1}(r)}{\int_0^r f^{n-1}(u)\,du} - (n-1)\frac{f'(r)}{f(r)} \tag{0.5}$$

0.3. **Cut-off phenomenon.** For fixed n, the Brownian motion X_n in M_f^n converges in law to $\mathcal{U}_n$, namely

$$X_n(t) \xrightarrow[t\to+\infty]{\mathcal{L}} \mathcal{U}_n.$$

Quantifying this convergence to equilibrium is relevant when the dimension n becomes large. This speed of convergence or mixing time, depends one the way the difference between the time marginal and the uniform distribution is measured. A cut-off

phenomenon in separation at time a_n is a kind of phase transition, namely the separation discrepancy between X_n and the equilibrium abruptly drops from the largest value 1 to the smallest one 0 on a small interval around a_n. More precisely, we say the family of diffusion processes $(X_n)_{n\in\mathbb{N}\setminus\{1\}}$ has a cut-off in separation with mixing times $(a_n)_{n\in\mathbb{N}\setminus\{1\}}$ when

$$\begin{aligned} \forall\, r>0, \qquad \lim_{n\to\infty} \mathfrak{s}(\mathcal{L}(X_n((1+r)a_n)),\mathcal{U}_n) = \lim_{n\to\infty} \mathbb{P}\left[\tau_n > (1+r)a_n\right] &= 0 \\ \forall\, r\in(0,1), \qquad \lim_{n\to\infty} \mathfrak{s}(\mathcal{L}(X_n((1-r)a_n)),\mathcal{U}_n) = 1 - \lim_{n\to\infty} \mathbb{P}\left[\tau_n \leq (1-r)a_n\right] &= 1 \end{aligned}$$

0.4. **Discusses other related work.** In the context of card shuffling, the cut-off phenomenon was discovered by Diaconis and Shahshahani [4] and Aldous and Diaconis [1]. Afterward, the cut-off phenomenon has been proven for a large variety of finite Markov chains, see e.g. Diaconis and Fill [5], Levin, Peres and Wilmer [8] and Ding, Lubetzky and Peres [6]. Nevertheless the literature on the cut-off phenomenon for Markov processes on a continuous state space is rather sparse. For example Saloff-Coste [11] has proven the cut-off phenomenon in total variation distance for the Brownian motions on the spheres $\mathbb{S}^n$ for high dimensions n, with a mixing time of order $\ln(n)/(2n)$, see also Méliot [10] for extensions to classical symmetric spaces of compact type. Their approach are based on complete knowledge of the spectral decomposition. It is shown in Hermon, Lacoin and Peres [7] that total variation and separation cut-off are not equivalent and neither one implies the other. Our goal here is to check that there is a cut-off phenomenon in separation for a large class of family of rotationally symmetric manifolds with non-negative Ricci curvature, including the case of spheres. We also give examples of rotationally symmetric manifolds with non-negative Ricci curvature where there is no cut-off in separation. Our results are connected with those of Salez, concerning sequences of irreducible Markov chains with symmetric support and non-negative coarse Ricci curvature that exhibit cut-off in total variation when an additional product condition hypothesis is satisfied, see [12] for the precise statement.

Our proof is based on two ingredients, the resort to the strong stationary times for X_n presented in [2] and the detailed quantitative estimates on the cover time of dual process that appear to be an one-dimensional diffusion processes in the case of rotationnaly symmetric manifolds. The concentration of volume phenomenon plays a crucial role to detect the scale on which the cut-off phenomenon occurs. This alternative point of view differs from the traditional approach based on spectral analysis and could be extended to other situations where spectral information is less available.

0.5. **Main result.** The following Theorem shows a phase transition (with respect to the parameter $\alpha\in(-1,+\infty)$ introduced below) for the cut-off phenomenon concerning the Brownian motions on the model M_f^n for high dimensions n, depending on the shape of the function f at $L/2$. Let us first introduce another set of assumptions on f:

$$\left\{\begin{array}{c} \forall\, s\in[0,L], \qquad f(L-s)=f(s), \\ \forall\, s\in[0,L/2), \qquad f'(s)>0, \\ \forall\, s\in[0,L]\setminus\{L/2\}, \qquad f''(s)\leq 0, \end{array}\right. \tag{0.6}$$

Theorem 0.3. *Consider a C^2 function f on $[0,L]\setminus\{L/2\}$ and C^1 in $[0,L]$, satisfying Assumptions* (0.1) *and* (0.6). *Assume there exist $\alpha\in(-1,+\infty)$ and $C>0$ such that for $h\neq 0$ small enough,*

$$f''(L/2-h) \;=\; -C|h|^{\alpha}+o(|h|^{\alpha}) \tag{0.7}$$

Let $X_n:=(X_n(t))_{t\geq 0}$ be the Brownian motion described in Definition 0.1.

- *if $\alpha\in(-1,0)$ then $(X_n)_{n\in\mathbb{N}\setminus\{1\}}$ has a cut-off in separation at time C_1/n, with $C_1=2\int_0^{L/2}\frac{f(s)}{f'(s)}$,*
- *if $\alpha=0$ then $(X_n)_{n\in\mathbb{N}\setminus\{1\}}$ has a cut-off in separation at time $C_2\ln(n)/n$, with $C_2=\frac{f(L/2)}{C}$,*
- *if $\alpha>0$ then $(X_n)_{n\in\mathbb{N}\setminus\{1\}}$ has no cut-off in separation,*

Remarks 0.4. *An instance where* (0.7) *is satisfied is when there exist $\alpha\in(-1,+\infty)$, $C>0$ and $\epsilon\in(0,L/2)$ such that $\forall\, h\in[-\epsilon,\epsilon]$, $\qquad f(L/2+h)=f(L/2)-C|h|^{2+\alpha}$.*

Corollary 0.5. *Let $X_n:=(X_n(t))_{t\geq 0}$ be the Brownian motion (0.1) in the sphere $\mathbb{S}^n$ and where $\widetilde{0}$ now stands for any point of $\mathbb{S}^n$. Then $(X_n)_n$ has a cut-off in separation with mixing times $(a_n)_n=\left(\frac{\ln(n)}{n}\right)$.*

Proof. Use Theorem 0.3, with $f=\sin$ and $L=\pi$, $M^n_{\sin}\sim\mathbb{S}^n$.

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INTERTWINING FOR INTERACTING PARTICLE SYSTEMS IN THE CONTINUUM

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Classification AMS 2020: 60J25; 60G55; 60K35

Keywords: duality and intertwining; algebraic method; interacting particle systems; measure-valued processes

1. BACKGROUND

Duality with respect to a function is a useful tool in the theory of Markov processes, in particular interacting particle systems, queuing theory and mathematical population genetics. Two processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ with state spaces E and F and semigroups $(P_t)_{t\geq 0}$ and $(Q_t)_{t\geq 0}$ are *dual with duality function* $H : E \times F \to \mathbb{R}$ if, for all $x \in E$, $y \in F$, and $t \geq 0$

$$\mathbb{E}_x\big[H(X_t, y)\big] = \mathbb{E}^y\big[H(x, Y_t)\big]$$

or equivalently

$$\int P_t(x, \mathrm{d}x')H(x', y) = \int Q_t(y, \mathrm{d}y')H(x, y').$$

A closely related concept is *intertwining*: A kernel $\Lambda(x, \mathrm{d}y)$ is an intertwiner for the semigroups if $P_t\Lambda = \Lambda Q_t$, i.e.,

$$\int P_t(x, \mathrm{d}y')\Lambda(y', B) = \int \Lambda(x, \mathrm{d}y)Q_t(y, B)$$

for all $x \in E$, measurable $B \subset F$, and $t \geq 0$. On finite state spaces we may think of P_t, Q_t, H as matrices and the duality and intertwining relation become

$$P_tH = HQ_t^{\mathsf{T}}, \quad P_t\Lambda = \Lambda Q_t.$$

Staying with finite state spaces, suppose that (Y_t) has a reversible measure μ with strictly positive masses $\mu(y) > 0$. Let D be the diagonal matrix with entries $\mu(y)$. Then $DQ_t = Q_t^{\mathsf{T}}D$ and one easily sees that H is a duality function if and only if $\Lambda(x, y) = H(x, y)\mu(y)$ is an intertwiner.

For interacting particle systems on lattices–e.g. the contact process, voter model or symmetric simple exclusion process—the state space is $\{0, 1\}^{\mathbb{Z}^d}$, possibly restricted to configurations $\eta = (\eta_x)_{x\in\mathbb{Z}^d}$ with finitely many particles, $N(\eta) = \sum_{x\in\mathbb{Z}^d} \eta_x < \infty$. One is interested in the long-time behavior of a system that starts with a very large number of particles and faced with the challenge of a formidably large state space. As a way out, one studies simpler quantities such as single-site occupation numbers $\mathbb{E}[\eta_x(t)]$ or two-point correlation functions $\mathbb{E}[\eta_x(t)\eta_y(t)]$. Duality sometimes allows to map the time-evolution, of, say, the two-point correlation functions for a system of one million particles to the time-evolution of a two-particle system, a considerable simplification.

Date: 12 August 2024.

On lattices, the technique is well-developed. Much less is known for particles in the continuum. It turns out that, if one seeks to generalize duality to the continuum, it is advantageous to work with intertwining instead. We prove a general theorem for consistent, conservative processes with finitely many particles. Such processes may be modleled either with labeled particles or with counting measures.

2. Intertwining for consistent particle systems

Let $(\mathbb{X}, \mathcal{X})$ be a well behaved measurable space (e.g. Polish) and $\mathbf{N}_{<\infty}$ the space of finite counting measures η on $\mathbb{X}$. Every counting measure is a sum of finitely many Dirac measures, $\eta = \delta_{x_1} + \cdots + \delta_{x_n}$. Let $(\eta_t)_{t\geq 0}$ be a continuous-time Markov process with state space $\mathbb{X}$ and let $(P_t)_{t\geq 0}$ be its semigroup. The process is *conservative* if the total number of particles stays constant in time and it is *consistent* if time evolution and random removal of particle, uniformly among the finitely many particles, commute. That is, introducing the annihilation operator

$$\mathcal{A}f\big(\delta_{x_1} + \cdots + \delta_{x_n}\big) = \sum_{i=1}^{n} f\Big(\sum_{j\neq i} \delta_{x_j}\Big)$$

we ask that $P_t\mathcal{A}f = \mathcal{A}P_t f$ for every measurable test function $f : \mathbf{N}_{<\infty} \to \mathbb{R}_+$. We assume that there is a compatible process for labelled configurations, i.e., for each n a process $(X_1^{(n)}(t), \ldots, X_n^{(n)}(t))_{t\geq 0}$ with state space $\mathbb{X}^n$ and semigroup $p_t^{(n)}$ such that (η_t) started in $\delta_{x_1}+\cdots+\delta_{x_n}$ equals $\delta_{X_1^{(n)}(t)}+\cdots+\delta_{X_n^{(n)}(t)}$ with $\boldsymbol{X}^{(n)}(t)$ started in $(x_1, \ldots, x_n)$. The factorial moment measures $\eta_t^{(n)}$ satisfy

$$\int f \mathrm{d}\eta_t^{(k)} = \sum_{\substack{1\leq i_1,\ldots,i_k\leq n:\\ \text{pairwise different}}} f\big(X_{i_1}^{(n)}(t), \ldots, X_{i_k}^{(n)}(t)\big)$$

for all measurable $f : \mathbb{X}^k \to \mathbb{R}_+$.

Theorem 2.1. [3] *Let $(\eta_t)_{t\geq 0}$ be a conservative and consistent Markov process with state space $\mathbf{N}_{<\infty}$, and $((\boldsymbol{X}^{(n)}(t))_{t\geq 0}$ a compatible family of processes with state space $\mathbb{X}^n$. Then*

$$\mathbb{E}_{\delta_{x_1}+\cdots+\delta_{x_n}}\Big[\int f \mathrm{d}\eta_t^{(k)}\Big] = \sum_{\substack{1\leq i_1,\ldots,i_k\leq n:\\ \textit{pairwise different}}} \mathbb{E}^{(x_1,\ldots,x_k)}\Big[f\big(X_1^{(k)}(t), \ldots, X_k^{(k)}(t)\big)\Big]$$

for all $k \in \mathbb{N}$, measurable $f : \mathbb{X}^k \to \mathbb{R}_+$, $n \in \mathbb{N}$, $(x_1, \ldots, x_n) \in \mathbb{X}^n$ and $t \geq 0$.

