SCIENTIFIC REPORTS

Singularities in Fluids and General Relativity

16 Dec 2024–10 Jan 2025

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STATIONARY SELF-SIMILAR PROFILES FOR THE TWO-DIMENSIONAL INVISCID BOUSSINESQ EQUATIONS

KEN ABE

Classification AMS 2020: 35Q31, 35Q35

Keywords: Self-similar solutions, Dubreil-Jacotin–Long equation, Minimax theorems, Singular elliptic problem

This talk is based on a joint work with D. Ginsberg and I.-J. Jeong [1]. We consider $(-\alpha)$-homogeneous solutions (stationary self-similar solutions of degree $-\alpha$) to the two-dimensional inviscid Boussinesq equations in a half-plane. We show their non-existence and existence with both regular and singular profile functions. More specifically, we demonstrate:

- Non-existence of rotational $(-\alpha)$-homogeneous solutions with regular profiles $(u,p,\rho)\in C^1(\overline{\mathbb{R}^2_+}\backslash\{0\})$ for $0\leq\alpha\leq 1$ and $(u,p,\rho)\in C^2(\overline{\mathbb{R}^2_+}\backslash\{0\})$ for $-1/2\leq\alpha<0$
- Existence of rotational $(-\alpha)$-homogeneous solutions with regular profiles $(u,p,\rho)\in C^2(\overline{\mathbb{R}^2_+}\backslash\{0\})$ for $\alpha>1$ and $(u,p,\rho)\in C^1(\overline{\mathbb{R}^2_+})$ for $\alpha<-2$
- Existence of rotational $(-\alpha)$-homogeneous solutions with x_1-symmetric singular profiles
  $(u,p,\rho)\in C^\infty(\overline{\mathbb{R}^2_+}\backslash\{x_1=0\}\cup\{x_2=0\})\cap C(\overline{\mathbb{R}^2_+})$ for $-1<\alpha<-1/2$ and $(u,p,\rho)\in C^\infty(\overline{\mathbb{R}^2_+}\backslash\{x_1=0\}\cup\{x_2=0\})$ for $-1/2\leq\alpha<1$

The $(-\alpha)$-homogeneous solutions with continuous profiles $(u,p,\rho)\in C^\infty(\overline{\mathbb{R}^2_+}\backslash\{x_1=0\}\cup\{x_2=0\})\cap C(\overline{\mathbb{R}^2_+})$ for $-1<\alpha<-1/2$ provide examples to self-similar weak solutions in $\mathbb{R}^2_+$ for the scaling exponent $\alpha\approx-0.657$, at which Wang et al. [2] numerically discovered the existence of backward self-similar solutions with smooth profile functions.

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Department of Mathematics, Graduate School of Science, Osaka Metropolitan University, 3-3-138 Sugimoto, Sumiyoshi-ku Osaka, 558-8585, Japan
Email address: kabe@omu.ac.jp

ENERGY ESTIMATES FOR SHOCK FORMATION TO THE FULL 3D COMPRESSIBLE EULER SYSTEM WITHOUT ASSUMING ADDITIONAL REGULARITY ON THE VORTICITY AND ENTROPY

LEONARDO ABBRESCIA

Classification AMS 2020: Primary: 35L67 - Secondary: 35L05, 35Q31, 74J40, 76N10

Keywords: Shock formation; Cauchy horizon; singular boundary; shock development problem; globally hyperbolic maximal development; compressible Euler equations; vectorfield method; eikonal function; null condition; null hypersurface; null structure

1. THE EQUATIONS AND MAIN RESULTS

This talk was concerned with the $3D$ compressible Euler equations, which may be expressed as

$$\mathbf{B}v^i = -c^2\partial_i \rho - \exp(-\rho)\overline{\varrho}^{-1}\frac{\partial p}{\partial s}\partial_i s, \qquad \mathbf{B}\rho = -\mathrm{div} v, \qquad \mathbf{B}s = 0, \tag{1.1}$$

where $v = (v^1, v^2, v^3)$ is the velocity field, $\rho = \ln(\varrho/\overline{\varrho})$ is the Logarithmic density and $\overline{\varrho}$ is a background density, s is the entropy, $p = p(\rho, s)$ is the pressure, $c = \sqrt{\overline{\varrho}^{-1}\exp(-\rho)\frac{\partial p}{\partial \rho}} > 0$ is the speed of sound, and $\mathbf{B} = \partial_t + v^i\partial_{x^i}$ is the material vectorfield. We denote the solution variables by $\vec{\Psi} = (v^1, v^2, v^3, \rho, s)$. We will analyze solutions on the spacetime $\mathbb{R} \times \Sigma$, where the spatial topology is given by $\Sigma = \mathbb{R} \times \mathbb{T}^2$. We denote the Euclidean coordinates on $\mathbb{R}\times\Sigma$ by $(t := x^0, x^1, x^2, x^3)$. The main result presented was the following.

Theorem 1.1. *Let $\vec{\Psi}^{\mathrm{PS}}|_{t=0} = \vec{\Psi}_0^{\mathrm{PS}}$ be isentropic ($s \equiv s_0$) plane-symmetric (constant in (x^2, x^3)) initial data for* (1.1) *whose fastest genuinely nonlinear characteristics form a shock singularity at time T_{shock}. Then there exists an N sufficiently large (which does not depend on $\vec{\Psi}_0$) and an open set of data that is not isentropic nor plane-symmetric satisfying $\|\vec{\Psi}_0 - \vec{\Psi}_0^{\mathrm{PS}}\|_{H^N(\{t=0\}\times\Sigma)} \lesssim \epsilon$, $\|\mathrm{curl} v, \partial s\|_{H^{N-1}(\{t=0\}\times\Sigma)} \lesssim \epsilon$ for which the perturbed solution launched by $\vec{\Psi}_0$ also forms a shock at time $T_{\text{shock}} + O(\epsilon)$ if ϵ is sufficiently small. In particular, the Cauchy data for vorticity $\omega = \mathrm{curl} v$ and gradient entropy $S^i := \partial_i s$ are one degree* less *regular than that of $\vec{\Psi}_0$.*

Theorem 1.1 is part of our much larger research program [1] to describe a full connected *portion* of the maximal globally hyperbolic development (MGHD) of the data up to its boundary in a vicinity near the shock, which is described below in Section 3. Describing the full MGHD remains an outstanding open problem in the field.

2. THE ACOUSTIC GEOMETRY AND WAVE–TRANSPORT FORMULATION

It has been known since the time of Riemann [12] that solutions to (1.1) develop shock singularities even from smooth initial data and in $1D$. However, it was not until Alinhac's foundational work [3, 4] '99 on quasilinear wave equations which fail the null condition, and Christodoulou's breakthrough '07 monograph for the relativistic Euler equations [6], that shock formation was provably constructed in multi-D and for data without symmetry. Shock formation is characterized by the infinitely dense accumulation of the genuinely nonlinear characteristics of (1.1), and hence also implies a Gradient blow-up of $\vec{\Psi}$. These results were proved in [3, 4, 6] for *irrotational* ($\text{curl}v = 0$) *isentropic* ($s \equiv s_0$) solutions of (1.1), which reduces the characteristics of the hyperbolic system to two and forces a quasilinear wave-like character to the flow.

For irrotational isentropic solutions, there exists a potential function ϕ such that $\partial\phi \sim \vec{\Psi}$ which satisfies the quasilinear wave equation $(\mathbf{g}^{-1})^{\alpha\beta}\partial_\alpha\partial_\beta\phi = 0$, where $\mathbf{g}$ is the *acoustical metric* that governs the geometry of propagating sound waves:

$$\mathbf{g} := -\mathrm{d}t \otimes \mathrm{d}t + c^{-2}\sum_{a=1}^{3}(\mathrm{d}x^a - v^a\mathrm{d}t) \otimes (\mathrm{d}x^a - v^a\mathrm{d}t). \tag{2.1}$$

We note that $\mathbf{g}$ is Lorentzian. One of the many innovations of Christodoulou [6] is that the gradient singularity can be "regularized" in a geometric coordinate system adapted to the characteristics. Namely, he introduced an acoustical eikonal function solving

$$(\mathbf{g}^{-1})^{\alpha\beta}\partial_\alpha u\partial_\beta u = 0, \tag{2.2a}$$

$$u|_{t=0} = -x^1. \tag{2.2b}$$

Importantly, the collapsing characteristics are given by level sets of u, which we denote by $\mathcal{P}_{u'} := \{u(t, x^1, x^2, x^3) \equiv u'\}$. By "regularized", we mean that $(\frac{\partial}{\partial t}, \frac{\partial}{\partial u}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3})\vec{\Psi}$ remains bounded in the *geometric coordinates* (t, u, x^2, x^3). One then recovers the shock singularity by composing the solution with the change of variables map $\Upsilon : (t, u, x^2, x^3) \mapsto (t, x^1, x^2, x^3)$. The shock singularity is characterized by the vanishing of the *inverse foliation density* $\mu := \frac{-1}{\mathbf{g}^{-1}(\mathrm{d}t,\mathrm{d}u)}$. Indeed, the chain rule implies $\partial_{x^\alpha}\vec{\Psi} \sim \frac{\partial}{\partial t}\vec{\Psi} + \frac{1}{\mu}(\frac{\partial}{\partial u} + \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3})\vec{\Psi}$ and so $\mu \to 0$ signifies gradient blowup.

Another major insight in Christodoulou's monograph [6] was to capitalize on the fact that variations $\delta\phi$ of ϕ (and hence at the level of the original solution variables) satisfy the *covariant* wave equation

$$\mu\square_{\mathbf{g}}\delta\phi = 0, \tag{2.3}$$

where $\square_{\mathbf{g}} = |\mathbf{g}|^{-1/2}\partial_\alpha|\mathbf{g}|^{1/2}(\mathbf{g}^{-1})^{\alpha\beta}\partial_\beta$ is the covariant wave operator of the acoustical metric. Crucially, because of the factor of μ in (2.3), Christodoulou observed that *geometric coordinates commute with* $\mu\square_{\mathbf{g}}$ ***without*** *introducing harmful factors of* μ^{-1}! Since his work in '07, every single known proof of shock formation which carefully tracks the characteristics, notably our joint work with Speck [1], Luk–Speck [9, 11], and Shkoller–Vicol [13], has used Christodoulou's pioneering ideas in a fundamental way: taking a variation of the main evolution equation, then multiplying by μ, then commuting with the geometric coordinates partial derivative vectorfields.

For the full compressible Euler system without irrotationality nor isentropicity assumptions, we cannot rely on (2.3) because no such potential function ϕ exists. We rely instead on the following geometric formulation of (1.1) which accounts for the advection of vorticity and entropy.

Theorem 2.1. *Let* $\omega^i := (\text{curl}v)^i$ *denote the vorticity and* $S^i := \partial_i s$ *denote the entropy gradient. Then* C^2 *solutions of* (1.1) *solve the following wave–transport system, written* *schematically:*

$$\Box_{\mathrm{g}} v^i \sim \mathrm{curl}(\omega)^i + \mathcal{Q}(\partial\vec{\Psi}, \partial\vec{\Psi}), \qquad \Box_{\mathrm{g}} \rho \sim \mathrm{div}(S) + \mathcal{Q}(\partial\vec{\Psi}, \partial\vec{\Psi}), \tag{2.4a}$$

$$\mathbf{B}\omega \sim \partial v \cdot \omega, \qquad \mathbf{B}S \sim \partial v \cdot S, \tag{2.4b}$$

where $\mathcal{Q}(\partial\vec{\Psi}, \partial\vec{\Psi})$ *are* g*-null forms* [10, 14]. *In fact, Speck–Yu* [15] *recently proved* (2.4) *implies* (1.1)*, and so the formulations of the equations are equivalent.*

It is easy to see that (2.4) naively suffers from a loss of derivatives because $\mathbf{B}\partial\mathrm{curl} \sim \partial^2 v$. In [11] and our work [1, 2] this was circumvented because $\mathrm{curl}(\omega)$ and $\mathrm{div}(S)$ solve a div–curl–transport system with miraculous null structure for which elliptic estimates are possible to derive. However, this requires ω and S to be *as regular as the velocity and entropy*. For the purposes of shock formation, this is not an issue. It *is* an issue the purposes of *shock development* which is the problem of describing the transition from a smooth solution, to one with a gradient singularity, to one where there is a *jump* in the solution variables across a shock hypersurface. The issue is that even though one can freely pose extra regularity assumptions on the ω and S on the Cauchy data to see the shock form, the regularity across the shock hypersurface of discontinuities is constrained by both the laws of thermodynamics and the Rankine–Hugoniot jump conditions and therefore not free. *This is why it is important that* ω *and* S *are one degree* less *regular than* $\vec{\Psi}$ *in Theorem 1.1*. Hence, we anticipate our Theorem 1.1 to be useful in the proof of shock development, which away from symmetry, has thus far only been solved by Christodoulou [7] for solutions which agree with Euler before the shock hypersurface, but for which the jump in vorticity and entropy is manually set to 0.

3. The maximal globally hyperbolic development

After the initial onset of a shock, due to finite speed of propagation for (1.1), it is natural to ask if one can solve the equations classically on other regions and if there exists a *largest* such region. The answer to this question is remarkably subtle and is deeply connected with the acoustical geometry of g. Since g is Lorentzian, the problem of classical uniqueness to the equations is well-posed on *globally hyperbolic* regions $\mathcal{M} \subset \mathbb{R} \times \Sigma$, namely, regions with a Cauchy hypersurface $\Sigma \subset \mathcal{M}$ on which the data is posed. We call $\mathcal{M}$ a *globally hyperbolic development* (GHD) of the data, and it is a *maximal globally hyperbolic development* (MGHD) if it is inextendible as a GHD of the data on Σ. From this definition, it is clear that spacetime points on which a shock singularities form constitute *boundary points* of the MGHD.

The notion of MGHD's was developed in the context of Einstein's equations of general relativity in Choquet-Bruhat–Geroch's historic work [5]. Roughly, they proved that if an initial data set satisfies the constraint equations, then there exists an MGHD of the data, and moreover, it is unique. To date, there is not a *single* result for *any* equation in *any* dimension which constructs an MGHD for smooth data which terminates due to the formation of shocks on its boundary. Worst yet, the breakthrough work of Eperon–Reall–Sbierski [8] gave examples of quasilinear wave equations for which there exists MGHD's $\mathcal{M}_1$ and $\mathcal{M}_2$ *of the same initial conditions* with $\mathcal{M}_1 \neq \mathcal{M}_2$. This means uniqueness is *not* guaranteed for quasilinear wave equations unlike [5]. The fundamental difference is that the compressible Euler equations (and the examples given in [8]) are posed on a fixed

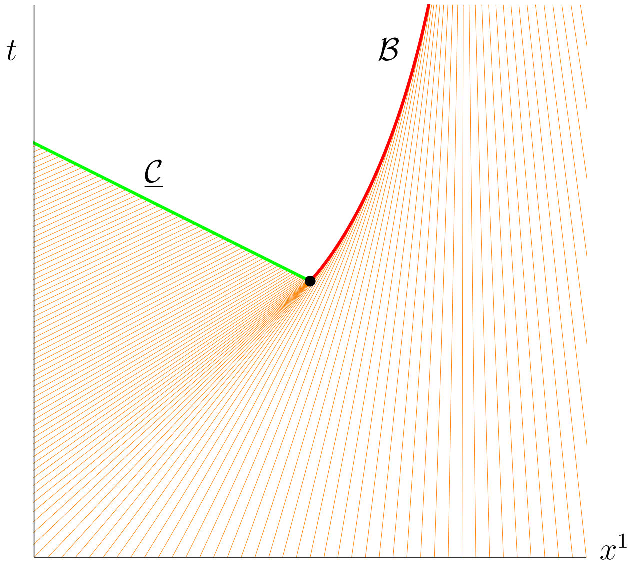

FIGURE 1. The MGHD for non-degenerate shock forming data with the (x^2, x^3) dimensions suppressed. The singular boundary $\mathcal{B}$ is depicted by the red curve and the Cauchy horizon is depicted by the green curve $\underline{\mathcal{C}}$. The crease is depicted by the black dot. The orange lines denote the level sets $\mathcal{P}_u$, which have infinite density along $\mathcal{P}_u$.

background manifold $\mathbb{R} \times \Sigma$, whereas for Einstein's equations, the differential structure of the spacetime is an unknown and solved uniquely alongside the equations [5].

For solutions launched by shock-forming data satisfying a non-degeneracy condition[1], the situation is not as bleak. In the same paper [8], they found that if an MGHD lies on one side of its boundary, then it is unique. We stress that this result is teleological in the sense that one must know the result holds on the full MGHD before one can prove uniqueness. Our prior work [1] with Speck proves that for solutions developed from an open set of non-degenerate shock forming data, there is a distinguished co-dimension 2 submanifold $\partial_-\mathcal{B}$ of spacetime called *the crease* on which the solution's gradient blows up (i.e. $\mu = 0$). Emanating from the crease in the shocking characteristic direction, we also proved in [1] that there is a hypersurface $\mathcal{B}$ called *the singular boundary* with dramatic causal degeneracies on which the solution's gradient continues to blow up. Finally, there is another hypersurface $\underline{\mathcal{C}}$ emanating from the crease in the remaining characteristic direction called *the Cauchy horizon* on which the solution remains bounded (i.e. $\mu > 0$), but is in the future causal domain of influence of the crease. Constructing the Cauchy horizon for the full $3D$ compressible Euler equations is ongoing work with Speck. The upshot is that $\mathcal{B} \cup \underline{\mathcal{C}} \cup \partial_-\mathcal{B}$ constitute a compact portion of the MGHD's boundary *which satisfies the geometric condition of* [8] that guarantees uniqueness, namely that the MGHD lies on one side of it, see Figure 1. Extending this result globally in order to truly prove uniqueness remains an outstanding open problem. Finally, we note that the first to predict the geometry of Figure 1 was Christodoulou in his '07 monograph [6].

