SCIENTIFIC REPORTS

Recent Developments in Algebraic Geometry, Arithmetic and Dynamics Part 2

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LOCAL DYNAMICS OF SKEW-PRODUCTS TANGENT TO IDENTITY (JOINT WORK WITH L. BOC THALER)

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Skew-products are holomorphic self-maps of $\mathbb{C}^2$ of the form

$$P(z, w) = (p(z), q(z, w)).$$

An important feature of these maps is that they preserve the set of vertical lines in $\mathbb{C}^2$. This means that we can view the restriction of P^n to a line $\{z\} \times \mathbb{C}$ as the composition of n entire functions on $\mathbb{C}$, which allows techniques from one-dimensional complex dynamics to be applied. The dynamics of skew-products is therefore in some ways reminiscent of the dynamics of one-variable maps; however, in recent years, several important results have shown that these maps have rich and interesting dynamics, see [8, 11, 12, 15]. For example, in [3], it was shown that there exists polynomial skew-products, i.e. P is a polynomial map, with *wandering Fatou components*, a dynamical phenomenon that is known not to occur for polynomial maps in one complex dimension. The proof of the main result in that paper involves the adaptation of *parabolic implosion* to the skew-product setting (see also [4, 5, 2] for further results on parabolic implosion in several complex variables). Polynomial skew-products were also used in [6] and [14] to construct *robust* bifurcations, i.e. open sets contained in the bifurcation locus of the family of endomorphisms of $\mathbb{P}^2$ of given algebraic degree $d \geq 2$.

Given a germ of a holomorphic self-map P of $\mathbb{C}^2$ that fixes the origin, we say that P is *tangent to the identity* if it is of the form $P = \mathrm{Id} + P_k(z, w) + O(\|(z, w)\|^{k+1})$, where $k \geq 2$ and $P_k : \mathbb{C}^2 \to \mathbb{C}^2$ is a non-trivial homogeneous polynomial map of degree k. The study of local dynamics of germs tangent to the identity has received significant attention over the last decades. For general germs of $(\mathbb{C}^2, 0)$ tangent to the identity, a complete description of the dynamics on a full neighborhood of the origin is for now far out of reach. Much effort has been instead devoted to investigating the existence of invariant manifolds or invariant formal curves on which the dynamics converges to the origin (see e.g. [7, 1], and more recently [10, 9]).

In this talk we investigate the local dynamics of skew-products P which are tangent to the identity and have a non-degenerate second order differential at the origin.

By this we mean holomorphic maps $P : \mathbb{C}^2 \to \mathbb{C}^2$ of the form

$$P(z, w) = \left(z + \sum_{i \geq 2} a_i z^i, w + \sum_{i+j \geq 2} b_{i,j} z^i w^j \right) \tag{0.1}$$

with $a_2 \neq 0$, $b_{2,0} \neq 0$ and $b_{0,2} \neq 0$.

Definition 0.1. *Let P be a holomorphic self-map of $\mathbb{C}^k$ with a parabolic fixed point at the origin. A* parabolic domain *of P is a maximal invariant connected domain $\mathcal{B}_P \subset \mathbb{C}^2$ such that the origin is contained in the boundary of $\mathcal{U}$ and the iterates $P^n_{|\mathcal{U}}$ converge locally uniformly on $\mathcal{U}$ to the origin. Moreover, when $k > 1$ we say that a parabolic domain is tangent to a direction $v \in \mathbb{CP}^{k-1}$ if and only if each point from the domain is attracted to the origin along trajectories tangent to v.*

We begin by discussing the existence of parabolic domains for maps of the form (0.1), which depends only on $b := b_{0,2}$:

Theorem 0.2. *Let P be a map of the form* (0.1)*. Then*

(1) If $b \in (\frac{1}{4}, +\infty)$, the map P has an invariant parabolic domain which is not tangent to any directions.
(2) If $b \in \mathbb{C}\backslash(\frac{1}{4}, +\infty)$, the map P has an invariant parabolic domain which is tangent to one of its non-degenerate characteristic directions.

Invariant parabolic domains which are not tangent to any direction are also sometimes called *spiral domains*. Such domains were first constructed by Rivi in her thesis [13, Proposition 4.4.4].

From now on we will assume that $b := b_{0,2} > \frac{1}{4}$, and we introduce the following notations:

$$c := \frac{\sqrt{4b-1}}{2}, \quad \alpha_0 := e^{\pi/c}. \tag{0.2}$$

Observe that since $b > \frac{1}{4}$, we have $c > 0$ and $\alpha_0 > 1$.

Theorem 0.3. *Let P be a map of the form* (0.1) *with $b > \frac{1}{4}$, and satisfying some explicit condition on cubic terms which we do not state here. Then P has wandering Fatou components which admit non-constant limit of iterates.*

In the case where P is quadratic, we can be more precise:

Theorem 0.4. *Let $P(z, w) := (z + z^2, w + w^2 + bz^2)$, with $b > \frac{1}{4}$. Then P has countably many different grand orbits of wandering domains, each of which admit non-constant limit of iterates.*

Definition 0.5. *Given real numbers $\alpha > 1$ and $\beta \in \mathbb{R}$, we say that a strictly increasing sequence of positive integers $(n_k)_{k\geq 0}$ is (α, β)-*admissible *if and only if its* phase sequence *$(\sigma_k)_{k\geq 0}$, defined by $\sigma_k := n_{k+1} - \alpha n_k - \beta \ln n_k$, is bounded. In the case where $\beta = 0$, we will simply call such a sequence α-*admissible.

The two theorems mentionned above are consequences of the following more technical result:

Main Theorem. *Let P be a map of the form* (0.1)*, satisfying the conditions of Theorem 0.3. Let $(n_k)_{k\geq 0}$ be an (α_0, β_0)-admissible sequence and let $(\sigma_k)_{k\geq 0}$ denote its phase sequence (β_0 is a constant depending explicitly on cubic terms of P). Then*

$$P^{n_{k+1}-n_k}(p^{n_k}(z), w) = (0, \mathcal{L}(\alpha_0, \sigma_k; z, w)) + o(1) \qquad (\text{ as } k \to +\infty)$$

with uniform convergence on compacts in $\mathcal{B}_p \times \mathcal{B}_{q_0}$*, where* $\mathcal{L} : \mathbb{C}^2 \times \mathcal{B}_p \times \mathcal{B}_{q_0} \to \mathbb{C}$ *is a holomorphic map called the* generalized Lavaurs map *of* P*.*

The usefulness of this Main Theorem (and of similar results, such as [[3], Proposition A] is that by applying it successively, one can estimate more and more precisely certain high iterates of P in terms of iterates of the maps $\mathcal{L}_z : w \mapsto \mathcal{L}(\alpha_0, \sigma_k; z, w)$. Therefore, one can transfer dynamical properties of $\mathcal{L}_z$ to obtain information on the dynamics of P. These maps $\mathcal{L}_z$ are quite complicated (they are non-explicit, transcendental maps, with infinitely many critical points and in general infinitely many critical values). However, by thinking of them as a one-parameter family of maps $(\mathcal{L}_z)_{z \in \mathcal{B}_p}$, we can use ideas from one-dimensional bifurcation theory to obtain information on the dynamics of $\mathcal{L}_z$ for certain values of z. Moreover, under the additional assumption that $\alpha_0 \in \mathbb{N}_{\geq 2}$, we prove that these maps are semi-conjugated to *finite type maps* in the sense of Epstein, which allow us to obtain a more precise understanding of their dynamics, and in turn, of the dynamics of P.

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COMPLEX TROPICAL CURRENTS

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In this presentation, we revisit fundamental concepts in tropical geometry, including tropical varieties and tropicalization. We also recall the interpretation of tropicalization concerning the trivial valuation from several angles:

- It can be seen as the intersection of corner loci of tropicalized polynomials.
- It serves as a logarithmic limit set.
- It represents a Chow cohomology class in compatible toric varieties.

For a comprehensive understanding, see [**?**]. As an important application, we review the proof of Read and Rota–Heron–Welsh conjecture in the realisable case by June Huh and Eric Katz in [**?**].

Next, we explore the definition of analytical constructs such as tropical currents, which serve as counterparts to tropical varieties in the context of positive closed currents. We also view tropicalization as a dynamic process. To this end, we define the mapping:

$$\begin{aligned} \Phi_m : (\mathbb{C}^*)^n &\longrightarrow (\mathbb{C}^*)^n \\ (z_1, \ldots, z_n) &\longmapsto (z_1^m, \ldots, z_n^m), \end{aligned}$$

We then present the following theorem (refer to [**?**]):

Theorem 0.1. *Let $Z \subseteq (\mathbb{C}^*)^n$ be an irreducible subvariety of dimension p. As m tends to infinity, we have:*

$$\frac{1}{m^{n-p}} \Phi_m^*[Z] \longrightarrow \mathscr{T}_{\textit{trop}(Z)},$$

Here, $\mathscr{T}_{\textit{trop}(Z)}$ represents the complex tropical current associated with trop(Z).

We also discuss a version that corresponds to Kajiwara–Payne tropicalization:

Theorem 0.2. *Let $Z \subseteq (\mathbb{C}^*)^n$ be an irreducible subvariety of dimension p, and let $\overline{Z}$ be the "tropical compactification" of Z in the compatible smooth toric variety X. As m approaches infinity, we observe:*

$$\frac{1}{m^{n-p}} \Phi_m^*[\overline{Z}] \longrightarrow \overline{\mathscr{T}}_{\textit{trop}(Z)},$$

Where $\Phi_m : X \longrightarrow X$ is the continuous extension of $\Phi_m : (\mathbb{C}^*)^n \longrightarrow (\mathbb{C}^*)^n$, and $\overline{\mathscr{T}}_{\textit{trop}(Z)}$ is the extension by zero of $\mathscr{T}_{\textit{trop}(Z)}$ to X.

We then proceed to explore various applications of the above results in recovering several theorems in tropical geometry. The presentation further delves into an equivalent version of the Hodge Conjecture formulated in the language of currents, as well as a stronger version of the Hodge conjecture for positive currents, as proposed by Demailly. We provide a brief overview of the strategy for finding counterexamples in [**?**] and [**?**] for the latter statement.

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CENTRAL LIMIT THEOREMS FOR COMPLEX HÉNON MAPS AND AUTOMORPHISMS OF COMPACT KÄHLER MANIFOLDS

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Keywords: Complex Hénon maps, Exponential mixing of all orders, Central Limit Theorem

Hénon maps are among the most studied dynamical systems that exhibit interesting chaotic behaviour. They were introduced by Michel Hénon in the real setting as a simplified model of the Poincaré section for the Lorenz model. Hénon maps are also actively studied in the complex setting, where complex analysis offers additional powerful tools. This talk was based on our work [2], where we prove that the measure of maximal entropy of any complex Hénon map is *exponentially mixing of all orders* with respect to Hölder observables. As a consequence, we also solve a long-standing question proving the Central Limit Theorem for all Hölder observables with respect to the maximal entropy measures of complex Hénon maps. A similar result holds for automorphisms of compact Kähler surfaces with positive entropy [3], and related versions are also true in higher dimension. We just focus on the case of Hénon maps in this report for simplicity.

The reader can find in the work of Bedford, Dinh, Fornaess, Lyubich, Sibony, Smillie fundamental dynamical properties of Hénon maps. In particular, by Bedford-Lyubich-Smillie, the measure of maximal entropy μ is Bernoulli, which implies that it is mixing of all orders. On the other hand, the control of the speed of mixing (i.e., the rate of the above convergence) for general dynamical systems and for regular enough observables is a challenging problem, and usually one can obtain it only under strong hyperbolicity assumptions on the system. Let us recall the following general definition.

Definition 0.1. *Let (X, f) be a dynamical system and ν an f-invariant measure. Let $(E, \|\cdot\|_E)$ be a normed space of real functions on X with $\|\cdot\|_{L^p(\nu)} \lesssim \|\cdot\|_E$ for all $1 \leq p < \infty$. We say that ν is* exponentially mixing of order $\kappa \in \mathbb{N}^*$ *for observables in E if there exist constants $C_\kappa > 0$ and $0 < \theta_\kappa < 1$ such that, for all $g_0, \dots, g_\kappa$ in E and integers $0 =: n_0 \leq n_1 \leq \cdots \leq n_\kappa$, we have*

$$\Big|\langle \nu, g_0(g_1 \circ f^{n_1}) \dots (g_\kappa \circ f^{n_\kappa})\rangle - \prod_{j=0}^{\kappa} \langle \nu, g_j \rangle\Big| \leq C_\kappa \cdot \Big(\prod_{j=0}^{\kappa} \|g_j\|_E\Big) \cdot \theta_\kappa^{\min_{0\leq j\leq \kappa-1}(n_{j+1}-n_j)}.$$

We say that ν is exponentially mixing of all orders *for observables in E if it is exponentially mixing of order κ for every $\kappa \in \mathbb{N}$.*

A recent major result by Dolgopyat, Kanigowski, and Rodriguez-Hertz ensures that, under suitable assumptions on the system, the exponential mixing of order 1 implies that the system is Bernoulli. In particular, it implies the mixing of all orders (with no control on the rate of decay of correlation). It is a main open question whether the exponential mixing of order 1 implies the exponential mixing of all orders.

Let now f be a complex Hénon map on $\mathbb{C}^2$. It is a polynomial diffeomorphism of $\mathbb{C}^2$. We can associate to f its unique measure of maximal entropy μ. It was established by Dinh that such measure is exponential mixing of order 1 for Hölder observables. Similar results were obtained by Liverani in the case of uniformly hyperbolic diffeomorphisms and Dolgopyat for Anosov flows.

Theorem 0.2. *Let f be a complex Hénon map and μ its measure of maximal entropy. Then, for every $\kappa \in \mathbb{N}^*$, μ is exponential mixing of order κ as in Definition 0.1 for $\mathcal{C}^\gamma$ observables ($0 < \gamma \leq 2$), with $\theta_\kappa = d^{-(\gamma/2)^{\kappa+1}/2}$.*

For endomorphisms of $\mathbb{P}^k(\mathbb{C})$, the exponential mixing for all orders for the measure of maximal entropy and Hölder observables was established by Dinh-Nguyen-Sibony. We recently proved such property for a large class of invariant measures with strictly positive Lyapunov exponents [1]. This was done by constructing a suitable (semi-)norm on functions that turns the so-called Ruelle-Perron-Frobenius operator (suitably normalized) into a contraction.

The exponential mixing of all orders is one of the strongest properties in dynamics. It was recently shown to imply a number of statistical properties. As an example, a consequence of Theorem 0.2 is the following result. Take $u \in L^1(\mu)$. As μ is ergodic, Birkhoff's ergodic theorem states that

$$n^{-1}S_n(u) := n^{-1}\big(u(x) + u \circ f(x) + \cdots + u \circ f^{n-1}(x)\big) \to \langle \mu, u \rangle \quad \text{for } \mu-\text{a.e. } x \in X.$$

This can be seen as a version of the law of large numbers for the sequence $\{u \circ f^j\}_{j\in\mathbb{N}}$, which can be interpreted as a sequence of non independent random variables with respect to μ. We say that u *satisfies the Central Limit Theorem* (CLT) with variance $\sigma^2 \geq 0$ with respect to μ if $n^{-1/2}(S_n(u) - n\langle \mu, u \rangle) \to \mathcal{N}(0, \sigma^2)$ in law, where $\mathcal{N}(0, \sigma^2)$ denotes the (possibly degenerate, for $\sigma = 0$) Gaussian distribution with mean 0 and variance σ^2. By a result of Björklund and Gorodnik [4], the following is then a consequence of Theorem 0.2.

Corollary 0.3. *Let f be a complex Hénon map and μ its measure of maximal entropy. Then all Hölder observables u satisfy the Central Limit Theorem with respect to μ with $\sigma^2 = \sum_{n\in\mathbb{Z}} \langle \mu, \tilde{u}(\tilde{u} \circ f^n) \rangle = \lim_{n\to\infty} \frac{1}{n} \int_X (\tilde{u} + \tilde{u} \circ f + \ldots + \tilde{u} \circ f^{n-1})^2 d\mu$, where $\tilde{u} := u - \langle \mu, u \rangle$.*

Theorem 0.2 and Corollary 0.3 hold also in the larger settings of Hénon-Sibony automorphisms (sometimes called regular, or regular in the sense of Sibony) of $\mathbb{C}^k$ in any dimension and invertible horizontal-like maps in any dimension.

Our method to prove Theorem 0.2 relies on pluripotential theory and on the theory of positive closed currents. The idea is as follows. By interpolation, we can reduce the problem to the case $\gamma = 2$. For simplicity, assume that $\|g_j\|_{\mathcal{C}^2} \leq 1$ for all j. The measure of maximal entropy μ of a Hénon map f of $\mathbb{C}^2$ of algebraic degree $d \geq 2$ is the intersection $\mu = T_+ \wedge T_-$ of the two Green currents T_+ and T_- of f. If we identify $\mathbb{C}^2$ to an affine chart of $\mathbb{P}^2$ in the standard way, these currents are the unique positive closed $(1,1)$-currents of mass 1 on $\mathbb{P}^2$, without mass at infinity, satisfying $f^*T_+ = dT_+$ and $f_*T_- = dT_-$.

Consider the automorphism F of $\mathbb{C}^4$ given by $F := (f, f^{-1})$. Such automorphism also admits Green currents $\mathbb{T}_+ = T_+ \otimes T_-$ and $\mathbb{T}_- = T_- \otimes T_+$. These currents satisfy $(F^n)^*\mathbb{T}_+ = d^2\mathbb{T}_+$ and $(F^n)_*\mathbb{T}_- = d^2\mathbb{T}_-$. Under mild assumptions on their support, other positive closed $(2,2)$-currents S of mass 1 of $\mathbb{P}^4$ satisfy the estimate

$$(0.1) \qquad |\langle d^{-2n}(F^n)_*(S) - \mathbb{T}_-, \Phi\rangle| \leq c_{S,\Phi} d^{-n}$$

when Φ is a sufficiently smooth test form and $c_{S,\Phi}$ is a constant depending on S and Φ.

We show that proving the exponential mixing for $\kappa + 1$ observables $g_0, \ldots, g_\kappa$ with $\|g_j\|_{\mathcal{C}^2} \leq 1$ can be reduced to proving the convergence (we assume that n_1 is even for simplicity)

$$(0.2) \qquad |\langle d^{-n_1}(F^{n_1/2})_*[\Delta] - \mathbb{T}_-, \Theta_{\{g_j\},\{n_j\}}\rangle| \lesssim d^{-\min_{0\leq j\leq \kappa-1}(n_{j+1}-n_j)/2},$$

where $\Theta_{\{g_j\},\{n_j\}} := g_0(w)g_1(z)(g_2 \circ f^{n_2-n_1}(z)) \ldots (g_\kappa \circ f^{n_\kappa-n_1}(z))\mathbb{T}_+$, $[\Delta]$ denotes the current of integration on the diagonal Δ of $\mathbb{C}^2 \times \mathbb{C}^2$, and (z, w) denote the coordinates on $\mathbb{C}^2 \times \mathbb{C}^2$. A crucial point here is that the estimate should not only be uniform in the g_j's, but also in the n_j's. Note also that the current $[\Delta]$ is singular and the dependence of the constant $c_{S,\Phi}$ in (0.1) from S makes it difficult to employ regularization techniques to deduce the convergence (0.2) from (0.1).

The key point here is to notice that, when $dd^c\Phi \geq 0$ (on a suitable open set), one can also get the following variation of (0.1):

$$(0.3) \qquad \langle d^{-2n}(F^n)_*(S) - \mathbb{T}_-, \Phi\rangle \leq c_\Phi d^{-n}.$$

With respect to (0.1), only the bound from above is present, but the constant c_Φ is now independent of S. This permits to regularize Δ and work as if this current were smooth. Note also that, although $\Theta_{\{g_j\},\{n_j\}}$ is not smooth, we can handle it using a similar regularization.

Working by induction, we show that it is possible to replace both $\Theta_{\{g_j\},\{n_j\}}$ and $-\Theta_{\{g_j\},\{n_j\}}$ in (0.2) with currents $\Theta^\pm$ satisfying $dd^c\Theta^\pm \geq 0$. This permits to deduce the estimate (0.2) from two upper bounds given by (0.3) for $\Theta^\pm$, completing the proof.

In the companion paper [3], we explain how to adapt the strategy above to get the exponential mixing of all orders and the CLT for automorphisms of compact Kähler manifolds with simple action on cohomology. The proof in that case requires the theory of super-potentials, which is not needed for Hénon maps.

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ON THE CHERN NUMBERS OF A SMOOTH THREEFOLD

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Keywords: Chern numbers, Minimal model program

The aim of this talk is to discuss the following question:

Question 0.1 (Kotschich [4]). *Let X be a smooth complex projective threefold. Are Chern numbers of X bounded by a number that depends only on the topology of the manifold underlying X?*

Note that this question is known to have a negative answer for non-Kähler complex threefolds [5] and for complex projective varieties of dimension greater than three [6]. On the other hand, the question has a positive answer in the case of Kähler varieties underlying a spin manifold [8].

In the case of a smooth projective threefold X, the only Chern numbers are $c_1^3(X)$, $c_1c_2(X)$ and $c_3(X)$. The last one coincides with the topological Euler characteristic of X and, in particular, it is a topological invariant. On the other hand, by Hirzebruch-Riemann-Roch theorem, we have that $|c_1c_2(X)|$ coincides with $|24\chi(\mathcal{O}_X)|$ and it is therefore bounded by an integer depending only on the sum of the Betti numbers of X. Therefore, it remains to bound $c_1^3(X)$ or, in other words, K_X^3.