Choosing, for example, $k = 1$ and $f = \mathbf{1}_A$ the indicator of a set $A \subset \mathbb{X}$, we find that the expected number of particles in A is

$$\mathbb{E}_{\delta_{x_1}+\cdots+\delta_{x_n}}\big[\eta_t(A)\big] = \sum_{i=1}^{n} p_t^{(1)}(x_i, A).$$

Thus, the time-evolution of the intensity measure of η_t is determined by the one-particle dynamics! That is exactly the kind of simplification one is after. In [3] we explain how the theorem may be recast as a self-intertwining relation with a signed kernel K, representing Lenard's K-transform, and how it relates to existing results on lattices.

3. Intertwining with orthogonal polynomials

Suppose in addition that the process (η_t) has a reversible measure ρ. Assume that the reversible measure is the law of a completely random measure, i.e., under ρ the occupation numbers $\eta(A)$ for disjoint observation windows are independent. This is the case, for instance, when ρ is the distribution of a Poisson point process. Let $\overline{\mathbf{P}_n} \subset L^2(\mathbf{N}_{<\infty}, \rho)$ be the closure of linear combinations of monomials

$$\eta \mapsto \eta(A_1) \cdots \eta(A_k) \quad k \leq n,\ A_1, \ldots, A_k \subset \mathbb{X}.$$

We define an orthogonalized version of the above monomial by subtracting the orthogonal projection onto $\overline{\mathbf{P}_{n-1}}$. The result is denoted with the Wick dots from mathematical physics as $: \eta(A_1) \cdots \eta(A_k) :$. Under our conditions on ρ, we have

$$: \eta(A_1) \cdots \eta(A_k) : = \prod_{i=1}^{k} : \eta(A_i) :$$

whenever $A_1, \ldots, A_k$ are disjoint. Below expressions like $: \eta_t(A_1) \cdots \eta_t(A_k) :$ should be read as follows: first, orthogonalize, this gives a function from $\mathbf{N}_{<\infty}$ to $\mathbb{R}$ (defined up to ρ-null sets); second, evaluate that function in η_t.

Theorem 3.1. [3] *In the setup of Theorem 2.1, assume in addition that (η_t) has a reversible measure ρ. Then*

$$\mathbb{E}_{\delta_{x_1}+\cdots+\delta_{x_n}}\Big[: \eta_t(A_1) \cdots \eta_t(A_k) :\Big] =: \int (p_t^{(k)} f_k) \mathrm{d}\eta^k : \big(\delta_{x_1} + \cdots \delta_{x_n}\big). \tag{3.1}$$

for all k, $A_1, \ldots, A_k$, and ρ-almost all $\mu = \delta_{x_1} + \cdots \delta_{x_n}$, with $f_k)(x_1, \ldots, x_k) = \prod_{i=1}^{k} \mathbf{1}_{A_i}(x_i)$.

Write $\langle \cdot \rangle_\rho$ for expectation with respect to the reversible measure ρ. When $k = 2$ and A_1 and A_2 are disjoint, the left side in Eq. (3.1) is a kind of centered moment

$$\mathbb{E}_{\delta_{x_1}+\cdots+\delta_{x_n}}\Big[\prod_{i=1}^{2} \big(\eta_t(A_i) - \langle \eta(A_i) \rangle_\rho\big)\Big].$$

For the right side in Eq. (3.1) , we proceed as follows:

(1) Define

$$F_t(x_1, x_2) := \mathbb{P}^{(x_1,x_2)}(X_1^{(2)}(t) \in A_1, X_2^{(2)}(t) \in A_2).$$

This is the probability that in the two-particle process started in (x_1, x_2), at time t particle no. 1 is in A_1 and particle no. 2 is in A_2.

(2) Consider the function $G_t : \mathbf{N}_{<\infty} \to \mathbb{R}_+$ given by

$$G_t(\delta_{x_1} + \cdots + \delta_{x_n}) = \sum_{1 \leq i,j \leq n} F_t(x_i, x_j).$$

The function G_t is in $\overline{\mathbf{P}_2}$ and we denote with $: G_t :$ its orthogonal version, i.e., G_t minus its projection onto $\overline{\mathbf{P}_1}$. The right side in (3.1) is simply

$$: G_t : (\delta_{x_1} + \cdots + \delta_{x_n}).$$

In [3] we explain how the theorem generalizes known dualities with orthogonal polynomials on the lattice. Applications of intertwining relations in the continuum are considered in [4]. In [2, 1] we explain the algebraic approach and we explore connections with concepts from quantum mechanics, current algebras, chaos

decompositions, non-Gaussian white noise analysis. In these articles we also consider concrete classes of distributions, processes, orthogonal polynomials. This brings in Poisson and negative binomial point processes, Gamma random measures, Meixner and Charlier polynomials, measure-valued processes, and spatial birth-death processes.

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ON THE INTERTWINING APPROACH FOR PROVING POINCARÉ TYPE FUNCTIONAL INEQUALITIES

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Classification AMS 2020: 60J60, 39B62, 37A30.

Keywords: Spectral gap; Poincaré type inequality; Neumann boundary condition; Log-concave probability measure; Convex body.

Let μ be some probability measure on a connected domain $\Omega \subset \mathbb{R}^d$ ($d \geq 2$) with smooth boundary $\partial\Omega$ (if not empty). We assume that its Lebesgue density is proportional to e^{-V}, where V is some smooth real-valued potential. Then the spectral gap $\lambda_1(\Omega, \mu)$ is defined as the smallest positive eigenvalue (if it exists, the first one being 0 associated to the constant eigenfunctions) of the self-adjoint extension in $L^2(\mu)$ of (minus) the diffusion operator

$$L = \Delta - \langle \nabla V, \nabla \rangle.$$

We endow this operator with Neumann boundary conditions on Ω: $\langle \nabla f, \eta \rangle = 0$, where η is the outer unit-normal vector. From a dual point of view, the spectral gap is nothing but the optimal constant $\lambda > 0$ in the following Poincaré inequality: for all sufficiently smooth function g,

$$\lambda \operatorname{Var}_\mu(g) \leq \int |\nabla g|^2 \, d\mu,$$

where the variance of g under μ is defined as

$$\operatorname{Var}_\mu(g) = \int g^2 d\mu - \left(\int g \, d\mu \right)^2.$$

Since in dimension higher than two the spectral gap is explicitly known only for few examples (*i.e.*, for some product measures for which the situation reduces to the one-dimensional case, such as the Gaussian measure on the whole Euclidean space, and for the uniform distribution on Euclidean balls or on specific triangles), a challenging question is to obtain relevant lower bounds depending on the quantities of interest of the problem (parameters of the underlying distribution, dimension, geometrical characteristics of the domain, etc.). Actually, the literature for estimating the spectral gap has been growing in the 40 last years, using modern tools of geometry, analysis and probability. It concerns many different approaches (localization, geometric and functional inequalities, isoperimetry, measure concentration, semigroup, coupling, optimal transport, etc.). The pioneers on this subject are Poincaré at the end of the 19th century and then Lichnerowicz, Payne-Weinberger, Bakry-Emery, Gromov-Milman, Ledoux and many others in the second part of the 20th century. See for instance [2] for a nice introduction to this topic, with precise references and credit. In the recent years, the convex setting has attracted a lot of attention, culminating in the famous KLS isoperimetric conjecture [6]: it states if the probability measure μ is log-concave (the set Ω and the potential V are convex), then the spectral gap $\lambda_1(\Omega, \mu)$ is of order the

inverse of the operator norm of its covariance matrix. See [7] for the latest (and sharpest) estimate appearing in the literature, which confirms the conjecture up to some logarithmic prefactor of the dimension.

Let us now introduce the notion of weighted intertwining which is at the heart of our work on Poincaré type inequalities and spectral gap estimates. The main idea, presented in [1], is based on the introduction in the classical intertwining of a matrix weight, depending smoothly of the space variable and corresponding to a notion of matrix Doob's transform. Such a transformation, which is familiar to probabilists when they compute the distribution of some conditioned Markov processes, leaves the *carré du champ* unchanged but modifies the invariant measure μ according to the weight. It gives an important degree of freedom through the choice of the weight in the convex framework and also it allows to compensate the possible lack of convexity of the potential V. As such, it allows to estimate the spectral gap beyond the scope of the KLS conjecture. Recall first that the classical intertwining between gradient and operator is an identity of the following form:

$$\nabla L = \widetilde{\mathcal{L}}\,\nabla,$$

where the operator $\widetilde{\mathcal{L}}$ acting on gradients or more generally on vector fields is of Schrödinger type, *i.e.*, $\widetilde{\mathcal{L}} = \mathrm{diag}(L) - \nabla^2 V$. Here $\mathrm{diag}(L)$ is a diagonal matrix diffusion operator with L on the diagonal and the Hessian matrix $\nabla^2 V$ acts as a multiplicative operator. Such an intertwining appeared a long time ago in Riemannian geometry through the Weitzenböck formula for differential forms and is reminiscent of the Γ_2 calculus of Bakry and Emery through the Bochner formula. To simplify the presentation, we present now a version of the weighted intertwining for which the matrix weight is a multiple of the identity, *i.e.*, of the form $w^{-1} I$ with w some smooth positive function. The general approach appears in [1]. If we denote $\widetilde{\mathcal{L}_w}$ the matrix operator acting on smooth vector fields as

$$\widetilde{\mathcal{L}_w} = w^{-1}\,\widetilde{\mathcal{L}}\,(w\,\cdot),$$

then starting from the classical intertwining yields the following intertwining with weight:

$$w^{-1}\,\nabla L = \widetilde{\mathcal{L}_w}\,(w^{-1}\,\nabla).$$

Similarly to the classical intertwining, the matrix operator $\widetilde{\mathcal{L}_w}$ is of Schrödinger type, meaning that it admits the decomposition $\widetilde{\mathcal{L}_w} = \mathrm{diag}(L_w) - M_w$, with the diffusion operator

$$L_w = L + 2w^{-1}\,\langle \nabla w, \nabla \rangle,$$

and the matrix M_w corresponding to the multiplicative operator is given by

$$M_w = \nabla^2 V - \frac{Lw}{w}\,I.$$

Since the operator $-L$ is (essentially) self-adjoint on $L^2(\mu)$, then so is the operator $-L_w$ on the space $L^2(\mu_w)$, where $\mu_w = w^2 \cdot \mu$ is its invariant measure. Note that there is no reason *a priori* for μ_w to have finite mass.

To understand how those quantities come naturally into the analysis, let us consider the one-dimensional case of the real line, for which we provide a nice probabilistic interpretation. Actually, the intertwining with weight w is a composition of the classical intertwining with Doob's h-transform (with $h = w$): if $(P_t)_{t\geq 0}$ is the semigroup

associated to L, then by the classical intertwining (we omit to write the dependence with respect to the space variable),

$$\partial_x P_t f = \mathbb{E}\left[\partial_x f(X_t)\,\exp\left(-\int_0^t \partial_x^2 V(X_s)\,ds\right)\right],$$

where $(X_t)_{t\geq 0}$ is the diffusion process with generator L. Hence by Girsanov theorem, the latter rewrites as

$$\partial_x P_t f = \mathbb{E}\left[\partial_x f(X_{w,t})\,\exp\left(-\int_0^t \partial_x^2 V(X_{w,s})\,ds\right)\,M_t^{(w)}\right],$$

where now $(X_t^{(w)})_{t\geq 0}$ stands for the diffusion process with generator

$$L_w = L + 2w^{-1}\partial_x w\,\partial_x = \partial_x^2 - \partial_x\left(V + \log(w^{-2})\right)\,\partial_x,$$

and $(M_t^{(w)})_{t\geq 0}$ is the Girsanov martingale

$$\begin{aligned} M_t^{(w)} &= \frac{w}{w(X_{w,t})}\,\exp\left(-\int_0^t L_w(w^{-1})\,w\,(X_{w,s})\,ds\right) \\ &= \frac{w}{w(X_{w,t})}\,\exp\left(+\int_0^t \frac{Lw}{w}\,(X_{w,s})\,ds\right). \end{aligned}$$

Therefore the intertwining with weight w rewrites at the level of the semigroups as

$$\begin{aligned} w^{-1}\partial_x P_t f &= \mathbb{E}\left[(w^{-1}\partial_x f)(X_{w,t})\,\exp\left(-\int_0^t \left(\partial_x^2 V - \frac{Lw}{w}\right)(X_{w,s})\,ds\right)\right] \\ &= P_{w,t}^{M_w}(w^{-1}\partial_x f), \end{aligned}$$

where $(P_{w,t}^{M_w})_{t\geq 0}$ is the (Feynman-Kac) semigroup associated to $\widetilde{L_w} = L_w - M_w$.