[1]Roughly, the data for the plane-symmetric background solution must satisfy $\min \partial_{x^1} \vec{\Psi}^{\mathrm{PS}} < 0$, the minimum occurs at a unique point x_*, and $\partial^3_{x^1} \vec{\Psi}^{\mathrm{PS}}(x_*) > 0$.

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Georgia Institute of Technology
Email address: abbrescia@math.gatech.edu

KINETIC SHOCKS PROFILES FOR THE LANDAU EQUATION

DALLAS ALBRITTON

Classification AMS 2020: 35Q20, 35Q70, 82B40, 35Q35

Keywords: Kinetic theory, Landau equation, shock profile, Chapman-Enskog expansion

Compressible Euler solutions develop jump discontinuities known as *shocks*. However, physical shocks are not, strictly speaking, discontinuous; the transition region has an internal structure, the so-called *shock profile*, connecting two end states, which satisfy the Rankine-Hugoniot conditions.

Commonly, the shock profile is modeled via the compressible Navier-Stokes equations

$$\begin{cases} - s\partial_x \varrho + \partial_x(\varrho u) = 0 \\ - s\partial_x(\varrho u) + \partial_x(\varrho u^2 + p) = \partial_x \tau \\ - s\partial_x E + \partial_x(u(E+p)) = \partial_x(\kappa \partial_x \theta) + \partial_x(\tau u)\,, \end{cases} \tag{0.1}$$

written in a frame co-moving with the shock at speed s. There is a well developed theory of shock profile solutions to (0.1), going back to [1, 2]. Therefore, it is quite interesting that Navier-Stokes shock profiles do not closely match the experimental measurements for $\mathrm{Ma} \gtrsim 2$ [8]. It has been proposed that in this regime it is more accurate to describe shocks at level of collisional kinetic equations

$$(v_1 - s)_x F = Q(F,F)\,, \tag{0.2}$$

where $F = F(x, v_1, v_2, v_3)$, $x \in \mathbb{R}$, is a particle distribution function, and $Q(F,F)$ is a collision kernel. The choice of collision kernel depends on the particle interactions, with the Boltzmann hard-sphere interaction a common assumption. For hard spheres (and, in one instance [3], hard cut-off potentials), (0.2) was investigated by various authors [6, 3, 4, 5].

With a view toward plasmas, we will be interested in the Coulomb interaction and the associated Landau collision kernel

$$Q(F,G)(v) = \nabla_v \cdot \left(\int_{\mathbb{R}^3} \phi(v-u) \left(F(u) \nabla_v G(v) - G(v) \nabla_u F(u) \right) \, du \right) \tag{0.3}$$

$$\phi^{ij}(v) = \left(\delta^{ij} - \frac{v^i v^j}{|v|^2} \right) |v|^{-1}\,, \tag{0.4}$$

in which case (0.2) is known as the Landau equation.

Unfortunately, the PDE theory of (0.2) has been restricted to *weak shocks*, characterized by a small jump ε between the end states. This is because weak shocks are asymptotically-in-ε well approximated by compressible Navier-Stokes shock profiles. With this information, (0.2) might be a considered a (very!) singular perturbation problem.

Date: March 3, 2025.

In joint work [7] with Matthew Novack (Purdue University) and Jacob Bedrossian (UCLA), we demonstrate the existence of weak shock profiles to the kinetic Landau equation.

To formulate the main theorem, we introduce the following conventions. Let $s_0 = \sqrt{5/3}$. We normalize the left end state μ_L to be the Maxwellian with $(\varrho_L, u_L, \theta_L) = (1, \varepsilon, 1)$ and $0 < \varepsilon \ll 1$. There is a unique right end-state μ_R with $(\varrho_R, u_R, \theta_R)(\varepsilon)$ satisfying the Rankine-Hugoniot conditions such that the jump between end states is $O(\varepsilon)$.

Theorem 0.1. *Let $N \geq 8$ and $0 \leq q_0 < 1$. Let $0 < \varepsilon \ll_{N,q_0} 1$. There exists a smooth shock profile solution F to the Landau equations*

$$(v_1 - s_0)_x F = Q(F, F) \tag{0.5}$$

satisfying the decomposition

$$F = F_{\text{NS}} + \mu_0^{\frac{1}{2}} f \,, \tag{0.6}$$

where F_{NS}, properly defined in [7]*, is a 'lifting' to the kinetic setting of the unique, up to spatial translations, compressible Navier-Stokes shock profile $(\varrho_{\text{NS}}, u_{\text{NS}}, \theta_{\text{NS}})$. For any $0 \leq \delta \ll_{N,q_0} 1$, there exists a constant $C_0 = C_0(N, q_0) > 0$ such that for any $\alpha \in \mathbb{N}$ and multi-index $\beta \in \mathbb{N}^3$ with $|\alpha| + |\beta| \leq N$, the remainder f satisfies*

$$\left\| e^{\delta \langle \varepsilon x \rangle^{\frac{1}{2}}} \mu_0^{-q_0} \varepsilon^{-\alpha} \partial_x^\alpha \partial_v^\beta f \right\|_{L^2} \leq C_0 \varepsilon^2 \,. \tag{0.7}$$

Our methods are inspired by those in [5]. The main difficulty is to invert the linearized operator $(v_1 - s_0)_x - Q(\mu_{\text{NS}}, F) - Q(F, \mu_{\text{NS}})$, which is technically not invertible; it has a one-dimensional kernel due to translations. A crucial point, shared with [5], is to extract the linearized compressible Navier-Stokes operator from this operator via a linearized Chapman-Enskog expansion.

Throughout, the Landau operator (0.3) is technically more demanding than the Boltzmann collision kernel for hard spheres, primarily because the inverse of the linearized collision kernel loses moments in v. This is the reason for the stretched exponential decay in (0.7).

While the present work is once again restricted to small shocks, we hope that the mathematical study of kinetic shock profiles will one day attain the strong shock regime.

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University of Wisconsin-Madison, Department of Mathematics, 480 Lincoln Dr, Madison, WI 53706, USA
Email address: dalbritton@wisc.edu

LOW REGULARITY ILL-POSEDNESS FOR ELASTIC WAVES AND FOR MHD SYSTEM IN 3D AND 2D

HAOYANG CHEN

Classification AMS 2020: 35L05, 35L67, 35R25, 35Q35.

Keywords: Elastic waves, compressible ideal MHD system, low regularity ill-posedness, shock formation.

In this talk, we discuss the low regularity ill-posedness problems for some physical systems with multiple wave speeds. The first system is the elastic wave equations. The equations of motion for the displacement of an isotropic, homogeneous, hyperelastic material form a quasilinear hyperbolic system:

$$\partial_t^2 U - c_2^2 \Delta U - (c_1^2 - c_2^2)\nabla(\nabla \cdot U) = N(\nabla U, \nabla^2 U), \tag{0.1}$$

with $U = (U^1, U^2, U^3)$ and wave speeds $c_1 > c_2 > 0$. The quadratic nonlinear terms $N(\nabla U, \nabla^2 U)$ are like $\nabla U \nabla^2 U$.

Another physical system is the compressible ideal MHD system, which is composed of the Euler equations and the coupled Maxwell's equations, and it forms a quasilinear hyperbolic system (3D ideal compressible MHD):

$$\begin{cases} \partial_t \varrho + \nabla \cdot (\varrho u) = 0, \\ \varrho\{\partial_t + (u \cdot \nabla)\}u + \nabla p + \mu_0 H \times \text{rot} H = 0, \\ \partial_t H - \text{rot}(u \times H) = 0, \\ \partial_t S + (u \cdot \nabla) S = 0, \\ \nabla \cdot H = 0. \end{cases} \tag{0.2}$$

Here, $\mu_0 \neq 0$ is a physical constant, ρ is the density, $u \in \mathbb{R}^3$ is the plasma velocity, $H \in \mathbb{R}^3$ is the magnetic field, S is the entropy and p is the pressure satisfying the polytropic equation of state $p = Ae^S \varrho^\gamma$ with $A > 0$ and $\gamma > 1$ positive constants.

Our research is inspired by studies on scalar wave equations. The sharp ill-posedness results for semilinear and quasilinear wave equations were established by Lindblad in [6][8] and in [7], respectively. Our results generalizes Lindblad's work on scalar wave to physical systems with multiple wave speeds. In particular, we proved in [1] the following result for 3D elastic waves:

Theorem 0.1. *The Cauchy problems of 3-dimensional elastic wave equations are ill-posed in H^3 in the following sense: There exists a class of compactly supported smooth initial data $(U_0^{(\eta)}, U_1^{(\eta)})$ with $\|U_0^{(\eta)}\|_{\dot{H}^3(\mathbb{R}^3)} + \|U_1^{(\eta)}\|_{\dot{H}^2(\mathbb{R}^3)} \to 0$ and a sequence of time $\{T_\eta\}$ such that T_η is the shock formation time, then $T_\eta \to 0$ as $\eta \to 0$.*

Moreover, in a spatial region $\Omega_{T_\eta^*}$ the H^2 norm of the solution to elastic waves (0.1) *blows up at shock formation time T_η^*:*

$$\|U_\eta(\cdot, T_\eta^*)\|_{H^2(\Omega_{T_\eta^*})} = +\infty.$$

For 3D compressible ideal MHD system, our result in [2] is stated as below:

Theorem 0.2. *The Cauchy problems of the 3D ideal compressible MHD equations* (0.2) *are ill-posed in* $H^2(\mathbb{R}^3)$ *in the following sense:*

There exists a family of compactly supported, smooth initial data $\Phi_0^{(\eta)}$ *satisfying*

$$\|\Phi_0^{(\eta)}\|_{\dot{H}^2(\mathbb{R}^3)} \to 0 \quad \text{as} \quad \eta \to 0,$$

where $\eta > 0$ *is a small parameter and it identifies the datum in this family. For each* η*, the Cauchy problem of the 3D ideal MHD system admits a solution that ceases to be regular in finite time. Let* T_η^* *be the largest time such that the solution* $\Phi_\eta \in C^\infty\big(\mathbb{R}^3 \times [0, T_\eta^*)\big)$*. The following statements hold:*

i) **(Instantaneous shock formation)** *Evolving from each initial datum* $\Phi_0^{(\eta)}$*, a shock forms at* T_η^**. And as* $\eta \to 0$*, we have* $T_\eta^* \to 0$*.*

ii) **(Blow-up of H^1-norm)** *For each solution* Φ_η*, with* $\Omega_{T_\eta^*}$ *being a spatial neighborhood of the first (shock) singularity, the* H^1*-norm* $\|\Phi_\eta\|_{H^1(\Omega_{T_\eta^*})}$ *blows up at* T_η^**. In particular,*

$$\|\partial_x u_1^{(\eta)}(\cdot, T_\eta^*)\|_{L^2(\Omega_{T_\eta^*})} = +\infty, \ \|\partial_x \varrho^{(\eta)}(\cdot, T_\eta^*)\|_{L^2(\Omega_{T_\eta^*})} = +\infty. \tag{0.3}$$

Furthermore, we extend the above results to 2D case in [3] and [4]. We prove the $H^{\frac{11}{4}}$ ill-posedness for the elastic waves and $H^{\frac{7}{4}}$ ill-posedness for ideal MHD system.

Moreover, for our MHD system, if the magnetic field vanishes, i.e., $H \equiv 0$, it reduces to the compressible Euler equations.

$$\begin{cases} \partial_t \varrho + \nabla \cdot (\varrho u) = 0, \\ \varrho\{\partial_t + (u \cdot \nabla)\}u + \nabla p = 0, \\ \partial_t S + (u \cdot \nabla) S = 0. \end{cases} \tag{0.4}$$

Our results above are also applicable for Euler equations, with instantaneous shock formation and H^1-norm inflation of the velocity u_1 and fluid density ϱ. By recent low regularity local well-posedness results of Disconzi-Luo-Mazzone-Speck [5], Wang [9], Zhang [10] and Zhang-Andersson [11], our ill-posedness results (H^2 for the 3D case, and $H^{\frac{7}{4}}$ for the 2D case) are sharp with respect to the regularity of the fluid velocity u and density ϱ.

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NATIONAL UNIVERSITY OF SINGAPORE
Email address: hychen@nus.edu.sg

FINITE TIME SINGULARITIES FOR INCOMPRESSIBLE FLUIDS.

DIEGO CÓRDOBA

Classification AMS 2020: 35Q31, 35B65, 76B03

Keywords: Euler equations, incompressible fluids, Boussinesq equations, Singularities

We begin by reviewing recent constructions of finite time singularities in the 3D incompressible Euler equations and the hypodissipative Navier-Stokes equations. Additionally, we present a mechanism for blow-up in the 2D Boussinesq equation, achieved through a multi-layer degenerate pendula with a uniform $C^{1,\alpha}$ forcing term.

1. Finite time blow-up results of classical solutions for 3D incompressible Euler equations.

Recently, various blow-up scenarios, where the singularity happens in the bulk of the fluid, have been developed within the locally well-posed regime $C^{1,\alpha}$ for the 3D incompressible Euler equations. Specifically, there are four known cases of blow-up involving finite energy and no boundary:

- The first blow-up result is due to Elgindi in [10] and Elgindi, Ghoul and Masmoudi in [11]. They construct self similar blow up for axi-symmetric flows and no swirl, focusing on velocity profiles within the $C^{1,\alpha}$ Hölder space, where α is chosen to be a very small number, and which are C^∞ almost everywhere.
- In our recent work in collaboration with Zheng [8], we presented a novel blowup mechanism for axi-symmetric flows without swirl that does not depend on self-similar profiles. These solutions remain smooth everywhere except at a singular point, where the solution is $C^{1,\alpha}$ and α being a very small value (see also [2]).
- In the recent study [6], we propose a blow-up mechanism for the forced 3D incompressible Euler equations, focusing specifically on non-axisymmetric solutions. We construct solutions in $\mathbb{R}^3$ within the function space $C^{3,\frac{1}{2}} \cap L^2$ over the time interval $[0,T)$, where $T > 0$ is finite, subject to a continuous force in $C^{1,\frac{1}{2}-\epsilon} \cap L^2$. This framework results in a blow-up: as time t approaches the critical value T, the integral $\int_0^t |\nabla u|\, ds$ diverges, indicating unbounded growth. Despite this, the solution remains smooth everywhere except at the origin. Importantly, our blow-up construction does not rely on self-similar coordinates and extends to solutions beyond the critical $C^{1,\frac{1}{3}+}$ regularity for global well-posedness in axi-symmetric flows without swirl. Furthermore, following the strategy of [6] we construct finite time singularities, in [9] in collaboration with Zheng, for the hypodissipative 3D Navier-Stokes equations with an external forcing which is in $L^1_t C^{1,\epsilon}_x \cap L^\infty_t L^2_x$.
- In a very recent study, Elgindi and Pasqualotto examined the case of axi-symmetric flows with swirl and the 2D Boussinesq system (both systems are

strongly related). In [13], they prove a self-similar blow-up profile for solutions in $C^{1,\beta}$ (with β very small), where the singularity occurs away from the origin. The solutions they considered belong to $H^{2+\delta}$, which places them above the critical regularity for axi-symmetric solutions, in a stronger sense than the usual $C^{1,\beta}$ solutions discussed in previous works on self-similar profiles.

2. Blow-up scenario for Boussinesq with forcing.

Here we consider the forced inviscid incompressible 2D Boussinesq system, which is given by

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla P - \rho \hat{e}_1 + f_u && \text{(momentum equation)}\,,\\
&\frac{\partial \rho}{\partial t} + u \cdot \nabla \rho = f_\rho && \text{(energy equation)}\,,\\
&\nabla \cdot u = 0 && \text{(incompressibility constraint)}\,.
\end{aligned}
\tag{2.1}
$$

These equations describe the evolution of the velocity field $u : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$, the hydrodynamic pressure $P : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ and the density fluctuations $\rho : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ of a compressible fluid subject to gravity, weak external heating $f_\rho : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ and external inertial forcing $f_u : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ in the limit of small Mach number and "powerful" gravity. In other words, these equations can be deduced from the compressible Euler equations through appropriate approximations (see [1] and [14] for more details).

Similarly to what happens in the incompressible Euler equations, taking the curl in the momentum equation, one can eliminate not only the pressure but also the incompressibility constraint. Indeed, taking the curl in (2.1) provides

$$
\begin{aligned}
&\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = \frac{\partial \rho}{\partial x_2} + f_\omega && \text{(vorticity equation)}\,,\\
&\frac{\partial \rho}{\partial t} + u \cdot \nabla \rho = f_\rho && \text{(energy equation)}\,,
\end{aligned}
\tag{2.2}
$$

where $\omega = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$ is the fluid vorticity, $f_\omega = \frac{\partial f_{u,2}}{\partial x_1} - \frac{\partial f_{u,1}}{\partial x_2}$ is a source term and the velocity field u can be recovered from the vorticity as $u = \nabla^\perp \Delta^{-1} \omega$.

Without further ado, we present our main result (see [7]).