Thanks to the Minimal Model Program, we know that a smooth projective threefold is birational to either a minimal model, i.e. a variety Y such that K_Y is nef or to a variety Y which admits a Mori fibre space, i.e. a fibration $\eta\colon Y \to Z$ such that $\dim Z < 3$, the general fibre of η is Fano and the relative Picard number $\rho(Y/Z)$ is one. We first want to bound K_Y^3. To this end, if Y is a minimal model then K_Y^3 concides with the volume of Y and by [2] this number is bounded by the Betti numbers of X. In the second case, instead, it follows from [7] that if the cubic form F_Y has non-zero discriminant Δ_{F_Y}, then K_Y^3 is bounded by a number that depends only on F_Y and the first Pontryagin class $p_1(Y)$ of Y.

We then need to show that these bounds hold also on X. By [3], we have that the number k of steps of an MMP

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \dots \dashrightarrow X_k = Y$$

starting from X and the singularities of the output Y of this MMP are both bounded by a number which depends only on the topology of the manifold underlying X. Thus, the primary challenge lies in bounding the variation of the topological invariants of the underlying varieties and the Chern number $K_{X_i}^3$ at each step $X_i \dashrightarrow X_{i+1}$ of this MMP. In the case of divisorial contractions, this problem was solved in [2], provided that the discriminant of the associated cubic form F_{X_i} is non-zero. In this talk, we discuss the case of flips:

Theorem 0.2. [1] *Let X be a smooth complex projective threefold and let*

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \ldots \dashrightarrow X_k = Y$$

be a K_X-MMP.

Then $|K_{X_i}^3 - K_{X_{i+1}}^3|$ is bounded by an integer which depends only on $b_2(X)$.

The remaining question is to understand how the cubic form varies after each flip. Indeed if $X_i \to X_{i+1}$ is a divisorial contraction to a curve then $K_{X_i}^3 - K_{X_{i+1}}^3$ depends on the cubic form of F_{X_i} of X_i and not only on its Betti numbers (e.g., consider the blowup of a rational curve of degree d in $\mathbb{P}^3$). More precisely, if $X_i \dashrightarrow X_{i+1}$ is a flip then we need to show:

(1) the equivalence class of $F_{X_{i+1}}$ belongs to a finite set which depends only on F_{X_i}; and
(2) if $\Delta_{F_{X_i}} \neq 0$ then $\Delta_{F_{X_{i+1}}} \neq 0$.

We have a partial solution about the finiteness of cubic forms.

Theorem 0.3. [1] *Let X be a smooth complex projective threefold and let*

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \ldots \dashrightarrow X_k = Y$$

be a K_X-MMP. Assume that $X_i \dashrightarrow X_{i+1}$ is a flip for some $i = 1, \ldots, k-1$, and let $\phi_i \colon X_i \to W_i$ be the corresponding flipping contraction.

Then the equivalence class of $F_{X_{i+1}}$ belongs to a finite set which depends only on $b_2(X)$, F_{X_i} and $\phi_i^* H^2(W_i, \mathbb{Z}) \subset H^2(X_i, \mathbb{Z})$

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VARIETIES OF GENERAL TYPE WITH MANY GLOBAL k-FORMS

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Classification AMS 2020: 14E05, 14J20

Keywords: Varieties of general type, global k-forms, canonical volumes, the canonical stability index

This talk aims to report my recent research advances with Zhi Jiang on the relevant topics (for details, please refer to [1]).

Let X be a smooth projective variety of general type. Define the canonical stability index

$$r_s(X) := \min\{l|\varphi_{m,X} \text{ is birational for all } m \geq l\}.$$

For any $n \geq 1$, define the n-th canonical stability

$$r_n := \max\{r_s(X)|\text{X is a smooth proj. } n\text{-fold of general type}\}$$

and the n-th minimal volume

$$v_n := \min\{\mathrm{vol}(X)|\text{X is a smooth proj. } n\text{-fold of general type}\}.$$

Conjecture 0.1. $v_3 = \frac{1}{420}$.

Iano-Fletcher had the following example:

Example 0.2. The general hypersurface

$$X_{46} \subset \mathbb{P}(4, 5, 6, 7, 23)$$

is a canoncal 3-fold with the canonical volume $\frac{1}{420}$.

Recall the first theorem on the lower bound of the canonical volume in dimension 3 as follows:

Theorem 0.3. (Chen-Chen [2, Theorem 3.11]) Let X be a smooth projective 3-fold of general type with $\chi(\mathcal{O}_X) \leq 1$. Then $\mathrm{vol}(X) \geq \frac{1}{420}$. The equality holds if and only if the weighted basket of X is $\mathbb{B}(X) = \{B_{420}, 0, 1\}$ where B_{420} is the Reid basket corresponding to Example 0.2.

Let X be a smooth projective 3-fold of general type. Recall the following two known results:

(i) If $p_g(X) > 0$, then one has $\mathrm{vol}(X) \geq \frac{1}{75}$ by Chen [3, Theorem 1.4] and Chen-Chen [4, Corollary 1.7];

(ii) If $q(X) > 0$, then $\mathrm{vol}(X) \geq \frac{3}{8}$ by Chen-Hacon and Jiang (see [5, Theorem 1.5]).

Assume $p_g(X) = q(X) = 0$. Then $\chi(\mathcal{O}_X) > 1$ if and only if $h^2(\mathcal{O}_X) > 0$. This is the main reason that we study a variety with many global k-forms ($k > 0$).

Lemma 0.4. (Chen-Jiang [1, Lemma 2.2]) Let $\mathcal{E}$ be a torsion-free sheaf of rank r over a projective variety X. Assume that $h^0(X, \mathcal{E}) > 0$. There exists a torsion-free subsheaf $\mathcal{F} \subset \mathcal{E}$ such that $h^0(X, \det \mathcal{F}) \geq \frac{h^0(X,\mathcal{E})}{r}$.

Now we take $\mathcal{E} := \Omega_X^k$ with $k > 0$. By Lemma 0.4, there exists a subsheaf $\mathcal{F}$ of Ω_X^k such that $h^0(X, \det \mathcal{F}) \geq \frac{h^0(X,\Omega_X^k)}{\binom{n}{k}}$. We may replace $\mathcal{F}$ by its saturation in Ω_X^k and denote by $\mathcal{Q}$ the corresponding quotient bundle. Set $H := \det \mathcal{F}$ and $L := \det \mathcal{Q}$. Then

$$\binom{n-1}{k-1} K_X \sim \det(\Omega_X^k) \sim H + L.$$

By Campana and Paun [6, Theorem 1.2], we know that L is pseudo-effective.

The first application is the direct consequence on 3-folds with $\chi(\mathcal{O}_X) > 1$.

Theorem 0.5. (Chen-Jiang [1, Theorem 2.6]) Let X be a smooth projective threefold of general type with $h^{2,0}(X) \geq 3$. Then $\mathrm{vol}(X) \geq \frac{1}{224}$.

About the number v_3, we have the following Theorem:

Theorem 0.6. (cf. Chen-Jiang [1, Theorem 1.2]) Let X be a smooth projective 3-fold of general type with $\chi(\mathcal{O}_X) \neq 2, 3$ and $h^{2,0}(X) \neq 1, 2$. Then $\mathrm{vol}(X) \geq \frac{1}{420}$. The equality holds if and only if the weighted basket of X is $\mathbb{B}(X) = \{B_{420}, 0, 1\}$ where B_{420} is the Reid basket corresponding to Example 0.2.

The second application of our key idea is some direct result on varieties with sufficiently many global 2-forms.

Theorem 0.7. (Chen-Jiang [1, Theorem 1.3, Theorem 1.4]) Let X be any smooth projective 3-fold of general type with either $h^{2,0}(X) \geq 108 \cdot 18^3 + 4$ or $\chi(\mathcal{O}_X) \geq 108 \cdot 18^3 + 5$. Then the m-canonical map $\varphi_{m,X}$ is birational for all $m \geq 3$.

Theorem 0.8. (Chen-Jiang [1, Theorem 1.6]) There exists a constant $H(4) > 0$ such that, for any nonsingular projective 4-fold X of general type with $h^0(X, \Omega_X^2) \geq H(4)$, $\varphi_{m,X}$ is birational for all $m \geq 5$.

We end up with the following conjecture:

Conjecture 0.9. (Chen-Jiang [1, Conjecture 7.3]) For any $n \geq 5$, there exists a constant $H(n) > 0$ such that, for every smooth projective n-fold X of general type with $h^{2,0}(X) \geq H(n)$, $|mK_X|$ induces a birational map for all $m \geq r_{n-2}$.

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HYPERPLANE RESTRICTION THEOREM AND APPLICATIONS

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Keywords: proper mapping, orthogonal map, hyperplane restriction theorem

In this talk, we will talk about a hyerplane restriction theorem for the local holomorphic mappings between projective spaces, which is inspired by the corresponding theorem of Green for homogeneous ideals in polynomial rings.

To state our hyperplane restriction theorem, we first bring out the fact that every positive integer A can be written as certain sums of binomial coefficients. For every $n \in \mathbb{N}^+$, there exist unique positive integers $a_n > a_{n-1} > \cdots > a_\delta$, where $\delta \geq 1$ and $a_j \geq j$ for every j, such that $A = \binom{a_n}{n} + \cdots + \binom{a_\delta}{\delta}$. This is called the n-th Macaulay's representation of A and its existence and uniqueness can be proved by a greedy algorithm. These representations originally appeared in Macaulay's work of homogeneous ideals in polynomial rings [Ma]. Using the n-th Macaulay representation of A, we define the operation $A^{-<n>} := \binom{a_n-1}{n-1} + \cdots + \binom{a_\delta-1}{\delta-1}$. In what follows, "span" means the projective linear span:

Theorem 0.1. *([GN]) Let $f : U \subset \mathbb{P}^n \to \mathbb{P}^M$ be a local holomorphic map such that $\dim(\mathrm{span}(f(U)) \geq N$. Then, for a general hyperplane H such that $H \cap U \neq \varnothing$, $\dim(\mathrm{span}(f(H \cap U)) \geq N^{-<n>}$.*

The equality in the theorem can hold, for example, when f is a rational map whose components are all the monomials of a fixed degree. Our theorem is obtained from combining Green's hyperplane restriction theorem (Theorem **??**) with a pair of combinatoric identities. It holds for any local holomorphic maps between projective spaces and we believe that it will find applications elsewhere.This theorem gives us a formula to estimate the dimension of the linear span of image from the dimension of image of a general linear subspace.

As an application, we will introduce a new coordinate-free approach to study the Cauchy-Riemann maps between the real hyperquadrics in the complex projective space.The study of holomorphic mappings between real hyperquadrics in the complex projective space is a very classical topic in Several Complex Variables, especially in the field of CR (Cauchy-Riemann) Geometry.The traditional approach to the study is based on Chern-Moser's normal form theory, in which the central theme is that one can choose good coordinates such that the CR manifolds and the relevant holomorphic maps take certain normal forms. This is a powerful method which has been used to solve many problems but usually it requires formidable calculation.On the other hand, we observe that when the CR manifolds being concerned are real hyperquadrics, there are a certain type of orthogonality and a number of related notions such as null spaces and orthogonal complements which interact well with the CR maps. Our approach to the study of real hyperquadrics is to work on these geometric objects directly. While our method might not be able to produce as much detail of the relevant maps as the traditional normal form theory, it has the advantages of being coordinate free and geometrically more transparent. It is especially suited for obtaining rigidities and general behaviors with shorter and easier arguments.

Theorem 0.2. *([GN1]) Let $U \subset \mathbb{B}^{r,s,t}$ be a connected open set such that $U \cap \partial\mathbb{B}^{r,s,t} \neq \varnothing$ and $f : U \to \mathbb{B}^{r',s',t'}$ be a proper map. Then f is either null or quasi-linear if one of the conditions below is satisfied:*

$$(i) \quad r, s \geq 2 \quad \textit{and} \quad \min\{r', s'\} \leq \min\{r, s\};$$
$$(ii) \quad t = 0 \quad \textit{and} \quad \min\{r', s'\} \leq 2\min\{r, s\} - 2;$$
$$(iii) \quad t = 0 \quad \textit{and} \quad r' + s' \leq 2\dim(\mathbb{P}^{r,s}) - 1.$$

From Theorem 0.2, we can deduce and generalize a number of well-known rigidity theorems for the holomorphic maps between real hyperquadrics, including those of Baouendi-Huang [BH] (from (i) and (iv)); Baouendi-Ebenfelt-Huang [BEH] (from (ii)); Faran [Fa1] (from (iii)); and Xiao-Yuan [XY] (from (iii)).

The other application of Hyperplane restriction Theorem is to study the structure of the set of rational proper maps between complex unit balls,which is a very classical topic in Several Complex Variables. Among the many unsolved problems in this topic, it is well-known that there is an interesting *gap phenomenon*, as follows. Fix an integer $n \geq 2$. For each $k \in \mathbb{N}^+$ such that $k(k+1)/2 < n$, define the closed interval $\mathcal{I}_k := [kn+1, (k+1)n - \frac{k(k+1)}{2} - 1]$. The classical theorem of Faran [Fa1] amounts to saying that when $N \in \mathcal{I}_1 = [n+1, 2n-2]$, any local holomorphic map sending an open piece of $\partial\mathbb{B}^n$ to $\partial\mathbb{B}^N$ actually maps $\partial\mathbb{B}^n$ to a linear section $\partial\mathbb{B}^n \subset \partial\mathbb{B}^N$. In other words, there are no "new" maps when N increases from n to $2n-2$. Then, it was discovered by Huang-Ji-Xu [HJX] that the same phenomenon holds for $N \in \mathcal{I}_2 = [2n+1, 3n-4]$ and later by Huang-Ji-Yin [HJY] for $N \in \mathcal{I}_3 = [3n+1, 4n-7]$. The *Gap Conjecture*, formulated in [HJY2], states that the gap phenomenon holds whenever $N \in \mathcal{I}_k$.

We are going to establish the existence of similar gaps for all levels *at once* and also to demonstrate the gap phenomenon actually holds for *all* generalized balls (whose definition will be recalled below).

Theorem 0.3. *(GN1) Let $k, n \in \mathbb{N}^+$ such that $n > k(k+1)$. For the local proper holomorphic maps between generalized balls, the gap phenomenon holds over the intervals*

$$\mathcal{J}_k := [kn + k, (k+1)n - (k^2+1)].$$

For the ordinary unit balls, Theorem 0.3 is understood as the usual way, as described above. However, we will see that formulating precisely the gap phenomenon for all generalized balls. Note that although the interval $\mathcal{J}_k$ in our theorem is smaller than the $\mathcal{I}_k$ in the original Gap Conjecture, this is to be expected since our theorem holds for *all generalized balls*. As a matter of fact, the lower bound for $\mathcal{J}_k$ is sharp in the present context.

Our proof for Theorem 0.3 consists of two main ingredients: the orthogonality preserved by the relevant proper maps; and a hyperplane restriction theorem for holomorphic mappings. Regarding the study of orthogonality, we proposed a coordinate free approach to the rigidity problems related to real hyperquadrics on the projecture space and generalized a number of well-known rigidity theorems by using rather simple arguments.

It is a joint work with Sui-Chung Ng.

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THE MUKAI-TYPE CONJECTURE

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Keywords: Fano manifolds, generalized pairs

First of all, we introduce he following famous conjecture by Shigeru Mukai:

Conjecture 0.1 ([M]). *Let X be a d-dimensional smooth Fano varieties, $\rho(X)$ the Picard number, and*

$$i_X := \max\{r \in \mathbb{Z} \mid -K_X \sim_{\mathbb{Z}} rH \text{ for some Cartier divisor } H\}$$

be the Fano index. Then it holds that

$$d + \rho(X) - i(X) \cdot \rho(X) \geq 0.$$

Moreover the above is equal if and only if $X \simeq \mathbb{P}^{i(X)-1} \times \cdots \times \mathbb{P}^{i(X)-1}$.

If we call $d + \rho(X) - i(X) \cdot \rho(X)$ *the Mukai complexity* , the above conjectures the complexities are non-negative and smallest one is the product of projective spaces. This observation is very closed to the Shokurov conjecture charactarizing the Toric varieties.

Then we propose a new approach to the above traditional conjecture by using more modern technique from the minimal model program and Toric geometry.

First we propose the new invariant and give a new conjecture of Mukai type.

Definition 0.2. *Let X be a Fano manifold. We define the total index γ_X of X as the maximal of $\sum a_i$ for a decomposition*

$$-K_X = \sum a_i L_i,$$

where L_i are nef line bundles which is not numerically trivial and $a_i \in \mathbb{Z}_{>0}$. Note that we allow $L_i = L_j$ for $i \neq j$.

Conjecture 0.3 (Mukai type conjecture). *It holds that*

$$\dim X + \rho_X - \gamma_X \geq 0$$

and the equality holds if and only if the product of projective spaces

Note that $\dim X + \rho_X - \gamma_X$ should be called by the Shokurov complexities for generalized pairs. The above Mukai type conjecture is more accessible than the original one since that is more direct related with the Shokurov complexities than the original Mukai conjecture by using the minimal model program techniques for generalized pairs. Indeed we proposed two approach to the Mukai type conjecture. First one is to use the Kawamata–Ambro effective non-vanishing ([Am], [Ka]):

Conjecture 0.4. *Let (X, B) be a projective klt pair and L a nef line bundle on X such that $L - (K_X + B)$ is ample.*

Then $H^0(X, L) \neq 0$.

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Indeed the we prove

Theorem 0.5. *Assume that Conjecture 0.4 holds. Then Conjecture 0.3 holds.*

The second approach is to generaralize the Shokurov conjecture (which is proved by [BMSZ]) to generalized pairs. Then we need to expand the notion of Shokurov's complexieies for generalized pairs. However that is known for only surfaces in [GM].

Moreover by comparing the Shokurov and Mukai complexities, we discuss when these have relationships. In particular, we discuss when Shokurov's one is smaller than and equal to Mukai's one. For this purpose, it seems that the key point to study of the effective and nef cone of the Fano manifolds. In particular, we need to investigate about the extremal contraction under the assumption of small Mukai complexities. If it is less than one, we expect to have only fiber type. On the other hand, we have very naive question about the effective cone of Fano manifolds such that all extremal contraction are fiber type. We shall ask whether the such cone is simplicial over $\mathbb{Z}$.

In the last, we include discussion the difference of the integer total index and rational total index.

Definition 0.6. *Let X be a Fano manifold. We define the total index γ_X of X as the maximal of $\sum a_i$ for a decomposition*

$$-K_X = \sum a_i L_i,$$

where L_i are nef line bundles which is not numerically trivial and $a_i \in \mathbb{Z}_{>0}$. Note that we allow $L_i = L_j$ for $i \neq j$.

It seems to be no direct relationship with the product of the Fano index and the Picard number although we propose Conjecture 0.3 motivated with Conjecture 0.1 ;

Example 0.7. *Let X be a three point blow-up of $\mathbb{P}^2$. Then $\gamma_X = 3$ but $\rho(X) = 4$ and the Fano index $i(X) = 1$. Thus $\gamma_X < \rho(X)i(X)$.*

Example 0.8. *Let X be a one point blow-up of $\mathbb{P}^2$. Then $\gamma_X = 3$ but $\rho(X) = 2$ and the Fano index $i(X) = 1$. Thus $\gamma_X > \rho(X)i(X)$.*

We also consider the rational version of γ_X:

Definition 0.9. *Let X be a Fano manifold. We define the total index $\gamma_{X,\mathbb{Q}}$ of X as the upperlimit of $\sum a_i$ for a decomposition*

$$-K_X = \sum a_i L_i,$$

where L_i are nef line bundles which is not numerically trivial and $a_i \in \mathbb{Q}_{>0}$. Note that we allow $L_i = L_j$ for $i \neq j$.

We have a naive question about whether the integral and rational total indexes coincide:

Question 0.10. *Let X be a Fano manifold. Then $\gamma_X = \gamma_{X,\mathbb{Q}}$?*

However, the answer is no to the following example by Atsushi Ito.

Example 0.11 (Atsushi Ito)**.** *Let $\pi : S \to \mathbb{P}^2$ be a general 4-points blow-up of the projective plane. Then $\gamma_S = 2$. Indeed let E_1, E_2, E_3, E_4 be the exceptional divisors of π and L be a line on $\mathbb{P}^2$. Since $\pi^*(2L) - \sum_{i=1}^4 E_i$ is semi-ample, the decomposition*

$$-K_S \sim_{\mathbb{Z}} (\pi^*(2L) - \sum_{i=1}^{4} E_i) + \pi^* L$$

gives $\gamma_S \geq 2$. On the other hand, if $\gamma_S \geq 3$, we have a decomposition $-K_S \sim_{\mathbb{Z}} L_1 + L_2 + L_3$ such that L_i is nef and not numerical trivial Cartier divisor. By the Reimann–Roch formula, we see that $H^0(X, L_i) \neq 0$. Thus we may assume that L_i is an effective divisor. $\pi_* L_i$ is equal to a line on $\mathbb{P}^2$. Now fix i, Since $\pi^*(\pi_* L_i) = L_i + \sum_j d_j E_j$ for some non negative integers d_j. Since L_i is nef, $\sum_j d_j E_j$ is the prime exceptional divisor. Thus we may assume that $\pi^*(\pi_* L_i) = L_i + \epsilon_i E_i$ (by changing the index of the exceptional divisors), where $\epsilon_i = 0$ or 1. But then $\sum L_i$ is not linear equivalent to $-K_S$. This is the contradiction. Thus we see that $\gamma_S = 2$. Now we see that $\gamma_{S,\mathbb{Q}} > 2$. Indeed, let

$$D_i = \pi^* L - E_i, D' = 2\pi^* L - E_1 - E_2 - E_3 - E_4.$$

Then it holds that

$$1/2(D_1 + D_2 + D_3 + D_4 + D') = -K_S.$$

This decomposition gives $\gamma_{S,\mathbb{Q}} \geq 5/2 > 2$. Thus $\gamma_S \neq \gamma_{S,\mathbb{Q}}$

Still, we are interested in the above question for the Toric varieties.