Once the intertwining has been introduced, let us give an idea of the results we are able to obtain through this approach. We do not give any details of the proof below but we can observe that the matrix M_w plays an important role for the estimation of the spectral gap. To fix the notation, J is the Jacobian matrix and $\rho(A)$ denotes the smallest eigenvalue of a given symmetric matrix A.

Theorem 0.1. *On a connected domain $\Omega \subset \mathbb{R}^d$ ($d \geq 2$) with smooth boundary $\partial\Omega$ and outer unit-normal η, we consider a probability measure μ whose Lebesgue density is proportional to e^{-V}, where V is some sufficiently smooth potential. Let w be some smooth positive function satisfying the two following assumptions:*

- (A_1) *For all $x \in \Omega$, the symmetric matrix $\nabla^2 V(x) - \frac{Lw(x)}{w(x)}\,I$ is bounded from below (in the sense of symmetric matrices) by some positive constant.*
- (A_2) *For all $x \in \partial\Omega$, the symmetric matrix $J\eta(x) + \frac{\langle \nabla w(x), \eta(x)\rangle}{w(x)}\,I$ (acting on the tangent space at point x) is non-negative.*

Then the following Poincaré type inequality holds: for all sufficiently smooth function g,

$$\mathrm{Var}_\mu(g) \leq \int \left\langle \nabla g,\, \left(\nabla^2 V - \frac{Lw}{w}\,I\right)^{-1} \nabla g \right\rangle d\mu.$$

In particular the spectral gap satisfies

$$\lambda_1(\Omega,\mu) \geq \inf_{x\in\Omega} \rho\left(\nabla^2 V(x) - \frac{Lw(x)}{w(x)}\,I\right).$$

The Poincaré type inequality above is called a generalized Brascamp-lieb inequality in [1] since it extends the classical Brascamp-Lieb inequality arising in convex geometry, cf. [5] (the usual one corresponding to the choice of a constant weight function w in our inequality).

Our result covers various situations as soon as we are able to find some function w satisfying the announced assumptions.

– On the one hand we are able to obtain convenient spectral gap estimates for interesting examples of log-concave probability measures on a convex body satisfying some positive curvature assumption, cf. [4]. Recall that a convex body is a compact, convex set of $\mathbb{R}^d$ with non-empty interior. A first case we have in mind deals with a vanishing potential V, so that L is the Laplacian and μ is the uniform distribution on Ω. Another case of interest is when the probability measure μ has some symmetries, for instance radial symmetry, meaning that the associated potential V only depends on the Euclidean norm $|\cdot|$ of the space variable.

– On the other hand we address some specific non-convex situations such as perturbed product measures on $\mathbb{R}^d$ arising in statistical mechanics, for which the interaction potential is far from being convex, cf. [3]. Moreover we are currently working on punctured convex domains, *i.e.*, convex domains with a convex hole, for which the method seems to be promising. It will be the matter of a forthcoming paper.

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ON FINITE INTERWEAVING RELATIONS

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Classification AMS 2020: 60J10, 60J22, 15B51, 15A20, 15A18.

Keywords: Intertwining relations, interweaving relations, finite state space transition kernels, generalised spectral decompositions, Jordan blocks, warming-up times, strong stationary times.

In [6], with Pierre Patie, we introduced interweaving relations, between the square Bessel semigroup $(Q_t^{(\beta)})_{t\geqslant 0}$ of index $\beta-1>-1$ (the associated semigroup is $\partial_x^2+\beta\partial_x$ for $x\in[0,+\infty)$) and the analogous birth-and-death semigroup $(\widetilde{Q}_t^{(\beta)})_{t\geqslant 0}$ whose generator is $(n+\beta)\partial_+ + n\partial_-$, for $n\in\mathbb{Z}_+$ and where ∂_+, ∂_- are respectively the jump generators to the left and right neighbour. More precisely, we have, for any $t\geqslant 0$,

$$\begin{aligned} Q_t^{(\beta)}\Lambda &= \Lambda\widetilde{Q}_t^{(\beta)} \\ \widetilde{Q}_t^{(\beta)}\widetilde{\Lambda} &= \widetilde{\Lambda}Q_t^{(\beta)} \\ \Lambda\widetilde{\Lambda} &= Q_1^{(\beta)} \\ \widetilde{\Lambda}\Lambda &= \widetilde{Q}_1^{(\beta)} \end{aligned}$$

where Λ and $\widetilde{\Lambda}$ are respectively the Markov kernels from $[0,+\infty)$ to $\mathbb{Z}_+$ and from $\mathbb{Z}_+$ to $[0,+\infty)$ given by:

- For any $x\in[0,+\infty)$, $\Lambda(x,\cdot)$ is the law of a Poisson random variable of parameter x.
- For any $n\in\mathbb{Z}_+$, $\widetilde{\Lambda}(n,\cdot)$ is the law of a standard Gamma random variable of shape parameter $n+\beta$.

The interest of these relations is that they enable to transfer informations from one of the semi-group to the other one, up to a deterministic warming time whose value is 1.

Our goal in this talk based on [5] is to investigate if such relations are common or not, in the finite context to begin with. Worse, we will only consider Markov kernels P and $\widetilde{P}$ defined on the same finite state space V.

We need some definitions:

Intertwining from P to $\widetilde{P}$:

$$P\Lambda = \Lambda\widetilde{P}$$

where the **link** Λ is another Markov kernel on V. When Λ is invertible, the relation is said to be **faithful**.

Bi-intertwining relation between P and $\widetilde{P}$, when in addition:

$$\widetilde{P}\widetilde{\Lambda} = \widetilde{\Lambda}P$$

with another link $\widetilde{\Lambda}$ from V to V.

Strengthening of bi-intertwining relations: **interweaving relations**, when furthermore there exists a probability distribution $q = (q_n)_{n\in\mathbb{Z}_+}$ on $\mathbb{Z}_+$ such that

$$\Lambda\widetilde{\Lambda} = \sum_{n\in\mathbb{Z}_+} q_n P^n$$

It is a **bi-interweaving relation**, when for a probability distribution $\widetilde{q} = (\widetilde{q}_n)_{n\in\mathbb{Z}_+}$ on $\mathbb{Z}_+$,

$$\widetilde{\Lambda}\Lambda = \sum_{n\in\mathbb{Z}_+} \widetilde{q}_n \widetilde{P}^n$$

These relations are said to be **faithful** when Λ and $\widetilde{\Lambda}$ are invertible.

We begin by presenting two results showing that interweaving relations relations are quite common:

Theorem 1. *Assume that P and $\widetilde{P}$ are irreducible and similar. Then there exists a faithful bi-interweaving relation between them, with a probability q whose support contains at most $m+1$ points, where m is the common period of P and $\widetilde{P}$. Thus when P is aperiodic, there exists a faithful bi-interweaving relation between P and $\widetilde{P}$ with a probability $q = \widetilde{q}$ having a support with at most two points. When in addition of aperiodicity, we assume that none of the eigenvalues of P vanishes, then there exists a faithful bi-interweaving relation between P and $\widetilde{P}$ with q a Dirac mass.*

Next, rather assume P and $\widetilde{P}$ are similar and non-transient kernels. Denote by C_1, C_2, ..., C_ℓ (respectively $\widetilde{C}_1$, $\widetilde{C}_2$, ..., $\widetilde{C}_\ell$) the irreducible classes of P (resp. $\widetilde{P}$). They are in the same number $\ell \in \mathbb{N}$, because this is the multiplicity of the eigenvalue 1. For all $l \in [\![\ell]\!] := \{1, 2, ..., \ell\}$, denote P_{C_l} (resp. $\widetilde{P}_{\widetilde{C}_l}$) the restriction of P (resp. $\widetilde{P}$) to C_l (resp. $\widetilde{C}_l$).

Theorem 2. *There exists a faithful bi-interweaving relation between P and $\widetilde{P}$ if and only if there exists a permutation $\sigma \in \mathcal{S}_\ell$ and a probability $q = \widetilde{q}$ on $\mathbb{Z}_+$ such that for any $l \in [\![\ell]\!]$, $|C_l| = |\widetilde{C}_{\sigma(l)}|$ and there is a faithful bi-interweaving relation between P_{C_l} and $\widetilde{P}_{\widetilde{C}_{\sigma(l)}}$ with the same probability q. It can furthermore be imposed that q has a finite support.*

By contrast, two non-transient Markov matrices P and $\widetilde{P}$ are similar if and only if there exists a faithful bi-intertwining relation between them. Thus there is a faithful bi-intertwining relation but no faithful bi-interweaving relation between

$$P := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 1/3 \end{pmatrix} \quad \widetilde{P} := \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$

The structure of the state space deduced from the Markov kernel is somewhat more preserved by interweaving relations than by intertwining relations.

Extending the proofs of the previous results to the "degenerate" framework where one of the Markov kernel is absorbing, we recover a result due to Matthews [4] relating the construction of strong stationary times and the spectrum, in the context of reversible Markov kernels (with non-negative eigenvalues). This point of view and the concept of spectral models are next extended under the hypothesis that the considered Markov kernels only admits non-negative eigenvalues (with Jordan blocks of any

dimension). Finally we end the talk by mentioning the extensions to the easier continuous-time framework.

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DISCRETE WHITTAKER PROCESSES

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Classification AMS 2020: primary 60J10, 60K35; secondary 05E05, 60B20, 82B23.

Keywords: Whittaker functions, Toda lattice, reverse plane partitions.

Given an integer partition λ, a reverse plane partition π with shape λ is a filling of λ with non-negative integers $(\pi_{ij},\ (i,j) \in \lambda)$ which is weakly increasing across rows and down columns. Let $\mathrm{RPP}(\lambda)$ denote the set of reverse plane partitions of shape λ. Fix λ, and consider the Markov chain on $\mathrm{RPP}(\lambda)$, defined as follows: for each $(i,j) \in \lambda$, subtract one from π_{ij} at rate

$$b_{ij}(\pi) = (\pi_{ij} - \pi_{i,j-1})(\pi_{ij} - \pi_{i-1,j}),$$

with the convention $\pi_{i,0} = \pi_{0,j} = 0$. The infinitesimal generator of this Markov chain is given by the difference operator

$$G^\lambda = \sum_{(i,j)\in\lambda} b_{ij}(\pi) D_{\pi_{ij}},$$

where D_n is the backward difference operator $D_n f(n) = f(n-1) - f(n)$.

If π is a Markov chain on $\mathrm{RPP}(\lambda)$ with generator G^λ and $\mu \subset \lambda$, then the restriction of π to μ is a Markov chain on $\mathrm{RPP}(\mu)$ with generator G^μ. In particular, the first row of π is a Markov chain in its own right, and it is natural to think of it as an interacting particle system on the non-negative integers: the values $n_j := \pi_{1j},\ j = 1, \dots, \lambda_1$ are the positions of λ_1 particles; the left-most particle at position n_1 jumps to the left at rate n_1^2, while for each $j > 1$, the particle at position n_j jumps to the left at rate $n_j(n_j - n_{j-1})$. In fact, this interacting particle system is closely related to a discrete (repulsive) delta Bose gas.

We may also consider restrictions of Markov chain on $\mathrm{RPP}(\lambda)$ with generator G^λ to certain skew diagrams λ/μ. For this we require that $\mu \subset \lambda^\circ$, where λ° denotes the set of $(i,j) \in \lambda$, such that $(i+1,j) \in \lambda$ and $(i,j+1) \in \lambda$. Remarkably, if the initial law on $\mathrm{RPP}(\lambda)$ is chosen correctly, then $\pi|_{\lambda/\mu}$ will evolve as a Markov chain in its own right.