Theorem 2.1. *Let $\alpha \in (0, \alpha_*)$, where $\alpha_* = \sqrt{\frac{4}{3}} - 1$. There are solutions (u, ρ) of the forced Boussinesq system* (2.1) *that satisfy:*

(1)
$$u(t,\cdot) \in C_c^\infty\left(\mathbb{R}^2\right), \quad \rho(t,\cdot) \in C_c^\infty\left(\mathbb{R}^2\right) \quad \forall t \in [0,1)\,.$$

(2)
$$f_\omega(t,\cdot) \in C_c^\infty\left(\mathbb{R}^2\right), \quad f_\rho(t,\cdot) \in C_c^\infty\left(\mathbb{R}^2\right) \quad \forall t \in [0,1)\,.$$

(3)
$$f_\omega(1,\cdot) \in C_c^\alpha\left(\mathbb{R}^2\right), \quad f_\rho(1,\cdot) \in C_c^{1,\alpha}\left(\mathbb{R}^2\right).$$

(4) *There is a finite-time singularity at $t = 1$, i.e.,*

$$\lim_{T \to 1^-} \int_0^T ||\nabla \rho(t,\cdot)||_{L^\infty(\mathbb{R}^2;\mathbb{R}^2)}\, \mathrm{d}t = \infty.$$

Remark 2.2. *The finite-time singularity presented in Theorem 2.1 takes place in the well-posedness regime of the Boussinesq system* (2.1)*, since we have local existence in $C^{1,\alpha}(\mathbb{R}^2)$ spaces.*

Remark 2.3 (Regularity of the solution at the blow-up time)**.** *At the instant of the blow-up, there is a regularity loss $r_{loss} \in (0,1)$ such that*

$$u(1,\cdot) \notin C^{\beta}\left(\mathbb{R}^2\right), \quad \rho(1,\cdot) \notin C^{\beta}\left(\mathbb{R}^2\right) \quad \forall \beta \in (1-r_{loss},1).$$

Choosing the parameters δ and μ of the solution small enough, this regularity loss r_{loss} can be made as small as we wish, independently of α. Nevertheless, it has an upper limit $r_{loss,\max}(\alpha)$, which does depend on α. We have

$$\lim_{\alpha\to 0} r_{loss,\max}(\alpha) \geq \frac{\alpha_*}{2} \approx 0.0774, \quad \lim_{\alpha\to\alpha_*} r_{loss,\max}(\alpha) = 0.$$

Remark 2.4 (Blow-up rate)**.** *Neither the vorticity ω nor the gradient of the density $\nabla\rho$ have a well defined blow-up rate. Nonetheless, one can find increasing time sequences $(t_{1,n})_{n\in\mathbb{N}}, (t_{2,n})_{n\in\mathbb{N}} \subseteq [0,1]$ such that $\lim_{n\to\infty} t_{1,n} = \lim_{n\to\infty} t_{2,n} = 1$ and so that, $\forall \varepsilon > 0$,*

$$\begin{aligned}
\frac{1}{(1-t_{1,n})^{\frac{1}{1-\gamma}}} \lesssim_{Y,\varepsilon,\delta,\gamma,\varphi} ||\omega(t_{1,n},\cdot)||_{L^\infty(\mathbb{R}^2)} \lesssim_{Y,\varepsilon,\delta,\gamma,\varphi} \frac{1}{(1-t_{1,n})^{\frac{1+\varepsilon}{1-\gamma}}},\\
\frac{1}{(1-t_{1,n})^{\frac{1}{1-\gamma}}} \lesssim_{Y,\varepsilon,\delta,\gamma,\varphi} \left|\left|\frac{\partial\rho}{\partial x_2}(t_{1,n},\cdot)\right|\right|_{L^\infty(\mathbb{R}^2)} \lesssim_{Y,\varepsilon,\delta,\gamma,\varphi} \frac{1}{(1-t_{1,n})^{\frac{1+\varepsilon}{1-\gamma}}}\\
\frac{1}{1-t_{2,n}} \lesssim_{Y,\varepsilon,\delta,\gamma,\varphi} ||\omega(t_{2,n},\cdot)||_{L^\infty(\mathbb{R}^2)} \lesssim_{Y,\varepsilon,\delta,\gamma,\varphi} \frac{1}{(1-t_{2,n})^{1+\varepsilon}},\\
\left|\left|\frac{\partial\rho}{\partial x_2}(t_{2,n},\cdot)\right|\right|_{L^\infty(\mathbb{R}^2)} = 0.
\end{aligned},$$

Y,δ,γ,φ are different parameters of the construction which will be introduced throughout the paper. For the moment, it is enough to bear in mind that the value of γ will lie very close to one and that $\gamma\to 1$ as $\alpha\to\alpha_*$. These huge differences in the blow-up rate of different sequences are due to the highly oscillatory behavior in time of both ω and $\frac{\partial\rho}{\partial x_2}$.

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INSTITUTO DE CIENCIAS MATEMATICAS, ICMAT, MADRID, SPAIN.
Email address: dcg@icmat.es

NONLOCAL CONSERVATION LAWS WITH BV KERNEL

GIANLUCA CRIPPA

Classification AMS 2020: 35L65, 35A01, 35R06

Keywords: nonlocal conservation laws in several space dimensions, models for pedestrian traffic, well-posedness, BV kernels, nonuniqueness, lack of selection

We consider the following Cauchy problem for nonlocal conservation laws in several space dimensions,

$$\begin{cases} \partial_t u + \operatorname{div}[uV(t,x,u*\eta)] = 0, \\ u(0,\cdot) = u_0, \end{cases}$$

where $u : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ is the unknown, $V : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^N \to \mathbb{R}^d$ is a Lipschitz continuous function, and $\eta \in L^1_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^N)$ is a convolution kernel. We denote by div the divergence computed with respect to the space variable only, whereas the symbol $*$ stands for the convolution with respect to the space variable only, that is

$$u * \eta(t,x) := \int_{\mathbb{R}^d} u(t,x-y)\eta(y)dy.$$

In [1] we establish local-in-time existence and uniqueness results for the above Cauchy problem under fairly weak differentiability assumption on η, in particular we focus on the case of Sobolev and BV (bounded total variation) regularity. We also consider an explicit example where the solution u experiences finite time blow up and show that, in general, solutions corresponding to different smooth approximations of η converge to different measures after the blow-up time.

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Departement Mathematik und Informatik, Universität Basel, Spiegelgasse 1, CH-4051 Basel, Switzerland

Email address: `gianluca.crippa@unibas.ch`

SUPPRESSION OF CHEMOTACTIC SINGULARITY BY NAVIER-STOKES FLOW WITH BUOYANCY FORCING

ZHONGTIAN HU

Classification AMS 2020: 35K20, 76D03, 76D05, 35Q92, 35Q35

Keywords: chemotaxis, Keller-Segel equation, blowup suppression, global regularity

In the presented work [22], we consider the parabolic-elliptic Keller-Segel equation in a periodic channel $\Omega = \mathbb{T} \times [0, \pi]$ subject to the influence of the buoyancy-driven Navier-Stokes equation:

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho - \Delta \rho + \operatorname{div}(\rho \nabla (-\Delta_N)^{-1}(\rho - \rho_m)) = 0, \\ \rho_m = \frac{1}{|\Omega|} \int_\Omega \rho(t,x) dx, \\ \partial_t u + u \cdot \nabla u - \frac{1}{\mathrm{Re}} \Delta u + \nabla p = \mathrm{Ra}\, \rho (0,1)^T, \\ \operatorname{div} u = 0. \end{cases} \tag{0.1}$$

We equip the system with initial data $\rho(0,x) = \rho_0(x)$, $u(0,x) = u_0(x)$, where ρ_0 is a nonnegative scalar function and u_0 is a divergence-free vector field. We also consider the following set of boundary conditions:

$$\partial_2 \rho = \nabla \rho \cdot n = 0, \quad u_2 = u \cdot n = 0, \quad \omega = \nabla^\perp \cdot u = 0, \qquad \text{on } \partial\Omega. \tag{0.2}$$

Here, $\mathbb{T} := [-\pi, \pi)$ is the one dimensional torus, and a function f defined on $\mathbb{T}$ means that f assumes the periodic boundary condition with period 2π; $n = (0,1)^T$ denotes the unit normal derivative along $\partial\Omega = \mathbb{T} \times \{0, \pi\}$; $\nabla^\perp$ denotes the differential operator $(-\partial_2, \partial_1)$; $-\Delta_N$ denotes homogeneous Neumann Laplacian.

The first equation in (0.1) is the classical parabolic–elliptic Keller-Segel equation with advection. This equation characterizes a population of bacteria with the density ρ that moves in response to an attractive chemical that the bacteria themselves secrete. Furthermore, chemotaxis usually takes place in ambient viscous fluids, which are classically modeled by Navier-Stokes equations with velocity u and pressure p. The main interaction on which we focus is the coupling by buoyancy, which originates from the variation of bacterial density in the domain. Mathematically, such interaction appears in the fluid equation through the forcing term $\mathrm{Ra}\, \rho(0,1)^T$, where Ra denotes the Rayleigh number measuring relative buoyancy strength due to density variation.

When the ambient fluid is absent (i.e., when $\mathrm{Ra} = 0$ and $u \equiv 0$ in (0.1)), we recover the classical parabolic-elliptic Keller-Segel equation:

$$\partial_t \rho - \Delta \rho + \operatorname{div}(\rho \nabla (-\Delta_N)^{-1}(\rho - \rho_m)) = 0, \quad \text{in } \Omega. \tag{0.3}$$

First introduced by Patlak [32], and Keller and Segel [27], (0.3) has been classically studied in various settings. We refer the interested readers to the following list of works: [1, 3, 4, 5, 6, 7, 8, 19, 20, 26, 30, 31]. A remarkable feature enjoyed by (0.3) is that the solution can form singularity in finite time when dimension is greater than 1. In dimension 2, (0.3) is L^1-critical. For any initial datum ρ_0 with finite second moment, the solution is globally regular if the initial mass $\|\rho_0\|_{L^1} < 8\pi$, see e.g. [3, 6, 8, 26, 36]. If the initial mass is strictly greater than 8π, a finite-time singularity forms, as seen in [6, 7, 26, 30]. A more careful analysis of such blowup solutions are also carried out [10, 11, 33, 34, 35].

It is also curious to understand the behavior of Keller-Segel equation under the influence of fluid advection, given the fact that most chemotactic processes take place in ambient fluid. The presence of fluid advection can bring complicated effects to chemotaxis. Among these effects, we would like to focus on the *regularization effects* induced by fluid advection. That is, we would like to understand how the transport term $u \cdot \nabla \rho$ could prevent potential singular behaviors in (0.3). For the past decade, much progress has been made to understand such regularization effects in the context of Keller-Segel equation.

In the regime of passive advection (i.e. the fluid velocity u is given and is not coupled to the Keller-Segel equation), Kiselev and Xu in [28] first demonstrate that given any initial datum ρ_0, there exists a relaxation-enhancing flow (see [12] for a precise definition) with sufficiently large amplitude that can suppress the singularity formation. This result is later generalized by [25] to a larger class of passive flows and more general aggregation equations. Moreover, in [2, 15], the authors exploit the enhanced dissipation phenomenon induced by strong monotone shear flows. Such flows effectively reduce the dimensionality of the problem, where in 2D the singularity can be suppressed. The fast-splitting scenario induced by hyperbolic flows are also explored in [17, 18].

There also have been numerous attempts to investigate the regularity properties of the Keller-Segel equation coupling to active fluid models, and many of which address the global regularity of solutions to such coupled systems. We highlight that, among those results, either there is smallness assumption on initial data (e.g. [9, 13, 29, 14]), or the global regularity of both the chemotaxis equation and fluid equation still hold if they are uncoupled (e.g. [37, 38]). We also note that the authors in [16, 39] study the blowup suppression mechanism of Keller-Segel-Navier-Stokes equation near a strong Couette flow. These results are almost linear in a sense that the main driven mechanism is still brought by a dominating passive background flow.

Recently in [23] by the author joining with Kiselev and Yao, they analyze how buoyancy effects in fluid equations suppress chemotactic singularities in a genuinely nonlinear setting. In [23], the authors investigated the Keller-Segel equation evolving in ambient porous media under the influence of buoyancy. The authors demonstrated that a coupling with the porous media equation via an arbitrarily weak buoyancy constant suffices to arrest any potential chemotactic blowup. The key argument in [23] is a careful analysis of a potential energy and the coercive term $\|\partial_1\rho\|_{H_0^{-1}}$ in its time derivative. The authors observed that this H_0^{-1} norm has to be small, and induces an anisotropic mixing effect along x_1-direction. This effect renders the system quasi-one-dimensional and therefore suppresses finite-time blowup.

It is interesting to understand whether a similar effect would happen in the Keller-Segel-Navier-Stokes system, which is a more classical and biologically relevant model. In the presented work [22], we affirmatively answer this question, at least in the setting where the buoyancy and the viscosity of the fluid is large. More precisely, our main result shows that the smooth solution of (0.1) with arbitrary large mass is in fact globally regular given both parameters Ra and Re sufficiently large, whose sizes only depend on initial data (ρ_0, u_0):

Theorem 0.1 (H., [22]). *Suppose initial data* (ρ_0, u_0) *with* $\rho_0 \in H^1$ *nonnegative and* $u_0 \in V$*, where* V *is the class of* H^1*, divergence-free vector fields that satisfy no-flux boundary condition. There exists a couple* $(\mathrm{Re}, \mathrm{Ra})$ *sufficiently large depending on initial data, so that* (0.1) *admits a unique, regular, and global-in-time solution.*

The main strategy of proving the main theorem relies crucially on the analysis of the following quasi-static model, which can be regarded as a leading order approximation of (0.1) in the parameter regime that we concern:

$$\partial_t \rho + u \cdot \nabla \rho - \Delta \rho + \operatorname{div}(\rho \nabla (-\Delta_N)^{-1}(\rho - \rho_m)) = 0, \tag{0.4a}$$

$$-\Delta u + \nabla p = g\rho(0,1)^T, \quad \operatorname{div} u = 0, \tag{0.4b}$$

$$\partial_2 \rho\big|_{\partial\Omega} = 0,\ u_2|_{\partial\Omega} = 0,\ \partial_2 u_1|_{\partial\Omega} = 0,\ \rho(0,x) = \rho_0(x) \geq 0. \tag{0.4c}$$

Using ideas deployed in [23], we are able to show the following damping property of (0.4):

Proposition 0.2 (Damping Property). *Consider problem* (0.4) *with nonnegative initial datum* $\rho_0 \in H^1$*. Assume that* $t_0 \geq 0$ *is inside the lifespan of the unique regular local solution* $\rho(t,x)$*. Then there exist* $N_0 = N_0(\rho_m, \|\rho_0 - \rho_m\|_{L^2})$*, a time* $T_* = T_*(N_0, \rho_m)$*, and* $g_0 := g_0(N_0, \rho_m) > 0$*, such that the following statement holds: if* $\|\rho(t_0) - \rho_m\|_{L^2}^2 \leq \frac{N_0}{2}$*, then for all* $g \geq g_0$*, the regular solution* $\rho(t,x)$ *can be continued in interval* $[t_0, t_0 + T_*]$ *with estimate*

$$\sup_{t \in [t_0, t_0 + T_*]} \|\rho(t) - \rho_m\|_{L^2}^2 \leq N_0. \tag{0.5}$$

Moreover, there exists a time instance $T \in \left[t_0 + \frac{T_*}{2}, t_0 + T_*\right]$ *such that*

$$\|\rho(T) - \rho_m\|_{L^2}^2 \leq \frac{N_0}{8}. \tag{0.6}$$

Finally, we combine Proposition 0.2 with a bootstrap argument to show that (0.4) is indeed an accurate approximation to the full problem (0.1) for all times, and from which we conclude the main Theorem 0.1. This argument involves various delicate estimates regarding the 2D Navier-Stokes equations.

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Department of Mathematics, Duke University, Durham, NC 27708, USA
Email address: zh126@duke.edu

FORMATION AND PERSISTENCE OF LARGE-SCALE VORTICES

IN-JEE JEONG

Classification AMS 2020: 35Q35

Keywords: Euler equations, vortex, variational principles

1. Incompressible Euler equations and stability

In the long-time limit of an inviscid, incompressible flow, one typically observes the emergence of a few coherent vortices. Inviscid flows, or nearly inviscid ones, often exhibit motions that, after long periods, remain stable and cannot be easily deformed. Explaining this phenomenon using the kinetic energy of the fluid goes back at least to the works of Kelvin in 1880 [1, 2]. Several mathematically rigorous results in this direction appeared after Kelvin's work.

Concrete research topics include:

- Global stability for large-scale vortices (e.g., Rankine vortices).
- The confinement problem: absence of dispersion.
- Desingularization problem: justification of point vortex dynamics (see below).

The Euler equations describe the motion of an inviscid fluid:

$$\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = 0, \tag{1.1}$$

where ω represents the vorticity, and u is the velocity field determined by the Biot-Savart law:

$$u = \nabla^{\perp} \psi, \quad \Delta \psi = \omega. \tag{1.2}$$

Steady states of the Euler equations correspond to critical points of the energy functional:

$$E(\omega) = -\frac{1}{2} \int_{\mathbb{R}^2} \omega \psi \, dx. \tag{1.3}$$

Together with the energy, one can use other conserved quantities, namely

- total circulation: $\int_{\mathbb{R}^2} \omega \, dx$.
- center of vorticity $\int \int_{\mathbb{R}^2} x\omega \, dx$.
- angular impulse $\int_{\mathbb{R}^2} |x|^2 \omega \, dx$.

Energy maximization principles have been used to establish the nonlinear stability of certain steady vortex configurations. Specifically, if ω_0 is the characteristic function of a ball (Rankine vortex) then it is stable under small perturbations. A precise statement and proof can be found in [3].

Date: February 16, 2025.

2. Point Vortex Dynamics and Desingularization

A fundamental model for concentrated vorticity in $\mathbb{R}^2$ is the point vortex system described by Helmholtz and Kirchhoff: for atomic measure vorticity of the form

$$\omega(t) = \sum_i \Gamma_i \delta_{x_i(t)},$$

the evolution of the points $x_i(t)$ is given by

$$\frac{dx_i}{dt} = \sum_{j \neq i} \Gamma_j K(x_i, x_j), \tag{2.1}$$

where $K(x_i, x_j)$ represents the velocity field induced by other vortices. Conserved quantities are:

- The total impulse: $I = \sum_i \Gamma_i x_i$.
- The angular momentum: $M = \sum_i \Gamma_i |x_i|^2$.
- The Hamiltonian: $H = -\sum_{i \neq j} \Gamma_i \Gamma_j \ln |x_i - x_j|$.