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AN UPPER BOUND FOR POLYNOMIAL VOLUME GROWTH OF AUTOMORPHISMS OF ZERO ENTROPY

FEI HU AND CHEN JIANG

ABSTRACT. Let X be a smooth complex projective variety of dimension d and f an automorphism of X. Suppose that the pullback $f^*|_{\mathsf{N}^1(X)_{\mathbf{R}}}$ of f on the real Néron–Severi space $\mathsf{N}^1(X)_{\mathbf{R}}$ is unipotent and denote the index of the eigenvalue 1 by $k+1$. We prove an upper bound for the polynomial volume growth $\operatorname{plov}(f)$ of f as follows:

$$\operatorname{plov}(f) \le (k/2+1)d.$$

Combining with the inequality $k \le 2(d-1)$ due to Dinh–Lin–Oguiso–Zhang, we obtain an optimal inequality that

$$\operatorname{plov}(f) \le d^2,$$

which affirmatively answers questions of Cantat–Paris-Romaskevich and Lin–Oguiso–Zhang.

Given a surjective endomorphism f of a smooth complex projective variety X of dimension d, Gromov [7] introduced in 1977 the so-called *iterated graph* $\Gamma_n \subset X^n$ of f, i.e., the graph of the morphism $(f, \dots, f^{n-1})\colon X \to X^{n-1}$, and bounded the topological entropy $h_{\mathrm{top}}(f)$ of f by the *volume growth* $\operatorname{lov}(f)$ of f and further by the *algebraic entropy* $h_{\mathrm{alg}}(f)$ of f as follows:

$$\begin{aligned} h_{\mathrm{top}}(f) \le \operatorname{lov}(f) &\coloneqq \limsup_{n\to\infty} \frac{\log \operatorname{Vol}(\Gamma_n)}{n} \\ &\le h_{\mathrm{alg}}(f) \coloneqq \log \max_{0\le i\le d} \lambda_i(f), \end{aligned}$$

where the volume $\operatorname{Vol}(\Gamma_n)$ is computed against the ample divisor on the product variety X^n induced from an arbitrary ample divisor H_X on X, and the *i-th dynamical degree* $\lambda_i(f)$ of f is defined by

$$\lambda_i(f) \coloneqq \lim_{n\to\infty} ((f^n)^* H_X^i \cdot H_X^{d-i})^{1/n}. \tag{1.1}$$

Combining with Yomdin's remarkable inequality $h_{\mathrm{alg}}(f) \le h_{\mathrm{top}}(f)$ (which resolves Shub's entropy conjecture; see [12]), the above Gromov's result yields the fundamental equality in higher-dimensional algebraic/holomorphic dynamics, saying that

$$h_{\mathrm{top}}(f) = \operatorname{lov}(f) = h_{\mathrm{alg}}(f).$$

Recently, there are many works on varieties with slow dynamics (see, e.g., [3, 10, 4, 6, 5, 11, 8]), and in particular, we are interested in automorphisms of zero entropy.

Let X be a normal projective variety of dimension d, H_X an ample divisor on X, and f an automorphism of X. Denote by $\mathsf{N}^1(X)_{\mathbf{R}}$ the real Néron–Severi space of Cartier

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Key words and phrases. automorphism, zero entropy, polynomial volume growth, Gelfand–Kirillov dimension, quasi-unipotency, restricted partition, unimodality, degree growth.

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divisors on X modulo numerical equivalence, which is a finite-dimensional $\mathbf{R}$-vector space. Suppose that f is of *zero entropy*, i.e., $f^*|_{\mathsf{N}^1(X)_\mathbf{R}}$ is *quasi-unipotent*. Namely, all eigenvalues of $f^*|_{\mathsf{N}^1(X)_\mathbf{R}}$ are roots of unity. Denote by $k+1$ the maximum size of Jordan blocks of (the Jordan canonical form of) $f^*|_{\mathsf{N}^1(X)_\mathbf{R}}$. Note that $k = 2r$ is an even (nonnegative) integer.

In their study of polynomial entropy in slow dynamics, Cantat and Paris-Romaskevich [4] introduced the *polynomial volume growth* $\operatorname{plov}(f)$ of f as follows:

$$\operatorname{plov}(f) \coloneqq \limsup_{n\to\infty} \frac{\log \operatorname{Vol}(\Gamma_n)}{\log n},$$

which turns out to be closely related to the Gelfand–Kirillov dimension of the twisted homogeneous coordinate ring associated with (X, f) (see [9, Proposition 6.11]). The coincidence of these two invariants in algebraic dynamics and noncommutative geometry was first noticed by Lin, Oguiso, and Zhang [11], where, among many other things, they also improved Keeler's upper bound $k(d-1)+d$ for $\operatorname{plov}(f)$ using dynamical filtrations introduced by their earlier joint work [5] with Dinh.

Our main result is a new upper bound for $\operatorname{plov}(f)$ which is almost the half of known upper bounds, and it affirmatively answers [11, Question 6.6] and [8, Question 2.10]. The proof involves combinatorics and representation theory (see Remark 1.4).

Theorem 1.1. *Let X be a normal projective variety of dimension d and f an automorphism of X. Suppose that $f^*|_{\mathsf{N}^1(X)_\mathbf{R}}$ is quasi-unipotent and the maximum size of Jordan blocks of $f^*|_{\mathsf{N}^1(X)_\mathbf{R}}$ is $k+1$. Then we have*

$$\operatorname{plov}(f) \le (k/2+1)d.$$

The upper bound in Theorem 1.1 is optimal when $k/2+1$ divides d. On the other hand, if $k=2$ and d is odd, then the actual optimal upper bound turns out to be $2d-1$ (see [8, Theorem 2.9] or [11, Theorem 4.2]).

Combining our Theorem 1.1 with [5, Theorem 1.1], we give affirmative answers to [4, Question 4.1] and [11, Question 1.5 (1)].

Corollary 1.2. *Let X be a normal projective variety of dimension d. Let f be an automorphism of zero entropy of X. Then one has*

$$\operatorname{plov}(f) \le d^2.$$

This upper bound is optimal by [11, Example 6.4].

Remark 1.3. *In noncommutative geometry, Artin, Tate, and Van den Bergh* [1, 2] *introduced in the 1990s the so-called twisted homogeneous coordinate ring $B \coloneqq B(X, f, \mathscr{L})$ associated with a normal projective variety X, an automorphism f of X, and an invertible sheaf $\mathscr{L}$ on X. It was proved by Keeler* [9] *that the Gelfand–Kirillov dimension $\operatorname{GKdim}(B)$ is finite (and equals $\operatorname{plov}(f)+1$) if and only if $f^*|_{\mathsf{N}^1(X)_\mathbf{R}}$ is quasi-unipotent (see also* [11]*). So our result provides an optimal upper bound d^2+1 for $\operatorname{GKdim}(B)$, too.*

Remark 1.4. *The idea of our proof of Theorem 1.1 is simple and goes back to* [2, 9]. *Precisely, without loss of generality, we may assume that $f^*|_{\mathsf{N}^1(X)_\mathbf{R}}$ is unipotent and hence can be written as $f^*|_{\mathsf{N}^1(X)_\mathbf{R}} = \operatorname{id} + N$, where N is a nilpotent operator on $\mathsf{N}^1(X)_\mathbf{R}$. Then by* [9, Proof of Lemma 6.13] *or* [11, Lemma 2.16]*, we will be interested in the vanishing of intersection numbers $N^{i_1}H_X \cdots N^{i_d}H_X$. Applying the projection formula and by induction,* *we obtain a homogeneous system of linear equations, where these intersection numbers are unknowns.*

To show the vanishing of certain intersection numbers, we shall prove that the matrix of coefficients of the above homogeneous system of linear equations, denoted by $A_{k,d,n}$ later, is of full column rank whenever $n > dk/2$. While handling a single matrix $A_{k,d,n}$ seems to be quite complicated, our key observation is that the family of such matrices (with n varying) can be naturally realized as representative matrices of Lefschetz operators of a representation of the Lie algebra $\mathfrak{sl}_2(\mathbf{C})$. This gives us a hard Lefschetz type result that the product matrix $A_{k,d,n+1}A_{k,d,n+2}\cdots A_{k,d,dk-n}$ is invertible for any $n < dk/2$.

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SLOPE INEQUALITY FOR FIBERED THREEFOLDS OVER CURVES

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Classification AMS 2020: 14D06, 14E30

Keywords: Fibration, threefolds, minimal model

A fibration always means a surjective morphism with connected fibers. Let $f : X \to B$ be a fibration from X to a curve B. We say that f is relatively minimal, if X is projective and normal, with at worst terminal singularities, and the divisor K_X is f-nef. Notice that Ohno [5, Theorem 1.4] has proved that this implies that $K_{X/B}$ is nef in characteristic zero. By an (a, b)-surface, we mean a minimal surface of general type with $K^2 = a$ and $p_g = b$.

In this talk, we will introduce the slope inequality for fibered varieties over curves. The classical slope inequality for fibered surfaces was established by M. Cornalba-J. D. Harris (c. f. [3]) and G. Xiao (c. f. [6]) independently. Motivated by their results, we are interested in the following question.

Question 0.1. *Let $f : X \to B$ be relatively minimal fibration from an n-fold X to a curve B such that the general fiber is of general type and that $\deg f_*\omega_{X/B} > 0$. What is the optimal lower bound of $K^n_{X/B}/\deg f_*\omega_{X/B}$?*

When $n = 2$, the optimal slope is obtained by [3] and [6]. When $n > 2$, some partial results are proved in [5], [1], [4] and [2].

Our first main result is the following theorem:

Theorem 0.2. *Let $f : X \to B$ be a relatively minimal fibration over a smooth projective curve. Then we have the following optimal inequality:*

$$K^3_{X/B} \geq \frac{4}{3} \deg f_*\omega_{X/B}.$$

This theorem removes the smoothness assumption in our previous work (c. f. [4]).

When the equality $K^3_{X/B} = \frac{4}{3} \deg f_*\omega_{X/B} > 0$, we prove that the general fiber of $f : X \to B$ is a $(1, 2)$-surface and X is Gorenstein minimal. It follows that X is locally factorial. We also give the explicit classification of geometric structure of f.

This is a joint work in progress with Tong Zhang.

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MINIMAL RATIONAL CURVES WHOSE VMRT AT A GENERAL POINT IS AN ADJOINT VARIETY

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Classification AMS 2020: 53C35, 14M27

Keywords: Minimal rational curves, adjoint variety, Variety of minimal rational tangents

A minimal rational curve on a projective manifold X is a rational curve through a general point $x \in X$ such that the family of all deformations of the rational curve passing through x is a projective family. We say that two minimal rational curves $C \subset X$ and $\widetilde{C} \subset \widetilde{X}$ have biholomorphic germs, if there are open neighborhoods in Euclidean topology, say, $C \subset U \subset X$ and $\widetilde{C} \subset \widetilde{U} \subset \widetilde{X}$, with a biholomorphic map $\varphi : U \to \widetilde{U}$ satisfying $\varphi(C) = \widetilde{C}$. The question we are interested in is how to check whether two minimal rational curves have biholomorphic germs.

One of the basic invariants of the germ of a minimal rational curve $C \subset X$ is its variety of minimal rational tangents. Pick a point $x \in C$ such that the family of all deformations of C fixing x is a projective family. Then the subset $\mathcal{C}_x \subset \mathbf{P}T_xX$ consisting of tangent directions to deformations of C fixing x is a projective subvariety called the variety of minimal rational tangents (abbr. VMRT) of C at x. If $\varphi : U \to \widetilde{U}$ is a biholomorphic map of germs of minimal rational curves $C \subset X$ and $\widetilde{C} \subset \widetilde{X}$, then the VMRT of C and $\widetilde{C}$ at points related by φ must be isomorphic as projective subvarieties. Thus VMRT is an invariant of the biholomorphic equivalence of germs.

We are interested in the cases when the isomorphism type of the VMRT at a general point of a minimal rational curve $C \subset X$ determines the germs. First we have the following example of germs whose VMRT at a general point is an arbitrarily given smooth projective variety $Z \subset \mathbf{P}^{n-1}$.

Example 1 Let $\mathbf{P}^{n-1} \subset \mathbf{P}^n$ be a hyperplane in the n-dimensional projective space and fix a submanifold $Z \subset \mathbf{P}^{n-1}$. Let X_Z be the blowup of $\mathbf{P}^n$ along Z. There is a family of minimal rational curves on X_Z whose general members are proper transformations of lines on $\mathbf{P}^n$ intersecting Z. Then its VMRT at any point on the open subset of X_Z corresponding to $\mathbf{P}^n \setminus \mathbf{P}^{n-1}$ is isomorphic to $Z \subset \mathbf{P}^{n-1}$.

The most interesting cases are when $Z \subset \mathbf{P}^{n-1}$ is a homogeneous projective submanifold. Then the germs of all general minimal rational curves in the above example are biholomorphic to one another. We have the following rigidity results for some classes of homogeneous submanifolds.

Theorem 1 [3] Let $Z \subset \mathbf{P}^{n-1}$ be the VMRT at a point of minimal rational curves on an irreducible Hermitian symmetric space of compact type. Suppose $C \subset X$ is a minimal rational curve in a projective manifold X of dimension n such that the VMRT at a general

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point of C is isomorphic to Z as projective submanifolds. Then the germ of C in X is biholomorphic to the germ of a general line in X_Z.

Theorem 2 [2] Let $Z \subset \mathbf{P}^{2n-1}$ be a homogeneous Legendrian submanifold. Suppose $C \subset X$ is a minimal rational curve in a projective manifold X of dimension $2n$ such that the VMRT at a general point of C is isomorphic to Z as projective submanifolds. Then the germ of C in X is biholomorphic to the germ of a general line in X_Z.

In a recent joint work with Qifeng Li, we studied the case when $Z \subset \mathbf{P}^{n-1}$ is an adjoint variety, namely, the unique closed orbit in $\mathbf{P}\mathfrak{g}$ of the adjoint representation on a complex simple Lie algebra $\mathfrak{g}$. In this case, we have the following examples in addition to Example 1.

Example 2 The VMRT at a general point of a general line on a smooth hyperplane section of the Grassmannian $\mathrm{Gr}(3; \mathbf{C}^6) \subset \mathbf{P}(\wedge^3 \mathbf{C}^6)$ is isomorphic to the adjoint variety for $\mathfrak{g}$ of type A_2.

Example 3 [1] The VMRT at a general point of a general minimal rational curve on the wonderful group compactification for $\mathfrak{g}$ of type different from A_ℓ is isomorphic to the adjoint variety for $\mathfrak{g}$.

The main result in the joint work with Qifeng Li is the following.

Theorem 3 Let $Z \subset \mathbf{P}\mathfrak{g}$ be the adjoint variety of a complex simple Lie algebra $\mathfrak{g}$ of type different from $\mathsf{A}_{\ell \geq 3}$. Suppose $C \subset X$ is a minimal rational curve in a projective manifold X of dimension equal to $\dim \mathfrak{g}$ such that the VMRT at a general point of C is isomorphic to Z as projective submanifolds. Then the germ of C in X is biholomorphic to the germ of a general line in Examples 1, 2, or 3.

The proofs of Theorems 1, 2 and 3 use differential geometric techniques. In particular, for Theorem 3, the classical theory of G-structures and affine symmetric spaces plays an essential role.

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WILD AUTOMORPHISMS OF COMPACT COMPLEX SPACES OF LOWER DIMENSIONS

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Keywords: Wild automorphism, complex torus, Inoue surface, entropy

This presentation is based on the paper [6].

Let X be a compact complex space. We will use the analytic Zariski topology on X whose closed sets are all analytic sets (cf. [4, Page 211]). An automorphism $\sigma \in \mathrm{Aut}(X)$ is called wild in the sense of Reichstein–Rogalski–Zhang ([9]) if for any non-empty analytic subset Z of X satisfying $\sigma(Z) = Z$, we have $Z = X$; or equivalently, for every point $x \in X$, its orbit $\{\sigma^n(x) \mid n \geq 0\}$ is Zariski dense in X.

The following two conjectures generalise [9, Conjecture 0.3] and [8, Conjecture 1.4] from the projective case to the Kähler case.

Conjecture 0.1 (cf. [9, Conjecture 0.3]). Assume that a compact Kähler space X admits a wild automorphism. Then X is isomorphic to a complex torus.

Conjecture 0.2 (cf. [8, Conjecture 1.4]). Every wild automorphism σ of a compact Kähler space X has zero entropy.

When X is a projective variety, wild automorphisms are related with the twisted homogeneous coordinate rings, which play a role in noncommutative algebraic geometry (see [9]). The study of wild automorphisms is also of interest from the viewpoint of dynamical systems (see [1]).

For a compact Kähler surface X with a wild automorphism σ, it is well-known that X is a complex torus, and σ is of the certain form, a priori, of zero entropy (see [9, Theorem 6.5] and [1, Theorem 6.10]).

In this article, we consider compact complex surfaces (not necessarily Kähler) with a wild automorphism. We give a characterisation of such surfaces and show that there do exist examples of non-Kähler surfaces that admit a wild automorphism.

Theorem 0.3. Let X be a compact complex space of dimension ≤ 2. Assume that X admits a wild automorphism σ. Then we have:

(1) X is either a complex torus or an Inoue surface of type $S_M^{(+)}$, and σ has zero entropy.
(2) Both cases in (1) occur: there are pairs (X', σ') where X' is a complex torus or an Inoue surface of type $S_M^{(+)}$ and σ' acts on X' as a wild automorphism.

We refer to §6 for the definition and the constructions of Inoue surfaces. In §6, we will construct examples of wild automorphisms of Inoue surfaces of type $S_M^{(+)}$. We remark that there are examples of wild automorphisms of complex abelian surfaces. More strongly, there are complex abelian surfaces with an automorphism of which all orbits are Euclidean dense (see [1, Example 6.6 and Lemma 6.7]).

Question 0.4 (cf. [2, Section 4.2]). Are there Inoue surfaces of type $S_M^{(+)}$ with an automorphism of which all orbits are Euclidean dense?

As a by-product of our argument, we obtain new results about the automorphism groups of Inoue surfaces, which might be of independent interest. Theorem 0.5 below gives a more refined structure than [5, Section 6]. We remark that Inoue surfaces are divided into three different types: S_M, $S_M^{(+)}$ and $S_M^{(-)}$.

Theorem 0.5. Let X be an Inoue surface.

(1) If X is either of type S_M or $S_M^{(-)}$, then the (biholomorphic) automorphism group $\mathrm{Aut}(X)$ is finite.
(2) If X is of type $S_M^{(+)}$, the neutral connected component $\mathrm{Aut}_0(X) \simeq \mathbb{C}^*$ and $\mathrm{Aut}(X)/\mathrm{Aut}_0(X)$ is finite.

Fujiki [3] has studied the automorphism groups of parabolic Inoue surfaces. It is worth noting that a parabolic Inoue surface has positive second Betti number, which distinguishes it from the usual Inoue surfaces.

We propose the following questions rather than conjectures due to the lack of evidence.

Question 0.6.

(1) Is a compact complex space in Fujiki's class $\mathcal{C}$ admitting a wild automorphism a complex torus?
(2) Does every wild automorphism of a compact complex space have zero entropy?

A compact complex space is called in Fujiki's class $\mathcal{C}$ if it is bimeromorphic to a compact Kähler manifold. In dimension two, a compact complex manifold is in Fujiki's class $\mathcal{C}$ if and only if it is Kähler, while starting from dimension three, the category of Fujiki's class $\mathcal{C}$ is strictly larger. In particular, the answers to both two questions are affirmative in dimension two due to Theorem 0.3.

In the rest of this article, we study Conjectures 0.1 and 0.2 in dimension three and four.

Theorem 0.7. Let X be a compact Kähler space of dimension three, and let σ be a wild automorphism of X. Then

(1) X is either a complex torus or a weak Calabi–Yau threefold;
(2) σ has zero entropy.

Here a smooth complex projective variety V is called

(1) a weak Calabi–Yau manifold, if $K_V \sim_{\mathbb{Q}} 0$ and $\pi_1(V)$ is finite;
(2) a Calabi–Yau manifold in the strict sense, if V is simply connected, $K_V \sim 0$ and $H^j(V, \mathcal{O}_V) = 0$ for $0 < j < \dim V$.

Let us remark that, X in Theorem 0.7 could not be a weak Calabi–Yau threefold if one assumes the generalised non-vanishing conjecture which predicts that any nef Cartier divisor on a Calabi–Yau threefold is effective ([8, Theorem 7.4], see also [7, Theorem 4.7]). The following proposition provides further evidence.

Proposition 0.8. Let X be a weak Calabi–Yau threefold, and let $c_2(X)$ be the second Chern class of X. Assume that either

(1) $c_2(X) \cdot D > 0$ for every non-torsion nef Cartier divisor D on X; or

(2) there exists a non-torsion semi-ample Cartier divisor D on X such that $c_2(X) \cdot D = 0$.

Then X has no wild automorphism.

Theorem 0.9. Conjecture 0.2 is true in all three cases below.

(1) $\dim X \leq 3$.
(2) $\dim X = 4$, and the Kodaira dimension $\kappa(X) \geq 0$,
(3) $\dim X = 4$, and the irregularity $q(X) \neq 1, 2$.

