Problems to accompany lectures by D. Flannery

Basic definitions. Let $\mathbb{F}$ be a field. A subgroup G of $\mathrm{GL}(n,\mathbb{F})$ is *irreducible* if the only subspaces of the n-dimensional vector space $\mathbb{F}^n$ that G leaves invariant (under matrix multiplication) are $\mathbb{F}^n$ and $\{\mathbf{0}\}$. G is *absolutely irreducible* if G stays irreducible as a subgroup of $\mathrm{GL}(n,\mathbb{E})$ for every field extension $\mathbb{E}/\mathbb{F}$; here $\mathrm{GL}(n,\mathbb{F})$ is viewed as a subgroup of $\mathrm{GL}(n,\mathbb{E})$.

A square matrix is *(upper) triangular* if it has zeros everywhere below the main diagonal; it is *(upper) unitriangular* if it has zeros everywhere below the main diagonal and 1s all down the main diagonal. Unitriangular matrices are unipotent (have all eigenvalues equal to 1), and it can be shown that a unipotent subgroup (subgroup with all elements unipotent) of $\mathrm{GL}(n,\mathbb{F})$ is conjugate to a group of unitriangular matrices.

The *transvection* $t_{i,j}(m) \in \mathrm{Mat}(n,\mathbb{F})$ has m in position (i,j), 1s down its main diagonal, and zeros everywhere else.

$G \leq \mathrm{GL}(n,\mathbb{F})$ is *monomial* if G is conjugate to a group of monomial matrices; a monomial matrix has exactly one non-zero entry in each row and column. The group of all monomial matrices in $\mathrm{GL}(n,\mathbb{F})$ is the semidirect product $D_n \rtimes S_n$, where D_n denotes the group of diagonal matrices and $S_n \cong \mathrm{Sym}(n)$ is the group of permutation matrices.

The *enveloping algebra* $\langle H\rangle_{\mathbb{F}}$ of $H \leq \mathrm{GL}(n,\mathbb{F})$ is the $\mathbb{F}$-linear span of H.

$\mathbb{Z}[1/\mu]$ denotes the subring $\{a/\mu^i \mid a \in \mathbb{Z}, i \geq 0\}$ of $\mathbb{Q}$ generated by $1/\mu$, μ a positive integer.

A group G is *residually X*, for some group-theoretic property X, if for each $g \in G \setminus \{1\}$ there exists a normal subgroup N of G such that G/N has property X and $g \notin N$.

A group G is *X-by-Y* if there is $N \trianglelefteq G$ with property X such that G/N has property Y.

A group is *virtually free* if it has a free subgroup of finite index, i.e., is free-by-finite.

For coprime positive integers m and μ, let $\varphi_m\colon \mathrm{GL}(n,\mathbb{Z}[1/\mu]) \to \mathrm{GL}(n,p)$ be the (congruence) homomorphism that reduces matrix entries modulo m.

Below, ‘dense’ means ‘Zariski-dense’.

1. Let $G \leq \mathrm{GL}(n,\mathbb{F})$. Prove that G has a normal unipotent subgroup that contains all normal unipotent subgroups of G.

2. Let $A \leq \mathrm{GL}(n,\mathbb{F})$ be abelian. Prove that if A is irreducible then the enveloping algebra $E := \langle A\rangle_{\mathbb{F}}$ is a field extension of $\mathbb{F}1_n$ of degree n. Prove that if A is absolutely irreducible then $n = 1$.

3. Prove that $G \leq \mathrm{GL}(n,\mathbb{F})$ is unipotent-by-abelian if and only if G is conjugate to a group of triangular matrices, possibly as a subgroup of $\mathrm{GL}(n,\mathbb{E})$ for some field extension $\mathbb{E}/\mathbb{F}$.

4. Prove that if $G \leq \mathrm{GL}(n,\mathbb{C})$ is irreducible, then $|G : Z(G)| \geq n^2$, where $Z(G)$ denotes the center of G.

5. (Minkowski.) Prove that if $m > 2$ then the kernel K of φ_m in $\mathrm{GL}(n,\mathbb{Z})$ is torsion-free, i.e., every non-identity element of K has infinite order.

6. Let $\mathcal{P}$ be any infinite set of primes. Prove that the intersection of the kernels of all congruence homomorphisms $\varphi_p\colon \mathrm{GL}(n,\mathbb{Z}) \to \mathrm{GL}(n,p)$ as p ranges over $\mathcal{P}$ is trivial. Deduce that $\mathrm{GL}(n,\mathbb{Z})$ is residually finite.