Point vortex dynamics exhibit interesting behaviors, such as uniform rotation around the center of mass and self-similar collapse in finite time. The stability of such configurations has been studied extensively. Desingularization methods replaces point vortices with localized vorticity distributions, with the goal of showing that such vorticity would retain essential dynamical features of the original system.

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1 Gwanakro, Seoul 08826, Republic of Korea
Email address: injee_j@snu.ac.kr

ON THE WELL-POSEDNESS OF α-SQG EQUATION IN A HALF-PLANE

JUNHA KIM

Classification AMS 2020: 76B47, 35Q35

Keywords: ill-posedness, norm inflation, surface quasi-geostrophic equation, fluid dynamics

This talk is based on a joint work with In-Jee Jeong and Yao Yao [14]. Let us consider the Cauchy problem for the α-SQG equation on the right half plane $\mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2; x_1 > 0\}$

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \\ u = -\nabla^\perp (-\Delta)^{-1+\frac{\alpha}{2}} \theta, \end{cases} \tag{α-SQG}$$

for $0 < \alpha \leq 1$. The Biot–Savart law is simply given by

$$u(t,x) = \int_{\mathbb{R}^2_+} \left[\frac{(x-y)^\perp}{|x-y|^{2+\alpha}} - \frac{(x-\tilde{y})^\perp}{|x-\tilde{y}|^{2+\alpha}} \right] \theta(t,y) \, \mathrm{d}y, \tag{0.1}$$

where $\tilde{y} := (-y_1, y_2)$ for $y = (y_1, y_2)$.

In domains with boundaries, (α-SQG) has recently attracted significant attention. The existence of weak solutions to (α-SQG) in $L^\infty_t L^2_x$ was established in [5, 20, 4]. In $\mathbb{R}^2$, the existence of weak solutions was studied in [21, 19, 1], while the non-uniqueness of the weak solutions was investigated in [2, 12, 3]. These results can be extended to the $\mathbb{R}^2_+$ case by considering solutions that are odd in one variable. Regarding half-plane patch solutions, we refer to [18, 11] for local well-posedness, [17] for finite-time singularity formation, [15, 16] for the absence of splash singularities, and [7] for the stability of a half-plane stationary solution.

In this talk, we focus on the local well-posedness of strong solutions to (α-SQG) that *do not* vanish on the boundary. In this case, unlike the 2D Euler equations, the two components of u exhibit different regularity properties, with one being only $C^{1-\alpha}$ near the boundary even when θ is smooth. This observation motivates us to introduce an anisotropic functional space, which plays a crucial role in establishing our first main result.

Definition 0.1. *For given $0 < \beta \leq 1$, let $X^\beta = X^\beta(\overline{\mathbb{R}^2_+})$ be a subspace of $C^\beta(\overline{\mathbb{R}^2_+})$ with anisotropic Lipschitz regularity: we say $f \in X^\beta$ if it is differentiable almost everywhere, and satisfies*

$$\|f\|_{X^\beta} := \|f\|_{L^\infty} + \|x_1^{1-\beta} \partial_1 f\|_{L^\infty} + \|\partial_2 f\|_{L^\infty} < \infty.$$

If $f \in X^\beta$ has a compact support, we denote $f \in X^\beta_c$.

Theorem 0.2. *Let $\alpha \in (0, \frac{1}{2}]$ and $\beta \in [\alpha, 1-\alpha]$. Then ($\alpha$-SQG) is locally well-posed in X^β_c: for any $\theta_0 \in X^\beta_c$, there exist $T = T(\|\theta_0\|_{X^\alpha}, |\operatorname{supp} \theta_0|) > 0$ and a unique solution θ to (α-SQG) in the class*

$$\mathrm{Lip}([0,T]; L^\infty) \cap L^\infty(0,T; X^\beta_c) \cap C([0,T]; C^{\beta'}),$$

for any $0 \le \beta' < \beta$*. Moreover, there exists a universal constant* $C > 0$ *such that* u *with* (0.1) *satisfies*

$$\|u_1(t)\|_{C^{1,1-\alpha}} + \|\partial_2 u_2(t)\|_{C^{1-\alpha}} + \left\| \frac{x_1^\alpha}{1+x_1^\alpha}\partial_1 u_2(t)\right\|_{L^\infty} \le C\|\theta(t)\|_{X^\beta} \tag{0.2}$$

for all $0 \le t < T$*. If the maximal time of existence* T^* *is finite, then*

$$\limsup_{t\to T^*} \left((T^*-t)^\eta \|\partial_2\theta(t,\cdot)\|_{L^\infty}\right) = \infty$$

holds for all $\eta \in (0,1)$*.*

We remark that (α-SQG) is well-posed in X_c^α, whereas it is ill-posed in the critical space $C^\alpha(\mathbb{R}^2)$ (and also in $H^{1+\alpha}(\mathbb{R}^2)$) ([13, 9, 10, 6]). Moreover, for $\alpha \in (0, \frac{1}{3})$, one can follow the same argument in [11, 17] to construct $\theta_0 \in C_c^\infty(\overline{\mathbb{R}^2_+})$ such that the solution leaves X^β in finite time. Recently, Zlatoš [22] obtained more precise and refined estimates, successfully extending the parameter regime to cover the entire range $\alpha \in (0, \frac{1}{2}]$ while also establishing the local well-posedness of (α-SQG).

Let us consider any smooth initial data $\theta_0 \in C_c^\infty(\overline{\mathbb{R}^2_+})$ which does not vanish at the boundary. Then, Theorem 0.2 implies for $\alpha \in (0, \frac{1}{2}]$ that (α-SQG) admits a unique solution in $L^\infty(0,T;X_c^{1-\alpha})$ for some $T > 0$. Our second main result shows that this regularity of solutions is sharp.

Theorem 0.3. *Let* $\alpha \in (0, \frac{1}{2}]$ *and* $\theta_0 \in C_c^\infty(\overline{\mathbb{R}^2_+})$ *do not vanish on the boundary. Then, the local-in-time solution* θ *to* (α-SQG) *given by Theorem 0.2 escapes* C^β *immediately for all* $\beta \in (1-\alpha, 1]$*.*

Let us deal with the case $\alpha > 1/2$. Our last main result shows the *nonexistence* of solutions not only in X^α but even in C^α.

Theorem 0.4. *Let* $\alpha \in (\frac{1}{2}, 1]$ *and* $\theta_0 \in C_c^\infty(\overline{\mathbb{R}^2_+})$ *do not vanish on the boundary. Then, there is no solution to* (α-SQG) *with initial data* θ_0 *belonging to* $L^\infty(0,\delta;C^\alpha(\overline{\mathbb{R}^2_+}))$ *for any* $\delta > 0$*.*

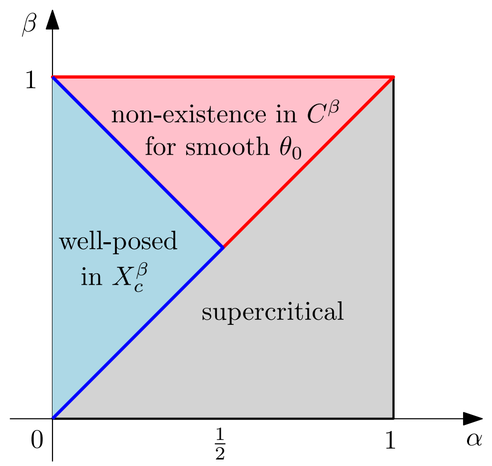

FIGURE 1. Illustration of well-posedness result of (α-SQG) in X_c^β spaces (in blue color), and non-existence results in C^β for smooth initial data (in red color).

Theorems 0.2-0.4 classify the well-posedness and ill-posedness of (α-SQG) in X_c^β for $\beta \in [\alpha, 1]$, where $\alpha \in (0, 1]$. We refer to [9, 22, 8] for the ill-posedness result of (α-SQG) in the supercritical regime.

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Department of Mathematics, Ajou University, Suwon, South Korea
Email address: `junha02@ajou.ac.kr`

SHOCK-TYPE SINGULARITY OF THE HYPERBOLIC-PARABOLIC CHEMOTAXIS SYSTEM

WOOJAE LEE

Classification AMS 2020: 35A21, 35B44, 35Q35, 35Q92

Keywords: Hyperbolic-parabolic chemotaxis, Finite-time blow-up, Blow-up profile, Shock-type singularity.

This talk is divided into two parts: the first part focuses on explaining the blow-up phenomenon in the hyperbolic-parabolic chemotaxis (HPC) system, and the second part discusses the construction of this blow-up profile using modulation analysis.

Chemotaxis describes the movement of cells toward or away from environmental chemical substances. Vasculogenesis, a phenomenon driven by chemotaxis, is the process by which new blood vessels form from endothelial cells rather than pre-existing vessels. During the initial stages of vascular formation, randomly distributed endothelial cells can spontaneously assemble into a vascular network, a crucial factor in tumor growth. To model this process, Gamba et al. [6] and Ambrosi et al. [1] introduced the HPC system. In this talk, we dicussed the blow-up profile in the following HPC system on $\mathbb{R}$:

$$\begin{aligned} \rho_t + (\rho u)_x =& 0, \quad x \in \mathbb{R}, \quad t > 0, \\ (\rho u)_t + (\rho u^2)_x + \frac{1}{\gamma}(\rho^\gamma)_x =& \rho\phi_x - \rho u, \\ \phi_t - \phi_{xx} =& \rho - \phi, \end{aligned} \tag{0.1}$$

where ρ and u are the density and the velocity of endothelial cells, respectively. ϕ is the concentration of the chemoattractant. $\gamma > 1$ is positive constant.

There is the blow-up profile of the HPC system closely aligns with experimental observations on endothelial cells. Numerical simulations by Filbet and Shu [5] and Filbet, Laurençot, and Perthame [4], suggest that in the HPC system, shock-type structures emerge before density implosion, indicating that density accumulation occurs near the edges of a network rather than at a single point. Notably, such shock-type structures are not observed in the Keller–Segel system.

We construct a blow-up profile, associated with the shock-type singularity:

Theorem 0.1. *For some smooth initial data (ρ_0, u_0, ϕ_0) with the maximally negative slope of (ρ_0, u_0) for sufficiently large, we construct the blow-up profile of the HPC system as follows.*

- *The blow-up profile is H^m stable prior to the singularity.*
- *The solution (ρ, u) has a unique blow-up point x^*:*

$$\rho_x(x^*, T^*) = u_x(x^*, T^*) = -\infty.$$

- *At the blow up point, the solution (ρ, u) has a cusp singularity with Hölder $C^{\frac{1}{3}}$ regularity.*
- *The HPC system is C^1 regular at any point $x \neq x^*$. In contrast, the solution ϕ is C^2 regular.*

In addition, the behavior of $\rho^{\frac{\gamma-1}{2}}$ and u near the blow-up point follows a similar pattern, characterized by $-(x-x^*)^{\frac{1}{3}}$.

To establish this blow-up profile, we draw inspiration from the approach proposed by Buckmaster, Vicol, and Shkoller [2, 3] for constructing shock solutions to the multidimensional isentropic compressible Euler equations. Their method is based on self-similar modulation analysis. They study the asymptotic stability of a self-similar variable W near the steady-state Burgers profile $\overline{W}$:

$$-\frac{1}{2}\overline{W} + \left(\frac{3}{2}y + \overline{W}\right)\overline{W}_y = 0, \quad \text{for all} \quad y \in \mathbb{R}.$$

First, we reformulate the HPC system into transport-type equations by introducing Riemann-type variables. Then, we apply a self-similar transformation along with modulation variables (τ, κ, ξ) to convert the finite-time blow-up problem into a global stability analysis. By constraining certain spatial derivatives of a self-similar variable W at 0, we define an ODE system for the modulation variables.

In this talk, we focus on closing the self-similar variable W_y near the spatial gradient of the steady-state Burgers profile $\overline{W}$ on the middle interval in the bootstrap argument. To achieve this, we derive a bound for the particle trajectory associated with the self-similar variable W. By leveraging this bound, the bootstrap assumptions, and a Gronwall-type lemma, we establish weighted estimates for W_y.

To ensure $H^m(\mathbb{R})$-stability before singularity formation, we prove local well-posedness and derive a continuation criterion for the system:

$$\sup_{t\in[0,T]} \left\|(\rho-\bar{\rho}, u_0, \phi_0-\bar{\phi})\right\|_{H^m} < \infty \Leftarrow \int_0^T \|(\rho_x, u_x)(t')\|_{L^\infty} dt' < \infty.$$

By constraining W at 0 through modulation variables, we can track the unique blow-up point. Finally, the previous mentioned weighted estimate $(1+y^2)^{\frac{1}{3}}(W_y - \overline{W}_y)$ implies that the unique blow-up point exhibits a cusp singularity.

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(Woojae Lee) Department of Mathematics
Yonsei University, 50 Yonsei-Ro, Seodaemun-Gu, Seoul 03722, Republic of Korea
Email address: woori0108@yonsei.ac.kr

SINGULARITY STRUCTURE OF FLRW SPACETIMES AT LOW REGULARITIES

JAN SBIERSKI

Classification AMS 2020: 83C75

Keywords: spacetime singularities, inextendibility

This talk considered the possible singularity structures occurring in the class of FLRW spacetimes. Let $(\overline{M}_K, \overline{g}_K)$ denote the unique simply connected and complete 3-dimensional Riemannian manifold of constant sectional curvature $K \in \{-1, 0, 1\}$. The class of *FLRW spacetimes* (M, g) is then defined by

$$M := (0, \infty) \times \overline{M}_K \;, \qquad g := -dt^2 + a(t)^2 \cdot \overline{g}_K$$

where t is the standard coordinate on the first factor of M and $a : (0, \infty) \to (0, \infty)$ is a smooth function with $a(t) \to 0$ for $t \to 0$, called the *scale factor*. A time-orientation is fixed by saying that ∂_t is future directed. If the quantity $\int_0^1 \frac{1}{a(t)}\, dt$ is finite, we say that the FLRW spacetime possesses *particle horizons* – if it is infinite, we say that the FLRW spacetime does not have particle horizons. Most of the spacetimes within this class of FLRW spacetimes are singular when $t \to 0$. (A notable exception is the Milne spacetime, for which $K = -1$ and $a(t) = t$.) This can be easily seen by computation of scalar curvature invariants:

$$R = 6\frac{\dot{a}^2 + \ddot{a}a + K}{a^2} \;, \qquad R_{\mu\nu\kappa\rho}R^{\mu\nu\kappa\rho} = 12\frac{a^2\ddot{a}^2 + (\kappa + \dot{a}^2)^2}{a^4} \;. \tag{0.1}$$

The strength of a singularity can be captured by the degree of regularity of the Lorentzian metric that is lost at the singularity. This is made precise by the notion of *inextendibility*:

Definition 0.1. *Let (M, g) be a smooth Lorentzian manifold and Γ a regularity class (e.g. C^0, $C^{0,1}_{\mathrm{loc}}$, C^2, ...). A* Γ-extension *of (M, g) is a smooth isometric embedding $\iota : M \hookrightarrow \tilde{M}$ of (M, g) into a Lorentzian manifold $(\tilde{M}, \tilde{g})$ of the same dimension as M with $\tilde{g} \in \Gamma$ and $\tilde{M} \supseteq \partial\iota(M) \neq \emptyset$. If a Γ-extension exists, we say that (M, g) is* Γ-extendible *– otherwise it is said to be* Γ-inextendible*.*

A *past Γ-extension* of an FLRW spacetime (M, g) is a Γ-extension such that there exists a smooth timelike curve in M along which $t \to 0$ and which has a smooth limit point in the extension. If no such extension exists, we say that (M, g) is *past Γ-extendible*. For a more detailed discussion of these notions we refer the reader to [12].

The main results discussed in this talk were the following statements:

Theorem 0.2 ([12])**.** *FLRW spacetimes with particle horizons are past $C^{0,1}_{\mathrm{loc}}$-inextendible for all $K \in \{-1, 0, 1\}$.*

Theorem 0.3 ([13])**.** *FRLW spacetimes without particle horizons, with $K = -1$ and for which the scale factor satisfies $\lim_{t\to 0} a(t) \exp\big(\int_t^1 \frac{1}{a(t')}\, dt'\big) = \infty$ are past C^0-inextendible.*

Note that the assumption on the scale factor complements the C^0-extendibility results of [2], [6] for Milne-like spacetimes.

Theorem 0.4 ([13]). *FLRW spacetimes without particle horizons and with $K = +1$ are past C^0-inextendible.*

There are still no low-regularity inextendibility results available for FLRW spacetimes without particle horizons and with $K = 0$. For a past C^0-inextendibility result for *past eternal* FLRW spacetimes with $K = 0$ see [7]. Low-regularity inextendibility results of FLRW spacetimes within symmetry classes are proven in [2], [5]. For low-regularity inextendibility results of other spacetimes see [11], [10], [3], [9], [4], [1], [8], [14].

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The University of Edinburgh, School of Mathematics, James Clerk Maxwell Building, Peter Guthrie Tait Road, Edinburgh, EH9 3FD, United Kingdom

Email address: Jan.Sbierski@ed.ac.uk

STABILITY OF GRAVITATIONAL COLLAPSE

MATTHEW SCHRECKER

Classification AMS 2020: 35Q85, 35B44, 34C05

Keywords: Singularity formation, Gravitational collapse, Euler-Poisson, non-linear PDE

The rigorous description of the collapse of a star under its own gravity is a fundamental mathematical and physical problem, described by the gravitational Euler-Poisson system. As well as physically significant families of steady states, the Euler-Poisson system admits both expanding and collapsing solutions. These latter solutions are important for understanding stellar gravitational collapse in the context of both the formation and the death of stars. In the astrophysical literature (for example, [**?**]), it is widely conjectured that as a star collapses under its own gravity, the star adopts an approximately self-similar form. This conjecture is known as the self-similarity hypothesis. The existence of self-similar collapse solutions for the Euler-Poisson system was first discovered numerically for the special case of isothermal gases by Larson [**?**] and Penston [**?**] in 1969 and constructed analytically in recent years by Guo–Hadžić–Jang [**?**].