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AN EFFECTIVE UPPER BOUND FOR ANTI-CANONICAL VOLUMES OF SINGULAR FANO 3-FOLDS

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Classification AMS 2020: 14J45, 14J30, 14J17.

Keywords: Fano threefolds, anti-canonical volumes, log canonical thresholds, boundedness

This report is an extended abstract of my talk at IMS. I will discuss my recent joint work with Yu Zou (Tsinghua University).

We work over the field of complex numbers $\mathbb{C}$.

A normal projective variety X is a *Fano* variety if $-K_X$ is ample. According to the minimal model program, Fano varieties form a fundamental class in the birational classification of algebraic varieties.

One recent breakthrough in birational geometry is the proof of the Borisov–Alexeev–Borisov conjecture by Birkar [2, 3], which states that for a fixed positive integer d and a positive real number ϵ, the set of d-dimensional Fano varieties with ϵ-klt singularities forms a bounded family. During the proof, one important step is to establish the upper bound for the anti-canonical volume $(-K_X)^d$ for an ϵ-klt Fano variety X of dimension d ([2, Theorem 1.6]).

Motivated by the explicit classification theory of algebraic varieties, we are interested in finding the explicit bound depending on d and ϵ for $(-K_X)^d$ for an ϵ-klt Fano variety X of dimension d.

When $d = 2$, we have a satisfactory answer.

Theorem 0.1 ([5])**.** *Fix a real number $\epsilon > 0$. Let X be an ϵ-klt Fano variety of dimension 2. Then*

$$(-K_X)^2 < \max\left\{9, \lfloor 2/\epsilon \rfloor + 4 + \frac{4}{\lfloor 2/\epsilon \rfloor}\right\}.$$

When $d = 3$ and $\epsilon = 1$, we have a recent progress.

Theorem 0.2 ([7])**.** *Let X be a canonical Fano 3-fold. Then $(-K_X)^3 \leq 324$.*

In the above theorem, conjecturally the upper bound should be 72, but this is the first effective upper bound.

Finally, let us consider the case when $d = 3$ and ϵ is abitrary. In this direction, Lai [8] gave an upper bound for those X which are $\mathbb{Q}$-factorial and of Picard rank 1, which is over $O((\frac{4}{\epsilon})^{384/\epsilon^5})$; later, the first author [6] showed the existence of a non-explicit upper bound; recently, Birkar [4] gave the first explicit upper bound, which is about $O(\frac{2^{1536/\epsilon^3}}{\epsilon^9})$. In the recent joint work with Yu Zou, we provide a reasonably small explicit upper bound with a sharp order, for the anti-canonical volume of an ϵ-klt Fano 3-fold.

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Theorem 0.3. *Fix a real number $0 < \epsilon < \frac{1}{3}$. Let X be an ϵ-klt Fano variety of dimension 3. Then*
$$(-K_X)^3 < \frac{3200}{\epsilon^4}.$$

The following example shows that the order $O(\frac{1}{\epsilon^4})$ in Theorem 0.2 is sharp. In fact, Ambro [1, Example 6.3] showed that for each positive integer q, there exists a projective toric 3-fold X such that X is $\frac{1}{q}$-lc Fano and $(-K_X)^3 > \frac{u_{4,q}}{q^4} = O(q^4)$.

According to Ambro's example, we may hope that the following conjecture is true.

Conjecture 0.4. *Fix a real number $\epsilon > 0$. Let X be an ϵ-klt Fano variety of dimension d. Then*
$$(-K_X)^d < O(\epsilon^{2^d-d-1}).$$

Unfortunately, very few is known in dimension at least 4 due to the lack of effective methods. See [4] for the case when $d = 4$ and $\epsilon = 1$.

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EISENSTEIN K3 SURFACES AND ANALYTIC TORSION (JOINT WORK WITH KEN-ICHI YOSHIKAWA)

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This is joint work with Ken-Ichi Yoshikawa [9].

Let $\rho = \frac{-1+\sqrt{-3}}{2}$ be a cubic root of unity. An *Eisenstein* $K3$ *surface* is a pair (X, σ) such that X is a $K3$ surface and σ is an automorphism of X with $\sigma^*(\eta) = \rho\,\eta$ for $\eta \neq 0 \in H^0(X, K_X)$. An *Eisenstein lattice* is a pair (T, g) such that T is an even lattice and $g\colon T \to T$ is an isometry with $g^2 + g + 1_T = 0$. If (X, σ) is an Eisenstein $K3$ surface, then $((H^2(X,\mathbb{Z})^\sigma)^\perp, \sigma^*)$ is an Eisenstein lattice, which we call the Eisenstein lattice of (X, σ). We write $T = (H^2(X,\mathbb{Z})^\sigma)^\perp$ and $g = \sigma^*$.

Let $T \otimes \mathbb{C} = T_{\mathbb{C}}(\rho) \oplus T_{\mathbb{C}}(\rho^2)$ be the eigenspace decomposition with respect to g. We set $\mathcal{B}_T := \{[\eta] \in \mathbb{P}(T(\rho)) \mid (\eta, \overline{\eta}) > 0\}$, which is a complex ball of dimension $\frac{\mathrm{rk}(T)}{2} - 1$. Let $U(T)$ be the group of isometries of T whose element commutes with g. Let $\mathcal{H}_T \subset \mathcal{B}_T$ denote the discriminant locus.

By the *type* of an Eisenstein $K3$ surface (X, σ), we mean the isometry class of its Eisenstein lattice. Artebani–Sarti [3] and Taki [14] show that there are 24 types of Eisenstein $K3$ surfaces and that such an Eisenstein lattice (T, g) is determined by $(r, a) := (\mathrm{rk}(T), \dim_{\mathbb{F}_3} A_T)$, where A_T is the discriminant group. Let X^σ be the fixed locus of X for σ. Then the 1-dimensional component $X^\sigma_{(1)}$ is the union of $\mathbb{P}^1$'s and a smooth projective curve C_g with $g \geq 0$, and g is determined by (r, a). Also, Artebani–Sarti [3], Taki [14], Dolgachev–Kondo [7], Ma–Ohashi–Taki [10] show that, via the period mapping, the coarse moduli space of Eisenstein $K3$ surfaces of type T is given by $\mathcal{M}^\circ_T := (\mathcal{B}_T \setminus \mathcal{H}_T)/U(T)$.

In general, for a compact Kähler manifold Y with a Kähler form κ and a finite group G acting on (Y, κ) holomorphically and isometrically, one can define G-equivariant analytic torsions $\tau_G(Y, \kappa)(g) \in \mathbb{R}_{>0}$ for each $g \in G$. When $g = 1$, $\tau_G(Y, \kappa)(1)$ is equal to the Ray–Singer analytic torsion $\tau(Y, \kappa)$ of (Y, κ). Ray–Singer analytic torsions and equivariant analytic torsions have been deeply studied by various authors such as Ray–Singer [13], Bismut–Gillet–Soulé [6], Bismut [4], Bismut–Lebeau [5], and Ma [11].

Let (X, σ) be an Eisenstein $K3$ surface of type T, and let κ be a σ-invariant Kähler form on X. Using the volumes of X and X^σ with respect to κ, $\langle\sigma\rangle$-equivariant analytic torsions, the Ray–Singer analytic torsion on the 1-dimensional component of X^σ with respect to κ, and some quantity related to κ that becomes 1 if κ is Ricci-flat, we introduce a quantity $\tau_T(X, \sigma) \in \mathbb{R}_{>0}$. Explicitly,

$$\tau_T(X,\sigma) := \mathrm{Vol}(X,\kappa)^{5-\frac{\mathrm{rank}(T)}{2}} \prod_{k=1}^{2} \tau_{\mu_3}(X,\kappa)(\sigma^k) \cdot \mathrm{Vol}(X^\sigma_{(1)}, \kappa|_{X^\sigma_{(1)}})^3 \tau(X^\sigma_{(1)}, \kappa|_{X^\sigma_{(1)}})^3 \cdot A_T(X,\sigma,\kappa),$$

where $\mu_3 \cong \langle\sigma\rangle$, $X^{\sigma}_{(1)}$ denotes the 1-dimensional component of X^{σ}, and $A_T(X, \sigma, \kappa)$ is the quantity that becomes 1 if κ is Ricci-flat. We remark that for a $K3$ surface with non-symplectic involution, such a quantity was constructed and deeply studied by Yoshikawa [15, 16, 17] and Ma–Yoshikawa [12].

We show that $\tau_T(X, \sigma)$ is independent of the choice of κ. It follows that τ_T is viewed as a function on the moduli space $\mathcal{M}^{\circ}_T$ of Eisenstein $K3$ surfaces of type T. By the pull-back of the projection $\mathcal{B}_T \setminus \mathcal{H}_T \to \mathcal{M}^{\circ}_T$, we obtain a function $\tau_{\mathcal{B}_T}$ on $\mathcal{B}_T \setminus \mathcal{H}_T$.

Studying behaviors of $\tau_{\mathcal{B}_T}$ near the discriminant locus $\mathcal{H}_T$ and the Torelli map $\mathcal{M}^{\circ}_T \in (X, \sigma) \to \mathrm{Jac}(X^{\sigma}_{(1)}) \in \mathcal{A}_g$ in detail, we obtain from $\tau_{\mathcal{B}_T}$ an automorphic form Ψ_T on $\mathcal{B}_T$ for $U(T)$. In many cases, we show that Ψ_T is a reflective modular form. As a corollary, in many cases, the moduli space $\mathcal{M}^{\circ}_T$ is quasi-affine.

For several Eisenstein lattices T, we obtain an explicit form of Ψ_T. Arguably the most interesting case is when $T = \mathbb{A}_2^+ \oplus \mathbb{A}_2^{\oplus 5}$, where $\mathbb{A}_2 = \left(\begin{smallmatrix} -2 & 1 \\ 1 & -2 \end{smallmatrix}\right)$ and $\mathbb{A}_2^+ = \mathbb{A}_2(-1)$. In this case, Allcock–Carlson–Toledo [1] and Dolgachev–van Geemen–Kondo [8] show that $\mathcal{M}^{\circ}_T$ is isomorphic to the moduli space of cubic surfaces. Further, Borcherds shows that there is an automorphic form χ_4 on $\mathcal{B}_T$ for $U(T)$ vanishing exactly on the discriminant locus $\mathcal{H}_T$ of order 1. In this case, we show that Ψ_T is equal to χ_4 up to a universal constant.

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1. ON DEFORMATIONS OVER NON-COMMUTATIVE BASE
2. ON NON-COMMUTATIVE DEFORMATIONS OF COMPLEX MANIFOLDS

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Keywords: non-commutative deformation, Hochschild cohomology, derived McKay correspondence.

1. On deformations over non-commutative base

We write "NC" for the abbreviation of "not necessarily commutative".

Let X be a fixed algebraic variety over a field k. We consider deformations of a coherent sheaf F on X over a parameter ring R which is not necessarily commutative (NC). In general, deformations of F are described by a differential graded algebra $A = R\mathrm{Hom}(F, F)$; the tangent space of the deformation functor is identified as $H^1(A) = \mathrm{Ext}^1(F, F)$ and the obstruction space for extending infinitesimal deformations is as $H^2(A) = \mathrm{Ext}^2(F, F)$. Since A is naturally NC, it is natural to consider deformations over an NC base ring.

We assume that F has a proper support, and R is a finite dimensional associative algebra with a two sided ideal M such that $R/M \cong k$, or an inverse limit of such rings $\hat{R} = \varprojlim \hat{R}/\hat{M}^n$.

The following definition is the same as the commutative case except that the base ring R is NC:

Definition 1.1. *A deformation $(\tilde{F}, \phi)$ of F over R is a pair consisting of a coherent sheaf on X, or an inverse limit of those, with a left R-module structure which is flat over R, and an isomorphism $\phi : k \otimes_R \tilde{F} \to F$. The deformation functor Def_F of F sends R to the set of isomorphism classes of deformations of F over R.*

The existence of the versal deformation is proved in the same way as in the commutative case.

Theorem 1.2. *Let $n_i = \dim Ext^i(F, F)$ for $i = 1, 2$. Then the parameter ring R of the versal NC deformation of F is described in the form*

$$R = k\langle\langle x_1, \dots, x_{n_1}\rangle\rangle/(f_1, \dots, f_{n_2})$$

where the f_i are possibly trivial NC formal power series of order at least 2.

The abelianization $R^{\mathrm{ab}} = R/[R, R] = k[[x_1, \dots, x_{n_1}]]/(f_1, \dots, f_{n_2})$ is the parameter algebra for commutative deformations.

More generally, we can consider *multi-pointed* NC deformations for a sheaf which has a direct sum decomposition such as $F = \bigoplus_{i=1}^r F_i$.

The commutative deformations yield a *moduli space* $\mathcal{M}$ for F, and the NC deformations give additional formal structure on $\mathcal{M}$.

2. On non-commutative deformations of complex manifolds

If we allow the base variety X to be deformed to an NC object, then it turns out that the base ring should be automatically commutative. Let X be a compact complex manifold and let R be a finite dimensional commutative local algebra with the maximal ideal M such that $R/M = k = \mathbf{C}$, or an inverse limit of such rings $\hat{R} = \varprojlim \hat{R}/\hat{M}^n$.

Definition 2.1. *An NC deformation $\tilde{X}$ of X over R is a pair $(\mathcal{A}, \phi)$ consisting of a sheaf of NC associative algebras $\mathcal{A}$ on X which has a structure of a flat R-module and an isomorphism of sheaves of algebras $\phi : k \otimes_R \mathcal{A} \to \mathcal{O}_X$. The deformation functor Def_X of X sends R to the set of isomorphism classes of deformations of X over R.*

The tangent space and the obstruction space for commutative deformations of X are given by $H^i(X, T_X)$ for $i = 1, 2$. There are more NC deformations than commutative ones, and the corresponding spaces are given by $T^i = \mathrm{Ker}(HH^{i+1}(X) \to H^{i+1}(X, \mathcal{O}_X))$, where HH denotes the Hochschild cohomology. By Rosenberg-Hochschild-Kostant isomorphism $HH^n(X) \cong \bigoplus_{p+q=n} H^q(X, \bigwedge^p T_X)$, we have $T^1 \cong H^0(X, \bigwedge^2 T_X) \oplus H^1(X, T_X)$ and $T^2 \cong H^0(X, \bigwedge^3 T_X) \oplus H^1(X, \bigwedge^2 T_X) \oplus H^2(X, T_X)$.

Theorem 2.2. *There exists a versal NC deformation of X whose parameter algebra is of the form*

$$R = k[[x_1, \dots, x_{n_1}]]/(f_1, \dots, f_{n_2})$$

for $n_i = \dim T^i$, where the f_i are possibly trivial formal power series of order at least 2.

We consider a special case where X is the minimal resolution of a surface quotient singularity of type A_n. We can construct an NC deformation of X over an algebraic ring instead of a formal ring in this case.

Let $G = \mathbf{Z}/(n+1)$ be a cyclic group which acts on an affine plane $\mathbf{C}^2$ by the action $g(x, y) = (\zeta x, \zeta^{-1} y)$ for generators $g \in G$ and $\zeta \in \mu_{n+1}$, where (x, y) are coordinates. The quotient space $Y = \mathbf{C}^2/G$ has an isolated singularity. Let $p : X \to Y$ be the minimal resolution. There is also a non-commutative resolution $\pi : [\mathbf{C}^2/G] \to Y$ from a quotient stack.

The set of simple objects in the abelian category of coherent sheaves $\mathrm{Coh}(X)$ on X over the singular point $y_0 \in Y$ corresponds bijectively to the exceptional set $p^{-1}(y_0)$, which is an infinite set. But the set of simple objects in the category of coherent sheaves $\mathrm{Coh}([\mathbf{C}^2/G])$ over $y_0 \in Y$ is a finite set of order $n+1$. In this sense, X is more geometric and $[\mathbf{C}^2/G]$ is more algebraic. The categories $\mathrm{Coh}(X)$ and $\mathrm{Coh}([\mathbf{C}^2/G])$ are quite different, but their derived categories $D^b(\mathrm{Coh}(X))$ and $D^b(\mathrm{Coh}([\mathbf{C}^2/G]))$ are equivalent (derived McKay correspondence).

Let $S = k[x, y]\#G$ be the twisted group ring, where the multiplication is given by $a_1 g_1 \cdot a_2 g_2 = a_1 g_1(a_2) g_1 g_2$ for $a_i \in k[x, y]$ and $g_i \in G$. S can be regarded as a coordinate ring of the quotient stack $[\mathbf{C}^2/G]$. The coordinate ring of Y is given by $\mathcal{O}_Y = eSe$ for $e = \sum_{g \in G} g/(n+1)$.

The versal algebraic NC deformations $\tilde{S}$ and $\tilde{\mathcal{O}}_Y$ of S and $\mathcal{O}_Y$, respectively, are given by [2]:

$$\tilde{S} = k[s_0, \dots, s_n]\langle x, y\rangle\#G/(xy - yx - \sum s_i g^i), \;\; \tilde{\mathcal{O}}_Y = e\tilde{S}e.$$

On the other hand, the geometric resolution X is covered by open subsets $U_i = \mathrm{Spec}\ k[x_i, y_i]$ for $i = 0, \dots, n$ with gluing transformations $x_i = x_{i-1}^2 y_{i-1}$ and $y_i = x_{i-1}^{-1}$. Let $\mathcal{A}_i = k[t_0, \dots, t_n]\langle x_i, y_i\rangle/(x_i y_i - y_i x_i - t_0)$ be associative algebras with gluing $x_i = x_{i-1}^2 y_{i-1} + s_i x_{i-1}$ and $y_i = x_{i-1}^{-1}$.

Then we can think that the algebras $\mathcal{A} = (\mathcal{A}_i)$ give an algebraic deformation $\tilde{X}$ of X over $\mathrm{Spec}\ k[t_0, \dots, t_n]$. We calculate the set of global functions $\Gamma(\tilde{X}, \mathcal{A})$ on $\tilde{X}$ and prove that $\Gamma(\tilde{X}, \mathcal{A}) \cong \tilde{\mathcal{O}}_Y$.

Moreover we can define an abelian category of coherent sheaves $\mathrm{Coh}(\mathcal{A})$ on $\tilde{X}$ and extend the derived McKay correspondence:

Theorem 2.3. *There is an equivalence of triangulated categories $D^b(Coh(\mathcal{A})) \cong D^b(mod\text{-}\tilde{S})$ under the linear change of coordinates $t_0 = (n+1)s_0$ and $t_0 + t_i = s_{n+1-i}$ for $i = 1, \dots, n$.*

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FACTORIZATION OF HOLOMORPHIC MATRICES

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1. Linear Algebra

Any matrix $A \in SL_n(\mathbb{C})$ is a product of elementary matrices of the form

$$Id + a_{ij}E_{ij} = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ 0 & a_{ij} & 1 & & 0 \\ \vdots & & \vdots & \ddots & \\ 0 & & 0 & & 1 \end{pmatrix}$$

or equivalently a product of upper and lower triangular unipotent matrices.

$$A = \begin{pmatrix} 1 & 0 \\ G_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & G_2 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & G_N \\ 0 & 1 \end{pmatrix}, where \;\; G_i \in \mathbb{C}^{n(n-1)/2}$$

Proof: Gauss elimination, it requires:
1.) Adding multiples of a row to another row
2.) Interchange of rows :

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

3.) multiplication of rows by constants:

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -a^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & a-1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & a^{-1}-1 \\ 0 & 1 \end{pmatrix}$$

(Whitehead lemma)

What if the matrix A depends on a parameter x (continuously, polynomially, holomorphically)? Can the upper and lower triangular unipotent matrices be chosen depending well on the parameter?

$$A(x) = \begin{pmatrix} 1 & 0 \\ G_1(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & G_2(x) \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & G_N(x) \\ 0 & 1 \end{pmatrix}$$

Now the G_i are maps $G_i : X \to \mathbb{C}^{n(n-1)/2}$.

- Let $R = \{f : X \to \mathbb{C}\}$ denote the ring of continuous/ polynomial /holomorphic functions on a topological space/ algebraic variety / complex space X.
- In the language of K-theory we are asking about factorization of $SL_n(R)$ (special linear group over the ring R) as product of elementary matrices over that ring.

- Given $m \geq 2$ and an associative, commutative, unital ring R, let $E_n(R)$ denote the set of those $n \times n$ matrices which are representable as products of unipotent matrices with entries in R. We ask about the relation of $E_n(R)$ and $SL_n(R)$.
- The obstruction to this factorization is called the special K_1-group of the ring R, (more precise the n-th, where n is the size of the matrices).

1.1. **Symplectic Notation.** Let I_n denote the $(n \times n)$ identity matrix and 0_n the $(n \times n)$ zero matrix.

Recall $Sp_{2n}(\mathbb{C}) := \{A \in Gl_{2n}(\mathbb{C}) \ : \ A^T J A = J\}$, where $J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$ is the standard symplectic form.

In the block notation

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{C}),$$

the symplectic condition $MJM^T = J$ gives rise to three simple types of J-symplectic matrices:

- (i): $\begin{pmatrix} I & B \\ 0 & I \end{pmatrix}$, upper triangular with symmetric $B = B^T$.
- (ii): $\begin{pmatrix} I & 0 \\ C & I \end{pmatrix}$, lower triangular with symmetric $C = C^T$.
- (iii): $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$, block diagonal with invertible $A \in \mathrm{GL_n}(\mathbb{C})$ and $D = (A^{-1})^T$.

We call those matrices of type (i) and (ii) *elementary symplectic matrices*.