7. Prove that an infinite simple linear group cannot be finitely generated.

8. Prove that $G = \mathrm{SL}(n, \mathbb{Z}[1/\mu])$ for $\mu > 1$ is not virtually free.

9. Prove that $\mathrm{GL}(n, \mathbb{Z})$ does not have the strong approximation property, i.e., $\mathrm{GL}(n, \mathbb{Z})$ does not surject onto $\mathrm{GL}(n, p)$ for almost all primes p.
Does $\mathrm{GL}(n, \mathbb{Z}[1/2])$ have the strong approximation property?

10. Let H be a finitely generated subgroup of $\mathrm{GL}(n, \mathbb{Q})$; so $H \leq \mathrm{GL}(n, \mathbb{Z}[1/\mu])$ for some μ. Prove that if $\varphi_p(H) \leq \mathrm{GL}(n, p)$ is absolutely irreducible for a prime p not dividing μ, then H is absolutely irreducible.

11. Suppose that $H \leq \mathrm{SL}(n, \mathbb{Z})$ surjects onto $\mathrm{SL}(n, p)$ modulo p for some prime p, where $p > 3$ if $n = 2$. Prove that H is not monomial (over $\mathbb{Q}$).

12. Let $H \leq G$, where G is a dense subgroup of $\mathrm{SL}(n, \mathbb{C})$. Prove that if H has finite index in G, then H is dense.

13. Prove that a solvable-by-finite subgroup of $\mathrm{SL}(n, \mathbb{Q})$ is not dense.

14. Let $\Gamma_{n,m}$ denote the principal congruence subgroup of level m in $\mathrm{SL}(n, \mathbb{Z})$, i.e., the kernel of the reduction modulo m congruence homomorphism $\varphi_m \colon \mathrm{SL}(n, \mathbb{Z}) \to \mathrm{SL}(n, \mathbb{Z}_m)$. Prove that $\Gamma_{n,m}/\Gamma_{n,m^2}$ is a finite abelian group of exponent dividing m.

15. Let H be a finite-index subgroup of $\mathrm{SL}(n, \mathbb{Z})$, $n \geq 3$. Since the congruence subgroup property holds for $\mathrm{SL}(n, \mathbb{Z})$, H contains a principal congruence subgroup of $\mathrm{SL}(n, \mathbb{Z})$ of least possible level, defined to be the level of H. Prove that if H has index at most m in $\mathrm{SL}(n, \mathbb{Z})$, then the level of H divides $m!$.

16. Prove that if k and m are coprime then φ_k surjects $\Gamma_{n,m}$ onto $\mathrm{SL}(n, \mathbb{Z}_k)$. Deduce from this that any congruence subgroup of $\mathrm{SL}(n, \mathbb{Z})$ is dense in $\mathrm{SL}(n, \mathbb{Q})$.

17. Let G be the subgroup $\langle t_{1,2}(a/b), t_{2,1}(a/b) \rangle$ of $\mathrm{SL}(2, \mathbb{Q})$, where a, b are positive coprime integers. Prove that G is dense by showing that the set $\Pi(G)$ of all primes p not dividing b such that $\varphi_p(G) \neq \mathrm{SL}(2, p)$ is equal to the set $\pi(a)$ of prime divisors of a.

18. Let H be a finitely generated dense subgroup of $\mathrm{SL}(n, \mathbb{Q})$ contained in $\mathrm{SL}(n, \mathbb{Z})$. Denote the congruence closure of H by $\mathrm{cl}(H)$, i.e., $\mathrm{cl}(H)$ is the intersection of all congruence subgroups of $\mathrm{SL}(n, \mathbb{Z})$ containing H. Prove that $\mathrm{cl}(H) = \Gamma_{n,\ell} H$ where ℓ is the level of $\mathrm{cl}(H)$.

19. Let $H \leq \mathrm{SL}(n, \mathbb{Z})$ be finitely generated dense. Prove that the following are equivalent: (i) $\varphi_m(H) = \mathrm{SL}(n, \mathbb{Z}_m)$ for all $m \geq 2$; (ii) $\mathrm{cl}(H) = \mathrm{SL}(n, \mathbb{Z})$.