Under the assumption of spherical symmetry, the isentropic Euler Poisson equations take the form

$$\partial_t\rho + \partial_r(\rho u) + \frac{2}{r}\rho u = 0, \tag{1}$$

$$\rho\big(\partial_t u + u\partial_r u\big) + \partial_r p + \frac{1}{r^2}\rho m = 0, \tag{2}$$

where the density ρ and radial velocity u are functions only of time t and radial distance $r = |x|$. The quantity $m(r)r^{-2}$ appearing in the conservation of momentum equation (**??**) corresponds to the action of the radial component of the gravitational force $\nabla\phi$ induced through the Poisson equation

$$\Delta\phi = 4\pi\rho, \qquad \lim_{|x|\to\infty}\phi(t,x) = 0.$$

Due to the effect of spherical symmetry, this Poisson equation is easily solved in terms of the the local mass function $m(r)$, which is determined directly from the density via

$$m(t,r) = 4\pi\int_0^r s^2\rho(t,s)\,\mathrm{d}s.$$

To close the system of equations (**??**)–(**??**), we require an equation of state for the pressure p, which we determine via a polytropic relation

$$p = p(\rho) = \kappa\rho^\gamma, \qquad \gamma \in \Big[1, \frac{4}{3}\Big), \quad \kappa > 0.$$

The case $\gamma = 1$ corresponds to the simpler case of an isothermal gas, while $\gamma > 1$ corresponds to more general polytropes. The range $\gamma \in [1, \frac{4}{3})$ is referred to as the *supercritical* range of exponents, and is the range for which we expect to find self-similar blow-up solutions from smooth initial data.

The reason for the restriction to the range $\gamma \in [1, \frac{4}{3})$ is easily explained. The earlier work [**?**] excludes such a blow-up for $\gamma > \frac{4}{3}$ under some mild assumptions on the solution. In the mass-critical case, $\gamma = \frac{4}{3}$, there is a famous family of solutions due to Goldreich and Weber, [**?**], which can either collapse or expand. These solutions are found using an effective separation of variables, allowing for the solution to be found as a time-modulated spatial profile satisfying a Lane-Emden type equation. In contrast, the solutions found in this work involve the careful balancing of all three main forces in the system: inertia, pressure and gravity.

In recent work, jointly with Yan Guo, Mahir Hadžić and Juhi Jang, [**?**], we have rigorously constructed exactly self-similar solutions to the Euler-Poisson system for the full range of supercriticial exponents. By allowing exponents $\gamma \in (\frac{6}{5}, \frac{4}{3})$, we are also able to prove the existence of solutions of finite energy, as predicted numerically by Yahil [**?**].

The main result of the paper [**?**], roughly stated, is

Theorem 1 (Existence Theorem, Rough Version)**.** *For all $\gamma \in [1, \frac{4}{3})$, there exists a smooth initial data pair $(\rho_0(r), u_0(r))$, defined on $[0, \infty)$, with $\rho_0(r) \to 0$ as $r \to \infty$ such that the system* (**??**)–(**??**) *with initial data $(\rho, u)|_{t=-1} = (\rho_0, u_0)$ has a smooth solution $(\rho(t, r), u(t, r))$ for $t \in (-1, 0)$ such that, at the spatial origin $r = 0$, the density $\rho(t, 0) \to \infty$ as $t \to 0^-$. For all $r > 0$, the limits of $\rho(t, r)$ and $u(t, r)$ exist as $t \to 0^-$ and define smooth functions $\rho(0, r), u(0, r)$ on $(0, \infty)$.*

As predicted by the self-similarity hypothesis, we construct these implosion solutions via a self-similar ansatz. Motivated by the invariance of the Euler-Poisson system (**??**)–(**??**) under the scaling, for $\lambda > 0$,

$$\rho(t, r) \mapsto \lambda^{-\frac{2}{2-\gamma}} \rho\Big(\frac{t}{\lambda^{\frac{1}{2-\gamma}}}, \frac{r}{\lambda}\Big), \quad u(t, r) \mapsto \lambda^{-\frac{\gamma-1}{2-\gamma}} u\Big(\frac{t}{\lambda^{\frac{1}{2-\gamma}}}, \frac{r}{\lambda}\Big), \tag{3}$$

we introduce a self-similar variable

$$y = \frac{r}{\sqrt{\kappa}(-t)^{2-\gamma}}$$

and make the ansatz

$$\rho(t, r) = (-t)^{-2}\tilde{\rho}(y), \quad u(t, r) = \sqrt{\kappa}(-t)^{1-\gamma}\tilde{u}(y).$$

A calculation then shows that such a pair of profiles $\tilde{\rho}$ and $\tilde{u}$ induces a solution of the original system of equations if the follpwing ODE system is satisfied:

$$\rho' = \frac{y\rho\left(2\omega^2 + (\gamma-1)\omega - \frac{4\pi\rho\omega}{4-3\gamma} + (\gamma-1)(2-\gamma)\right)}{\gamma\rho^{\gamma-1} - y^2\omega^2}, \tag{4}$$

$$\omega' = \frac{4-3\gamma-3\omega}{y} - \frac{y\omega\left(2\omega^2 + (\gamma-1)\omega - \frac{4\pi\rho\omega}{4-3\gamma} + (\gamma-1)(2-\gamma)\right)}{\gamma\rho^{\gamma-1} - y^2\omega^2}. \tag{5}$$

Here we have defined a new unknown

$$\omega(y) = \frac{\tilde{u}(y) + (2-\gamma)y}{y}$$

and dropped the $\sim$ notation. It is clear that any smooth solution of (**??**) with $\rho(0) > 0$ and $\rho(y) \to \infty$ as $y \to \infty$ gives a collapsing solution of the original system (**??**)–(**??**) with density blowup at the origin at time $t = 0$.

A key difficulty in solving this ODE system rigorously is the presence of singularities in the system. As well as the regular singular point at the origin (due to the radial symmetry assumption), there is an *a priori* unknown further singularity whenever $\gamma\rho^{\gamma-1} - y^2\omega^2 = 0$. At such a point, the relative speed $y\omega$ is exactly the speed of sound in the gas, $\sqrt{p'(\rho)}$. It turns out that physically meaningful solutions (that is, with bounded density at the origin and density decaying to zero at spatial infinity) must, provided they are continuous, have at least one sonic point. Handling this issue leads to many of the key difficulties of the proof.

In the isothermal case, $\gamma = 1$, the existence of the Larson–Penston gravitational collapse solution by Guo–Hadžić–Jang [**?**] was proved with a delicate shooting argument. The authors first constructed a local solution around candidate sonic points and then conducted the shooting argument both outwards to spatial infinity and inwards to the origin in order to select a global, smooth solution. The analysis was assisted by the fact that, when $\gamma = 1$, the system (**??**) simplifies significantly, and a less refined analysis is required to demonstrate the existence of the solutions. Even so, the basic skeleton of the proof for general $\gamma > 1$ is still based on such a shooting argument. Significant developments and refinements of these general arguments have also been successful in showing the existence of non-isentropic collapse solutions to Euler-Poisson [**?**] and of naked singularity solutions to the Einstein-Euler system [**?**]. Moreover, using a different set of analytic techniques, Sandine [**?**] has constructed a further family of isothermal blow-up solutions, known as the Hunter solutions [**?**].

In the case of the Euler equations without gravity, the introduction of a self-similar ansatz leads to autonomous ODE dynamics for collapse solutions. In this setting, Merle–Raphaël–Rodnianski–Szeftel, [**?**] were able to employ a sophisticated phase portrait analysis to show the existence of smooth, imploding solutions to the isentropic Euler equations. For the full Euler system, the use of the autonomous ODE system and associated techniques was also recently used by Jang–Liu–Schrecker to prove the existence of the Guderley imploding/exploding shock wave solutions, [**?**]–[**?**].

The implosion solutions of [**?**] were shown by the same authors to be finite co-dimension stable within the class of radially symmetric solutions, [**?**]. Later numerical work, [**?**] suggests that the finite co-dimension is positive, i.e. these solutions are unstable to generic radial perturbations. Certain restrictions on the exponents γ have been removed by the subsequent work [**?**] and the proof of finite co-dimension stability against general perturbations outside radial symmetry has been established also, [**?**].

In contrast, it is widely expected that the Larson-Penston and Yahil solutions are in fact stable in the class of radial solutions, based on numerical investigations, see [**?**]. The smoothness of the underlying self-similar profile appears to be essential for the stability properties, both for the full stability of the gravitational collapse and the finite co-dimensional stability of the gas flows.

In the isothermal case, the second main result of this note is the stability of the Larson-Penston isothermal collapse solution.

Theorem 2 ([**?**]). *The Larson-Penston solution of the Euler-Poisson system is nonlinearly stable in the class of radially symmetric solutions.*

Acknowledgements

M. Schrecker's research is supported by the EPSRC Post-doctoral Research Fellowship EP/W001888/1.

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Department of Mathematics, University of Bath, Claverton Down, Bath, UK
Email address: mris21@bath.ac.uk

SHOCK FORMATION FOR COMPRESSIBLE EULER EQUATIONS AND RELATED SYSTEMS VIA SELF-SIMILAR APPROACH

WENZE SU

Classification AMS 2020: 35L67, 35Q31, 76L05

Keywords: Shock formation, Euler equations, modulation method

The isentropic compressible Euler equations in $\mathbb{R}^n$ are given by

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}_x(\rho u \otimes u) + \nabla_x p = 0, \\ p = \dfrac{1}{\gamma}\rho^\gamma. \end{cases} \tag{0.1}$$

A fundamental question in fluid dynamics is whether smooth solutions to the compressible Euler equations can develop singularities in finite time, leading to shock formation. In one dimension, this phenomenon is well-understood via Riemann invariants [9, 10, 13]. However, the problem becomes significantly more challenging in multiple dimensions.

Early results on singularity formation in multiple dimensions were obtained by Sideris [19], who proved a blow-up result. The first rigorous proof of shock formation for multi-dimensional compressible Euler equations was achieved by Christodoulou in the context of relativistic fluids [6], and later by Christodoulou-Miao for non-relativistic irrotational flows [7]. These works utilized geometric methods, exploiting the wave-like structure of the Euler equations in irrotational regimes. For irrotational flows, the system reduces to a scalar second-order quasilinear wave equation, making techniques developed for wave equations applicable, such as those by Alinhac [1, 2] for the 2D setting, and by Speck-Holzegel-Luk-Wong[21], Speck[20], Miao-Yu[18] for the 3D setting. The first shock formation result for compressible Euler equations admitting non-zero vorticity was established by Luk and Speck [11, 12].

Building upon the comprehensive investigation of Burgers blow-up self-similarity in [8], Buckmaster, Shkoller, and Vicol [3, 4, 5] pioneered an alternative approach for proving shock formation using self-similar ansatz and modulation methods. This method, inspired by studies of singularity formation in nonlinear Schrödinger equations (e.g., [14, 15, 16]) and nonlinear heat equations (e.g., [17]), constructs approximate solutions that are asymptotically self-similar near the singularity. By dynamically rescaling the equations and introducing modulation variables to track key shock formation parameters like location, amplitude, and steepness, the problem is transformed into proving global existence in self-similar coordinates. Buckmaster, Shkoller, and Vicol first applied this modulation method to construct shock solutions for the 2D Euler equations with azimuthal symmetry [3], and later extended it to the 3D case without symmetry assumptions [4].

In this talk, we focus on the 2D isentropic compressible Euler equations and a simplified shallow water system, utilizing the self-similar Burgers ansatz to construct shock solutions in 2D without symmetry assumptions. Our main result for the 2D compressible Euler equations is:

Theorem 0.1. Consider the 2D isentropic compressible Euler equations (0.1). For any sufficiently small $\varepsilon > 0$, there exists initial data $(u_0, \rho_0) \in H^k$ with $k \geq 18$ such that $|\nabla(u_0, \rho_0)| = \mathcal{O}(1/\varepsilon)$, and the corresponding solution (u, ρ) develops a shock-type singularity in finite time $T \sim \mathcal{O}(\varepsilon)$. Moreover, the shock is a point shock with non-zero vorticity and exhibits asymptotic self-similarity.

Next, we consider the simplified shallow water model with Coriolis force:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u_1) = 0, \\ \partial_t u_1 + u_1 \partial_x u_1 + \partial_x \rho - u_2 = 0, \\ \partial_t u_2 + u_1 \partial_x u_2 + u_1 = 0. \end{cases} \tag{0.2}$$

Applying a similar approach, we construct smooth initial data leading to finite-time shock formation for this system, proving:

Theorem 0.2. For any sufficiently small $\varepsilon > 0$, there exists smooth initial data (u_0, ρ_0) for (0.2) such that $|\nabla(u_0, \rho_0)| = \mathcal{O}(1/\varepsilon)$ and the corresponding solution (u, ρ) develops a shock-type singularity within time $\mathcal{O}(\varepsilon)$.

The isentropic compressible Euler equations on a Riemannian manifold (M, g) have the form

$$\begin{cases} \partial_t \rho + \nabla_v \rho + \rho \operatorname{div} v = 0, \\ \partial_t v + \nabla_v v + \frac{\nabla p}{\rho} = 0, \\ \rho = \frac{1}{\gamma} \rho^\gamma. \end{cases} \tag{0.3}$$

We present a shock formation result for (0.3) on the two-dimensional sphere $M = \mathbb{S}^2$.

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Department of Mathematics, Faculty of Sciences, National University of Singapore, 10 Lower Kent Ridge Road, Singapore 119076

Email address: wenzesu@nus.edu.sg

A NEW PHASE TRANSITION IN COSMOLOGICAL FLUID DYNAMICS

ZOE WYATT

Classification AMS 2020: 35Q76, 35Q31

Keywords: Einstein–Euler equations, relativistic fluids, cosmology, shock formation

We are interested in the dynamics of fluid–filled cosmological spacetimes. Under the cosmological principle of isotropy and local homegeneity, the relevant spacetimes have topology $\mathbb{R}_+ \times M$ with Lorentzian metric

$$g = -(\mathrm{d}t)^2 + a(t)^2 g_0 \,. \tag{0.1}$$

Here (M, g_0) is a 3-dimensional Riemannian manifold with constant sectional curvature and $a : \mathbb{R}_+ \to \mathbb{R}_+$ is a monotonically increasing function called the scale factor.

Under these strong symmetry assumptions, the Einstein equations significantly restrict the form of the energy-momentum tensor to that of an idealised (constant entropy, viscous-free) fluid. The divergence of this energy-momentum tensor leads to the following relativistic Euler equations

$$\begin{aligned} u^\mu \nabla_\mu \rho + (\rho + P)\nabla_\mu u^\mu = 0, \\ (\rho + P)u^\mu \nabla_\mu u^\nu + (g^{\mu\nu} + u^\mu u^\nu)\partial_\mu P = 0 \,. \end{aligned} \tag{0.2}$$

Here, the functions ρ and P denote the energy density and pressure of the fluid, respectively, while u^ν is the velocity vector field of the fluid normalised as $u^\nu u_\nu = -1$. Standard cosmological models, such as ΛCDM, use a linear barotropic equation of state

$$P = K\rho \,, \tag{0.3}$$

where $K = c_S^2$ is the square of the speed of sound of the fluid taking values $K \in [0, 1/3]$. Solutions (ρ, u), where $\rho = \rho(t)$ and $u^\mu = (1, 0, 0, 0)$ are referred to as quiet fluid states.

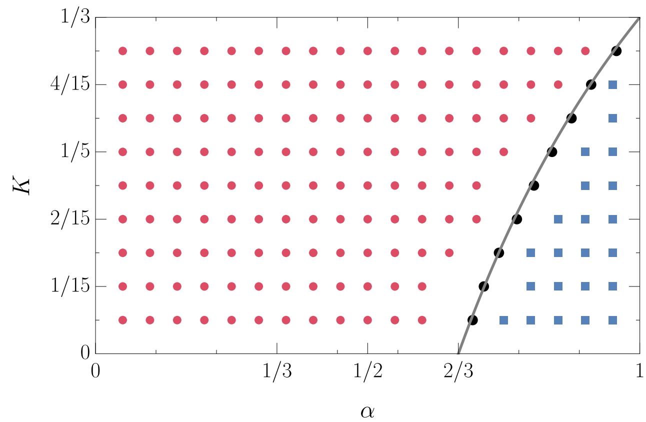
We can study the dynamics of quiet fluid states through an initial value problem for the nonlinear Euler equations (0.2) on a fixed Lorentzian geometry, or under the full Einstein–Euler equations (which we sometimes refer to as gravitational backreaction). It is known by work of Christodoulou that in the absence of expansion (i.e. on a fixed Minkowski spacetime where $a(t) \equiv 1$) irrotational fluid perturbations generically form shocks (i.e. gradient blow-up of the fluid variables) in finite time [2]. In particular, this holds for solutions which arise from initial data arbitrarily close to a quiet fluid state, and so we refer to this scenario in the following as *unstable*.

Brauer, Rendall and Reula were the first to study the dynamics of quiet fluid solutions in the regime of exponential expansion, i.e. $a(t) = \exp(\sqrt{\Lambda} t)$, albeit in a Newton–Cartan theory with gravitational backreaction [1]. They showed that the fluid solutions arising from initial data sufficiently close to a quiet fluid state, exist globally in the future expanding direction and do not form shocks. Building on work of Ringström [13], Rodnianski and Speck [14] initiated a long series of works establishing the future global stability of quiet fluid solutions under the full Einstein–Euler equations in the regime of exponential expansion [9, 10, 11, 12, 15]. We refer to this behaviour of the fluid as *stable*. The broad overview is that fast spacetime expansion leads to dissipative terms in the PDEs (essentially coming from $\ddot{a}/a = \Lambda > 0$), which regularise the fluid and prevent shocks from forming.