The simplest non-trivial example is related to Vandermonde's identity

$$\binom{n+m}{n} = \sum_k \binom{n}{k}\binom{m}{k}.$$

Let $\lambda = (2,1)$, $\mu = (1)$, and write $\pi_{11} = k$, $\pi_{12} = n$, $\pi_{21} = m$. In this notation,

$$G^{(2,1)} = k^2 D_k + n(n-k)D_n + m(m-k)D_m.$$

Suppose that, at time zero, $\pi_{12} = n$, $\pi_{21} = m$ and π_{11} is chosen at random according to the probability distribution

$$p_{n,m}(k) = \binom{n+m}{n}^{-1}\binom{n}{k}\binom{m}{k}, \qquad 0 \le k \le n \wedge m.$$

Then, if π evolves according to $G^{(2,1)}$, the restriction $\pi|_{\lambda/\mu} = (\pi_{12}, \pi_{21})$ is also a Markov chain, in its own filtration, with generator

$$L = \frac{n^3}{n+m} D_n + \frac{m^3}{n+m} D_m.$$

More generally, if λ is the staircase shape $\delta_{r+1} = (r, r-1, \ldots, 1)$ and $\mu = \delta_r$, then the restriction $\pi|_{\lambda/\mu}$ represents the 'boundary values' $n_i = \pi_{i,r-i+1}$, $i = 1, \ldots, r$. Let us write $G^r = G^{\delta_{r+1}}$ in this case.

Theorem 0.1 ([1]). *Suppose that, initially, the conditional law of $\pi|_{\delta_r}$, given the boundary values $(\pi_{1r}, \ldots, \pi_{r1})$, is proportional to*

$$W_r(\pi) = \prod_{(i,j)\in\delta_r} \binom{\pi_{i,j+1}}{\pi_{ij}} \binom{\pi_{i+1,j}}{\pi_{ij}}. \tag{0.1}$$

Then, if π evolves according to G^r, the boundary values $(\pi_{1r}, \ldots, \pi_{r1})$ will evolve as a Markov chain on $\mathbb{Z}_+^r$ with generator

$$L^r = \sum_{i=1}^{r} \frac{A_r(n - e_i)}{A_r(n)} n_i^2 D_{n_i},$$

where

$$A_r(n) = \sum_{\pi\in \mathrm{RPP}(\delta_r):\ \pi|_{\delta_r} = n} W_r(\pi).$$

In [1] it is also shown that the Markov chain with generator G^r has a unique entrance law starting from $\pi_{ij} = +\infty$ for all $(i,j) \in \lambda$, and that the conclusion of the above theorem remains valid under this entrance law.

The generator L^r is closely related to a discrete quantisation of the $(r+1)$-particle Toda chain, and the numbers $A_r(n)$ are series coefficients of a particular fundamental $SL(r+1)$-Whittaker function, see [1] for more details.

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SPECTRUM OF SOME MARKOV SEMIGROUPS ON STEP-2 CARNOT GROUPS USING INTERTWINING

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Classification AMS 2020: Primary 35P05, 37A30, 60G51, 58J65; Secondary 60J25

Keywords: Carnot Groups, Generalized Fourier Transform, Hypoelliptic operator, Kirillov's Orbit Method, Lévy processes, Ornstein-Uhlenbeck semigroup, sub-Laplacian, Horizontal heat semigroup

In this work we used the intertwining relationship to develop spectral theory for some Markov processes defined on Carnot groups of step 2. One can define a sub-Riemannian structure on G such that the horizontal vector fields are given by the first layer of the Lie algebra $\mathfrak{g}$. The sub-Laplacian on G is defined as $\Delta_{\mathcal{H}} = \sum_{i=1}^{n} X_i^2$, where $X_1, \ldots, X_n$ is a basis of the first layer of $\mathfrak{g}$. In this work, we considered two classes of Markov semigroups. First, we introduced a family of non-local operators obtained by perturbation of $\Delta_{\mathcal{H}}$. For any Lévy process with generator A, we defined $\Delta_A = \Delta_{\mathcal{H}} + A_{\mathcal{V}}$, where $A_{\mathcal{V}}$ is the Lévy operator on G acting only in the vertical directions. We showed that any left-translation invariant Markov process on G whose horizontal projections coincides with the Brownian motion on euclidean spaces, must have the generator of the form Δ_A. Using the Fourier analysis on nilpotent Lie groups, we obtained intertwining relationship between Δ_A and some operators defined on euclidean spaces. This enabled us to get a complete description of the spectrum of Δ_A. Subsequently, we considered linear perturbations $\mathbb{L}_A = \Delta_A + D$, where D denotes the generator of the dilation semigroup on G. Denoting $\mathbb{L}_A$ by $\mathbb{L}$ when $A \equiv 0$, it is known that $\mathbb{L}$ is the generator of the Ornstein-Uhlenbeck (OU) semigroup on G. Francoise Lust-Piquard [1] proved that the spectrum of $\mathbb{L}$ is the set of all negative integers, a phenomenon also exhibited by OU operators on euclidean spaces. This surprising connection motivated us to search for intertwining relationships between the OU operators on G and euclidean spaces. One of our main results in this direction is the following:

Theorem 0.1. *The operator $\mathbb{L}_A$ introduced above generates an ergodic Markov semigroup on G. Moreover, there exists a Markov operator Γ_A such that $\mathbb{L}\Gamma_A = \Gamma_A\mathbb{L}_A$.*

With the above result, we proved isospectrality of the perturbations, that is, $\sigma(\mathbb{L}_A) = -\mathbb{N}_0$ for any A. In addition to that, we proved existence of Markov operators Π and Λ_A such that $\mathbb{L}_A\Pi = \Pi\widetilde{L}$ and $L\Lambda_A = \Lambda_A\mathbb{L}_A$, where $\widetilde{L}$ and $\widetilde{L}$ are diffusive OU operators on some euclidean spaces.

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MIXING & SCALING LIMITS OF THE AVERAGING PROCESS

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The averaging process on a graph is a continuous-space Markov chain, which is commonly interpreted as an opinion dynamics, a distributed algorithm, or an interface moving through a randomized sequence of deterministic local updates. Its dynamics goes as follows. Attach i.i.d. Poisson clocks to edges, and assign real values to vertices; at the arrival times of these clocks, update the values with their average. As time runs, the averaging process converges to a flat configuration, and one major problem in the field is that of quantifying the speed of convergence to its degenerate equilibrium in terms of characteristic features of the underlying graph. In this talk, after reviewing some basic properties, intertwining relations, and recent results on mixing times for the averaging process on general graphs, we then focus on the discrete d-dimensional torus, and on some finer properties of the process in this setting. We discuss some quantitative features (e.g., limit profile, early concentration and local smoothness), and look at nonequilibrium fluctuations, a particularly interesting problem in this degenerate context lacking a non-trivial notion of local equilibrium.

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INTERTWINING OF NON-SELF-ADJOINT MARKOV SEMIGROUPS

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Classification AMS 2020: 37A30, 47D07, 60G18, 35P05, 41A60, 60E07, 42C15, 40E05, 30D05, 44A20.

Keywords: excursion theory, intertwining, Krein theory, non-selfadjoint Markov semigroups, self-similarity, spectral theory, convergence to equilibrium, infinite divisibility, Laguerre polynomials, Bernstein functions, special functions

In this talk we consider general intertwining of Markov semigroups and the implications thereof. We discuss questions such as the simultaneous intertwining of different boundary conditions, the preservation of the local time and the validity of the weak Krein property in the case one of the intertwining semi-groups is a generalized diffusion. We also present the intertwining of the class of all generalized one-sided Laguerre semigroups with the classical Laguerre semigroup with emphasis on which properties easily transfer from the latter diffusion to the whole class of non-selfadjoint semigroups and which properties require additional efforts and we illustrate how we overcome these difficulties.

Here, we always assume that we have two Markov semigroups $P = (P_t)_{t\geq 0}$ and $Q = (Q_t)_{t\geq 0}$ on the same Lusin space $(E, \mathcal{E})$ pertaining to two strong Markov processes $X = (X_t)_{t\geq 0}, Y = (Y_t)_{t\geq 0}$. In addition, we take for granted the existence of two excessive measures m_P, m_Q for these semigroups which allow for their extension to the Hilbert spaces $L^2(m_P), L^2(m_Q)$ and guarantee the existence of adjoint semigroups $\hat{P}, \hat{Q}$ connected to two moderate Markov processes $\hat{X} = (\hat{X}_t)_{t\geq 0}, \hat{Y} = (\hat{Y}_t)_{t\geq 0}$. Also, we assume the existence of $\Lambda : \mathcal{D}_\Lambda \subseteq L^2(m_Q) \to L^2(m_P)$, which is bounded, linear, order preserving operator such that $\overline{\mathcal{D}_\Lambda} = L^2(m_Q), \overline{\Lambda\mathcal{D}_\Lambda} = L^2(m_P)$, i.e. Λ has dense domain and dense range. Regardless of whether $\mathbf{1}_E \in L^2(m_Q)$ or not we consider Λ to be mass preserving, i.e. $\Lambda\mathbf{1}_E = \mathbf{1}_E$. This Λ will typically be an intertwiner between semigroups which in this case would read off as: for all $t \geq 0$

$$P_t\Lambda = \Lambda Q_t \text{ on } \mathcal{D}_\Lambda \cup \mathbf{1}_E. \tag{0.1}$$

Finally, let $b \in E$ be regular for itself, i.e. with

$$T_b^X = \inf\{t > 0 : X_t = b\} \text{ and } T_b^Y = \inf\{t > 0 : Y_t = b\}$$

then

$$\mathbb{P}_b(T_b^X = 0) = \mathbb{P}_b(T_b^Y = 0) = 1.$$

We set for $q \geq 0, x \in E$,

$$\varphi_q^X(x) = \mathbb{E}_x[e^{-qT_b^X}] \text{ and } \varphi_q^Y(x) = \mathbb{E}_x[e^{-qT_b^Y}].$$

Since b is regular there are local and inverse local times[1] at b

$$\mathfrak{l}^X = (\mathfrak{l}_s^X)_{s\geq 0} \text{ and } \mathfrak{l}^Y = (\mathfrak{l}_s^Y)_{s\geq 0}$$

[1] $\tau_t^{\cdot} = \inf\{s > 0 : \mathfrak{l}_s^{\cdot} > t\}$

$$\tau^X = (\tau_t^X)_{t\geq 0} \text{ and } \tau^Y = (\tau_t^Y)_{t\geq 0}.$$

Since τ^X, τ^Y are potentially killed subordinators

$$\mathbb{E}_b[e^{-q\tau_1^X}] = e^{-\Phi^X(q)} \text{ and } \mathbb{E}_b[e^{-q\tau_1^Y}] = e^{-\Phi^Y(q)} \text{ for } q \geq 0.$$

Φ^X, Φ^Y are Bernstein functions, i.e. for X

$$\Phi^X(q) = \underbrace{\delta^X}_{\text{killing rate}} + \underbrace{\gamma^X q}_{\text{elasticity parameter}} + \int_0^\infty (1-e^{-qs}) \underbrace{\mu^X(ds)}_{\text{jump measure}}$$

with $\delta_X \geq 0, \gamma_X \geq 0$ and μ^X the jump measure of the subordinator. The same is valid for Φ^Y. Define the killed semigroups $P^\dagger, Q^\dagger$ via

$$P_t^\dagger f(x) = \mathbb{E}_x[f(X_t), t < T_b^X] \text{ and } Q_t^\dagger f(x) = \mathbb{E}_x[f(Y_t), t < T_b^Y],\ x \in E.$$

Then we can formulate our first theorem which is a slightly narrower than our result in [2, Theorem 2.1]

Theorem 0.1. *Under the hypothesis for Λ above and with the additional assumptions*

$$\Lambda \mathbf{1}_b(x) = \mathbf{1}_b(x); \hat{\Lambda}\mathbf{1}_b(x) = \mathbf{1}_b(x); \Lambda Q_t f(b) = Q_t f(b), \hat{\Lambda}\hat{P}_t f(b) = \hat{P}_t f(b),$$

where $\hat{\Lambda}$ is the adjoint of Λ and f ranges across the respective domains of $\Lambda, \hat{\Lambda}$ we have the equivalence of the following relations

(1) *$P_t^\dagger \Lambda f = \Lambda Q_t^\dagger f$ for all $t \geq 0$ and $f \in \mathcal{D}_\Lambda \cup \{\mathbf{1}_E\}$;*
(2) *$P_t \Lambda f = \Lambda Q_t f$ for all $t \geq 0$ and $f \in \mathcal{D}_\Lambda \cup \{\mathbf{1}_E\}$.*

They imply

(3) *$\varphi_q^X(x) = \Lambda \varphi_q^Y(x) - m_P$ a.e.;*
(4) *$\Phi^X \equiv \Phi^Y$, i.e. the local times at $\{b\}$ coincide.*

Here, we remark that the intertwining implies the identity of the inverse local times of the two Markov processes and since the killed semigroups can be thought of as minimal semigroups then the stated equivalence implies the simultaneous intertwining of different boundary conditions which are reflected in the potentially different P, Q that correspond to the same minimal semigroups. This is in sharp contrast for higher dimensional intertwining of Laplacians and the related semigroups pertaining to killed Brownian motions.