2. History of the factorization problem

2.1. **Algebraic results.**

- $SL_n(\mathbb{C}[z_1])$ factorizes, more generally for Euclidean rings R $SL_n(R)$ factorizes
- $SL_2(\mathbb{C}[z_1, z_2, \ldots, z_n])$ does not factorize for $n \geq 2$ counterexample found by Cohn [1]

$$\begin{pmatrix} 1 - z_1 z_2 & z_1^2 \\ -z_2^2 & 1 + z_1 z_2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}[z_1, z_2])$$

- Suslin [9] proved that $SL_n(\mathbb{C}[z_1, z_2, \ldots, z_m])$ does factorize for all m and all $n \geq 3$

2.2. **Algebraic symplectic results.**

- $Sp_{2n}(\mathbb{Z}[z_1, z_2, \ldots, z_m])$ does factorize for all m and all $n \geq 2$, [2]
- $Sp_{2n}(\mathbb{C}[z_1, z_2, \ldots, z_m])$ does factorize for all m and all $n \geq 2$, [6], [7]

2.3. **Topological results.**

- $SL_n(Cont(\mathbb{R}^3))$ factorizes, [10]
- A general observation:

$$A_t(x) = \begin{pmatrix} 1 & 0 \\ tG_1(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & tG_2(x) \\ 0 & 1 \end{pmatrix} \ldots \begin{pmatrix} 1 & tG_N(x) \\ 0 & 1 \end{pmatrix} t \in [0,1]$$

gives a homotopy of the map $A : X \to SL_m(\mathbb{C})$ to a constant map. Such maps are called null-homotopic. **If a map factorizes, then it is necessarily null-homotopic.**

2.4. **Continuous result.**

Theorem 2.1 (Vaserstein, [11]). *For any natural number n and an integer $d \geq 0$ there is a natural number K such that for any finite dimensional normal topological space X of dimension d and null-homotopic continuous mapping $A\colon X \to SL_n(\mathbb{C})$ the mapping can be written as a finite product of no more than $K = K(d,n)$ unipotent matrices. That is, one can find continuous mappings $G_l\colon X \to \mathbb{C}^{n(n-1)/2}$, $1 \leq l \leq K$ such that*

$$A(x) = \begin{pmatrix} 1 & 0 \\ G_1(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & G_2(x) \\ 0 & 1 \end{pmatrix} \dots \begin{pmatrix} 1 & G_K(x) \\ 0 & 1 \end{pmatrix}$$

for every $x \in X$.

L. Vaserstein, Reduction of a matrix depending on parameters to a diagonal form by addition operations, *Proc. Amer. Math. Soc.* **103** (1988), no. 3, 741–746

Let

$$U_n(x_1, \dots, x_{n(n+1)/2}) = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ x_2 & x_{n+1} & \dots & x_{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_n & x_{2n-1} & \dots & x_{n(n+1)/2} \end{pmatrix}.$$

Given a map $G\colon X \to \mathbb{C}^{n(n+1)/2}$ let $U_n(G(x)) = U_n(G_1(x), \dots, G_{n(n+1)/2}(x))$ where the G_j's are components of the map G.

2.5. **Symplectic continous result.**

Theorem 2.2. [5] *Let X be a d-dimensional normal topological space and $f\colon X \to \mathrm{Sp}_{2n}(\mathbb{C})$ be a continuous mapping that is null-homotopic. Then there exist a natural number $K = K_{cont}(n,d)$ and continuous mappings $G_1, \dots, G_K\colon X \to \mathbb{C}^{n(n+1)/2}$ such that*

$$f(x) = \begin{pmatrix} I_n & 0_n \\ U_n(G_1(x)) & I_n \end{pmatrix} \begin{pmatrix} I_n & U_n(G_2(x)) \\ 0_n & I_n \end{pmatrix} \dots \begin{pmatrix} I_n & U_n(G_K(x)) \\ 0_n & I_n \end{pmatrix}$$

3. The main results

Theorem 3.1. [4] *Let X be a finite dimensional reduced Stein space and $A\colon X \to SL_n(\mathbb{C})$ be a holomorphic mapping that is null-homotopic. Then there exist a natural number $K = K(dimX, n)$ and holomorphic mappings $G_1, \dots, G_K\colon X \to \mathbb{C}^{n(n-1)/2}$ such that A can be written as a product of upper and lower diagonal unipotent matrices*

$$A(x) = \begin{pmatrix} 1 & 0 \\ G_1(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & G_2(x) \\ 0 & 1 \end{pmatrix} \dots \begin{pmatrix} 1 & G_K(x) \\ 0 & 1 \end{pmatrix}$$

for every $x \in X$.

Theorem 3.2. [8], [5] *Let X be a d-dimensional reduced Stein space and $f\colon X \to \mathrm{Sp}_{2n}(\mathbb{C})$ be a holomorphic mapping that is null-homotopic. Then there exist a natural number $K_{symp} = K_{symp}(n,d)$ and holomorphic mappings $G_1, \dots, G_K\colon X \to \mathbb{C}^{n(n-1)}$ such that*

$$f(x) = \begin{pmatrix} I_n & 0_n \\ U_n(G_1(x)) & I_n \end{pmatrix} \begin{pmatrix} I_n & U_n(G_2(x)) \\ 0_n & I_n \end{pmatrix} \dots \begin{pmatrix} I_n & U_n(G_K(x)) \\ 0_n & I_n \end{pmatrix}$$

3.1. **K-theorists use another symplectic form.** If block A in type (iii) $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$, block diagonal with invertible $A \in \mathrm{GL_n}(\mathbb{C})$ and $D = (A^{-1})^T$ is upper triangular, then $D = (A^{-1})^T$ is lower triangular. In fact, A and D are simultaneously upper or lower triangular in another basis.

This new basis can be obtained from the old one by reversing the order of the last n basis elements, giving a Gramian matrix

$$\tilde{J} = \begin{pmatrix} 0 & L \\ -L & 0 \end{pmatrix}, \tag{3.1}$$

where L is the $n \times n$ matrix with 1 along the skew-diagonal. Notice that symplectic matrices of type (i) and (ii) remain upper or lower triangular with respect to $\tilde{J}$, respectively.

If we allow these matrices too the number of factors will be denoted by $\tilde{K}_{symp}(n, d)$

3.2. **Number of factors.** Analytic techniques (Ivarsson-K.) can be used to show:

$$K(2, 1) = 4 \text{ and } K(2, 2) = 5$$

K-theory arguments (due to Dennis, Vaserstein, Vavilov, Smolenskii, Sury, generalized by Huang, Kutzschebauch, Schott [3] guarantee $K(n, d) \geq K(n + 1, d)$, $\tilde{K}_{symp}(n, d) \geq \tilde{K}_{symp}(n + 1, d)$ and one can prove that the optimal numbers satisfy

$$K(n, 1) = 4 \text{ for all } n,$$
$$4 \leq K(n, 2) \leq 5 = K(2, 2) \text{ for all } n, \text{ and}$$
$$\text{for each } d, \text{ there exists } n(d) \text{ such that } K(n, d) \leq 6 \text{ for all } n \geq n(d)$$
$$\tilde{K}_{symp}(n, 1) = 4 \text{ for all } n,$$
$$4 \leq \tilde{K}_{symp}(n, 2) \leq 5 = \tilde{K}_{symp}(1, 2) \forall n$$

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POSITIVITY OF THE TANGENT BUNDLE OF SMOOTH PROJECTIVE SURFACES AND FANO THREEFOLDS

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Classification AMS 2020: 14J45, 14J30, 14J40, 14E30

Keywords: tangent bundle, pseudo-effective, total dual VMR, Fano threefold, projective surface

Throughout this talk we will work over the field of complex numbers. This presentation is based on two joint works, a joint work with Hosung Kim and Jeong-Seop Kim, a joint work with Jia Jia and Guolei Zhong.

A well-known theorem of Mori [8] asserts that if the tangent bundle of a smooth projective variety is ample, then it is a projective space, which gives a solution to Hartshorne's conjecture. Since then, the study of smooth projective varieties whose tangent bundles admit some positivity properties has attracted a lot of attention and such properties are usually expected to impose strong restrictions on the geometry of the underlying varieties.

Definition 0.1. *Let X be a smooth projective variety. Given a vector bundle E on X, we denote by $\mathbb{P}(E)$ the Grothendieck projectivisation of E with $\mathcal{O}_{\mathbb{P}(E)}(1)$ denoting the relative hyperplane section bundle. Recall that E is ample (resp. nef, big, pseudo-effective) if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is ample (resp. nef, big, pseudo-effective) on $\mathbb{P}(E)$.*

Following the program of Campana and Peternell [1], a smooth Fano variety with nef tangent bundle is conjectured to be a rational homogeneous space, and this conjecture has been intensively studied. Starting from this aspect, it is natural to classify smooth projective varieties with other positivity properties, e.g., with big or pseudo-effective tangent bundles. In general it is difficult to give a numerical characterization for pseudo-effective or bigness of the tangent bundle, even in low dimension with low rank of Picard group.

It has been shown by Hsiao [4] that the tangent bundle of a toric variety is big, and this result is generalized by Liu [7] when an algebraic group G acts on X with a dense orbit. In the past few years, there are also many results in this direction, especially when X is a Fano manifold. For example, Höring, Liu and Shao [3] shows that the tangent bundle of a smooth del Pezzo surface of degree d is big (resp. pseudo-effective) if and only if $d \geq 5$ (resp. $d \geq 4$). Also in the paper [3], they solve these problems for del Pezzo threefolds. In [2], Höring and Liu consider Fano manifolds X with Picard number one, and they prove that if X admits a rational curve with trivial normal bundle and with big T_X then X is isomorphic to the del Pezzo threefold of degree five.

These all results indicate that assuming bigness should lead to strong restrictions on Fano manifolds, and lead us to consider naturally Fano threefolds with Picard number 2. In [6] we prove the following main theorem.

Theorem 0.2. [6] *We determine the bigness of the tangent bundle T_X of whole 36 deformation types of Fano threefolds X with Picard number 2. In particular, the tangent bundle T_X is big if and only if $(-K_X)^3 \geq 34$.*

We note that Fano threefolds of $(-K_X)^3 \geq 34$ (No. 26-36 in [9, Table 2]) have infinite automorphism groups.

Besides our main theorem, we also obtain the following.

Theorem 0.3. [6] *Let X be the blow-up of $\mathbb{P}^3$ along a smooth curve Γ. If Γ is a degenerate curve then T_X is big, and if Γ is a nondegenerate curve then T_X is big if and only if Γ is a twisted cubic curve.*

As smooth projective varieties with big tangent bundles are known to be uniruled, one may ask if there exist many non-uniruled projective varieties sitting in the "boundary", i.e., admitting a pseudo-effective but non-big tangent bundle. When X is smooth projective variety of general type, then by the semi-stability of the tangent sheaf T_X with respect to the canonical divisor K_X, T_X is not pseudo-effective. Apart from the trivial example of abelian varieties, a product of an abelian variety and any smooth projective variety becomes another example coming to our mind. To the best knowledge of ourselves, up to a finite étale cover, there seems no more other example which has been explored before.

In [5], we complete determine non-uniruled surfaces whose tangent bundle is pseudo-effective

Theorem 0.4. [5] *Let S be a non-uniruled smooth projective surface. Then the following assertions are equivalent.*

(1) *The tangent bundle T_S is pseudo-effective;*
(2) *S is minimal and the second Chern class vanishes, i.e., $c_2(S) = 0$.*

Moreover, if one of the above equivalent conditions holds, then the Kodaria dimension $\kappa(\mathbb{P}(T_S), \mathcal{O}(1)) = 1 - \kappa(S)$, and there is a finite étale cover $S' \to S$ such that S' is either an abelian surface or a product $E \times F$ where E is an elliptic curve and F is a smooth curve of genus ≥ 2.

From the above theroem, the pseudo-effectiveness of the tangent bundle forces the surface to be minimal, i.e., the canonical divisor is nef. However, this is no longer true in the higher dimensional case.

Further, as a consequence of the above theorem, we obtain the following corollary.

Corollary 0.5. [5] *Let S be a non-uniruled smooth projective surface. If the tangent bundle T_S is pseudo-effective, then there is some integer m such that $H^0(S, \mathrm{Sym}^m T_S) \neq 0$; in particular, the tautological line bundle of $\mathbb{P}(T_S)$ is $\mathbb{Q}$-linearly equivalent to an effective divisor.*

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ZARISKI DENSE ORBIT CONJECTURE ON AUTOMORPHISMS OF PROJECTIVE THREEFOLDS

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Keywords: Zariski dense orbit conjecture, weak Calabi-Yau, rationally connected, special MRC fibration, Zariski dense of periodic points.

Let X be a projective variety over an algebraically field k of characteristic zero and $f : X \dashrightarrow X$ be a dominant rational self-map. Denote by $k(X)^f$ the field of f-invariant rational functions on X. Let $X_f(k)$ be the set of $x \in X(k)$ whose orbit $\mathcal{O}_f(x)$ is well-defined.

The following Zariski dense orbit conjecture (ZDO for short) was proposed by Medvedev and Scanlon [5, Conjecture 5.10], by Amerik, Bogomolov and Rovinsky [1] and strengthens a conjecture of S.-W. Zhang [9].

Conjecture 0.1. *Let X be a projective variety over an algebraically closed field k of characteristic zero and $f : X \dashrightarrow X$ a dominant rational self-map. Then either $k(X)^f \neq k$ or there is a point $x \in X_f(k)$ whose orbit $\mathcal{O}_f(x)$ is Zariski dense in $X(k)$.*

Remark 0.2. *For the historical note of ZDO, we refer to* [8, Section 1.1.1] *or* [4, Section 1.2].

Below is our main result of ZDO for automorphisms of projective threefolds.

Theorem 0.3. *Let f be an automorphism of a normal projective threefold X with only klt singularities. Suppose $K_X \sim_{\mathbb{Q}} 0$ or $\kappa(X) = -\infty$. Then we may reduce ZDO for (X, f) to the following three cases:*

(1) *X is weak Calabi-Yau and f is primitive;*
(2) *X is a rationally connected threefold;*
(3) *X is a uniruled threefold admitting a special MRC fibration over an elliptic curve.*

Note that a birational automorphism f on a minimal Calabi-Yau threefold X of Picard number $\rho(X) \geq 2$ is primitive if the action $f^*|_{\mathrm{NS}_{\mathbb{Q}}(X)}$ is irreducible over $\mathbb{Q}$ (cf. [6, Corollary 1.3]). This motivates the following question.

Question 0.4. *Let f be a birational automorphism of a weak Calabi-Yau variety X with $\rho(X) \geq 2$. Suppose that $f^*|_{\mathrm{NS}_{\mathbb{Q}}(X)}$ is irreducible over $\mathbb{Q}$. Then is ZDO true for (X, f)?*

Motivated by Chen, Lin and Oguiso's explicit examples of Zariski dense orbits on irregular smooth varieties (cf. [2, Theorem 1.6]), we show the following result.

Theorem 0.5. *Let f be an automorphism of a normal projective variety X with positive dimension. Then ZDO is true for (X, f) if $q(X) \geq \dim X - 1$.*

In [3, Theorem 5.1], Fakhruddin proved the Zariski density of periodic points if f is a polarized endomorphism of a projective variety. Notice that Xie proved in [7, Proposition 6.2] that an automorphism f of a projective variety X has finite order if $f^*|_{\mathrm{NS}_{\mathbb{R}}(X)} = \mathrm{id}$ and the periodic points of f are Zariski dense. We may extend his result as follows.

Theorem 0.6. *Let f be an automorphism of a projective variety X with positive dimension. If the periodic points are Zariski dense and $f^*D \equiv D$ for some big $\mathbb{R}$-divisor D. Then f is of finite order. In particular, f does not have any Zariski dense orbit.*

The following proposition is noticed by Sheng Meng.

Proposition 0.7. *Let f be an automorphism of a projective variety X with positive dimension and $d_1(f) = 1$. Suppose the periodic points of f are Zariski dense and the pseduoeffective cone $\mathrm{PEC}(X)$ is polyhedral. Then f is of finite order.*

Finally, we give a result of projective varieties X with Picard number $\rho(X) = 1$ as follows.

Proposition 0.8. *Let f be a surjective endomorphism of a normal projective variety X in dimension ≥ 1 with at most klt singularities $\dim X \geq 1$ and $\rho(X) = 1$. Then the following statements hold.*

(1) *If $d_1(f) = 1$ and the periodic points of f are Zariski dense, then f is of finite order.*
(2) *Suppose $d_1(f) > 1$. Then to prove ZDO, we may assume that X is Fano.*

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DYNAMICAL FILTRATIONS – BEYOND ZERO ENTROPY

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Classification AMS 2020: 14J50, 32M05, 32H50, 37B40.

Keywords: Automorphisms of compact Kähler manifolds, Tits alternative, virtually solvable groups, zero entropy automorphisms.

Let X be a compact Kähler manifold of dimension $d \geq 1$, endowed with a holomorphic group action $G \circlearrowleft X$. The induced action $G \circlearrowleft H^\bullet(X, \mathbf{C})$ preserves the grading and the Hodge decomposition of the cohomology of X. As a consequence of Tits alternative, the image $G_{|H^{1,1}(X)}$ of $G \to \mathrm{GL}(H^{1,1}(X))$ satisfies one of the following properties:

- either $G_{|H^{1,1}(X)}$ contains a non-abelian free group;
- or $G_{|H^{1,1}(X)}$ is virtually solvable.

Our work concerns the study of the group actions $G \circlearrowleft H^\bullet(X, \mathbf{C})$ and $G \circlearrowleft X$ when $G_{|H^{1,1}(X)}$ is virtually solvable. Such a study was initiated by Dinh–Sibony [4], D.-Q. Zhang [6], and continued by many others (e.g. [5, 1, 2]).

1. Dynamical rank

One of the starting points is the following structural theorem.

Theorem 1.1 (D.-Q. Zhang [6]). *Assume that $G_{|H^{1,1}(X)}$ is virtually solvable. Then there exists a finite-index subgroup $G' \leq G$ such that:*

(1) *The subset $N(G')$ of zero-entropy elements of G' is a normal subgroup of G'.*
(2) *$G'/N(G') \simeq \mathbf{Z}^r$ for some integer $r \leq \dim X - 1$.*

In the statement, we recall that as a consequence of the Gromov–Yomdin theorem, zero-entropy automorphisms (i.e. biholomprhic maps) $f : X \circlearrowleft$ are characterized by the property that for all $p = 0, \ldots, \dim X$, the spectral radii of $f^* : H^{p,p}(X) \circlearrowleft$ are all equal to 1. We also recall that in general, the subset $N(G)$ of zero-entropy elements of $G \circlearrowleft X$ is not a subgroup of G.

The integer r in Theroem 1.1 is independent of the choice of the subgroup $G' \leq G$; we call it the *dynamical rank* of $G \circlearrowleft X$ and set $r(G) := r$.

2. Polynomial growth rate of zero-entropy automorphism action

Let $f : X \circlearrowleft$ be a zero-entropy automorphism. We have

$$\| f^* : H^{1,1}(X) \circlearrowleft \| \asymp n^{k(f)}$$

for some $k(f) \in \mathbf{Z}_{\geq 0}$. Together with T.-C. Dinh, K. Oguiso, and D.-Q. Zhang, we prove the following result.

Theorem 2.1 ([2]). *The integer $k(f)$ is even and satisfies*

$$k(f) \leq 2 \dim X - 2.$$

Such an upper bound is optimal, in the sense that for each integer $d \geq 1$, there exsits a zero-entropy automorphism $f : X \circlearrowleft$ of a compact Kähler manifold of dimension d which satisfies $k(f) = 2d - 2$.

For every $G \circlearrowleft X$, we define

$$k(G) = \max \{ \, k(g) \mid g \in N(G) \, \} \in 2\mathbf{Z}_{\geq 0}.$$

This could serve as a measure of the size of $N(G)$. For instance, suppose that $N(G)$ is a subgroup of G, then one can show that $k(G) = 0$ if and only if $N(G)_{|H^{1,1}(X)}$ is finite. We also note that for any finite-index subgroup $G' \leq G$, we have $k(G') = k(G)$.

3. Compensation between $r(G)$ and $k(G)$

From now on, we assume that $G_{|H^{1,1}(X)}$ is virtually solvable and that $N(G)$ is a subgroup of G.

The first compensation phenomenon between $r(G)$ and $k(G)$ was discoverd by Dinh–Hu–Zhang in [1], where they show that $r(G) = \dim X - 1$ implies that $N(G)_{|H^{1,1}(X)}$ is finite, or equivalently, $k(G) = 0$. We generalize this statement in a work in progress as follows.

Theorem 3.1. *We have*

$$r(G) + \frac{k(G)}{2} \leq \dim X - 1.$$

Note that Theorem 3.1 also generalizes the upper bounds in Theorems 1.1 and 2.1.

The main inputs we need to prove Theorem 3.1 (as well as Theorems 1.1 and 2.1) are the fact that $G \circlearrowleft H^{\bullet}(X)$ preserves various positive cones (e.g. the nef cone, the pseudoeffective cone, and their analogues in higher codimension), and the mixed Hodge–Riemann theorem [3]. The proof of each of these theorems unveils new structures of $G \circlearrowleft H^{\bullet}(X)$, which we can forumulate in terms of quasi-nef sequences (for Theorem 1.1), dynamical filtrations (for Theorems 2.1), and dynamical towers (for Theorem 3.1) as we presented in the talk.

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THE AMPLENESS CONJECTURE ON K-TRIVIAL FOURFOLDS

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Keywords: strictly nef divisors, Calabi–Yau manifolds, hyperkähler manifolds, K-trivial fourfolds, the abundance conjecture, the ampleness conjecture, Serrano's conjecture.