A natural question is whether there exists a critical expansion rate $a(t) = t^{\alpha_{\text{crit}}}$ where the stabilizing effect is too weak to regularize the fluid. Remarkably, for the case of linear expansion $a(t) = t$ it was shown by Speck that radiation fluids $(K = 1/3)$ are unstable and dust $(K = 0)$ is stable [16]. The stability of dust with backreaction was shown in [6]) while some of the present authors showed that fluids with $K \in (0, 1/3)$ are stable [5, 7] even in the presence of backreaction. These results give the first hint that the stabilisation effect depends on both the expansion rate and the speed of sound of the fluid. Moreover, our early universe is believed to have undergone an epoch of decelerated expansion, where $\dot{a} > 0$ yet $\ddot{a} < 0$, and so we aim to probe such regimes.

We present two recent works concerning a novel phase transition of fluid behaviour [3, 4]. For brevity, we outline here only [4]. We consider perturbations of a quiet fluid propagating under (0.2) on a fixed power-law geometry $\mathbb{R}_+ \times \mathbb{T}^3$ with metric $g = -(\mathrm{d}t)^2 + (t^\alpha)^2(\mathrm{d}\mathbf{x})^2$. Decelerated expansion corresponds to $\alpha < 1$. Our high-precision numerical scaling analysis of (0.2) for initial data with $\mathbb{T}^2$−symmetry provides strong evidence that the critical expansion rate, where the fluid behaviour changes from stable to unstable, is given (below left) by

(0.4) $$\alpha_{\text{crit}} = \frac{2}{3(1-K)}, \quad K \in (0, 1/3)\,.$$



In the above right, we show the numerical results in the (K, α) parameter space, with stable behaviour indicated by blue squares, unstable behaviour by red circles and the critical line (0.4) marked in black. To complement these numerical results, we are able to rigorously prove the stable behaviour in the region $\alpha > \alpha_{\text{crit}}$. The theorem below states the precise result. It requires a rescaling of the system (0.2). For that purpose we introduce the expansion-normalized variables

(0.5) $$v^i = \frac{t^\alpha u^i}{\sqrt{1 + t^{2\alpha} u^2}} \qquad \text{and} \qquad L = \log(t^{3\alpha(1+K)}\rho).$$

In these variables, (0.2) read

(0.6) $$\begin{aligned} \partial_t v^i = & -\frac{\alpha(1-3K)}{t} v^i - \frac{K}{1+K}\frac{t^{1-\alpha}}{t}(1-v^2)\partial_i L - \frac{t^{1-\alpha}}{t} v^j \partial_j v^i\,, \\ & + \frac{t^{1-\alpha}}{t}\left(1 - \frac{1-K}{1-Kv^2}\right) v^i \partial_j v^j + \frac{t^{1-\alpha}}{t}\frac{1-K}{1+K}\left(1 - \frac{1-K}{1-Kv^2}\right) v^i v^j \partial_j L \end{aligned}$$

$$+\frac{\alpha(1-3K)}{t}\frac{1-K}{1-Kv^2}v^2v^i\,.$$

$$\partial_t L = -\frac{t^{1-\alpha}}{t}\frac{(1+K)}{1-Kv^2}\partial_j v^j - \frac{t^{1-\alpha}}{t}\frac{(1-K)}{1-Kv^2}v^j\partial_j L + \frac{\alpha(1+K)}{t}\frac{1-3K}{1-Kv^2}v^2 \tag{0.7}$$

The main theorem then reads as follows, where $\overline{v}$ and $\overline{L}_0$ denote the respective spatial mean values.

Theorem ([4]) *Let* $0 < K < 1/3$ *with* $\alpha > \frac{2}{3(1-K)}$. *Let* $\mu > 0$ *be such that* $\alpha(1-3K)-\mu > 2(1-\alpha)$. *Let* $(v_0, L_0) \in H^3(\mathbb{T}^3) \times H^3(\mathbb{T}^3)$ *be a vector field and a function, respectively. Then there exists an* $\varepsilon > 0$ *such that for*

$$\overline{v}_0 + \|\nabla v_0\|_{H_2} + \|\nabla L_0\|_{H_2} < \varepsilon$$

the solution $(v(t), L(t))$ *to the system* (0.6), (0.7) *with initial data* (v_0, L_0) *exists future-global in time and the following decay rates hold.*

$$\begin{aligned} |\overline{v}(t)| &\leq C\varepsilon t^{(-\alpha(1-3K)+\mu)/2} \\ \|L(t) - \overline{L}_0\|_{L^\infty} &\leq C\varepsilon \\ \|\nabla v(t)\|_{H_2} + \|\nabla L(t)\|_{H_2} &\leq C\varepsilon t^{(-\alpha(1-3K)+\mu)/2} \end{aligned} \tag{0.8}$$

The theorem implies that the expansion-normalized variables remain close to the background, while the normalized velocity even decays. This implies orbital stability of the corresponding homogeneous fluid solutions. This constitutes the first result on fluid stabilization in the regime of decelerated expansion, with numerous further lines of investigation now being pursued.

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Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Wilberforce Road, Cambridge, CB3 0WA, UK.
Email address: zoe.wyatt@maths.cam.ac.uk

EFFECTIVENESS OF LITTLEWOOD-PALEY THEORY IN THE STUDY OF TURBULENCE AND MACHINE LEARNING

TSUYOSHI YONEDA

Classification AMS 2020: 42-04

Keywords: Fourier uncertainty principle, realtime filter, Bayesian optimization, Markov algorithm

In the IMS program on "Singularities in Fluids and General Relativity", I talked on recent works on turbulence (both physics and mathematics) and machine learning in terms of scale decomposition. In this report, we focus on the machine learning study, more precisely, we focus on a new method to find a suitable realtime filter for discrete time series data (note that the Python code of the realtime filter is available in the repository [2], see also [1]).

To find the appropriate filter, we use Bayesian optimization and the following weight function of the moving average:

$$\Psi(t) = \underbrace{\left(d_1 \cos\left(\frac{t}{\pi r_1}\right) + d_2 \cos\left(\frac{t}{\pi r_2}\right)\right)}_{passing\ freq} \underbrace{\frac{(w-t)^c}{w^c}}_{band\ width} \quad \text{for} \quad t \in [0, w],$$

where $d_1, d_2, r_1, r_2, c, w > 0$ are the parameters of the corresponding objective function (0.3). In what follows, we explain how to obtain this objective function. Let $\Omega \subset \mathbb{Z}$ be a prescribed finite sequence. Then we apply the weighted moving average to the original data $y : \Omega \to \mathbb{R}$ as follows:

$$\text{filtered data:} \quad y^*(t) := \sum_{t'=0}^{w} y(t-t')\Psi(t') \quad \text{for} \quad t \in \Omega.$$

We emphasize that since the weight function Ψ is localized in the timeline, in particular, it does not contain any future information, this bandwidth cannot have a unique threshold due to the Fourier uncertainty principle. Here is a rough explanation of the bandwidth. If c is close to zero, then the Fourier transform of the corresponding weight function is close to the sinc function, which represents narrow bandwidth (compared to the case when c is large). To the contrary, if c tends to infinity, then the Fourier transform of the corresponding weight function tends to identically one (since Ψ tends to the Dirac-delta function), which represents broad bandwidth (compared to the case when c is small). The main philosophy of this filter method is to effectively reduce the "indefinite factor" (such as Gaussian noise) without using the conventional Fourier transform, since the time domain of the training data Ω is, of course, always finite, and, applying the Fourier transform to such finite elements is no longer mathematically rigorous. To effectively reduce the indefinite factor, we use the idea of "dictionary", in other word, "key-value pairs". First, we discretize the range of the filtered data y^* as follows: For an integer K which is greater than 1, we choose $\{a_k\}_{k=1}^K \subset \mathbb{R}$ such that $a_1 < a_2 < \cdots < a_K$.

Remark 0.1. *To determine the positions of $\{a_k\}_{k=1}^K$, we have applied the cumulative distribution function to y^*, more precisely, $\#\{t \in \Omega : a_k \leq y^*(t) < a_{k+1}\}$ $(k = 1, 2, \cdots, K-1)$, $\#\{t \in \Omega : y^*(t) < a_1\}$ and $\#\{t \in \Omega : a_K \leq y^*(t)\}$ to be almost the same integer.*

Using this $\{a_k\}_{k=1}^K$, we classify the filtered data y^* of each function value as follows:

$$\tilde{y}(t) := \text{argmin}_{a \in \{a_k\}_{k=1}^K} |y^*(t) - a| \quad \text{for} \quad t \in \Omega. \tag{0.1}$$

To perform the pattern classification, we first choose the maximum and minimum length of the pattern a-priori: L_{max} and L_{min} ($L_{max} > L_{min} \geq 1$), and let σ_n^L ($n = 1, 2, \cdots, N_L$) be a permutation operator (we say "key") such that

$$\sigma_n^L : \{1, 2, \cdots, L\} \to \{a_1, a_2, \cdots, a_{K-1}, a_K\} \qquad (\ell \mapsto \sigma_n^L(\ell))$$

for $L \in [L_{min}, L_{max}] \cap \mathbb{Z}$ with $\sigma_n^L \neq \sigma_{n'}^L$ ($n \neq n'$). Then there exists a suitable N_L such that the permutation operators $\{\sigma_n^L\}_{n=1}^{N_L}$ satisfy the following two properties:

$$\begin{cases} \text{For any } t \in \Omega, \text{ there is } n \in \{1, \cdots, N_L\} \text{ such that} \\ \qquad \sigma_n^L(\ell) = \tilde{y}(t-\ell) \text{ for } \ell = 1, 2, \cdots, L, \\ \text{For any } n \in \{1, \cdots, N_L\} \text{ there is } t \in \Omega \text{ such that} \\ \qquad \sigma_n^L(\ell) = \tilde{y}(t-\ell) \text{ for } \ell = 1, 2, \cdots, L. \end{cases} \tag{0.2}$$

Note that $N_L \leq K^L$ due to the sequence with repetition. We now define sets of discrete-time $\Sigma_n^L \subset \Omega$ such that

$$\Sigma_n^L := \{t \in \Omega : \sigma_n^L(\ell) = \tilde{y}(t-\ell) \quad \text{for} \quad \ell = 1, 2, \cdots, L\}.$$

Then we can reduce the indefinite factor by using $\{\tilde{y}(t)\}_{t \in \Sigma_n^L}$ which is called "value". Let

$$\Pi_n^L(k) := \{t \in \Omega : \sigma_n^L(\ell) = \tilde{y}(t-\ell) \quad \text{for} \quad \ell = 1, 2, \cdots, L \quad \text{and} \quad \tilde{y}(t) = a_k\},$$

and we maximize the following objective function under the condition that the autocorrelation between the original and the filterd data is not small:

$$\frac{\displaystyle\sum_{L=L_{min}}^{L_{max}} \# \left\{ n : \text{there is a } k \in \{1, 2, \cdots, K\} \text{ such that } \frac{\#\Pi_n^L(k)}{\#\Sigma_n^L} \geq \gamma \right\}}{\displaystyle\sum_{L=L_{min}}^{L_{max}} N_L}, \tag{0.3}$$

where $\gamma \in (0, 1]$ is the match rate of a_k for each σ_n^L. Here, γ, as well as L_{max} and L_{min} are prescribed parameters in the numerical implementation. γ is close to 1. L_{max} and L_{min} are determined so that the length of frequent patterns are sufficiently covered.

Remark 0.2. *The well-known Kalman filter is also a type of realtime filter. The Kalman filter is based on the following assumption:*

- *Observational variable can be decomposed into unobservable variable and Gaussian noise.*

While this assumption is suitable for predicting planetary behavior, it might not be suitable for multiscale turbulence problems, where the "closure problem" is always present. The realtime filters in this report are nonparametric and thus do not require any assumptions on the noise.

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Graduate School of Economics, Hitotsubashi University, Tokyo 186-8601, Japan
Email address: `t.yoneda@r.hit-u.ac.jp`

LOW-REGULARITY LOCAL WELL-POSEDNESS FOR THE ELASTIC WAVE SYSTEM

SIFAN YU

Classification AMS 2020: 35Q74, 35L15, 35L72

Keywords: Elastic wave equations, Low regularity, Local well-posedness, Strichartz estimates, Vector field method

In this paper, we initiate the study of low-regularity local well-posedness for elastic waves in three spatial dimensions. For homogeneous isotropic hyperelastic materials, the motion of displacement $\vec{U} = (U^1, U^2, U^3)$ is governed by a *quasilinear* wave system with *multiple wave-speeds*:

$$\partial_t^2 \vec{U} - c_2^2 \Delta \vec{U} - (c_1^2 - c_2^2)\nabla(\nabla \cdot \vec{U}) = \vec{\mathcal{G}}(\nabla \vec{U}, \nabla^2 \vec{U}). \tag{0.1}$$

Here, c_1, c_2 are two constants representing respectively the speeds of longitudinal and transverse waves with $c_1 > c_2 > 0$. The nonlinear form $\vec{\mathcal{G}}(\nabla \vec{U}, \nabla^2 \vec{U})$ is linear in $\nabla^2 \vec{U}$ and is fixed by different materials.

To derive equations (0.1), we start from considering the motion of a 3D elastic body, which can be described by time-dependent orientation-preserving diffeomorphisms, denoted by $\vec{P} : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$. Here, $\vec{P} = \vec{P}(\vec{x}, t)$ verifies $\vec{P}(\vec{x}, 0) = \vec{x} = (x^1, x^2, x^3)$. The deformation gradient is further defined as $F := \nabla P$ with $F^i_j := \partial P^i / \partial x^j$. For a homogeneous isotropic hyperelastic material, a physical quantity called the stored energy W depends solely on the principal invariants of FF^T. Here, FF^T is called the Cauchy-Green strain tensor. With $\vec{P}$ and W, we can define the action functional $\mathcal{S}$:

$$\mathcal{S} := \int \int_{\mathbb{R}^3} \left\{ \frac{1}{2} |\partial_t \vec{P}|^2 - W(FF^T) \right\} \mathrm{d}x \mathrm{d}t. \tag{0.2}$$

Applying the least action principle, the resulting Euler-Lagrange equations are given by:

$$\frac{\partial^2 P^i}{\partial t^2} - \frac{\partial}{\partial x^l} \frac{\partial W(FF^T)}{\partial F^i_l} = 0. \tag{0.3}$$

Let $\vec{U} := \vec{P} - \vec{x}$ denote the displacement. Via (0.3), we then derive the elastic wave system (0.1).

0.1. **Main Result.** In this paper, we focus on the elastic wave equations for the admissible harmonic elastic materials, which belong to the class of harmonic elastic materials *described by John* in [2, 3] and are associated with remarkable decoupling properties. The equations that govern the motion of the admissible harmonic materials take the form:

$$\partial_t^2 \vec{U} - c_2^2 \Delta \vec{U} - (c_1^2 - c_2^2)\nabla(\nabla \cdot \vec{U}) = \mathrm{div} \left\{ G(\partial \vec{U}) I + b \det(F)(F^T)^{-1} \right\}. \tag{0.4}$$

Here, I is the identity matrix, b is a constant related to the material, F^T is the transpose of F, $\det(F)$ is the determinant of F, and $G(\partial \vec{U})$ is a smooth scalar function of $\partial \vec{U}$. Both

(0.1) and (0.4) are multi-wave-speeds systems. For equation (0.4), we now state the main theorem of this article.

Theorem 0.1 (Main theorem). *We study the admissible harmonic elastic wave system* (0.4) *with initial data $\vec{U}|_{\Sigma_0} := \vec{\phi}|_{\Sigma_0} + \vec{\psi}|_{\Sigma_0}$. Here, $\vec{\phi}$ corresponds to the "divergence-part" and $\vec{\psi}$ corresponds to the "curl-part" of the displacement. Let D be a fixed positive constant and $i = 1, 2, 3$. Let $\breve{\mathcal{R}}$ be the region of hyperbolicity. For any real number $3 < N < 7/2$, assume that $\vec{\phi}$ and $\vec{\psi}$ satisfy the following initial conditions:*

(1) $\|\phi^i\|_{H^N(\Sigma_0)} + \|\psi^i\|_{H^{N+1}(\Sigma_0)} \leq D$*, with D being a fixed positive constant and $i = 1, 2, 3$.*

(2) *The initial data functions lie within the interior of $\breve{\mathcal{R}}$.*

Then, for (0.4)*, the solution's time of classical existence $T > 0$ can be bounded from below in terms of D. Furthermore, the Sobolev regularity of the data is preserved during the propagation of the solution throughout the time slab of classical existence. Specifically, this means that*[1]

$$\left\|\phi^i\right\|_{L_t^\infty H_x^N([0,T]\times\mathbb{R}^3)} \lesssim 1, \tag{0.5a}$$

$$\left\|\psi^i\right\|_{L_t^\infty H_x^{N+1}([0,T]\times\mathbb{R}^3)} \lesssim 1. \tag{0.5b}$$

In addition, ϕ and ψ satisfy the following Strichartz estimates[2]*:*

$$\left\|\boldsymbol{\partial}^2\phi^i\right\|_{L_t^2 L_x^\infty([0,T]\times\mathbb{R}^3)} \lesssim 1, \tag{0.6a}$$

$$\left\|\boldsymbol{\partial}^3\psi^i\right\|_{L_t^2 L_x^\infty([0,T]\times\mathbb{R}^3)} \lesssim 1. \tag{0.6b}$$

Remark 0.2. *For general 3D elastic wave system, in* [1]*, An-Chen-Yin proved the H^3 ill-posedness, which is driven by instantaneous shock formation. The "divergence-part" in* [1] *is in H^3, and the "curl-part" in* [1] *is smooth. Therefore, the*[3] *H^{3+} local well-posedness (for the divergence part) obtained in Theorem 0.1 is the desired result for the admissible harmonic materials.*

Theorem 0.1 establishes optimal low-regularity local well-posedness for the admissible harmonic elastic wave system. This will be fundamental for studying this model both mathematically and numerically. Under minimal low-regularity initial conditions, the above theorem shows that singularity formation, including shock formation from sound wave compression, can be avoided at least for short times. Proving the above theorem requires a thorough understanding of the solution dynamics and the associated acoustic geometries of both the "divergence part" and the "curl part," which feature different characteristic speeds. Meanwhile, although this is a local result, to work with optimal regularity, we need to rescale the spacetime with respect to a large frequency λ and prove a decay estimate over the long rescaled time. This decay estimate is essential for proving a frequency-localized Strichartz estimate, which yields (0.6) via Duhamel's principle.