Next, we assume that Q is a generalized diffusion and $E = [0, \infty)$ with:

(1) generator $G = \frac{d}{dm_Q}\frac{d}{dx}$ where $m_Q(0) = 0, m_Q(\infty) = \infty$ and m_Q being non-decreasing;
(2) 0 is a regular boundary for Q.
(3) The killed semigroup $Q^\dagger = (Q_t^\dagger)_{t\geq 0}$ is defined as

$$Q_t^\dagger f(x) = \mathbb{E}_x[f(Y_t), T_0^Y > t].$$

(4) $Q^\dagger$ is self-adjoint with $\sigma(Q^\dagger) \subseteq \mathbb{R}$.

Then $Q^\dagger$ extends to $L^2(m_Q)$ and possesses the Krein property:

(1) for $f \in L^2(m_Q)$ and $t \geq 0$

$$Q_t^\dagger f = \int_{\sigma(Q^\dagger)} e^{-qt} dE_q^Y f,$$

where $E^Y = (E^Y_q)_{q\in\mathbb{R}}$ are orthogonal projections[2] on $L^2(m_Q)$;

(2) for $f, g \in L^2(m_Q)$

$$< dE^Y_q f, g >_{m_Q} = < f, h_q >_{m_Q} < g, h_q >_{m_Q} \nu_Y(dq).$$

We note that in [2] the possible generalized diffusions have less stringent conditions but for the sake of clarity and conciseness we have imposed the restrictions above.

For generalized diffusions the Bernstein function enjoys

$$\begin{aligned} -\ln \mathbb{E}_0(e^{-\tau^Y_1}) &= \Phi^Y(q) \\ &= \delta^Y + \gamma^Y q + \int_0^\infty (1 - e^{-qs})\mu^Y(ds) \end{aligned}$$

with

$$\mu^Y(ds) = m^Y(s)ds = \int_0^\infty e^{-sq}\nu^Y(dq)ds.$$

Then the following corollary is taken from [2, Corollary 2.6] and reflects the presence of the weak Krein property.

Corollary 0.2. *Let $P^\dagger_t\Lambda = \Lambda Q^\dagger_t$ as in the main theorem. Let in addition for all $q \in \sigma(Q^\dagger), E^Y_q : \mathcal{D}_\Lambda \mapsto \mathcal{D}_*$. Then for $f \in \mathcal{D}_\Lambda, g \in L^2(m_P)$*

$$\begin{aligned} P^\dagger_t \Lambda f &= \int_{\sigma(P^\dagger)} e^{-qt} dE^X_q \Lambda f \\ < dE^X_q \Lambda f, g >_{m_P} &= < f, h_q >_{m_Q} < g, \Lambda h_q >_{m_P} \nu^Y(dq). \end{aligned}$$

In the talk we also consider the spectral expansion of one-sided generalized Laguerre semigroups as developed in [3]. To introduce briefly the setting recall that the Laguerre semigroup $Q = (Q_t)_{t\geq 0}$ is generated by

$$Gf(x) = xf''(x) + (1 - x)f'(x), x > 0.$$

$m_Q(x) = \mathbf{1}_{x>0}e^{-x}dx$ is stationary law and

$$Q_t f = \sum_{n\geq 0} e^{-nt} < f, \mathcal{L}_n >_{m_Q} \mathcal{L}_n, \ \forall f \in L^2(m_Q), \ t \geq 0.$$

$(\mathcal{L}_n)_{n\geq 0}$ are the Laguerre polynomials

$$\mathcal{L}_n(x) = e^x \frac{1}{n!} \frac{d^n}{dx^n}(x^n e^{-x})$$

and form an orthonormal basis in $L^2(m_Q)$.

The generalized Laguerre semigroups have the properties

(1) $P_t : \mathbb{C}_0(\mathbb{R}^+) \mapsto \mathbb{C}_0(\mathbb{R}^+), \forall t \geq 0$, with $P_0 f = f$.

(2) $\exists\, m_P$ stationary law such that

$$m_P P_t f = m_P f, t \geq 0.$$

[2] $\lim_{q\uparrow r} E^Y_q = E^Y_r, \lim_{q\downarrow -\infty} E^Y_q = 0, \lim_{q\uparrow\infty} E^Y_q = Id, E^Y_q E^Y_r = E^Y_{\min\{q,r\}}$

(3) The deterministic space-time transform

$$(K_t)_{t\geq 0} = (P_{\log(1+t)} \circ d_{(1+t)})_{t\geq 0}$$

defines a Feller semigroup[3], where $d_c f(x) = f(cx)$.

Here, we assume in addition that the pertaining Markov process can jump downwards only. One of the main results taken from [3] states.

Theorem 0.3. *For any one-sided generalized Laguerre semigroup there exists a order preserving, bounded, injective, non-invertible linear operator* $\Lambda : L^2(m_Q) \mapsto L^2(m_P)$ *with dense range such that*

$$P_t\Lambda = \Lambda Q_t, \ \forall t \geq 0.$$

$\Lambda f(x) = \mathbb{E}[f(xI_\phi)]$*, where* $I_\phi = \int_0^\infty e^{-\eta_s} ds$*.*

We remark that I_ϕ is defined in [3] and is a special exponential functional of a subordinator related to the process behind the generalized Laguerre semigroup.

For the rest of the talk we presented how the semigroup P can be expanded with the help of the generalized Laguerre polynomials and the difficulties the description of the co-eigenfunctions posed. We discuss topics such hypocoercivity, threshold of spectral expansion and techniques employed in the derivation of the results. From technical perspective the biggest role is played by the so-called Bernstein-gamma functions which are thoroughly studied in [1]

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[3] K are semigroups of the positive 1-self-similar Markov processes $(X_t^{cx}) \stackrel{d}{=} (cX^x_{t/c})_{t\geq 0}$ with infinite life-time and an entrance law from zero m_P.

DUALITIES AND INTERTWININGS IN POPULATION GENETICS DIFFUSIONS AND BEYOND

DARIO SPANO

Mathematical population genetics has been an incredible culture broth for the recent developments of the modern theory of stochastic duality. Duality in genetics clarifies the intrinsic link between forward-in-time dynamics of a population's allele frequencies evolution and backward-in-time dynamics of the same population's ancestry. It has yielded a probabilistic insight into the spectral properties of both processes, and helped significantly the tractability of such processes for simulation and inference. I will review some aspects of stochastic duality - and related intertwining operators - playing a key role in the analysis of Wright-Fisher diffusion processes of population genetics and in some of their (not necessarily diffusive) generalisations, for which some open problems will be discussed.

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ANOTHER PERSPECTIVE FOR LIE SYMMETRIES WITH A VIEW TOWARDS SCALING LIMITS

FILIP STOJANOVIC

Classification AMS 2020: Primary 22D10, 35B06, 47D03; Secondary 35P05, 81S05

Keywords: Operator algebras, representation theory, self-adjoint operators, Sobolev spaces

1. MOTIVATION

This talk is based on the work [3] of Patie and the ongoing joint work [6] of the author with Patie.

The squared Bessel process of index 0 is the unique solution $(X_t)_{t\geq 0}$ to the SDE

$$dX_t = 2(dt + \sqrt{X_t}dW_t), \;\; t \geq 0,$$

and it is well-known that it possesses the following stochastic invariance properties

(1) $(g^{-1}X_{gt})_{t\geq 0} \stackrel{d}{=} (X_t)_{t\geq 0}$ for all $g > 0$ (self-similarity),

(2) $(t^2 X_{1/t})_{t\geq 0} \stackrel{d}{=} (X_t)_{t\geq 0}$ (time-inversion).

A natural question is if these (and, of course, the Markov property) are all of the invariance properties of the squared Bessel process.

This question has an affirmative answer, which is provided by Lie group symmetry methods as employed in [1] to obtain a complete description of the invariance properties of the Brownian motion. While powerful, these methods are limited to the analysis of diffusions because of the requirement of locality of the generator.

A dual, and perhaps more natural, question to ask is if given an invariance property, can one determine all Markov processes possessing this symmetry. Patie observed that when this question is considered from the viewpoint of Hilbert spaces, the analysis is not limited to diffusions, and it even generalizes beyond Markov processes. In the sequel, we provide this precise viewpoint and the answer to this dual question when the invariance property considered is self-similarity.

It is classical that a Feller process with semigroup $(P_t)_{t\geq 0}$ acting on $C_0(0,\infty)$ is *(1)-self-similar* if it satisfies the semi-invariance property

$$P_t = d_g^{-1} P_{g^{-1}t} d_g \tag{1.1}$$

for all $t \geq 0$, $g > 0$, where $d_g f(x) = f(gx)$ is the group of dilations; this goes back to the work of Lamperti [2], which also contains a characterization of all self-similar Feller processes on $(0,\infty)$. It was a fundamental observation of Patie in [3] that this invariance property (1.1) should be realized on the Hilbert space $\mathrm{L}^2(\mathbb{G})$ with $\mathbb{G} = (0,\infty)$, whereupon the group of dilations, subject to an appropriate normalization, will constitute a unitrary representation of the multiplicative group $(\mathbb{G},\times)$ on $\mathrm{L}^2(\mathbb{G})$. Deep results in respresentation theory, operator algebras, and spectral theory can then be used to answer the dual question above.

2. Main Results in the Self-Adjoint Setting

For a Markov process whose associated semigroup admits a self-adjoint generator B acting on $\mathrm{L}^2(\mathbb{G})$, [3] shows that the self-similarity of the process is equivalent to its generator B satisfying the fixed point property

$$B = \pi_g g^{-1} B \pi_g^* \tag{2.1}$$

for all $g \in \mathbb{G}$, where $\pi_{\mathbb{G}} = (\pi_g)_{g\in\mathbb{G}}$ is the strongly continuous unitary representation of $\mathbb{G}$ on $\mathrm{L}^2(\mathbb{G})$ given by $\pi_g f(x) = g^{-1/2} f(g^{-1}x)$. The fixed point property (2.1), which we remark interpolates between the Weyl commutation relation and the canonical commutation relation (appropriately translated from $\mathrm{L}^2(\mathbb{R})$ to $\mathrm{L}^2(\mathbb{G})$), can be satisfied by arbitrary linear operators on $\mathrm{L}^2(\mathbb{G})$, and we say that any operator satisfying (2.1) is self-similar. Our main results consist of the complete description of self-adjoint self-similar linear operators on $\mathrm{L}^2(\mathbb{G})$.

The representation $\pi_{\mathbb{G}}$ induces an action of $\mathbb{G}$ on the set $\mathbf{S}(\mathrm{L}^2(\mathbb{G}))$ of possibly unbounded self-adjoint operators on $\mathrm{L}^2(\mathbb{G})$

$$s_{\pi_{(\cdot)}} : \mathbb{G} \to \mathrm{Aut}(\mathbf{S}(\mathrm{L}^2(\mathbb{G})))$$

given by

$$s_{\pi_g}(B) = \pi_g g^{-1} B \pi_g^*$$

for all $g \in \mathbb{G}$ and $B \in \mathbf{S}(\mathrm{L}^2(\mathbb{G}))$. The self-similar self-adjoint operators on $\mathrm{L}^2(\mathbb{G})$ are those fixed by $s_{\pi_{\mathbb{G}}}$, and the set of all such operators is denoted

$$\mathbf{F}^{\mathbf{S}}_{\pi_{\mathbb{G}}} = \{B \in \mathbf{S}(\mathrm{L}^2(\mathbb{G})); \;\; s_{\pi_{\mathbb{G}}}(B) = B\}.$$

Their complete description is obtained in Theorem 2.5, which fundamentally relies on the classical Stone-von Neumann theorem and novel spectral theoretic decompositions of the unitary representation $\pi_{\mathbb{G}}$, its unitary commutant, and certain unitary intertwiners (see Proposition 2.4).