K-trivial manifolds are projective manifolds with linearly trivial canonical divisors and without irregularity. Two typical examples are strict Calabi–Yau manifolds and simple hyperkähler manifolds. The ampleness conjecture predicts that any strictly nef divisors on K-trivial manifolds are ample.

Conjecture 0.1 (Ampleness conjecture). *Let X be a K-trivial manifold. Then, any strictly nef divisor L on X is ample.*

A $\mathbb{Q}$-Cartier divisor L on a normal projective variety X is called *strictly nef* if $L \cdot C > 0$ for any curve C on X. In general, a strictly nef divisor is not necessarily ample. However, as Serrano's conjecture and generalized abundance conjecture predicted, strictly nef divisors should be ample on projective varieties with numerically trivial divisors. By the Beauville–Bogomorov–Yau decomposition, this prediction can be reduced to our ampleness conjecture 0.1 and is essentially equivalent.

It is well known that the ampleness conjecture holds in dimension 2. In dimension 3, it is also proved for most of the cases by Wilson, Peternell, Oguiso, Serrano, Campana–Chen–Peternell and so on, except the case that $L^3 = c_2(X) \cdot L = 0$. Recently, Svaldi and the speaker improved their results a little bit in [3]. In this workshop, the speaker presented a result jointed work with Shin-ichi Matsumura that the ampleness conjecture holds in dimension 4:

Theorem 0.2 ([2, Theorem 1.2]). *Let X be a K-trivial fourfold. Then, any strictly nef divisor L on X is ample.*

This theorem is divided into the abundance part and the non-vanishing part. For the abundance part, there is a more general log version result:

Theorem 0.3 ([2, Theorem 1.6]). *Let X be a K-trivial manifold of dimension ≤ 4. Let Δ be a non-zero effective divisor and L be a strictly nef divisor on X. Then, $\Delta + tL$ is ample for $t \gg 1$.*

Its proof uses the log minimal model program and induction of dimension. After sacrificing strict nefness, we can reduce the problem to the surface:

Theorem 0.4 ([2, Theorem 3.1]). *Let (S, Δ) be a log canonical pair of dimension 2 and L be an almost strictly nef divisor on S, i.e., there exists a birational morphism $f\colon S \to S^*$ and a strictly nef divisor L^* on S^* such that $L = f^*L^*$. Then, $K_X + \Delta + tL$ is big for $t > 3$.*

For the non-vanishing part, the proof is again divided into two parts according to the numerical dimension $\nu(L)$ of L.

Theorem 0.5 ([2, Theorem 4.1]). *Let X be a K-trivial fourfold and L be a nef Cartier divisor with $\nu(L) = 3$. Then $\kappa(L) = \nu(L) = 3$. In particular, $\nu(L) \neq 3$ if L is strictly nef.*

Theorem 0.6 ([2, Theorem 4.7]). *Let X be a K-trivial fourfold and L be a strictly nef divisor with $\nu(L) \leq 2$. Then $\kappa(L) \geq 0$.*

The proof of Theorem 0.5 is standard by using the Kawamata–Viehweg vanishing and the Riemann–Roch formula; the proof of Theorem 0.6 uses analytic methods introduced and developed by Demailly [1], Lazić, Oguiso, Peternell [4, 5] and so on.

Finally, we proposed some related questions as follows:

Question 0.7. *Does Serrano's conjecture hold in positive characteristic?*

Question 0.8. *Does Serrano's conjecture hold for algebraic integrable foliation?*

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THE STRICT ARAKELOV INEQUALITIES FOR A SEMI-STABLE FIBRATION

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Abstract. In this note, we briefly review the Arakelov inequality for a semi-stable family over a smooth projective curve as well as its generalizations. Moreover, we try to understand the equality or the strictness of these Arakelov type inequalities.

We work over the complex number $\mathbb{C}$. The study of the Arakelov inequality goes back to the conjecture of Shafarevich [Sha63] for fibrations of curves of genus $g \geq 2$. The proof of Shafarevich's conjecture given by Parshin [Par68] and Arakelov [Ara71] consists of two parts: 'boundedness' and 'rigidity'. The Arakelov inequality aims to the boundedness. Roughly speaking, it gives an upper bound on the degree of the Hodge bundle in terms of the base space.

Theorem 0.1 (classical Arakelov inequality)**.** *Let $f : S \to B$ be a semi-stable surface fibration of genus $g \geq 1$ and $\Upsilon \to \Delta$ be the singular locus. Then*

$$\deg f_*\omega_{S/B} \leq \frac{g}{2}\big(2g(B) - 2 + \#\Delta\big). \tag{0-1}$$

The above classical Arakelov inequality can be improved and generalized to families of higher dimensional varieties (e.g., abelian varieties) using the Hodge theory. We refer to [Vie09] and the references therein for a beautiful introduction to this subject. We briefly summarize as follows.

Theorem 0.2 (Arakelov type inequalities)**.** *Let $f : X \to B$ be a semi-stable fibration of varieties of dimension $n \geq 1$. Assume that $\Upsilon \to \Delta$ is the singular locus with $2g(B) - 2 + \#\Delta > 0$. Then for any non-zero subsheaf $\mathcal{E} \subseteq f_*\big(\omega_{X/B}^{\otimes \nu}\big)$, it holds*

$$\mu(\mathcal{E}) \leq \frac{n\nu}{2}\big(2g(B) - 2 + \#\Delta\big), \tag{0-2}$$

where $\mu(\mathcal{E}) := \frac{\deg \mathcal{E}}{\operatorname{rank} \mathcal{E}}$ is the slope of $\mathcal{E}$. If $n = 1$ and $0 \neq \mathcal{E} \subseteq f_*\omega_{X/B}$, then

$$\mu(\mathcal{E}) \leq \frac{1}{2}\big(2g(B) - 2 + \#\Delta_{nc}\big), \tag{0-3}$$

where $\Upsilon_{nc} \to \Delta_{nc}$ is the locus of singular fibers with non-compact Jacobians.

The above theorems are mainly due to Deligne, Faltings, and Viehweg-Zuo. It is a natural question to ask whether the above Arakelov inequalities strict or not. It is generally believed that any of the above Arakelov equalities would give a very strong restriction on the geometry of the fibrations. In the case when f is a surface fibration, we have

Theorem 0.3. *Let $f : S \to B$ be a semi-stable surface fibration of genus $g \geq 1$.*

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(1) *(Beauville* [Bea81]*) Suppose $g = 1$, and the equality holds in* (0-1)*. Then the family f is modular.*
(2) *(Tan* [Tan95]*) Suppose $g \geq 2$. Then the Araklelov inequality* (0-1) *is strict.*
(3) *(Möller* [Mol06]*) Suppose $g \geq 2$, and the equality holds in* (0-2) *for $\nu = 1$. Then the subsheaf $\mathcal{E}$ is an invertible subsheaf and the family f is Teichmüller.*
(4) *(Lu-Zuo* [LZ19]*) Suppose the equality holds in* (0-3) *for $\mathcal{E} = f_*\omega_{S/B}$. Then $g < 12$ and the family f is Shimura.*

Recall that the Hodge bundle $f_*\omega_{S/B}$ admits a so-called Fujita decomposition [Fuj78]

$$f_*\omega_{S/B} = \mathcal{A} \oplus \mathcal{U},$$

where $\mathcal{A}$ is ample while $\mathcal{U}$ is unitary. It is conjectured that, if the genus $g \gg 0$, the Arakelov type inequality (0-3) should be strict for $\mathcal{E} = \mathcal{A} \subseteq f_*\omega_{S/B}$ being the ample part in the Fujita decomposition. In fact, this is related to the Coleman-Oort conjecture, which predicts that there exists no positive dimensional Shimura subvariety contained generically in the Torelli locus of smooth curves of genus g when g is sufficiently large. We refer to [MO13] (and the references therein) for a thorough discussion on the Coleman-Oort conjecture. In [CLZ21], when the general fiber is superelliptic (including the hyperelliptic case), it is proved that the Arakelov type inequality (0-3) is strict for $\mathcal{E} = \mathcal{A} \subseteq f_*\omega_{S/B}$ being the ample part if $g \geq 8$.

When the fiber is of higher dimension, no two much is known. Most are restricted to the cases when the Kodaira dimension of the general fiber equals zero or n.

Theorem 0.4. *Let $f : X \to B$ be as in Theorem 0.2.*

(1) *(Sun-Tan-Zuo* [STZ03]*) If the general fiber is a K3 surface, and the equality holds in* (0-2) *for $\mathcal{E} = f_*\omega_{X/B}$, then the family f is Shimura.*
(2) *(Viehweg-Zuo* [VZ04]*) If the general fiber is an abelian varieties, and the equality holds in* (0-2) *for $\mathcal{E} = f_*\big(\omega_{X/B}^{\otimes\nu}\big)$, then the family f is Shimura of Mumford type.*
(3) *(Viehweg-Zuo* [VZ06] *for $\nu = 1$, Lu-Yang-Zuo* [LYZ22] *for the general case) If the general fiber is of general type, and the subsheaf $\mathcal{E} \subseteq f_*\big(\omega_{X/B}^{\otimes\nu}\big)$ defines a birational B-map $X \dashrightarrow \mathbb{P}_B(\mathcal{E})$ for some $\nu \geq 1$, then the Arakelov inequality* (0-2) *is strict.*

Remark that the Simpson correspondence [Sim90] is a key point in the proofs. To end this note, we would like propose the following conjecture

Conjecture 0.5 ([LYZ22])**.** *Let $f : X \to B$ be as in Theorem 0.2.*

(1) *If the equality holds in* (0-2) *for $0 \neq \mathcal{E} = f_*\big(\omega_{X/B}^{\otimes\nu}\big)$ for some $\nu \geq 1$, then the general fiber is of Kodaira dimension zero.*
(2) *Suppose that the general fiber is of general type. Then for any subsheaf $\mathcal{E} \subseteq f_*\big(\omega_{X/B}^{\otimes\nu}\big)$ with $\operatorname{rank}\mathcal{E} \geq 2$, the Arakelov inequality* (0-2) *is strict.*

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MINIMAL LOG DISCREPANCIES OF QUOTIENT SINGULARITIES

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The minimal log discrepancy (MLD) is an invariant of singularity defined in the context of the minimal model program. In this talk, we discussed the minimal log discrepancies of quotient singularities:

- ACC conjecture for quotient singularities (0.1.5).
- LSC conjecture for hyper-quotient singularities (0.2.5).
- PIA conjecture for quotient singularities (0.3.3).
- Shokurov's index conjecture for quotient singularities (0.4.8).

This is joint work with Kohsuke Shibata (see [24, 25, 26, 27]).

Conjecture 0.1 (ACC conjecture, Shokurov). *Let d be a positive integer and $I \subset [0,1]$ a DCC set. Then, the set*

$$\{\mathrm{mld}_x(X, \Delta) \mid \dim X = d, \Delta \in I, x \in X\}$$

satisfies the ACC.

The ACC conjecture is known to be true in the following cases:

(0.1.1) When $d \leq 2$ [1, 29].
(0.1.2) When X is smooth three-fold and the mld is in the interval $[1,3]$ [14, 13].
(0.1.3) When the Gorenstein index of X is bounded and I is a finite set [23].
(0.1.4) When X is a canonical three-folds and I is a finite set [23].
(0.1.5) When X has only quotient singularities and $I = \{0\}$ [24].
(0.1.6) See [3, 18, 16, 9, 11, 19, 21, 10] for other developments related to the ACC conjecture

Conjecture 0.2 (LSC conjecture, Ambro). *Let (X, Δ) be a log pair. Then, the function*

$$|X| \to \mathbb{R}_{\geq 0} \cup \{\infty\}; \quad x \mapsto \mathrm{mld}_x(X, \Delta)$$

is lower-semi-continuous.

The LSC conjecture is known to be true in the following cases:

(0.2.1) When $\dim X \leq 3$ [2].
(0.2.2) When X is smooth [6].
(0.2.3) More generally, when X is a normal local complete intersection variety [7].
(0.2.4) When X has only quotient singularities [22].
(0.2.5) When X has only hyper-quotient singularities [24, 25, 27].

Conjecture 0.3 (PIA conjecture). *Let (X, Δ) be a log pair and $S \subset X$ a normal Cartier divisor such that $S \not\subset \mathrm{Supp}(\Delta)$. Then, for a closed point $x \in S$, it follows that*

$$\mathrm{mld}_x(X, \Delta + S) = \mathrm{mld}_x(S, \Delta|_S).$$

The PIA conjecture is known to be true in the following cases:

(0.3.1) When X is smooth [6].
(0.3.2) More generally, when X is a normal local complete intersection variety [7].
(0.3.3) When X has only quotient singularities [24, 25, 27].

Conjecture 0.4 (Shokurov's index conjecture, cf. [15, Question 5.2])**.** *For any $n \in \mathbb{Z}_{>0}$ and $a \in \mathbb{R}_{\geq 0}$, there exists a positive integer $r(n, a)$ with the following condition.*

- *If an n-dimensional $\mathbb{Q}$-Gorenstein variety X and a closed point $p \in X$ satisfy $\mathrm{mld}_p(X) = a$, then the Cartier index of K_X at p is at most $r(n, a)$.*

Conjecture 0.4 is known to be true in the following cases:

(0.4.1) When $(n, a) = (2, 0)$ [28].
(0.4.2) When $n = 2$ [5]. They proved Conjecture 0.4 for pairs (see [5, Conjecture 6.3] for the generalized formulation).
(0.4.3) When $(n, a) = (3, 0)$ [12, 8].
(0.4.4) When X is a terminal threefold [17, 10].
(0.4.5) When X is a canonical threefold [15].
(0.4.6) When X has toric singularities [4].
(0.4.7) When X has only quotient singularity and a is sufficiently small [20].
(0.4.8) When X has only quotient singularity (without any assumption on a) [26].

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ON THE MOTIVIC CLASS OF THE MODULI STACK OF ADMISSIBLE G-COVERS

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For a finite group G and an integer $g \geq 2$, the locus $M_g(G)$ (in the moduli space M_g) of smooth curves that have an effective action by G plays an important role in several fields, for example: in the description of the singularities of M_g ([12], [5]), in the study of Shimura varieties ([18], [11]), of totally geodesic subvarieties of M_g [15], and also in the classification of higher dimensional varieties ([16], [6], [17]).

For several purposes it is more natural to consider the stack $\mathcal{M}_g(G)$ of genus g compact Riemann surfaces with an effective action by G. Then $M_g(G)$ can be obtained as the image of a natural morphism $\mathcal{M}_g(G) \to M_g$. We refer to [20] for the definition and some properties of $\mathcal{M}_g(G)$. The connected components of $\mathcal{M}_g(G)$ are in one-to-one correspondence with the topological types of G-actions on compact oriented surfaces of genus g ([5]). These topological types have been investigated and classified, by means of certain numerical invariants, since the work of Nielsen [19], for more recent results we refer to [14], [21], [13], [7], [8], [9].

The seminar was based on the work [3], where we consider the compactification of the stack $\mathcal{M}_g(G)$ given by admissible covers. The aim of our work is to describe the motivic class (in the Grothendieck group) of the stack of admissible covers in order to compute their topological invariants, e.g. their Betti numbers.

Let us recall the following definition from [1].

Definition 0.1. *An admissible G-cover is given by the following data:*

- *an n-pointed nodal curve of genus g, $(C, p_1, \dots, p_n)$;*
- *a finite morphism $\phi\colon D \to C$, where D is a nodal curve, ϕ maps every node of D to a node of C and it is étale over $C_{\text{gen}} := C \setminus \{$nodes and marked points$\}$;*
- *locally over a node of C, ϕ is given by $\phi_|\colon Spec\frac{\mathbb{C}[\xi,\eta]}{(\xi\eta)} \to Spec\frac{\mathbb{C}[x,y]}{(xy)}$, $\phi_|^*(x) = \xi^e$, $\phi_|^*(y) = \eta^e$, for some $e \geq 1$;*
  locally over a marked point of C, ϕ is given by $\phi_|\colon Spec\mathbb{C}[\xi] \to \mathbb{C}[x]$, $\phi_|^*(x) = \xi^e$, for some $e \geq 1$;
- *an action of G on D that commutes with ϕ and such that $\phi_|\colon D_{\text{gen}} \to C_{\text{gen}}$ is a principal G-bundle, where $D_{\text{gen}} = \phi^{-1}(C_{\text{gen}})$.*

The admissible G-cover is called balanced if, at each node $q \in D$, the action of $\mathrm{Stab}_q(G)$ on T_qD is balanced, i.e. $\zeta(\xi,\eta) = (\zeta\xi, \zeta^{-1}\eta)$.

In our work we consider only balanced admissible G-covers, so from now on by an admissible G-cover we mean a balanced one. Admissible G-covers of n-pointed nodal curves of genus g form a stack, which is denoted $\mathcal{A}dm_{g,n}(G)$.

Admissible G-covers can be described in terms of (balanced) twisted stable maps $(\mathcal{C}, \Sigma_1, \dots, \Sigma_n, f\colon \mathcal{C} \to BG)$, where $(\mathcal{C}, \Sigma_1, \dots, \Sigma_n)$ is a twisted nodal n-pointed curve and f is a stable representable map to the classifying stack BG of G. For more details on twisted stable maps we refer to [10], [2]. The stack of twisted n-pointed stable maps from curves of genus g to BG is denoted $\mathcal{B}^{\mathrm{bal}}_{g,n}(G)$, it is a Deligne-Mumford stack with projective coarse moduli space, furthermore it is isomorphic to $\mathcal{A}dm_{g,n}(G)$ [1].

In the following we will denote with $\mathcal{B}^{\mathrm{sm}}_{g,n}(G) \subseteq \mathcal{B}^{\mathrm{bal}}_{g,n}(G)$ the open locus of twisted n-pointed stable maps from smooth curves $\mathcal{C}$.

For each twisted stable map $(\mathcal{C}, \Sigma_1, \dots, \Sigma_n, f \colon \mathcal{C} \to BG)$, the restriction of f to Σ_i (which is a gerbe structure at the i-th marked point) yields an object of $\bar{\mathcal{I}}_\mu(BG) := \sqcup_r \bar{\mathcal{I}}_{\mu_r}(BG)$, the *stack of cyclotomic gerbes* in BG ([2, Definition 3.3.6]). In this way we obtain a morphism $\mathrm{ev}^i_n \colon \mathcal{B}^{\mathrm{bal}}_{g,n}(G) \to \bar{\mathcal{I}}_\mu(BG)$, which is called the i-th evaluation map. Notice that ev^i_n restricts to a morphism from $\mathcal{B}^{\mathrm{sm}}_{g,n}(G)$ to $\bar{\mathcal{I}}_\mu(BG)$. To simplify the notation, in the following we will denote $\bar{\mathcal{I}}_\mu(BG)$ with $\bar{\mathcal{I}}_\mu$.

To state out main result we need the following definition.

Definition 0.2 (Ekedahl). *For any positive integer n, let $\mathbb{S}_n$ be the symmetric group of degree n. The Grothendieck group $K_0^{\mathbb{S}_n}(\mathit{AlgSt}_{\mathbb{C}})$ is the abelian group generated by the isomorphism classes $\{\mathcal{X}\}$ of algebraic $\mathbb{C}$-stacks with an action of $\mathbb{S}_n$ subject to the relations*

1. *$\{\mathcal{X}\} = \{\mathcal{Y}\} + \{\mathcal{X} \setminus \mathcal{Y}\}$ if $\mathcal{Y}$ is a closed substack of $\mathcal{X}$ invariant under the action of $\mathbb{S}_n$, and*
2. *$\{\mathcal{E}\} = \{\mathbb{A}^r \times \mathcal{X}\}$ if $\pi \colon \mathcal{E} \to \mathcal{X}$ is an $\mathbb{S}_n$-equivariant vector bundle of rank r, where the $\mathbb{S}_n$-action on $\mathbb{A}^r \times \mathcal{X}$ is the extension of the given one on $\mathcal{X}$ by the trivial action.*

The ring structure on $K_0^{\mathbb{S}_n}(\mathit{AlgSt}_{\mathbb{C}})$ is given by defining $\{\mathcal{X}\} \cdot \{\mathcal{Y}\} := \{\mathcal{X} \times \mathcal{Y}\}$.

Remark 0.3. *For any algebraic $\mathbb{C}$-stack $\mathcal{X}$, with an action of $\mathbb{S}_n$, the class $\{\mathcal{X}\} \in K_0^{\mathbb{S}_n}(\mathit{AlgSt}_{\mathbb{C}})$ is called the motivic class of $\mathcal{X}$. The Betti numbers of $\mathcal{X}$ are determined by $\{\mathcal{X}\}$ via the Hodge-Poincaré characteristic* [4].

The following result is our main theorem, for any $n \geq 3$, it expresses $\{\mathcal{B}^{\mathrm{bal}}_{0,n}(G)\}$ in terms of $\{\mathcal{B}^{\mathrm{sm}}_{0,n}(G)\}$ and $\{\mathcal{B}^{\mathrm{bal}}_{0,k}(G)\}$, where $3 \leq k \leq n-1$. Since $\{\mathcal{B}^{\mathrm{bal}}_{0,3}(G)\} = \{\mathcal{B}^{\mathrm{sm}}_{0,3}(G)\}$, we deduce a recursive procedure to compute $\{\mathcal{B}^{\mathrm{bal}}_{0,n}(G)\}$ in terms of $\{\mathcal{B}^{\mathrm{sm}}_{0,k}(G)\}$, for $k = 3, \dots, n$. Notice also that $\mathcal{B}^{\mathrm{bal}}_{0,n}(G) = \emptyset$, if $n < 3$.