[1]In this article, $A \lesssim B$ means $A \leq C \cdot B$ for some universal constant C.

[2]We denote the schematic spatial partial derivatives by ∂ and the schematic space-time partial derivatives by $\boldsymbol{\partial}$.

[3]Here, H^{3+} means $H^{3+\varepsilon}$ for arbitrarily small ε.

To establish Theorem 0.1, we first explore the notable structures of the admissible harmonic elastic materials, which yield decoupling properties (not necessarily present in elastic materials in general) and regularity gains for both the faster-wave part and the slower-wave part.

Compared to previous low-regularity results, we face new challenges arising from the multiple wave-speed nature of the system. Specifically, the acoustic metric g of the faster-wave depends on both the faster-wave part and the slower-wave parts, requiring us to carefully analyze the corresponding geometric quantities associated with g, such as the connection coefficients. Additionally, controlling the dynamics of the faster-wave part requires a higher regularity for the "curl-part". In particular, the Ricci curvature associated with the faster-wave is one derivative rougher than that of the slower-wave dynamics. This phenomenon is common for physical quasilinear wave systems (e.g., the compressible Euler equations) featuring multiple characteristic speeds and would be a major obstacle to proving low-regularity local well-posedness results if the two parts with different traveling speeds do not exhibit strong (enough) decoupling properties or if the "curl-part" lacks the necessary structure for better regularity gains.

For the admissible harmonic elastic materials, we overcome those difficulties by fully exploiting the associated geometric structure. Specifically, we show that the "divergence-part" maintains to represent the faster-wave in the entire time of the existence of the solution. This property provides us that characteristic hypersurfaces of the faster-wave are spacelike with respect to the slower-wave. For the "curl-part", this results in a crucial coercive flux energy along the geometric cone of the faster-wave divergence part. Moreover, we carefully go through the spacetime energy estimates, a rescaled version of Strichartz estimate, a frequency-localized decay estimate, and conformal energy estimates. Meanwhile, we precisely trace and address the impact of the "curl-part" on the faster-wave "divergence-part" dynamics by a careful control of the geometry associated with the faster-wave. This includes computing complicated geometric structure equations satisfied by various appropriate geometric quantities and deriving mixed-norm estimates for those quantities.

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10 Lower Kent Ridge Rd, National University of Singapore

Email address: sifanyu@nus.edu.sg

LOW MACH NUMBER LIMIT OF NON-ISENTROPIC IDEAL MHD EQUATIONS WITH A PERFECTLY CONDUCTING BOUNDARY

JUNYAN ZHANG

Classification AMS 2020: Compressible ideal MHD, Low Mach number limit, Perfectly conducting wall.

Keywords: 35L60, 35Q35, 76M45, 76W05.

1. Introduction

The system of compressible ideal magnetohydrodynamics (MHD) describes the motion of a compressible conducting fluid in an electro-magnetic field. We consider the non-isentropic compressible ideal MHD equations in a low-Mach regime. Following the non-dimensionalization procedures as in Majda [21, Chapter 2.4], the dimensionless system in low-Mach regime is written as follows.

$$\begin{cases} aD_tq + \varepsilon^{-1}\nabla\cdot u = 0 & \text{in } [0,T]\times\Omega, \\ \rho D_t u + \varepsilon^{-1}\nabla q + \frac{1}{2}\nabla|B|^2 - B\cdot\nabla B = 0 & \text{in } [0,T]\times\Omega, \\ D_t B = B\cdot\nabla u - B(\nabla\cdot u) & \text{in } [0,T]\times\Omega, \\ \nabla\cdot B = 0 & \text{in } [0,T]\times\Omega, \\ D_t S = 0 & \text{in } [0,T]\times\Omega, \\ a = a(\varepsilon q, S) > 0, \quad \rho = \rho(\varepsilon q, S) > 0 & \text{in } [0,T]\times\bar{\Omega}. \\ u_d = B_d = 0 & \text{on } [0,T]\times\Sigma, \\ (q,u,B,S)|_{t=0} = (q_0,u_0,B_0,S_0) & \text{on } \{t=0\}\times\Omega. \end{cases} \tag{1.1}$$

Here we set $\Omega = \mathbb{R}^d_+ := \{x\in\mathbb{R}^d : x_d > 0\}$ for $d = 2,3$ with boundary $\Sigma := \{x_d = 0\}$. $\nabla := (\partial_1,\cdots,\partial_d)^{\mathrm{T}}$ is the spatial derivative. $D_t := \partial_t + u\cdot\nabla$ is the material derivative. The dimensionless unknowns, namley the fluid velocity, the magnetic field, the fluid density, the (modified) fluid pressure and the entropy, are denoted by $u = (u_1,\cdots,u_d)^{\mathrm{T}}$, $B = (B_1,\cdots,B_d)^{\mathrm{T}}$, ρ, q and S respectively. Note that the equation of S is derived from the equation of total energy and Gibbs relation. Throughout this manuscript, Einstein's summation convention is adopted and repeated indices range from 1 to d.

Defined as the ratio of characteristic fluid velocity to the sound speed, the Mach number ε is a *dimensionless* parameter that measures the compressibility of the fluid. The coefficients a,ρ in (1.1) are smooth functions satisfying

$$\rho = \rho(\varepsilon q, S) \geq \bar{\rho}_0 > 0,\ \frac{\partial\rho}{\partial p} > 0, \quad \text{in } \bar{\Omega}, \tag{1.2}$$

for some positive constant $\bar{\rho}_0$ and $a = a(\varepsilon q, S) := \frac{1}{\rho}\frac{\partial\rho}{\partial p} > 0$, where $p := 1+\varepsilon q$ is the dimensionless fluid pressure (not the modified one). For instance, we have ideal fluids

Date: February 26, 2025.

$\rho(p, S) = p^{1/\gamma}e^{-S/\gamma}$ with $\gamma > 1$ for a polytropic gas. (1.2) also guarantees the hyperbolicity of system (1.1).

The initial and boundary conditions of system (1.1) are

$$(q, u, B, S)|_{t=0} = (q_0, u_0, B_0, S_0) \text{ in } [0, T] \times \Omega, \tag{1.3}$$

$$u_d = 0, \ B_d = 0 \text{ on } [0, T] \times \Sigma, \tag{1.4}$$

where the boundary condition for u_d is the slip boundary condition, and the boundary condition for B_d shows that the plasma is closed off from the outside world by a perfectly conducting wall Σ. This together with $\nabla \cdot B = 0$ in Ω are both constraints for initial data so that the MHD system is not over-determined. As stated in [8, Chapter 4.6], this model is appropriate for the study of equilibrium, waves and instabilities of confined plasma as used in thermonuclear research and is also served as the simplest, the most relevant model to describe confined plasmas. Such configurations refer to tokamaks.

To make the initial-boundary-value problem (1.1)-(1.4) solvable, we need to require the initial data satisfying the compatibility conditions up to certain order. For $m \in \mathbb{N}$, we define the m-th order compatibility conditions to be

$$B_d|_{t=0} = 0 \text{ and } \partial_t^j u_d|_{t=0} = 0 \quad \text{on } \Sigma, \ 0 \leq j \leq m. \tag{1.5}$$

It should be noted that (1.5) indicates $\partial_t^j B_d|_{t=0}$ on Σ for $0 \leq j \leq m$ and we refer to Trakhinin [35, Section 4.1] for the proof.

1.1. **An overview of previous results.** As is well-known in fluid dynamics, we can formally derive the incompressible fluid equations from the compressible ones, which corresponds to passing to the limit in certain dimensionless form as the Mach number goes to zero. In particular, for inviscid fluids, taking low Mach number limit is a singular limit process of a hyperbolic system with large parameter, for example the coefficients ε^{-1} in (1.1). To study such singular limit, we shall classify the initial data to be two types.

- Well-prepared initial data: $\nabla \cdot u_0 = O(\varepsilon^k), \ \nabla q_0 = O(\varepsilon^k)$ for $k \geq 1$. In such case, the compressible data is exactly a slight perturbation of a given incompressible data. Uniform estimates in low Mach regime immedately give the strong convergence thanks to the uniform boundedness of first-order time derivatives.
- General initial data: $\nabla \cdot u_0 = O(1), \ \nabla q_0 = O(1)$. In such case, the compressible data includes a large perturbation which is actually a highly oscillatory acoustic wave that propagates with a speed of $O(1/\varepsilon)$. One has to filter such acoustic wave and find suitable function spaces for the *strong* convergence.

The low Mach number limit of Euler equations in $\mathbb{R}^d, \mathbb{T}^d$ or a fixed domain has been studied extensively and we refer to [17, 18, 7, 26, 27] for the case of well-prepared data and [39, 3, 12, 11, 28, 32, 22, 23, 1] for the case of general initial data and references therein. However, the study of singular limits in MHD are much less developed than that of Euler equations due to the strong coupling among the fluids, sound waves and magnetic fields, and there are still many unsolved problems since Majda raised open problems in this area in [21, pp. 71-72]. The existing literature mainly focuses on the Cauchy problem in $\mathbb{R}^d$, including the multi-scale singular limits and we refer to [13, 5, 6]. In contrast, the singular limits of ideal MHD in a domain with boundaries become even more subtle. Under the perfectly conducting wall condition, Ohno-Shirota [24] proved that the linearized problem near a non-zero magnetic field is *ill-posed* in standard Sobolev spaces. The well-posedness results are referred to Yanagisawa-Matsumura [41] and Secchi [29, 30, 31, 33] under the setting of anisotropic Sobolev spaces, which were first introduced by Shuxing Chen [4]. In contrast, the corresponding incompressible problem is still well-posed in standard Sobolev spaces [9].

The low Mach number limit for this problem was first proved by Ju-Wang [16] in the case of isentropic general data and non-isentropic well-prepared data by using a suitable closed subspace of standard Sobolev space (introduced in [30]), but more restrictive constraints for the boundary value of initial data are required whose physical interpretation is unclear. We also refer to a very recent result by Secchi [34] about the isentropic problem with general initial data by using another anisotropic Sobolev spaces defined in [33].

We also remark that the direction of the magnetic field is crucially important for the study of ideal MHD in a domain with boundaries even if one only studies the local existence. For example, when the magnetic field is not tangential to the boundary, one can use the transversality of the magnetic field to compensate the loss of normal derivative and we refer to [40] for the local existence in standard Sobolev spaces. See also the results about the singular limits in [15, 14].

By observing a key structure, we prove the incompressible limit for the non-isentropic problem with well-prepared initial data [36], and the function spaces that we used are exactly the same as the ones for their well-posedness, that is, the anisotropic Sobolev norms converge to an energy functional defined in standard Sobolev spaces. The framework of [36] is also generalized to the study of free-boundary problems [43, 44] by the author. But *none of the existing works applies to the case of non-isentropic problems with general initial data.*

The aim of our presenting work [38] is to rigorously justify the singular limit in low Mach number regime for ideal MHD with the perfectly conducting wall condition *in the non-isentropic case with general initial data*. Our framework can cover the existing works and does not require extra boundary constraints as in [16]. The proof is based on the combination of several key observations and techniques: a special structure in vorticity analysis that illustrates the "mismatch" between the anisotropic norms and the standard Sobolev norms, enhanced regularity of the entropy in the direction of the magnetic field, the usage of material derivative D_t instead of ∂_t when defining the energy functional and the application of "modified Alinhac good unknowns" to overcome the difficulty brought by the anisotropy of the function spaces.

1.2. **Anisotropic Sobolev spaces.** Before stating our results, we should first define the anisotropic Sobolev space $H^m_*(\Omega)$ for $m \in \mathbb{N}$ and $\Omega = \mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x_d > 0\}$, which was first introduced by Shuxing Chen [4]. Let $\omega = \omega(x_d)$ be a cutoff function[1] on $[0, +\infty)$ defined by $\omega(x_d) = \frac{x_d}{1+x_d}$. Then we define $H^m_*(\Omega)$ for $m \in \mathbb{N}^*$ as follows

$$H^m_*(\Omega) := \left\{ f \in L^2(\Omega) \middle| (\omega\partial_d)^{\alpha_{d+1}} \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} f \in L^2(\Omega),\ \forall \alpha \text{ with } \sum_{j=1}^{d-1} \alpha_j + 2\alpha_d + \alpha_{d+1} \le m \right\},$$

[1]The choice of $\omega(x_d)$ is not unique. We just need $\omega(x_d)$ vanishes on Σ, comparable to the distance function near Σ and comparable to 1 far away from Σ.

equipped with the norm

$$\|f\|^2_{H^m_*(\Omega)} := \sum_{\sum_{j=1}^{d-1}\alpha_j+2\alpha_d+\alpha_{d+1}\le m} \|(\omega\partial_d)^{\alpha_{d+1}}\partial_1^{\alpha_1}\cdots\partial_d^{\alpha_d} f\|^2_{L^2(\Omega)}. \tag{1.6}$$

For any multi-index $\alpha := (\alpha_0, \alpha_1, \cdots, \alpha_d, \alpha_{d+1}) \in \mathbb{N}^{d+2}$, we define

$$\partial_*^\alpha := \partial_t^{\alpha_0}(\omega\partial_3)^{\alpha_{d+1}}\partial_1^{\alpha_1}\cdots\partial_d^{\alpha_d},\ \langle\alpha\rangle := \sum_{j=0}^{d-1}\alpha_j + 2\alpha_d + \alpha_{d+1},$$

and define the **space-time anisotropic Sobolev norm** $\|\cdot\|_{m,*}$ to be

$$\|f\|^2_{m,*} := \sum_{\langle\alpha\rangle\le m}\|\partial_*^\alpha f\|^2_{L^2(\Omega)} = \sum_{\alpha_0\le m}\|\partial_t^{\alpha_0} f\|^2_{H_*^{m-\alpha_0}(\Omega)}. \tag{1.7}$$

We also denote the interior Sobolev norm to be $\|f\|_s := \|f(t,\cdot)\|_{H^s(\Omega)}$ for any function $f(t,x)$ on $[0,T]\times\Omega$ and denote the boundary Sobolev norm to be $|f|_s := |f(t,\cdot)|_{H^s(\Sigma)}$ for any function $f(t,x)$ on $[0,T]\times\Sigma$.

From now on, we assume the dimension to be $d=3$, that is, $\Omega = \mathbb{R}^3_+ = \{x\in\mathbb{R}^3 : x_3 > 0\}$ and $\Sigma = \{x_3 = 0\}$. The 2D case follows in the same manner as in [36, Section 3.5].

1.3. **Main results.** Now, we establish a local-in-time estimate that is uniform in Mach number ε for general initial data.