Definition 2.1. *We make use of the following notation for conjugation by an injective operator:*

$$\mathtt{Ad}_{\Lambda}(T) := \Lambda^{-1} T \Lambda,$$

where $T : \mathcal{D}\mathrm{om}(T) \subseteq \mathrm{H} \to \mathrm{H}$ is a densely defined operator on a Hilbert space H and $\Lambda : \mathrm{H} \to \mathrm{H}'$ is an injective operator between the Hilbert spaces H and H'.

Definition 2.2. *Let $B \in \mathbf{S}(\mathrm{L}^2(\mathbb{G}))$ be a self-adjoint operator with a spectral representation*

$$B = \mathtt{Ad}_U(\mathrm{M}_{f(\mathbb{R})}),$$

where $U : \mathrm{L}^2(\mathbb{G}) \to \mathrm{L}^2(\mathbb{R})$ is unitary and $\mathrm{M}_{m(\mathbb{R})} : \mathcal{D}\mathrm{om}(\mathrm{M}_{m(\mathbb{R})}) \subseteq \mathrm{L}^2(\mathbb{R}) \to \mathrm{L}^2(\mathbb{R})$ is the multiplication operator

$$\mathrm{M}_{m(\mathbb{R})} f(\xi) = m(\xi) f(\xi)$$

with $m : \mathbb{R} \to \mathbb{R}$ measurable and whose essential image is the spectrum of B. The functional calculus for B is the map

$$\mathrm{L}^\infty(\mathbb{R}) \ni u \mapsto u(B) := \mathtt{Ad}_{\mathfrak{O}}(\mathrm{M}_{u \circ m(\mathbb{R})}) \in \mathbf{B}(\mathrm{L}^2(\mathbb{G})),$$

which is independent of the choice of spectral representation for B.

Theorem 2.3 ([3]). *There is, up to composition by a unitary multiplication operator on* $\mathrm{L}^2(\mathbb{R})$*, a unique unitary operator* $\mathfrak{O} : \mathrm{L}^2(\mathbb{G}) \to \mathrm{L}^2(\mathbb{R})$ *such that*

$$\mathfrak{O}\pi_g f(\xi) = g^{i\xi}\mathfrak{O}f(\xi)$$

for all $f \in \mathrm{L}^2(\mathbb{G})$*,* $g \in \mathbb{G}$*,* $\xi \in \mathbb{R}$*. Moreover, the unitary representation is generated by the self-adjoint operator*

$$D = \mathtt{Ad}_{\mathfrak{O}}(\mathrm{M}_{\mathbb{R}}) \in \mathbf{S}(\mathrm{L}^2(\mathbb{G}));$$

that is,

$$\pi_g = g^{iD} \quad \forall g \in \mathbb{G}.$$

The Mellin transform is the unique operator $\mathfrak{O}$ exhibited in the above theorem. By restricting the operator onto $C_c^\infty(\mathbb{G})$, we recover the classical representation of the Mellin transform as an integral operator:

$$\mathfrak{O}|_{C_c^\infty(\mathbb{G})} f(\xi) = \int_0^\infty g^{i\xi - 1/2} f(g) dg.$$

It is also shown in [3] that the group of unitary operators that commute with $\pi_{\mathbb{G}}$ and the set of unitary operators that intertwine $\pi_{\mathbb{G}}$ with its adjoint are in bijection with the following set of unimodular functions on $\mathbb{R}$:

$$\mathbb{S}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C};\ \ f \text{ is measurable, } |f| = 1\}.$$

Proposition 2.4 ([3]). *The unitary commutant of* $\pi_{\mathbb{G}}$ *is the subgroup of unitary operators on* $\mathrm{L}^2(\mathbb{G})$ *commuting with* $\pi_{\mathbb{G}}$*:*

$$\mathbb{C}_\pi = \{\Lambda \in \mathbb{U}(\mathrm{L}^2(\mathbb{G}));\ \ \pi_{\mathbb{G}}\Lambda = \Lambda\pi_{\mathbb{G}}\}.$$

It is explicitly characterized using the functional calculus for D*:*

$$\mathbb{C}_\pi = \{u(D);\ \ u \in \mathbb{S}(\mathbb{R})\}.$$

The set of unitary intertwiners of $\pi_{\mathbb{G}}$ *and its adjoint group is the set of unitary operators that intertwine* $\pi_{\mathbb{G}}$ *with* $\pi_{\mathbb{G}}^*$*:*

$$\mathbf{I}_{\pi,\pi^*} = \{\mathcal{H} \in \mathbb{U}(\mathrm{L}^2(\mathbb{G})) : \pi_{\mathbb{G}}\mathcal{H} = \mathcal{H}\pi_{\mathbb{G}}^*\}.$$

It is the right (or, equivalently, left) coset of $\mathbb{C}_\pi$ *by the unitary involution* $J \in \mathbb{U}(\mathrm{L}^2(\mathbb{G}))$ *given by* $Jf(x) = x^{-1}f(x^{-1})$*:*

$$\mathbf{I}_{\pi,\pi^*} = \mathbb{C}_\pi J = J\mathbb{C}_\pi.$$

Theorem 2.5 ([3]). (1) *Let* $\mathcal{H}_0$ *denote the Hankel transform of order 0, the unitary operator on* $\mathrm{L}^2(\mathbb{G})$ *given by*

$$\mathcal{H}_0 = J\mathtt{Ad}_{\mathfrak{O}}(\mathrm{M}_{u_0(\mathbb{R})})$$

where $u_0 \in \mathbb{S}(\mathbb{R})$ *is the function*

$$u_0(\xi) = \frac{\Gamma(1/2 + i\xi)}{\Gamma(1/2 - i\xi)}$$

for all $\xi \in \mathbb{R}$*. The Hankel transform* $\mathcal{H}_0$ *belongs to* $\mathbf{I}_{\pi,\pi^*}$ *and it diagonalizes the generator of the squared Bessel process of order 0:*

$$B_0 = \mathtt{Ad}_{\mathcal{H}_0}(\mathrm{M}_{\mathbb{G}}).$$

(2) *The set of self-adjoint self-similar operators on* $\mathrm{L}^2(\mathbb{G})$ *is the orbit through of* B_0 *of* $\mathbb{C}_\pi$ *acting by conjugation:*

$$\mathbf{F}^{\mathbf{S}}_{\pi_{\mathbb{G}}} = \{\mathtt{Ad}_\Lambda(B_0);\ \ \Lambda \in \mathbb{C}_\pi\}.$$

Equivalently, it is the orbit through $\mathrm{M}_{\mathbb{G}}$ *of the partial group* $\mathbf{I}_{\pi,\pi^*}$ *acting by conjugation:*

$$\mathbf{F}^{\mathbf{S}}_{\pi_{\mathbb{G}}} = \{\mathtt{Ad}_{\mathcal{H}}(\mathrm{M}_{\mathbb{G}});\ \ \mathcal{H} \in \mathbf{I}_{\pi,\pi^*}\}.$$

Moreover, $\mathbf{F}^{\mathbf{S}}_{\pi_{\mathbb{G}}}$ *is in bijection with the equivalence classes of* $\mathbb{S}(\mathbb{R})$ *under the equivalence relation*

$$u \sim v \overset{def}{\iff} \exists \lambda \in \mathbb{C} \text{ with } |\lambda| = 1 \text{ such that } u = \lambda v.$$

(3) *Set* $\mathcal{H}_u = J\mathtt{Ad}_{\mathfrak{D}}(\mathrm{M}_{u(\mathbb{R})})$ *for some* $u \in \mathbb{S}(\mathbb{R})$*. Define the operator*

$$B_u = \mathtt{Ad}_{\mathcal{H}_u}(\mathrm{M}_{\mathbb{G}}) \in \mathbf{F}^{\mathbf{S}}_{\pi_{\mathbb{G}}}$$

and the function

$$\Psi_u(\xi) = \overline{u}(-\xi - i/2)u(-\xi - i/2)$$

for all $\xi \in \mathbb{R}$*, where we assume that* u *has an analytic continuation on a domain containing the strip* $\{z \in \mathbb{C} : |\Im(z)| \leq 1/2\}$*. It follows that*

$$\mathfrak{D}_- B_u f(\xi) = \Psi_u(\xi)\mathfrak{D}_+ f(\xi) \quad \text{on } \mathcal{D}\mathrm{om}(B_u),$$

where $\mathfrak{D}_\pm : \mathrm{L}^2(\mathbb{G}, g^{\mp 1}dg) \to \mathrm{L}^2(\mathbb{R})$ *are unitary operators that diagonalize the representations of* $\mathbb{G}$ *on* $\mathrm{L}^2(\mathbb{G}, g^{-1}dg)$ *and* $\mathrm{L}^2(\mathbb{G}, gdg)$*, respectively, given by*

$$\pi^+_g f(x) = f(g^{-1}x)$$

for all $f \in \mathrm{L}^2(\mathbb{G}, g^{-1}dg)$ *and*

$$\pi^-_g f(x) = g^{-1} f(g^{-1}x)$$

for all $f \in \mathrm{L}^2(\mathbb{G}, gdg)$*. More precisely,* $\mathfrak{D}_\pm$ *are characterized by the identities*

$$\mathfrak{D}_\pm \pi^\pm_g f(\xi) = g^{i\xi}\mathfrak{D}_\pm f(\xi)\ \ \forall f \in \mathrm{L}^2(\mathbb{G}, g^{\mp 1}dg).$$

(4) *Assuming the hypotheses of (3),* B_u *generates a Markov semigroup if and only if*

$$u(\xi) = \frac{W_\phi(i\xi + i/2)}{W_\phi(-i\xi + 1/2)}$$

for all $\xi \in \mathbb{R}$*, where* ϕ *is a Bernstein function and* $W_\phi(i\xi + 1) = \phi(i\xi)W_\phi(i\xi)$ *is the corresponding Bernstein-Gamma function introduced in* [5]*. Moreover, in the notation of (3),*

$$\Psi_u(\xi) = \phi(i\xi)\phi(-i\xi) = \Psi_u(0) + \frac{\sigma_u^2}{2}\xi^2 + \int_{\mathbb{R}^\times}(1 - \cos(\xi y))\Pi_u(dy),$$

where $\Psi_u(0), \sigma_u \geq 0$ *and* Π_u *is a symmetric Lévy measure.*

(5) *(*[2]*) Assuming the hypothesis of (3) and using the notation of (4), if* B_u *generates a Markov semigroup, then*

$$B_u|_{C^\infty_c(\mathbb{G})} f(x) = \frac{\sigma_u^2}{2} B_0 f(x) + \int_{\mathbb{R}^\times} f(xy)\frac{\Pi_u(dy)}{x} - \frac{\Psi_u(0)}{x} f(x)$$

but $C^\infty_c(\mathbb{G})$ *need not be a core for* B_u*.*

This provides a complete description of self-adjoint self-similar operators on $\mathrm{L}^2(\mathbb{G})$ in terms of their spectral representations along with classical representations (i.e. as pseudo-differential and integro-differential operators) of those operators in $\mathbf{F}^{\mathbf{S}}_{\pi_{\mathbb{G}}}$ that admit such descriptions. It is important to observe that the spectral representation of self-similar operators may be restricted to obtain other classical representations of the operators, whereas it is not possible in general to lift these classical representations to obtain a spectral representation. Indeed, the integro-differential representation of B_u in item (5) in general does not even uniquely define the operator because it fails to even specify it on a core, so recovering a spectral representation is impossible.

We also remark that [3] develops a rich theory of scaling limits and universality for self-adjoint self-similar operators.

3. Main Results in the Non-Self-Adjoint Setting

A complete description of the non-self-adjoint self-similar operators on $\mathrm{L}^2(\mathbb{G})$ analogous to the one obtained in [3] is currently unavailable. However, much progress has been made in [4], which shows that the generators of self-similar Markov semigroups on $\mathbb{G}$ are self-similar pseudo-differential operators weakly similar to the generator of the squared Bessel semigroup. The authors also leverage this intertwining to obtain a spectral representation of these self-similar operators on at least a dense subspace of their domains.

We pursue the description of non-self-adjoint self-similar operators on $\mathrm{L}^2(\mathbb{G})$ in [6] by making use of the algebraic insights developed in [3]. The representation theoretic and operator theoretic observations therein generalize to the non-self-adjoint setting, which enables us to construct non-self-adjoint self-similar operators directly in terms of their spectral representations, extending the results in [4] along the lines of a generalization of the Stone-von Neumann theorem for non-self-adjoint operators.