Theorem 0.4 ([3]). *Let $n \geq 3$, then the following relation holds true in $K_0^{\mathbb{S}_n}(AlgSt_{\mathbb{C}})$:*

$$\begin{aligned}
\{\mathcal{B}^{\mathrm{bal}}_{0,n}(G)\} \;=\; & \sum_{m=3}^{n} \left\{ \left(\mathcal{B}^{\mathrm{sm}}_{0,m}(G) \times_{\bar{\mathcal{I}}_\mu^m} \coprod_{\substack{\underline{k} \in \mathbb{N}^m \\ |\underline{k}| = n}} Sh(\underline{k}) \times_{i=1}^{m} \left((\bar{\mathcal{I}}_\mu)_{k_i} \coprod \mathcal{B}^{\mathrm{bal}}_{0,k_i+1}(G) \right) \right) / \mathbb{S}_m \right\} \\
& + \left\{ \left(\coprod_{\substack{\underline{k} \in \mathbb{N}^2 \\ |\underline{k}| = n}} Sh(\underline{k}) \times \mathcal{B}^{\mathrm{bal}}_{0,k_1+1}(G) \times_{\bar{\mathcal{I}}_\mu^2} \mathcal{B}^{\mathrm{bal}}_{0,k_2+1}(G) \right) / \mathbb{S}_2 \right\} \\
& - \left\{ \coprod_{\substack{\underline{k} \in \mathbb{N}^2 \\ |\underline{k}| = n}} Sh(\underline{k}) \times \mathcal{B}^{\mathrm{bal}}_{0,k_1+1}(G) \times_{\bar{\mathcal{I}}_\mu^2} \mathcal{B}^{\mathrm{bal}}_{0,k_2+1}(G) \right\}
\end{aligned}$$

where, for a multi-index $\underline{k} = (k_1, \ldots, k_m) \in \mathbb{N}^m$ *of modul* $|\underline{k}| = \sum_{i=1}^m k_i = n$, $Sh(\underline{k}) \subset \mathbb{S}_n$ *is the set of shuffles. In the first summand:* $(\bar{\mathcal{I}}_\mu)_{k_i} = \emptyset$, *if* $k_i > 1$ *or* $k_i = 0$, *and* $(\bar{\mathcal{I}}_\mu)_1 = \bar{\mathcal{I}}_\mu$; *the fibered product is with respect to the evaluation maps on the first factor, and the identity on* $(\bar{\mathcal{I}}_\mu)_1$ *or the evaluation map at the last marked point on* $\mathcal{B}^{\mathrm{bal}}_{0,k_i+1}(G)$. *In the second and third summands the fibered product is with respect to the evaluation map on* $\mathcal{B}^{\mathrm{bal}}_{0,k_1+1}(G)$ *and the composition of the involution* $\bar{\mathcal{I}}_\mu \to \bar{\mathcal{I}}_\mu$ *induced by the map* $G \to G$ *that sends* $x \mapsto x^{-1}$ *with the evaluation map on* $\mathcal{B}^{\mathrm{bal}}_{0,k_2+1}(G)$.

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FINITE TORSORS ON PROJECTIVE SCHEMES DEFINED OVER A DISCRETE VALUATION RING

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Classification AMS 2020: 14F06, 14F35, 14L15, 14G20, 14L30.

Keywords: discrete valuation rings, group schemes, Tannakian categories, coherent sheaves.

Given a Henselian and Japanese discrete valuation ring A and a flat and projective A-scheme X, we follow the approach of Biswas and dos Santos [3] to introduce a full subcategory of coherent modules on X which is then shown to be Tannakian. We then prove that, under normality of the generic fibre, the associated affine and flat group is pro-finite in a strong sense (so that its ring of functions is a Mittag-Leffler A-module) and that it classifies finite torsors $Q \longrightarrow X$. This establishes an analogy to Noris theory of the essentially finite funda- mental group. In addition, we compare our theory with the ones recently developed by MehtaSubramanian and AnteiEmsalemGasbarri. Using the comparison with the former, we show that any quasi-finite torsor $Q \longrightarrow X$ has a reduction of the structure group to a finite one.

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SUBMULTIPLICATIVE NORMS ON SECTION RINGS

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Keywords: projective manifolds, section rings, extension theorems

We fix a complex projective manifold X of dimension n and an ample line bundle L over X. Denote by $R(X, L)$ the *section ring*, defined as follows

$$R(X, L) := \oplus_{k=1}^{\infty} H^0(X, L^k).$$

A graded norm $N = \sum N_k$, $N_k = \| \cdot \|_k$, over $R(X, L)$ is called *submultiplicative* if for any $k, l \in \mathbb{N}^*$, $f \in H^0(X, L^k)$, $g \in H^0(X, L^l)$, we have $\|f \cdot g\|_{k+l} \leq \|f\|_k \cdot \|g\|_l$.

For a metric h^L on L, denote the induced graded L^∞-norm on $R(X, L)$ by $\mathrm{Ban}^\infty(h^L) = \sum \mathrm{Ban}_k^\infty(h^L)$. This norm, $\| \cdot \|_{L^\infty(h^L)} := \mathrm{Ban}^\infty(h^L)$, defined as $\|f\|_{L^\infty(h^L)} := \sup_{x \in X} |f(x)|_{h^L}$, $f \in H^0(X, L^k)$, is clearly submultiplicative. The main result of this report says that all submultiplicative norms are essentially of this form.

In order to state the result precisely, we associate for any norm N_k on $H^0(X, L^k)$ a metric $FS(N_k)$ on L^k as follows: for any $x \in X$, $l \in L_x^k$, we put $|l|_{FS(N_k)} = \inf \|s\|_k$, where the infimum is taken over all $s \in H^0(X, L^k)$, verifying $s(x) = l$ (the set of such sections s is non-empty for k big enough by the ampleness of L). Clearly, if $N = \sum N_k$ is submultiplicative, $FS(N_k)$, $k \in \mathbb{N}^*$, is submultiplicative as well, i.e. for any $k, l \in \mathbb{N}^*$, $FS(N_{k+l}) \leq FS(N_k) \cdot FS(N_l)$. By Fekete's lemma, the sequence of metrics $FS(N_k)^{\frac{1}{k}}$ on L converges, as $k \to \infty$, to an upper semi-continuous metric (possibly only bounded from above and even null), which we denote by $FS(N)$.

Theorem [7, Theorem 1.1] *Assume that a graded norm $N = \sum N_k$ over $R(X, L)$ is submultiplicative, and $FS(N)$ is continuous and non-null everywhere. Then $N \sim \mathrm{Ban}^\infty(FS(N))$, where the latter equivalence means that for any $\epsilon > 0$, there is $k_0 \in \mathbb{N}$, such that for any $k \geq k_0$, we have* $\mathrm{Ban}_k^\infty(FS(N)) \leq N_k \leq \exp(\epsilon k) \cdot \mathrm{Ban}_k^\infty(FS(N))$.

This result is a projective analogue of Gelfand representation theorem for commutative Banach algebras. It is also a complex-geometric analogue of a theorem from non-Archimedean geometry due to Boucksom-Jonsson [3, Theorem D]. Below we explain some of the consequences of the Theorem and highlight the role of submultiplicative norms in functional analysis and complex/algebraic geometry.

Example 1. We denote by $\mathscr{O}(1)$ the hyperplane bundle over the projective space $\mathbb{P}^{n-1}$. Then $R(\mathbb{P}^{n-1}, \mathscr{O}(1))$ is the symmetric tensor algebra $\mathrm{Sym}\mathbb{C}^n = \sum \mathrm{Sym}^k \mathbb{C}^n$, and we can identify it with the algebra of complex polynomials in n variables graded by the homogeneous degree. For $P \in \mathrm{Sym}^k \mathbb{C}^n$, $P(x_1, \cdots, x_n) = \sum_{|\alpha|=k} a_\alpha x^\alpha$, we define the norm $\|P\|_k := \sum_{|\alpha|=k} |a_\alpha|$. The induced norm on $R(\mathbb{P}^{n-1}, \mathscr{O}(1))$ is clearly submultiplicative. It satisfies the assumptions of the Theorem, and the Theorem specializes to the following statement: for any $\epsilon > 0$, there is $k_0 \in \mathbb{N}$, such that for any

$k \geq k_0$, $a_\alpha \in \mathbb{C}$, $\alpha \in \mathbb{N}^n$, $|\alpha| = k$, we have

$$\exp(-\epsilon k) \cdot \sum_{|\alpha|=k} |a_\alpha| \leq \sup_{\substack{x_1,\dots,x_n \in \mathbb{C} \\ |x_1|,\dots,|x_n| \leq 1}} \Big| \sum_{|\alpha|=k} a_\alpha x^\alpha \Big| \leq \sum_{|\alpha|=k} |a_\alpha|.$$

The study of general projective tensor norms on symmetric algebras also falls into the scope of the Theorem, see [7, Theorem 3.13 and Remark 3.14].

Example 2. Consider an embedding $Y \hookrightarrow X$ of a compact complex manifold Y. The restriction operator $\mathrm{Res} : R(X, L) \to R(Y, L)$ for any norm $N_X = \| \cdot \|_X$ on $R(X, L)$ induces a norm $N_Y = \| \cdot \|_Y$ on $H^0(Y, L^k) \ni q$ by $\|q\|_Y := \inf \big\{ \|f\|_X : \mathrm{Res}(f) = q \big\}$ for big k (by ampleness, such sections f exist for big k, so the norm N_Y is well-defined on elements of high degree). If N_X is submultiplicative, then N_Y is clearly submultiplicative as well. We use this observation for $N_X = \mathrm{Ban}^\infty(h^L)$ for some smooth positive metric h^L. Then Theorem for N_Y gives us that for any $\epsilon > 0$, there is $k_0 \in \mathbb{N}$, such that for any $k \geq k_0$, $f \in H^0(Y, L^k)$, there is a holomorphic extension $\tilde{f}$ of f to X, such that

$$\exp(-\epsilon k) \cdot \sup_{x \in X} |\tilde{f}(x)|_{h^L} \leq \sup_{y \in Y} |f(y)|_{h^L} \leq \sup_{x \in X} |\tilde{f}(x)|_{h^L}.$$

This result was previously established by Zhang [19], Bost [2] and Randriambololona [14] in various degrees of generality. In [5, Theorems 1.1 and 1.10], author refined these statements by giving an explicit asymptotic formula for the extension $\tilde{f}$, which implied that one can replace $\exp(-\epsilon k)$ by $1 - \frac{C}{\sqrt{k}}$ for some $C > 0$ in the estimate above.

We note that we use Examples 1, 2 in our proof of the Theorem, and so *stricto sensu* both examples should not be considered as consequences of the Theorem.

Example 3. We fix two graded norms $N_i = \sum N_k^i$, $\| \cdot \|_i := N_i$, $i = 0, 1$, over the section ring $R(X, L)$. One can then define the complex interpolation $N_t = \sum N_k^t$, $t \in [0, 1]$, between N_0 and N_1 in the following manner. We let $f \in H^0(X, L^k)$ and define its norm, $\|f\|_t$, $\| \cdot \|_t = N_k^t$, as $\|f\|_t := \sup\{ \|g(i\theta)\|_0, \|g(1 + i\theta)\|_1, \theta \in \mathbb{R} \}$, where the supremum is taken over all holomorphic functions $g : \{ z \in \mathbb{C} : 0 < \Re z < 1 \} \to H^0(X, L^k)$, $g(t) = f$, continuous over $\{ z \in \mathbb{C} : 0 \leq \Re z \leq 1 \}$.

A trivial verification shows that if both N_i, $i = 0, 1$, are submultiplicative, then for any $t \in [0, 1]$, the norm N_t is submultiplicative. In particular, for any smooth positive metrics h_i^L, $i = 0, 1$, the above construction for $N_i := \mathrm{Ban}^\infty(h_i^L)$ yields a submultiplicative norm N_t, $t \in [0, 1]$. The results of Phong-Sturm [11] and Berndtrsson [1] show that $FS(N_t)$ is continuous and can be identified with the Mabuchi geodesic h_t^L connecting h_0^L and h_1^L, see Mabuchi [10]. When the Theorem is applied for N_t, it refines the results of Phong-Sturm and Berndtrsson by proving $N_t \sim \mathrm{Ban}^\infty(h_t^L)$, see [6, Theorem 1.8].

Example 4. We fix a graded decreasing filtration $\mathcal{F}$ on $R(X, L)$, $\mathbb{Z} \ni \lambda \mapsto \mathcal{F}^\lambda R(X, L) \subset R(X, L)$, which is submultiplicative, i.e. for any $\lambda, \mu \in \mathbb{Z}$, we have

$$\mathcal{F}^\lambda R(X, L) \cdot \mathcal{F}^\mu R(X, L) \subset \mathcal{F}^{\lambda + \mu} R(X, L).$$

For example, $\mathcal{F}$ might be associated with a divisor $D \subset X$, where $\mathcal{F}^\lambda R(X, L)$ consists of sections vanishing up to order at least λ along D. Define a ray of norms $N_t = \| \cdot \|_t$, $t \in [0, +\infty[$, as $\|f\|_t = \inf \{ \sum \|f_i\| \cdot \exp(-t\lambda_i) \}$, where the infimum is taken over all possible decompositions $f = \sum f_i, f_i \in \mathcal{F}^{\lambda_i} R(X, L)$. If both N and $\mathcal{F}$ are submultiplicative, then N_t, $t \in [0, +\infty[$, is clearly submultiplicative. From the work of Phong-Sturm [13], when the filtration $\mathcal{F}$ is finitely generated and $N = \mathrm{Ban}^\infty(h^L)$ for some smooth positive metric h^L, the lower-semicontinuous regularization of the ray of metrics $FS(N_t)$ on L coincides with the Mabuchi geodesic ray h^L_t, $h^L_0 = h^L$, associated with the test configuration induced by $\mathcal{F}$. See Witt Nyström [17] and Székelyhidi [16] for the correspondence between test configurations and filtrations; see Phong-Sturm [12] for the correspondence between test configurations and Mabuchi geodesic rays.

In [8, Theorem 4.1], author proved the uniform version (in t) of the Theorem for N_t. As a consequence of it, the above works of Phong-Sturm and the uniform version of Tian-Bouche-Catlin-Zelditch asymptotic expansion of Bergman kernel, in [8, Theorem 1.1] author established a relation between the geometry at infinity of the space of positive metrics on L and the asymptotic properties of submultiplicative filtrations on $R(X, L)$, conjectured by Darvas-Lu [4, p. 3 and 7] and Zhang [18, Remark 6.12]. The latter result refines some previous results of Witt Nyström [17], Hisamoto [9] and Reboulet [15].

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HIGHER DIMENSIONAL ARAKELOV INEQUALITIES

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Classification AMS 2020: 14D06, 14D23, 14E05, 14D07.

Keywords: Families of manifolds, flat families, variation of Hodge structures, Arakelov-type inequalities

While numerical invariants play a central role in classification in all fields of mathematics, it is often very difficult to compute their exact value. As a result we opt for the next best thing: try to give estimates by finding upper or lower bounds. In algebraic geometry, and in particular in the construction of moduli spaces, giving bounds for certain invariants provides a fundamnatal tool. Without such bounds it would be extremely difficult to find reasonably-behaved moduli spaces; for example, we could not even hope for such spaces to be of finite type.

One of the early examples of such bounds, with an eye towards the construction of moduli spaces of higher dimensional varieties, is Matsusaka's Big Theorem. Boundedness questions are present in many other more or less related questions, such as Mordell's Conjecture, Lang's Conjecture, or Shafarevich's Conjecture. The latter, and its more modern generalizations, are the most relevant to the present work.

Shafarevich conjectured that there are only finitely many non-isotrivial families of smooth projective curves of fixed genus (≥ 2) over a fixed curve. Parshin [5] and Arakelov [1] proved this conjecture in two steps: *boundedness*, that is, there are only finitely many deformation types of such families, and *rigidity*; those families are actually rigid, so each one is the *only one* in its deformation type.

Boundedness can be roughly translated to some associated parameter scheme being of finite type. These parameter spaces are often constructed via an appropriate Hilbert scheme and hence being of finite type is closely related to bounding the degree of an ample line bundle. In fact, already Arakelov used this idea to prove boundedness in order to prove Shafarevich's conjecture in the curve case.

More generally, we consider a smooth projective family of canonically polarized varieties $\pi : U \to V$. Then V maps to a moduli space parametrizing the fibers. This target moduli space is equipped with an ample line bundle. The pullback of this line bundle to V is $\det \pi_* \omega^m_{U/V}$ (for some well-chosen $m > 0$ and up to a suitable power). Therefore, in order to carry out the above sketched plan for the boundedness problem, one would need to uniformly bound the degree of this line bundle.

This is exactly what Arakelov did. He established such a universal bound for all families of curves of genus at least 2 over base spaces of dimension one [1]. More precisely, he showed that, for every sufficiently large $m \in \mathbb{N}$, there is a polynomial function $b_{m,g} \in \mathbb{Z}_{>0}[x_1, x_2]$, depending only on m and a fixed integer $g \in \mathbb{N}$, $g \geq 2$, such that the inequality

$$(\star) \qquad \deg(\det f_* \omega^m_{X/B}) \leq b_{m,g}\big(g(B), \deg(D)\big)$$

holds for any smooth compactification $f : X \to B$ of *any* non-isotrivial smooth projective family $f_U : U \to V$ of curves of genus g over a one dimensional base V, where $D := B\backslash V$. In fact Arakelov showed that the coefficients of $b_{m,g}$ are themselves purely g-dependent functions of m and $r_m := \mathrm{rank}(f_* \omega^m_{X/B})$.

Bedulev and Viehweg [2] proved a generalization of Arakelov's inequality for families of canonically polarized manifolds, *still* over curves. Other, more Hodge theoretic analogues of ($\star$) were also established by Deligne and Peters.

The equation ($\star$) became known as Arakelov's inequality. Based on [3], Viehweg and others speculated that the inequality ($\star$) should have analogues over higher dimensional base spaces. In fact, at the end of his survey [6] Viehweg explains how a higher dimensional Arakelov inequality would be useful.

Definition 0.1 (Higher dimensional Arakelov type inequalities)**.** *Let V be a smooth quasi-projective variety of dimension d and B a smooth compactification of V such that $B\backslash D \simeq V$, with D being a reduced divisor on B having simple normal crossing support. Further let $f_U : U \to V$ be a smooth family of projective varieties and let X be a smooth compactification of U such that there exists a projective morphism $f : X \to B$ with $f|_U = f_U$. We will refer to these by saying that (the pair) (B, D)* is a smooth compactification of V *and that $f : X \to B$ is a* smooth compactification of $f_U : U \to V$.

Still working with the above notations, let H be an ample Cartier divisor on B and set $Sm_{n,\nu}$ to denote the class of smooth projective families, $f_U : U \to V$, of canonically polarized varieties of dimension n and canonical volume $\nu = vol(K_{U_t}) := K^n_{U_t}$ over V. Members of a subclass of $S \subseteq Sm_{n,\nu}$ will be said to satisfy an Arakelov inequality*, if for all sufficiently large and divisible $m \in \mathbb{N}$, there exists a function $b_{m,n,\nu} \in \mathbb{Z}_{>0}[x_1, x_2]$, depending only on m, n, and ν, for which the inequality*

$$\deg_H \left(\det f_* \omega^m_{X/B} \right) \leq b_{m,n,\nu} \left(\deg_H(K_B + D), \deg_H(D) \right) \tag{0.1}$$

holds for any smooth compactification $f : X \to B$ of any family $(f_U : U \to V) \in S$. Here for any divisor Δ and line bundle $\mathscr{L}$ on B, we define $\deg_H(\Delta) := \Delta \cdot H^{d-1}$ and $\deg_H(\mathscr{L}) := c_1(\mathscr{L}) \cdot H^{d-1}$.

Theorem 0.2 (cf. [4])**.** *In the setting of 0.1, if $K_B + D$ is pseudo-effective, then all members of $Sm_{n,\nu}(V)$ satisfy an Arakelov inequality.*

We note that when $d = 1$, this higher dimensional Arakelov-type inequality fully recovers the original one for curves in ($\star$).

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ARITHMETIC DEGREES OF DOMINANT RATIONAL SELF-MAPS

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Classification AMS 2020: Primary 37P15; Secondary 37P05, 37P30, 37P55.

Keywords: arithmetic degree, dynamical degree, Zariski dense orbit.

Let X be a smooth projective variety and let $f : X \dashrightarrow X$ be a dominant rational self-map, defined over an algebraically closed field of characteristic 0. The *k-th dynamical degree* of f is defined as

$$\delta_k(f) = \lim_{i\to\infty} \left((f^i)^*(H^k) \cdot H^{\dim X-k}\right)^{1/i}.$$

where H is an ample line bundle on X. This limit exists and is independent of the choice of ample line bundle H by the works of Dinh–Sibony, Truong, Dang. In general, there are p and q such that

$$1 = \delta_0(f) < \cdots < \delta_p(f) = \cdots = \delta_q(f) > \cdots > \delta_{\dim X}(f).$$

Definition 0.1. *We say that f is $(p$-$)$cohomologically hyperbolic, if there is a positive integer $1 \le p \le \dim X$ such that*

$$\delta_p(f) > \delta_i(f) \text{ for all } i \in \{1, 2, \dots, \dim X\} \setminus \{p\}.$$

Cohomologically hyperbolic maps have been investigated extensively from the viewpoint of complex dynamics. In this talk, I will present the study joint with Yohsuke Matsuzawa on the arithmetic degrees of cohomologically hyperbolic maps ([MW22]).