Theorem 1.1 ([38], **Uniform-in-ε estimate**). *Let $\varepsilon > 0$ be fixed. Let $(q_0, u_0, B_0, S_0) \in H^8(\Omega)\times H^8(\Omega)\times H^8(\Omega)\times H^9(\Omega)$ be the initial data of* (1.1) *satisfying the compatibility conditions* (1.5) *up to 7-th order and*

$$E(0) \le M \tag{1.8}$$

for some $M > 0$ independent of ε. Then there exists $T > 0$ depending only on M, such that the solution to (1.1) *satisfies the energy estimate*

$$\sup_{t\in[0,T]} E(t) \le P(E(0)), \tag{1.9}$$

where $P(\cdots)$ is a generic polynomial in its arguments, and the energy $E(t)$ is defined to be

$$E(t) := \sum_{l=0}^{4} E_{4+l}(t), \quad E_{4+l}(t) := \sum_{k=0}^{4-l}\left\|(\varepsilon D_t)^{k+2l}(q,u,B,S,\rho^{-1}B\cdot\nabla S)\right\|^2_{4-k-l}. \tag{1.10}$$

Remark 1.2 (Relations with anisotropic Sobolev space). *The energy functional $E(t)$ above is considered as a variant of $\|\cdot\|_{8,*}$ norm at time $t>0$. In fact, the slip boundary condition implies that D_t must be a tangential derivative on the boundary Σ, and so the "anisotropic order" of $\|D_t^{k+2l}f\|_{4-k-l}$ is at most $(k+2l)+2\times(4-k-l) = 8-k \le 8$. The Mach number weights are determined according to the number of material derivatives such that the energy estimate for the above "modified $\|\cdot\|_{8,*}$ norm" is uniform in $\varepsilon > 0$. In particular, the first-order time derivatives of u, q are not bounded uniformly in ε, which corresponds to the case of general initial data.*

Remark 1.3 (Well-posedness and regularity). *For the initial data given in Theorem 1.1, the local well-posedness of* (1.1) *in $H^8_*(\Omega)$ for a fixed $\varepsilon > 0$ has been proven in* [31, Theorem 2.1'] *or* [36, Theorem 1.1], *where the latter one also gives the uniform-in-ε estimates for well-prepared initial data. We assume the initial data belong to $H^8(\Omega)$ instead of $H^8_*(\Omega)$*

only because we must guarantee the initial energy $E(0) < \infty$. There is no loss of regularity in the energy estimates. We choose $H^8_*(\Omega) \hookrightarrow H^4(\Omega)$ (instead of $H^3(\Omega)$ as in Euler equations or elastodynamics [37]*) because there are lots of terms in vorticity analysis of compressible ideal MHD that requires the L^∞ boundedness of the second-order derivatives.*

Remark 1.4 (Enhanced regularity of the entropy). *We assume $S_0 \in H^9(\Omega)$ (actually $H^9_*(\Omega)$) because we need the $H^8_*(\Omega)$-regularity of the directional derivative $(\rho^{-1}B\cdot\nabla)S$ to control the vorticity instead of the full $H^9_*(\Omega)$ regularity. This enhanced "directional" regularity of S can be propagated from initial data because D_t commutes with $(\rho^{-1}B\cdot\nabla)$.*

The next main theorem concerns the low Mach number limit. We consider the inhomogeneous MHD equations together with a transport equation satisfied by (u^0, B^0, π, S^0):

$$
\begin{cases}
\varrho(\partial_t u^0 + u^0\cdot\nabla u^0) - B^0\cdot\nabla B^0 + \nabla(\pi + \frac{1}{2}|B^0|^2) = 0 & \text{in } [0,T]\times\Omega,\\
\partial_t B^0 + u^0\cdot\nabla B^0 - B^0\cdot\nabla u^0 = 0 & \text{in } [0,T]\times\Omega,\\
\partial_t S^0 + u^0\cdot\nabla S^0 = 0 & \text{in } [0,T]\times\Omega,\\
\nabla\cdot u^0 = \nabla\cdot B^0 = 0 & \text{in } [0,T]\times\Omega,\\
u^0_3 = B^0_3 = 0 & \text{on } [0,T]\times\Sigma.
\end{cases}
\tag{1.11}
$$

Theorem 1.5 ([38], **The low Mach number limit**). *Under the hypothesis of Theorem 1.1, we assume that $(u_0, B_0, S_0) \to (u^0_0, B^0_0, S^0_0)$ in $H^4(\Omega)$ as $\varepsilon \to 0$ with $\nabla\cdot B^0_0 = 0$ in Ω and $u^0_{0d} = B^0_{0d} = 0$ on Σ, and that there exist positive constants N_0 and σ such that S_0 satisfies*

$$
|S_0(x)| \le N_0|x|^{-1-\sigma}, \quad |\nabla S_0(x)| \le N_0|x|^{-2-\sigma}.
\tag{1.12}
$$

Then it holds that

$$(q,u,B,S) \to (0,u^0,B^0,S^0) \quad \text{weakly-* in } L^\infty([0,T];H^4(\Omega)) \text{ and strongly in } L^2([0,T];H^{4-\delta}_{\text{loc}}(\Omega))$$

for $\delta > 0$. $(u^0, B^0, S^0) \in C([0,T];H^4(\Omega))$ solves (1.11) *with initial data $(u^0,B^0,S^0)|_{t=0} = (w_0, B^0_0, S^0_0)$, that is, the incompressible MHD system together with a transport equation of S^0, where $w_0 \in H^4(\Omega)$ is determined by*

$$
w_{0d}|_\Sigma = 0, \quad \nabla\cdot w_0 = 0, \quad \nabla\times(\rho(0,S^0_0)w_0) = \nabla\times(\rho(0,S^0_0)u^0_0).
\tag{1.13}
$$

Here ϱ satisfies the transport equation

$$\partial_t\varrho + u^0\cdot\nabla\varrho = 0, \ \varrho|_{t=0} = \rho(0,S^0_0).$$

The function π satisfying $\nabla\pi \in C([0,T];H^3(\Omega))$ represents the fluid pressure for incompressible MHD (1.11)*. In the case of $d = 2$, $\nabla\times(\rho(0,S^0_0)w_0) = \nabla\times(\rho(0,S^0_0)u^0_0)$ should be replaced by $\nabla^\perp\cdot(\rho(0,S^0_0)w_0) = \nabla^\perp\cdot(\rho(0,S^0_0)u^0_0)$ with $\nabla^\perp = (-\partial_2,\partial_1)^{\mathrm{T}}$.*

Remark 1.6. *In the proof of the Theorem 1.1, the uniform estimates can be established regardless of the boundedness of the domain. However, in the proof of the Theorem 1.5, the unboundedness of the domain, for example assuming Ω to be the half-space or the exterior of a compact smooth domain in $\mathbb{R}^d$, is necessary due to the (global) dispersion property for the wave equation. The strong convergence in time can be obtained by proving local energy decay via the defect measure technique* [22, 1]*. The entropy decay condition* (1.12) *is also needed due to* [10, Theorem 17.2.8] *used in the proof of strong convergence.*

2. DIFFICULTIES AND STRATEGIES

(1.1)-(1.4) is a first-order hyperbolic system with characteristic boundary conditions of constant multiplicity, for which there is a potential of normal derivative loss. For Euler equations and elastodynamics inside a rigid wall with the slip boundary condition, the vorticity can be controlled in standard Sobolev spaces [27, 42], which could compensate the loss of normal derivative. However, for compressible ideal MHD, one encounters a loss of normal derivative in vorticity analysis caused by the simultaneous appearance of compressibility and the magnetic field. That is exactly one has to use the anisotropic Sobolev spaces to prove the local existence.

2.1. **Motivation to define $E(t)$: a special structure in vorticity analysis.** We are inspired by a special structure of Lorentz force in vorticity analysis to define the energy functional $E(t)$ as in (1.10). Let us take the estimate of $\|\nabla\times u\|_3^2$ in the control of $\|u\|_4^2$ for example.

In the case of non-isentropic problem with general initial data, one has to replace the coefficient $\rho(\varepsilon q, S)$ by $\rho_0 := \rho(0, S)$ to derive the evolution equation of the vorticity and the current as stated in Métivier-Schochet [22]. Or else, there must be singular terms like $\partial^\alpha\rho(\nabla\times D_t u)$ whose coefficient $\partial^\alpha\rho$ is $O(1)$ size. The advantage of such substituiton is that the unbounded terms are avoided and the extra-generated term has small coefficient and its leading-order part is curl-free.

2.1.1. *A weighted anisotropic structure contributed by the Lorentz force.* In the analysis of $\|\nabla\times(\rho_0 u)\|_3^2$, we must encounter a fifth-order term

$$\mathcal{K} := -\int_\Omega \partial^3\nabla\times(\rho_0 B)\cdot\partial^3\nabla\times(\rho_0\rho^{-1}B\nabla\cdot u)\,\mathrm{d}x,$$

which is uncontrollable in the setting of H^4. But if we insert the continuity equation $\nabla\cdot u = -\varepsilon a D_t q$, commute $\nabla\times$ with D_t and insert the momentum equation $-\nabla q = \varepsilon(\rho D_t u + B\times(\nabla\times B))$, we can see that

$$\partial^3\nabla\times(\rho_0\rho^{-1}B\nabla\cdot u) = -\varepsilon^2 a\rho^{-1}B\times\left[B\times D_t(\partial^3\nabla\times(\rho_0 B))\right] + \varepsilon^2\rho_0 aB\times(\partial^3 D_t^2 u) + \cdots$$

On the right side, the first term contributes to an energy term $-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega a\rho^{-1}|\varepsilon B\times(\partial^3\nabla\times(\rho_0 B))|^2\,\mathrm{d}x$ that gives the regularity of the Lorentz force, while the second term exhibits a "weighted" anisotropic structure that indicates us to trade one normal derivative (in the curl operator $\nabla\times$) for the second-order tangential derivative $\varepsilon^2 D_t^2$. As $\varepsilon\to 0$, this anisotropic part converges to 0, and so the incompressible problem can be studied in standard Sobolev spaces.

2.1.2. *The whole reduction procedures.* Apart from vorticity, we also need to consider divergence. The continuity equation indicates us to trade one normal derivative in $\nabla\cdot u$ for one tangential derivative $\varepsilon D_t q$. Then we can always invoke the momentum equation to convert a spatial derivative in ∇q to $\varepsilon D_t u$ and the estimate of current $\nabla\times B$ until there is no spatial derivative falling on q and no spatial derivative falling on u. All tangential derivatives generated in the above reduction procedures are material derivatives, which explains why we write D_t instead of ∂_t when defining $E(t)$.

Now, we can explain why we define $E(t)$ to be

$$\sum_{l=0}^{4}\sum_{k=0}^{4-l}\left\|(\varepsilon D_t)^{k+2l}(q,u,B,S,\rho^{-1}B\cdot\nabla S)\right\|_{4-k-l}^2$$

Let $l\in\{0,1,2,3\}$ be the number of times that we do the vorticity estimates. After such l times of reduction for $\|u,B\|_4$, we have at most $4-l$ normal derivatives left, but we also obtain $(\varepsilon D_t)^{2l}$ falling on each variable. This step corresponds to the $(\varepsilon D_t)^{2l}$-part in the energy functional. Then for each fixed $l\in\{0,1,2,3\}$, $(\varepsilon D_t)^k$-part corresponds to the reduction of pressure and divergence, which is parallel to the study of Euler equations.

2.2. **Enhanced "directional" regularity of the entropy.** We require the directional derivative $(\rho^{-1}B\cdot\nabla)S$ has the same regularity as the other variables. This fact is easy to prove, as *we observe that D_t commutes with $(\rho^{-1}B\cdot\nabla)$* which then leads to $D_t((\rho^{-1}B\cdot\nabla)S)=0$ and the desired enhanced regularity. In fact, this is even easier to be observed if we study the free-boundary MHD in Lagrangian coordinates, e.g., [19], in which the material derivative becomes ∂_t and $(\rho^{-1}B\cdot\nabla)$ becomes *time-independent*! Without the enhanced directional regularity, there must be a loss of derivative in the control of vorticity and current. In contrast, such difficulty never appears in the study of Euler equations or any simpler cases for MHD. On the other hand, our observation exactly resolves this issue. Similar argument also applies to the recent work by the author [37] about neo-Hookean elastodynamics.

2.3. **Usage of material derivatives and "modified" Alinhac good unknowns.** Now, we turn to explain why we shall use D_t instead of ∂_t to define $E(t)$. If we decompose the material derivative D_t into ∂_t, tangential spatial derivative $\overline{\partial}=\partial_1$ or ∂_2 and the co-normal part $\omega(x_3)\partial_3$, then we must separately do the tangential estimate for each of them as in [36]. However, this will again produce singular terms such as $((\varepsilon\overline{\partial})^8\rho)(D_tu)$ and we cannot substitute ρ by ρ_0 as in the vorticity control. To overcome this difficulty, we must ensure every tangential derivative to be the material derivative D_t such that we could completely eliminate the singular terms by using $D_tS=0$.

A new difficulty appears: D_t does not commute with the ∇. Although we can explicitly compute the commutators $[\partial,D_t^m]f$, an essential difficulty still exists: the top-order commutator $[\partial,D_t^8]f$ contains $(\partial D_t^7u)(\partial f)$ and $8(\partial u)(\partial D_t^7 f)$ whose $L^2(\Omega)$ norms cannot be directly controlled when $\partial=\partial_3$. To overcome this difficulty, we adopt the idea of Alinhac good unknowns [2], which reveals that the essential leading-order part in $D_t^8\partial_i f$ is not simply $\partial_i D_t^8 f$ but actually the ∂_i derivative of the Alinhac good unknowns (denoted by $\mathbf{F}^8$)

$$D_t^8\partial_i f\neq\partial_i D_t^8 f+L^2(\Omega)\text{-controllable terms}\Rightarrow D_t^8\partial_i f=\partial_i\mathbf{F}^8+L^2(\Omega)\text{-controllable terms}.$$

In other words, the core idea of using Alinhac good unknowns is to rewrite the uncontrollable terms to be the form $\partial_i(L^2$-controllable) terms and merge the terms in the parenthesis into the leading-order term, such that the covariance of the system is still preserved after being differentiated by variable-coefficient derivatives. In view of the singular limit problems, this step is used to preserve the anti-symmetric structure $E_0(U)\partial_tU+\varepsilon^{-1}\nabla U=f$ for the differentiated system.

Under the setting of standard Sobolev spaces, $\mathbf{F}^8$ is defined to be $D_t^8f-D_t^7u\cdot\nabla f$, see for example [9]. Under the setting of anisotropic Sobolev spaces, the author [19] observed that the merged terms must be modified according as the variable f. After careful and delicate calculations, the "modified" Alinhac good unknowns used in our work [38] are

$$\mathbf{U}_i^8 := D_t^8 u_i - 9 D_t^7 u \cdot \nabla u_i, \qquad \mathbf{Q}^8 := D_t^8 \check{q} - D_t^7 u \cdot \nabla \check{q}, \quad \check{q} := q + \frac{\varepsilon}{2}|B|^2.$$

Such difference is due to the fact that we must preserve the divergence structure for the velocity but there is no such structure for the pressure.

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10 Lower Kent Ridge Road, Department of Mathematics, National University of Singapore, Singapore 119076.
Email address: `zhangjy@nus.edu.sg`

FINITE-TIME BLOWUP FOR KELLER-SEGEL-NAVIER-STOKES SYSTEM IN THREE DIMENSIONS

TAO ZHOU

Classification AMS 2020: Analysis of PDEs

Keywords: Keller-Segel equation, Navier-Stokes equation, buoyancy, self-similar blowup, stability

This talk focuses on the existence of finite-time blowup solution to $3D$ coupled Keller-Segel-Navier-Stokes system with buoyancy force

$$\text{(KS-NS)}\qquad \begin{cases} \partial_t \rho + u \cdot \nabla \rho = \Delta \rho - \nabla \cdot (\rho \nabla c), \\ -\Delta c = \rho, \\ \partial_t u + u \cdot \nabla u = \Delta u - \nabla \pi - \rho e_3, \\ \nabla \cdot u = 0, \end{cases}$$

where ρ represents the cell density, while c denotes the concentration of the self-emitted chemical substance, $e_3 = (0,0,1)$ and the vector field u, representing the ambient fluid flow, satisfies the Navier-Stokes equation with an additional force term $-\rho e_3$ on the right-hand side, which denotes the cell's reaction force to the buoyancy exerted by the fluid.

In literature, the standard Keller-Segel equation

$$\text{(KS)}\qquad \begin{cases} \partial_t \rho = \Delta \rho - \nabla \cdot (\rho \nabla c), \\ -\Delta c = \rho, \end{cases}$$

has been widely studied, especially for the existence of finite-time blowup solution and the related singularity formation. In particular, the $3D$ Keller-Segel equation (KS) admits an explicit given self-similar blowup solution with form

$$\text{(Profile)}\qquad \rho(t,x) = \frac{1}{T-t} Q\left(\frac{x}{\sqrt{T-t}}\right), \quad \text{with} \quad Q(x) = \frac{4(6+|x|^2)}{(2+|x|^2)^2},$$

whose radial stability has been clarified in [1].

However, it is still an open problem whether the coupled system (KS-NS) enjoys finite-time blowup with finite mass. Unluckily, with the highly coupled fact, it is no longer easy to deal with it by using the classical method like introducing the second moment to derive a contradiction (it has been applied to (KS), see [2] for details), and we need to introduce more robust method to handle this problem.

Precisely, we choose the self-similar solution Q to (KS) given in (Profile) as an approximate solution to (KS-NS) and study the evolution of perturbation near Q in the self-similar coordinate. And the main result is as follows:

Theorem 0.1 ([3] Existence of smooth finite-time blowup solution with nonnegative density and finite mass)**.** *For any integer $s \geq 3$ and any divergence-free vector field $u_0 \in$*

$H^\infty_\sigma(\mathbb{R}^3)$ *fixed, there exists non-negative* $\rho_0 \in C^\infty_0(\mathbb{R}^3)$*, such that the smooth solution to* (KS-NS) *with initial data* (ρ_0, u_0) *blows up at some time* $t = T < \infty$*. Moreover, we have*

$$\rho(t,x) = \frac{1}{T-t}\left[Q\left(\frac{x}{\sqrt{T-t}}\right) + \varepsilon\left(t, \frac{x}{\sqrt{T-t}}\right)\right], \quad \text{for } x \in \mathbb{R}^3,\ t \in [0,T), \tag{0.1}$$

where Q *is given by* (Profile) *and* $\lim_{t\to T^-} \|\varepsilon(t)\|_{H^s(\mathbb{R}^3)} = 0$.

As for the idea of the proof, we observe that the fluid part can treated perturbatively, based on the crucial observation that the Navier-Stokes equation is subcritical in scaling with respect to the Keller-Segel equation. Hence the main task can be simplified into obtaining decay of linearized flow near Q for the Keller-Segel part.

Due to the anisotropic nature of buoyancy in (KS-NS), the radial stability of (Profile) studied in [1] is not sufficient for our argument, and we have to extend to the nonradial setting. The main idea is to use the compactness theory to split the linearized operator

$$-\mathcal{L} = \mathcal{L}_0 + \mathcal{L}', \quad \text{with } \mathcal{L}_0 f = -\Delta f + \frac{1}{2}\Lambda f \ \text{ and } \ \mathcal{L}' f = -\nabla\cdot(f\nabla\Delta^{-1}Q) - \nabla\cdot(Q\nabla\Delta^{-1}f), \tag{0.2}$$

into the form

$$-\mathcal{L} = A_0 - \frac{1}{16} + K,$$

where A_0 is maximally dissipative operator and $K : H^k \to H^k$ is a compact operator, which yields the semigroup decay modulo finite unstable directions. Then based on linear theory of $\mathcal{L}$, we are able to handle the nonlinear stability and prove the finite-codimension stability of (Profile).

Nevertheless, out of technical limitation, the finite-codimension stability result restricts us to cut off Q super far away from the origin so that it will stay in the stable manifold. Hence to find a finite-time blowup solution to (KS-NS) with finite mass, we need to modify the unstable modes and introduce modified unstable/stable subspace, which almost preserves the original spectral properties.

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Department of Mathematics, Faculty of Science, National University of Singapore, Singapore

Email address: zhoutao@u.nus.edu