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A NUMERICAL SEARCH FOR INTERTWINING RELATIONS

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Classification AMS 2020: 60J10; 60J27; 65F99;

Keywords: Markov chain, probability kernel, intertwining.

We say that a probability kernel Q is intertwined on top of another probability kernel P if there exists a third probability kernel K (the "intertwiner") such that

$$PK = KQ. \tag{0.1}$$

Given a square probability kernel P of size $d \times d$, we are interested in the problem of finding probability kernels Q and K of the same size as P such that Q is intertwined on top of P with intertwiner K, and moreover Q is "as simple as possible". The basic idea is to define K_0 to be the identity matrix and then inductively define probability kernels $K_1, K_2, K_3, \ldots$ by

$$K_{t+1} := K_t + PK_t - K_t\mathcal{Q}(P, K_t) \qquad (t \geq 0), \tag{0.2}$$

for some cleverly chosen function $\mathcal{Q}$. If we are lucky, then there exist probability kernels K and Q such that

$$K_t \underset{t\to\infty}{\longrightarrow} K \quad \text{and} \quad \mathcal{Q}(P, K_t) =: Q_t \underset{t\to\infty}{\longrightarrow} Q, \tag{0.3}$$

which implies

$$PK - KQ = \lim_{t\to\infty} \bigl(PK_t - K_tQ_t\bigr) = \lim_{t\to\infty} \bigl(K_{t+1} - K_t\bigr) = 0, \tag{0.4}$$

leading to a solution of (0.1). We focus on a function $\mathcal{Q}$ whose choice is based on the following two guiding principles:

- We want the kernel Q to be as simple in possible, in the sense that as many as possible of its off-diagonal elements are zero.
- We put restrictions on K by requiring some of its off-diagonal elements to be zero.

More precisely, we fix a set

$$Z \subset \bigl\{(x, y) \in \{1, \ldots, d\}^2 : x \neq y\bigr\} \tag{0.5}$$

that has the interpretation that these are the off-diagonal elements of K that we want to be zero, and define $\mathcal{Q} = \mathcal{Q}_Z$ as follows. First, we set

$$\begin{aligned}\mathcal{K}_Z := \bigl\{K : K \text{ is a probability kernel of size } d \times d \\ \text{such that } K(x, y) = 0 \text{ for all } (x, y) \in Z\bigr\},\end{aligned} \tag{0.6}$$

and we let $\mathcal{C}_Z(P, K)$ denote the set

$$\begin{aligned}\mathcal{C}_Z(P, K) := \bigl\{Q : Q \text{ is a probability kernel of size } d \times d \\ \text{such that } K' := K + PK - KQ \in \mathcal{K}_Z\bigr\}.\end{aligned} \tag{0.7}$$

Date: August 22, 2024.

For given probability kernels P, K of size $d \times d$ such that $K \in \mathcal{K}_Z$, we then define $\mathcal{Q}_Z(P, K)$ by setting

$$\mathcal{Q}_Z(P, K) := \text{the unique minimiser of } Q \mapsto \sum_{x \neq y} Q(x, y) \text{ in } \mathcal{C}_Z(P, K), \tag{0.8}$$

where the sum runs over all $1 \leq x, y \leq d$ such that $x \neq y$. The idea of minimising this function is that we want as many as possible of the off-diagonal elements of Q to be zero. It is should be noted that a priori, it is not clear that this is a good definition since in general we do not know whether such a minimiser exists (since $\mathcal{C}_Z(P, K)$ could be empty) or whether the minimiser is unique.

In my talk, I presented numerical data for the evolution in (0.2) for probability kernels P on sets of the form $\{1, \ldots, d\}$ that satisfy:

(1) $P(x, y) = 0$ for all $|x - y| \geq 2$,
(2) $P(d, d) = 1$,
(3) the spectrum of P is contained in $[0, 1]$.

I considered two possible choices of the set Z from (0.5), namely

(1) $Z = \big\{(x, x + 1) : 1 \leq x < d\big\}$,
(2) $Z = \big\{(x, x + 1) : 1 \leq x < d\big\} \cup \big\{(x, x - 2) : 2 < x \leq d\big\}$.

For the first choice of Z, I recovered a discrete-time analogue of an intertwining for continuous-time Markov chains first described in [1]. For the second choice of Z I discovered an intertwining that seems to be new. A more precise description of the results and a collection of scripts that can be used to simulate (0.2) can be found in [2].

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PROJECTIONS OF THE ALDOUS CHAIN ON BINARY TREES: INTERTWINING AND CONSISTENCY

MATTHIAS WINKEL

Abstract: Consider the Aldous Markov chain on the space of rooted binary trees with n labelled leaves in which at each transition a uniform random leaf is deleted and reattached to a uniform random edge. Now, fix $1 \leq k \leq n$ and project the leaf mass onto the subtree spanned by the first k leaves. This yields a binary tree with edge weights that we call a "decorated k-tree with total mass n". We introduce label swapping dynamics for the Aldous chain so that, when it runs in stationarity, the decorated k-trees evolve as Markov chains themselves, and are projectively consistent over k. The construction of projectively consistent chains is a crucial step in our construction of the Aldous diffusion on continuum trees, which is the $n \to \infty$ continuum analogue of the Aldous chain. This is joint work with Noah Forman, Soumik Pal and Douglas Rizzolo.

Classification AMS 2020: 60C05, 60J80, 60J10.

Keywords: Aldous diffusion, down-up chain, intertwining, Markov chain on binary trees, Rémy tree growth, trees with edge-weights.

1. THE ALDOUS CHAIN

Denote by $\mathbb{T}_n^{\mathrm{pl}}$ the set of rooted binary n-leaf planar trees (and $(n-1)$ branchpoints of degree 3) and consider the following down-up transitions that define the "Aldous chain":

1. Remove a uniform leaf (together with the adjacent edge and branchpoint).
2. Reinsert at a uniform edge (reversing the above steps at the chosen edge).

Proposition 1.1 (Aldous)**.** *This Markov chain has uniform stationary distribution on* $\mathbb{T}_n^{\mathrm{pl}}$.

While this observation is elementary, Aldous [3] and Schweinsberg [10] actually showed that the relaxation time is of order n^2. As Aldous [1] had established a metric space scaling limit for uniform n-leaf trees, the Brownian continuum random tree (BCRT), and after finding some diffusive limits of projections, he conjectured the following.

Conjecture 1.2 (Aldous [2])**.** *This Markov chain has a diffusive scaling limit when taking* n^2 *steps per unit time, with the distribution of the BCRT as stationary distribution.*

This conjecture is delicate as the BCRT has a complicated local structure. One way to capture this is to consider contour functions of uniform planar trees. Their scaling limit is a Brownian excursion. Informally, the BCRT can be seen as the tree whose contour function is a Brownian excursion, where each local minimum of the Brownian excursion corresponds to a branch point of the BCRT, resulting in a dense set of branch points.

A version of this conjecture was resolved in [7], for unrooted trees and using a new space of algebraic trees that does not capture metric distances in the trees, just tree structure. Our approach to this conjecture [5] constructs the diffusively evolving BCRT

Date: 13 August 2024.

as a (simple) Markov process with continuous paths in the usual Gromov–Hausdorff–Prokhorov space of metric measure trees. [5] also includes a scaling limit result for a continuous-time version of the Aldous chain. Following the spirit of Aldous's approach to continuum random trees, a key element of our construction is to first build a system of projectively consistent evolution of trees spanned by $k \in \mathbb{N}$ leaves. The purpose of this talk, based on [4], is to build the corresponding projectively consistent Markov chains associated with the Aldous chain for all $k \in [n] = \{1, \dots, n\}$ and fixed $n \in \mathbb{N}$.

Aldous actually studied a version of this Markov chain on cladograms, which are leaf-labelled trees. Indeed, we will eventually also work in the set $\mathbb{T}_{[n]}$ of (non-planar) rooted binary trees with n leaves labelled $1, \dots, n$, but it is instructive to begin without labels.

2. PROJECTIONS

Fix a branchpoint and record the numbers of leaves ("sizes") in the component containing the root ("root component") and in the other two components, "left"/"right", or "top". This gives rise to induced transitions on vectors of length 3 with sum n:

1A. Reduce by 1 a component with probability proportional to its size.
2A. Increase by 1 a component with probability proportional to $2\times$size± 1.

Note that the root component has $2\times$size$+1$ edges, while the other two components have $2\times$size-1 edges, and this is the meaning we attach to the ± 1.

Problem 2.1. *Eventually, a top component reaches size 0.*

This corresponds to the branchpoint around which we decompose being removed. We address this by further decomposing the root component along the path between the root and the branchpoint ("spine"). In addition to 1A. and 2A., the induced transitions are:

1B. If a spinal component reaches size 0, remove it.
If a top component reaches size 0, replace it by the top spinal component.
2B. Or create a new size-1 component in each gap with probability proportional to 1.

Here "gap" refers to a position before the first, between two adjacent or after the last component in the vector capturing the sizes of spinal components. The gaps correspond to the edges on the spine, which are now not in any of the spinal components.

Problem 2.2. *Eventually, a top component reaches size 0, with 0 spinal components.*

From this degenerate state (all mass in one component), we can continue in a Markovian way (start again from a state chosen suitably at random). More generally, $\mathbb{T}^{\circ(n)}_{[k]} = \bigcup_{\mathbf{s}\in\mathbb{T}_{[k]}} \{\mathbf{s}\} \times \big\{((n_i)_{i\in[k]}, (n_{E,j})_{1\le j\le \ell_E, E\in \mathrm{IntEdge}(\mathbf{s})}) : n_i, n_j^E \ge 1, \ell_E \ge 0, \sum n_i + \sum n_j^E = n\big\}$ is the set of possible vectors of component sizes when decomposing a tree in $\mathbb{T}_{[n]}$ around the branchpoints of the tree spanned by $[k]$ and along the paths between branch points and to the root. We denote by $\rho_k^{\circ(n)} \colon \mathbb{T}_{[n]} \to \mathbb{T}^{\circ(n)}_{[k]}$ the natural projection and by $\kappa_n^{\{i,j\}}$ the distribution of $\rho_2^{\circ(n)}(T)$ for $T \sim \mathrm{Unif}(\mathbb{T}_{[n]})$ with labels 1 and 2 relabelled i and j.

1C. If a component n_h is reduced to 0 with 0 spinal component on the parent edge E, swap h with another label j and resample label j. Skip Step 2A.–2B.

Here, "swap" means swapping labels h and $j = \max\{a, b, h\}$ before the down-step, where a and b are the smallest labels on the subtrees in $\mathbf{s}$ respectively adjacent to h and adjacent to the parent edge of h in $\mathbf{s}$. We do this swap in order to achieve consistency of these

transition rules as k varies: if $h \le k' < a \le k$, then the component of h does not reduce to 0 in the process for k', and if $h \le k' < b \le k$, then there is a top spinal component. "Resample" means label j is reinserted in an up-step by increasing a component or a gap in the usual way. If the chosen component is a spinal component, turn it into the top component labelled j. If the chosen component is top component i, split it using $\kappa_{n_i}^{\{i,j\}}$.

For $k = n$, this gives a labelled version of the Aldous chain. Indeed, $\mathbb{T}_{[n]}^{\circ(n)} \cong \mathbb{T}_{[n]}$.

Proposition 2.3. *This $\mathbb{T}_{[n]}$-valued chain has uniform stationary distribution.*

We call it the "uniform chain". Note that all labels are treated differently according to their rank, but the stationary distribution turns out to still be uniform on $\mathbb{T}_{[n]}$.

Theorem 2.4. *Let $(T(m))_{m\ge0}$ be a stationary uniform chain. Then $(\rho_k^{\circ(n)}(T(m)))_{m\ge0}$ is a stationary Markov chain, which we call the "decorated k-tree chain".*

Furthermore, the Markov chains $(\rho_k^{\circ(n)}(T(m)))_{m\ge0}$, as $k \in [n]$ varies, form a system that is projectively consistent via the natural projections $\mathbb{T}_{[k+1]}^{\circ(n)} \to \mathbb{T}_{[k]}^{\circ(n)}$, $k \in [n-1]$.

3. Intertwining and the proof of Theorem 2.4

While the statement of Theorem 2.4 appears to call for an intertwining relation, $(T(m))_{m\ge0}$ and $(\rho_k^{\circ(n)}(T(m)))_{m\ge0}$ are actually not intertwined for $2 \le k \le n-2$. It turns out that the labels involved in the projection are important. The intermediate space $\mathbb{T}_{[k]}^{[n]} = \bigcup_{\mathbf{s}\in\mathbb{T}_{[k]}} \{\mathbf{s}\} \times \big\{((B_i)_{i\in[k]}, (B_j^E)_{1\le j\le \ell_E, E\in \mathrm{IntEdge}(\mathbf{s})}) : i \in B_i, \text{and } B_i \text{ and } B_j^E \text{ partition } [n]\big\}$ captures these labels in the natural projection $\rho_k^{[n]} \colon \mathbb{T}_{[n]} \to \mathbb{T}_{[k]}^{[n]}$.

Lemma 3.1. *$(T(m))_{m\ge0}$ and $(\rho_k^{[n]}(T(m)))_{m\ge0}$ are intertwined using the link kernel $\Lambda_{[k]}^{[n]}$ that grafts to $\mathbf{s}$ independent uniform trees labelled B_i and B_j^E.*

Proof of Theorem 2.4. By Lemma 3.1 and the Rogers–Pitman [9] intertwining criterion, $(\rho_k^{[n]}(T(m)))_{m\ge0}$ is a stationary Markov chain, and by the Kemeny–Snell/Dynkin criterion [6], the further projection to $\mathbb{T}_{[n]}^{\circ(n)} \cong \mathbb{T}_{[n]}$ is also a stationary Markov chain. □

4. Rémy tree growth

To prove Lemma 3.1, we appeal to Rémy tree growth [8], which uses up-steps from $\mathbb{T}_{[m]}$ to $\mathbb{T}_{[m+1]}$ to insert leaf $m+1$ as in Step 2. of the Aldous chain. Rémy exploited this:

Proposition 4.1. *The marginals of the Rémy tree growth chain are uniform on $\mathbb{T}_{[m]}$, $m \in \mathbb{N}$.*

If we run Rémy's tree growth chain starting from $\mathbf{s} \in \mathbb{T}_{[k]}$ and observe the growth of components, we find that each component follows the up-steps of Rémy tree growth hence growing uniform trees that are conditionally independent given their label sets.

Sketch proof of Lemma 3.1. Starting the uniform chain from $T(0) \sim \Lambda_{[k]}^{[n]}(\mathbf{t}_k, \cdot)$ for some $\mathbf{t}_k \in \mathbb{T}_{[k]}^{[n]}$ and making one transition to obtain $T(1)$, we have to show that the conditional distribution of $T(1)$ given $\rho_k^{[n]}(T(1)) = \mathbf{t}'_k \in \mathbb{T}_{[k]}^{[n]}$ is $\Lambda_{[k]}^{[n]}(\mathbf{t}'_k, \cdot)$.

Indeed, the conditional independence and uniformity of subtrees labelled B_i and B_j^E is preserved under down- and up-steps in all cases, the delicate case being when a label $j \in [k]$ resamples in Step 1C., but then it resamles into a component that is uniform and $\kappa_n^{\{i,j\}}$ splits it into more uniform components if resampling is into top component i. □

5. THE ALDOUS DIFFUSION

Let us finally return to our approach in [5] to construct the Aldous diffusion via a consistent system of trees spanned by $k \in \mathbb{N}$ leaves. In [5], we use spaces of decorated k-trees $\mathbb{T}_{[k]}^{\mathrm{cont}} = \bigcup_{\mathbf{s}\in\mathbb{T}_{[k]}} \{\mathbf{s}\} \times \Big\{((x_i)_{i\in[k]}, (\beta_E)_{E\in \mathrm{IntEdge}(\mathbf{s})}) \colon \sum x_i + \sum_E \sum_{U\in\beta_E} \mathrm{Leb}(U) = 1\Big\}$, where $x_i \geq 0$ can be seen as the proportion of the continuum of leaves in a component labelled i and β_E captures the proportions of an ordered collection of typically infinitely many spinal components in an interval partition with blocks U of corresponding lengths (Lebesgue measure $\mathrm{Leb}(U)$).

Making any of this more precise or indeed using the projectively consistent system of such decorated k-tree evolutions to obtain a continuous evolution of continuum trees is well beyond the scope of this talk. However, the role that intertwining plays in this setting, in combination with Dynkin's criterion is unchanged. The decorated $(k+1)$-tree evolution is intertwined with a decorated k-tree evolution that is enriched by marking the block to which $k+1$ is projected, and the latter evolution projects to the decorated k-tree evolution via Dynkin's criterion, for all $k \in \mathbb{N}$. For details, see [5].

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IMPURITY IN THE TOTALLY ASYMMETRIC SIMPLE EXCLUSION PROCESS

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Classification AMS 2020: 82C22, 60K35

Keywords: Interacting particle systems, Asymmetric Simple Exclusion Process (ASEP), impurity particle, coupling methods.

Since its introduction in the 1970s, the exclusion process has been a fundamental model for interacting particle systems [1], with significant importance in non-equilibrium statistical physics, including applications to transport phenomena such as traffic flow and molecular biology [2]. While the simplest case of the exclusion process, involving only identical particles, has been widely studied, much less is known when multiple interacting species of particles are present.

In this work, we focus on understanding the effect of the presence of a single particle of a different type, called an impurity particle. This question is motivated by both theoretical and application perspectives. We define our model where a usual particle hops forward to an empty site with rate 1, while an impurity hops to an empty site with rate α. Additionally, a particle can overtake an impurity with rate β. Denoting a particle by $\bullet$, an empty site by $\circ$, and an impurity by $*$, the dynamics of the model can be schematically represented by the following exchanges with their corresponding rates:

$$\bullet\circ \xrightarrow{1} \circ\bullet \qquad *\circ \xrightarrow{\alpha} \circ * \qquad \bullet * \xrightarrow{\beta} * \bullet\,. \tag{0.1}$$

For $\alpha = \beta = 1$, the impurity behaves as a usual second-class particle, which has well-known properties. A TASEP with a single second-class particle can be viewed as a TASEP with an extra site, where the site containing the second-class particle is replaced by a pair of sites, consisting of a hole on the left and a particle on the right. The second-class particle moves forward if the particle in the pair jumps forward, and moves backward if the hole is overtaken by another particle. This pair representation is thoroughly discussed in [3]. One of our contributions is a generalization of this pair representation to the case of an impurity. In a nutshell: One can replace an impurity by a pair consisting of a "special" hole followed by a "special" particle, where the special hole has a (backward) hopping rate β, meaning that if a particle encounters it, it will overtake it with rate β, while the special particle has a (forward) hopping rate α. Unlike the case with unity rates, this representation is highly non-intuitive. A major difference is that the special hole and the special particle, initially neighbors, will soon move away from each other. Yet surprisingly, if we examine the hole-particle pair in the new system, it exhibits the same statistics as the impurity in the original process. This tool will be defined rigorously in a future publication. We mention below two results obtained using it.

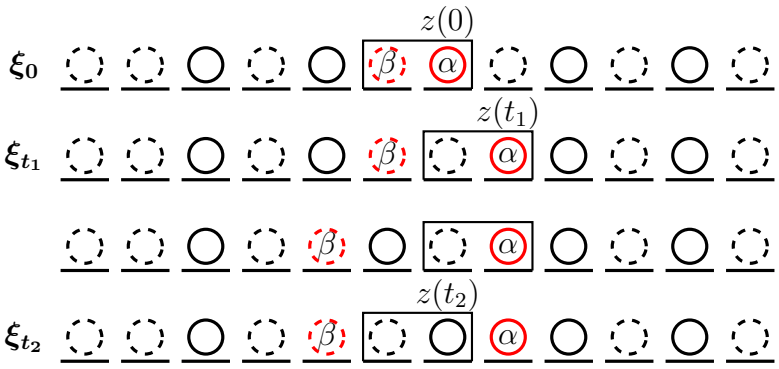

FIGURE 1. An illustration of the pair representation. The position of the box (denoted by $z(t)$) can be mapped to the position of an impurity with the same initial condition (outside the box)

Theorem 0.1. *Consider TASEP on the line with an initial uniform Bernoulli product measure with density ρ and a single impurity particle initially located at the site X_0. Define the asymptotic speed of the impurity particle as $v_* = \lim_{t\to\infty} \frac{X_t}{t}$. This limit has a deterministic value given by:*

$$v_* = \min\{\alpha, 1-\rho\} - \min\{\beta, \rho\}. \tag{0.2}$$

This result is already known in the literature and has been proved using integrable techniques such as the Matrix Product Ansatz [4] and Bethe Ansatz [5]. Our novelty is to provide a purely probabilistic, simpler proof based on the pair representation previously mentioned.

If we place the impurity at the interface separating two different densities, little is known about the system's behavior, except in the case $\alpha = \beta = 1$, where the impurity selects an asymptotic speed sampled from a uniform distribution in the interval $[1 - 2\rho^L, 1 - 2\rho^R]$, where ρ^L and ρ^R are the densities to the left and right of the impurity, respectively [6]. For general α and β, the following distinction holds depending on the value of $\alpha + \beta$:

- If $\alpha+\beta > 1$, the impurity does not impact the macroscopic behavior of the system but may still have a non-deterministic asymptotic speed.
- If $\alpha + \beta < 1$, the impurity induces a macroscopic perturbation in the TASEP environment, attempting to impose a density $1 - \alpha$ on its left and β on its right. It will have a deterministic speed.

In the following theorem, We generalise the result in [6] for the case of an impurity, in a special case of $\rho^L = 1$ and $\rho^R = 0$ and $\alpha + \beta < 1$

Theorem 0.2. *Consider TASEP on the line with only particles on the negative sites and only holes on the positive sites. Additionally, let an impurity particle with rates $\alpha, \beta \leq 1$ and $\alpha + \beta > 1$ be located initially at the origin. Let X_t denote its position at time t. Then,*

$$\frac{X_t}{t} \xrightarrow{P} U \quad \text{as} \quad t \to \infty,$$

where U is a random variable uniformly distributed in $[1 - 2\beta, 2\alpha - 1]$.

For the case $\alpha > 1$ in a $1 - 0$ step initial condition, the impurity particle has a non-zero probability of escaping the rarefaction fan from the right and moving freely with velocity

α in a void environment. We computed this escaping probability, denoted by R_1:

$$R_1 = \frac{\alpha - 1}{\alpha + \beta - 1} \mathbb{1}_{\alpha > 1}. \tag{0.3}$$

A similar phenomenon can occur from the left if $\beta > 1$, and the associated escaping probability is given by:

$$L_1 = \frac{\beta - 1}{\alpha + \beta - 1} \mathbb{1}_{\beta > 1}. \tag{0.4}$$

The remaining question is what happens to the impurity if does not escape the rarefaction fan. This is the subject of the following conjecture.

Conjecture 0.3. *Consider TASEP on the line with only particles on the negative sites and only holes on the positive sites. Let an impurity particle with rates $\alpha > 1$ and/or $\beta > 1$ be located initially at the origin. Conditioned on non-escaping, the impurity particle has an asymptotic speed sampled from a uniform distribution in the interval $[-1, 1]$.*

For arbitrary densities ρ^L and ρ^R with $\rho^L > \rho^R$, the distribution of the asymptotic speed is still an open question. However, it will have the following support:

$$U \in \left[\max(1 - 2\rho^L, 1 - 2\beta), \min(2\alpha - 1, 1 - 2\rho^R)\right].$$

This remark holds only for $\alpha + \beta > 1$. If $\alpha + \beta < 1$, the impurity will have a deterministic asymptotic speed, and the more interesting question concerns the macroscopic behavior of the TASEP environment, which will generally be impacted by the presence of the impurity. A rich phenomenology is observed as a function of the four parameters α, β, ρ^L, and ρ^R. The full classification will be available in a future publication. An initial preprint can be found in Chapter 5 of [7].

It is also possible to define the same model on the line where each site can either contain a usual particle with probability $\rho_\bullet$ or an impurity with probability ρ_*. The hydrodynamic behavior of this model was studied in [8], where the macroscopic limit shape was computed starting from a step initial condition. Numerical simulations [7] suggest an intriguing relationship between the macroscopic limit shape of this model in the limit of a vanishing density of impurities and the macroscopic limit shape of TASEP with a single impurity particle. The two coincide for some parameter values and differ for others. Explaining this phenomenon remains an open question.

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