Let X be a smooth projective variety and let $f : X \dashrightarrow X$ be a dominant rational self-map, defined over $\overline{\mathbb{Q}}$. Let $X_f(\overline{\mathbb{Q}})$ be the set of points $x \in X(\overline{\mathbb{Q}})$ whose f-orbit $O_f(x) = \{f^i(x)\}_{i\ge 0}$ is well-defined. For $x \in X_f(\overline{\mathbb{Q}})$, we define the *lower* and the *upper arithmetic degree* of f at x by

$$\underline{\alpha}_f(x) = \liminf_{i\to\infty} \max\left\{h_H(f^i(x)), 1\right\}^{1/i},$$

$$\overline{\alpha}_f(x) = \limsup_{i\to\infty} \max\left\{h_H(f^i(x)), 1\right\}^{1/i},$$

where H is an ample line bundle on X and h_H is a Weil height function associated with H. These quantities are independent of the choice of ample line bundle H and height function h_H. We also write

$$\alpha_f(x) = \lim_{i\to\infty} \max\left\{h_H(f^i(x)), 1\right\}^{1/i}$$

if the limit exists, and call it the *arithmetic degree* of f at x.

Conjecture 0.2 (Kawaguchi–Silverman, [KS16a]). *Let X be a smooth projective variety over $\overline{\mathbb{Q}}$. Let $f : X \dashrightarrow X$ be a dominant rational self-map and $x \in X_f(\overline{\mathbb{Q}})$.*

(a) *We have $\underline{\alpha}_f(x) = \overline{\alpha}_f(x)$. In other words, the following limit exists:*

$$\alpha_f(x) = \lim_{i\to\infty} \max\left\{h_H(f^i(x)), 1\right\}^{1/i}.$$

(b) *The arithmetic degree* $\alpha_f(x)$ *is an algebraic integer.*
(c) *The collection of arithmetic degrees* $\{\alpha_f(y) : y \in X_f(\overline{\mathbb{Q}})\}$ *is a finite set.*
(d) *If the forward* f*-orbit* $O_f(x)$ *is Zariski dense in* X*, then*

$$\alpha_f(x) = \delta_1(f).$$

When $f : X \to X$ is a surjective morphism, the first three claims (a), (b) and (c) were shown in [KS16b], while the last one (d) is still open. We refer to [MZ23] and references therein for known results.

When $f : X \dashrightarrow X$ is an arbitrary dominant rational self-map, the problem is much more complicated, because of the deficiency of functoriality of height functions. Part (c) is not true due to [LS20]. Part (b) is not true either; see Corollary 0.6 below. Part (a) and part (d) are still open for birational self-maps of $\mathbb{P}^2$, or self-morphisms of $\mathbb{A}^2$.

Let us now state our main results ([MW22]). Recall that an orbit $O_f(x)$ is called *generic*, if it is infinite and the intersection with any proper Zariski closed subset is finite. Equivalently, $O_f(x)$ is infinite and any infinite subsequence of $O_f(x)$ is Zariski dense. Clearly "generic" implies "Zariski dense", while the converse is predicted by the *dynamical Mordell–Lang conjecture* (see e.g., [Xie22, Conjecture 2.5]).

Theorem 0.3. *Let* $f : X \dashrightarrow X$ *be a dominant rational self-map of a smooth projective variety* X *defined over* $\overline{\mathbb{Q}}$*. Assume that* f *is* p*-cohomologically hyperbolic.*

(1) *For every* $x \in X_f(\overline{\mathbb{Q}})$ *with generic orbit* $O_f(x)$*, we have*

$$\underline{\alpha}_f(x) \geq \frac{\delta_p(f)}{\delta_{p-1}(f)}.$$

(2) *There is a sequence* $\{x_i\}_{i\geq 1} \subset X_f(\overline{\mathbb{Q}})$ *of* $\overline{\mathbb{Q}}$*-points, such that*

$$\lim_{i\to+\infty} \underline{\alpha}_f(x_i) \geq \frac{\delta_p(f)}{\delta_{p-1}(f)}.$$

As a direct corollary, we have

Corollary 0.4. *Assume that* f *is* 1*-cohomologically hyperbolic.*

(1) *For every* $x \in X_f(\overline{\mathbb{Q}})$ *with generic orbit* $O_f(x)$*,* $\alpha_f(x)$ *exists and* $\alpha_f(x) = \delta_1(f)$*.*
(2) *There is a sequence* $\{x_i\}_{i\geq 1} \subset X_f(\overline{\mathbb{Q}})$ *of* $\overline{\mathbb{Q}}$*-points, such that* $\lim_{i\to+\infty} \underline{\alpha}_f(x_i) = \delta_1(f)$*.*

As an application of the study of arithmetic degrees, we show the existence of Zariski dense orbits following an idea from [JSXZ21].

Theorem 0.5. *Let* X *be a smooth projective variety defined over* $\overline{\mathbb{Q}}$*. Let* $f\colon X \dashrightarrow X$ *be a* 1*-cohomologically hyperbolic dominant rational self-map. Then there is* $x \in X_f(\overline{\mathbb{Q}})$ *such that the* f*-orbit* $O_f(x)$ *is Zariski dense in* X*. Moreover, we can take such* x *so that* $\alpha_f(x)$ *exists and* $\alpha_f(x) = \delta_1(f)$*.*

For the background about the above result, we refer to [Xie22]. Combining with [BDJ20], we obtain the first example of transcendental arithmetic degree.

Corollary 0.6. *There is a dominant rational self-map* $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ *defined over* $\overline{\mathbb{Q}}$ *and a* $\overline{\mathbb{Q}}$*-point* $x \in \mathbb{P}^2_f(\overline{\mathbb{Q}})$ *such that* $\alpha_f(x)$ *exists and* $\alpha_f(x) = \delta_1(f)$ *is a transcendental number.*

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POLARIZED ENDOMORPHISM AND FROBENIUS LIFTABILITY

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Classification AMS 2020: 14G17

Keywords:Abelian varieties; toric varieties; numerically flat; Frobenius splitting

In the paper, we work over an algebraically closed field k in characteristic $p \geq 0$.

First, I introduce polarized endomorphism and a generalization of polarized endomorphism.

Definition 0.1. *Let X be a projective variety over k and $f\colon X \to X$ a finite endomorphism.*

- *We say that f is polarized if there exists an ample line bundle L and an integer $q \geq 2$ such that $f^*L \simeq L^{\otimes q}$.*
- *We say that f is int-amplified if there exists an ample line bundle L such that $f^L \otimes L^{-1}$ is also ample.*

Polarized endomorphism is a classical and natural notion, which is deeply related to dynamics of endomorphisms of a projective space. Furthermore, int-amplified endomorphism is a natural generalization from the viewpoint of classification theory in algebraic geometry. We are interested in a classification of smooth projective varieties admitting an int-amplified endomorphism. The related result is in [3], [4], [5].

In characteristic zero, it is known that if a smooth projective variety has an étale int-amplified endomorphism, then it is an étale quotient of an Abelian variety by [2]. The main result of the talk is an analog of such result in positive characteristic as follows.

Theorem 0.2. *Let X be a smooth projective variety over k. We assume that p is positive and X is F-split, that is, the homomorphism $\mathcal{O}_X \to F_*\mathcal{O}_X$ induced by the Frobenius morphism on X splits as $\mathcal{O}_X$-modules. We further assume that X has an étale int-amplified endomorphism $f\colon X \to X$. Then X is an étale quotient of an Abelian variety.*

In the situation of the above theorem, we can see that the tangent bundle T_X of X is numerically flat. Therefore, it is a corollary of the following result.

Theorem 0.3. [1] *Let X be a smooth projective variety over k with numerically flat tangent bundle. We assume that p is positive and X is F-split. Then X is an étale quotient of an Abelian variety.*

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AUTOMORPHISM GROUPS OF SMOOTH HYPERSURFACES

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Classification AMS 2020: 14J50, 14J70, 20E99.

Keywords: automorphisms, hypersurfaces, finite groups.

In this talk, I discuss some recent results about classifying automorphism groups of smooth hypersurfaces in the projective space over the complex number field $\mathbb{C}$. This talk is based on joint works with Keiji Oguiso, Li Wei, Song Yang, and Zigang Zhu.

Let (n, d) be a pair of integers satisfying $n \geq 2$, $d \geq 3$, and $(n, d) \neq (2, 4)$. We say a finite group G is an (n, d)-*group* if G is isomorphic to a subgroup of the automorphism group of a smooth hypersurface in $\mathbb{P}^{n+1}$ of degree d. Matsumura–Monsky [7] proved that for a smooth hypersurface $X_{n,d} \subset \mathbb{P}^{n+1}$ of degree d, its automorphism group $\mathrm{Aut}(X_{n,d})$ is a finite group, and

$$\mathrm{Aut}(X_{n,d}) = \{\phi \in \mathrm{PGL}(n+2, \mathbb{C}) \mid \phi(X_{n,d}) = X_{n,d}\}.$$

Two smooth hypersurfaces of dimension n and degree d are isomorphic if and only if they are projectively equivalent, that is, their defining equations are the same up to linear change of coordinates. Therefore, classifying all (n, d)-groups is equivalent to classifying all finite subgroups of $\mathrm{PGL}(n+2, \mathbb{C})$ preserving smooth homogeneous polynomials of degree d. By solving the latter problem in the classical invariant theory, Oguiso–Yu [8] classified all possible groups acting faithfully on a smooth quintic threefold, a most basic example of Calabi-Yau threefolds.

Theorem 0.1 ([8])**.** *For a finite group G, the following two conditions are equivalent to each other:*

(i) G has a faithful action on a smooth quintic threefold, and

(ii) G is isomorphic to a subgroup of one of the 22 groups below:

$C_5^4 \rtimes S_5$, $C_4 \times (C_5^3 \rtimes S_3)$, $(C_5^2 \times C_4^2) \rtimes C_2$, $C_{16} \times (C_5^2 \rtimes C_2)$, $S_3 \times (C_5^3 \rtimes S_3)$, $C_5 \times C_{16} \times C_4$, $C_{64} \times C_5$, $C_5^2 \times C_4 \times S_3$, $(C_5^2 \rtimes C_2) \times (C_{13} \rtimes C_3)$, $C_{16} \times (C_5 \times S_3)$, C_{256}, $C_4 \times C_5 \times (C_{13} \rtimes C_3)$, $C_5 \times (C_{51} \rtimes C_4)$, $(C_5^2 \times C_3^2) \rtimes D_8$, $C_{205} \rtimes C_5$, $C_5 \times S_3 \times (C_{13} \rtimes C_3)$, $C_5 \times ((\mathrm{SL}(2,3).C_2) \rtimes C_2)$, $\mathrm{SL}(2,3) \rtimes C_4$, $C_5 \times (C_3 \rtimes Q_8)$, $C_5 \times D_{24}$, $C_5 \times S_5$, $C_{32} \times C_2$.

The work of Oguiso–Yu [8] gives a systematic (and computer-aided) method for classifying all possible (n, d)-groups for prescribed integers n and d. Based on this method, Wei–Yu [10] completed the classification of all possible groups acting faithfully on a smooth cubic threefold, an important counterexample to the three-dimensional Lüroth problem ([1]).

Theorem 0.2 ([10])**.** *For a finite group G, the following two conditions are equivalent to each other:*

(i) G has a faithful action on a smooth cubic threefold, and

(ii) G is isomorphic to a subgroup of one of the 6 groups below:

$C_3^4 \rtimes S_5$, $((C_3^2 \rtimes C_3) \rtimes C_4) \times S_3$, C_{24}, C_{16}, $\mathrm{PSL}(2, 11)$, $S_5 \times C_3$.

Following the approach of Oguiso–Yu's work, Yang–Yu–Zhu [11] complete the classification of all $(5,3)$-groups and $(4,3)$-groups by introducing two new notions, *partitionability* and *characteristic sets* of homogeneous polynomials.

Theorem 0.3 ([11])**.** *A finite group G can act faithfully on a smooth cubic fivefold if and only if G is isomorphic to a subgroup of one of the following 20 groups:*
$C_3^6 \rtimes S_7$, $((C_3^2 \rtimes C_3) \rtimes C_4) \times (C_3^3 \rtimes S_4)$, $C_8 \times (C_3^3 \rtimes S_3)$, $S_5 \times (C_3^3 \rtimes S_3)$, $C_{48} \times S_3$, $\mathrm{PSL}(2,11) \times (\mathrm{C}_3^2 \rtimes \mathrm{C}_2)$, $((C_3^2 \rtimes C_3) \rtimes C_4)^2 \rtimes C_2$, $((C_3^2 \rtimes C_3) \rtimes C_4) \times C_8$, $S_5 \times ((C_3^2 \rtimes C_3) \rtimes C_4)$, C_{96}, $C_{63} \rtimes C_6$, $C_3.M_{10}$, $S_7 \times C_3$, $C_3 \times ((C_8 \times C_2) \rtimes C_2)$, $C_3 \times (\mathrm{PSL}(3,2) \rtimes C_2)$, $C_3.A_7$, $C_3 \times \mathrm{GL}(2,3)$, $((C_3^2 \rtimes C_3) \rtimes Q_8) \rtimes C_3$, C_{64}, $C_{43} \rtimes C_7$.

Theorem 0.4 ([11])**.** *A finite group G can act faithfully on a smooth cubic fourfold if and only if G is isomorphic to a subgroup of one of the following 15 groups:*
$C_3^5 \rtimes S_6$, $((C_3 \times (C_3^3 \rtimes C_3)) \rtimes C_3) \rtimes (C_4 \times C_2)$, $C_8 \times (C_3^2 \rtimes C_2)$, $S_5 \times (C_3^2 \rtimes C_2)$, C_{48}, $\mathrm{PSL}(2,11) \times \mathrm{C}_3$, $((C_3 \times (C_3^2 \rtimes C_3)) \rtimes C_3) \rtimes (C_4^2 \rtimes C_2)$, C_{32}, $C_{21} \rtimes C_6$, M_{10}, S_7, $(C_8 \times C_2) \rtimes C_2$, $\mathrm{PSL}(3,2) \rtimes C_2$, $\mathrm{GL}(2,3)$, $(C_3^2 \rtimes Q_8) \rtimes C_3$.

Remark 0.5. *Explicit examples of smooth hypersurfaces acted on by the maximal groups in the classifications in Theorems 0.1, 0.2, 0.3-0.4 are given in* [8], [10], [11] *respectively.*

Remark 0.6. *The study of the automorphism groups $\mathrm{Aut}(X)$ of smooth cubic hypersurfaces X has a long and rich history. All possible groups acting faithfully on cubic surfaces have been classified (see* [9], [5], [2]*). For $\dim(X) = 4$, recently Laza–Zheng* [6] *classified the symplectic automorphism groups $\mathrm{Aut}^s(X)$ of cubic fourfolds and proved that the Fermat cubic fourfold has the largest possible order for $|\mathrm{Aut}(X)|$. For some partial results on abelian subgroups of automorphism groups of smooth cubic hypersurfaces of arbitrary dimension, see* [3], [12], [4].

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NOETHER INEQUALITY FOR IRREGULAR THREEFOLDS OF GENERAL TYPE

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Classification AMS 2020: 14J30, 14J10, 14E30.

Keywords: Irregular varieties, Threefolds of general type, Noether inequality

Throughout this extended abstract, we work over an algebraically closed field of characteristic zero, and all varieties are projective.

1. Motivation

Classifying algebraic varieties is a central problem in algebraic geometry. One way to attack this problem, especially for varieties of general type, is to study the relation among their birational invariants first, and then to obtain explicit classifications using these numerical information. This is often refered as the geography of algebraic varieties in the literature.

It is via this approach that many classification results for varieties of general type have been proved. As a typical example, it was proved by M. Noether that every minimal surface S of general type satisfies the following optimal inequality:

$$K_S^2 \geq 2p_g(S) - 4, \tag{1.1}$$

which is now called the Noether inequality. In [8], Horikawa completely classified surfaces of general type attaining the Noether equality.

The Noether inequality problem in dimension three also has attracted lots of attentions (e.g. [13, 2, 4]). Recently, J. Chen, M. Chen and Jiang [5] proved that every minimal 3-fold X of general type with $p_g(X) \geq 11$ satisfies the following optimal Noether inequality:

$$K_X^3 \geq \frac{4}{3}p_g(X) - \frac{10}{3}. \tag{1.2}$$

They also asked [loc. cit.] whether 3-folds attaining the equality can be classified, and this question has been solved by the authors in a recent work [11].

It is a fundamental question to what extent does the irregularity have an influence on the distribution of birational invariants. In fact, if a minimal surface S of general type satisfies the equality in (1.1), then $q(S) = 0$ [1]. Therefore, there must be a sharper Noether inequality for irregular surfaces of general type. This problem was studied extensively by Bombieri [1, §10] and later solved by Debarre [7]. More precisely, Debarre [loc. cit.] proved that every minimal irregular surface S of general type satisfies the following optimal Noether inequality:

$$K_S^2 \geq 2p_g(S). \tag{1.3}$$

Moreover, he proved that if the equality holds, then $1 \leq q(S) \leq 4$. Based on the work of Horikawa [9], Debarre [loc. cit.], Catanese-Ciliberto-Mendes Lopes [3] and Ciliberto-Mendes Lopes-Pardini [6], a complete classification of irregular surfaces of general type attaining the equality has been established.

Very similar to the surface case, it is proved recently that if a 3-fold X of general type satisfies the Noether equality in (1.2), then $q(X) = 0$ [10]. Thus it is natural to ask:

Question. What is the optimal Noether inequality for irregular 3-folds of general type? Once the desired inequality is obtained, can one classify the 3-folds attaining the equality?

2. Main results

Our main results give an answer to the above question for "almost all" irregular 3-folds of general type.

Theorem 2.1. [12, Theorem 1.1] Let X be a minimal irregular 3-fold of general type. Then we have the following optimal Noether inequality:

$$K_X^3 \geq \frac{4}{3} p_g(X), \tag{2.1}$$

provided one of the following conditions holds:

(1) X is Gorenstein;
(2) $q(X) = 1$ and $p_g(X) \geq 17$;
(3) $q(X) \geq 2$.

Moreover, if the equality in (2.1) holds for X which satisfies any of the above three conditions (1)–(3), then $1 \leq q(X) \leq 2$.

Note that minimal 3-folds X with $\mathrm{vol}(X) < \frac{4}{3} p_g(X) \leq \frac{64}{3}$ form a bounded family [14, Corollary 2]. Thus Theorem 2.1 shows that the Noether inequality (2.1) holds for all minimal and irregular 3-folds of general type except possibly finitely many families.

We would like to remark on the optimality of the Noether inequality (2.1). Indeed, the authors in [10] classified all minimal irregular 3-folds X of general type with $\mathrm{vol}(X) = \frac{4}{3}\chi(\omega_X)$ and showed that these 3-folds must satisfy $q(X) = 1$ and $h^2(X, \mathcal{O}_X) = 0$. Thus we deduce that

$$K_X^3 = \frac{4}{3}\left(p_g(X) - h^2(X, \mathcal{O}_X) + q(X) - 1\right) = \frac{4}{3} p_g(X)$$

for any such 3-fold X.

The next theorem shows that "almost" all minimal irregular 3-folds attaining the equality (2.1) are of the above form.

Theorem 2.2. [12, Theorem 1.2] Let X be a minimal irregular 3-fold of general type with $p_g(X) \geq 17$. Then $K_X^3 = \frac{4}{3} p_g(X)$ if and only if $K_X^3 = \frac{4}{3}\chi(\omega_X)$. As a result, minimal irregular 3-folds X of general type with $p_g(X) \geq 17$ and $K_X^3 = \frac{4}{3} p_g(X)$ can be explicitly classified.

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STRUCTURE OF PROJECTIVE VARIETIES WITH CERTAIN POSITIVE TANGENT SHEAVES

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Keywords: Tangent sheaves, Almost nef sheaves, Rationally connected varieties, Abelian varieties, Toric varieties, MRC fibrations, Numerically flat vector bundles.

ABSTRACT. Positivities of tangent sheaves are expected to impose rather restrictive geometry on the underlying space. In this talk, we consider a projective klt variety X with an almost nef tangent sheaf T_X, i.e., $T_X|_C$ is nef for any curve $C \subseteq X$ not lying in a given countable union of proper closed subvarieties. We establish a structure theorem towards the fundamental building blocks of such varieties. Then we apply this structure to the classification of surfaces and threefolds with certain positive tangent sheaves. Furthermore, we will discuss a few questions on the equivariant minimal model program with respect to certain positive tangent sheaves. This is based on some part of my joint work [IMZ23] with Masataka Iwai and Shin-ichi Matsumura.

First of all, we review some basic definitions and after that we will summarize our main results in this talk.

For a normal projective variety X, its tangent sheaf $\mathcal{T}_X$ of X is defined by the reflexive hull $\mathcal{T}_X := (j_*\mathcal{T}_{X_{\text{reg}}})^{\vee\vee}$ of the direct image sheaf $j_*\mathcal{T}_{X_{\text{reg}}}$, where $j : X_{\text{reg}} \to X$ is the natural inclusion and $\mathcal{T}_{X_{\text{reg}}}$ is the tangent bundle on X_{reg}. Similarly, the reflexive cotangent sheaves of degree p is defined by $\Omega_X^{[p]} := (j_*\Omega^p_{X_{\text{reg}}})^{\vee\vee}$.

Definition 0.1. *Let X be a normal projective variety of dimension n and $\mathcal{E}$ be a sheaf on X.*

(1) *$\mathcal{E}$ is said to be ample (resp. strictly nef, nef) if the tautological line bundle $\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(1)$ is ample (resp. strictly nef, nef) on the projectivization $\mathbb{P}_X(\mathcal{E}) :=$ Proj(Sym$^\bullet\mathcal{E}$).*

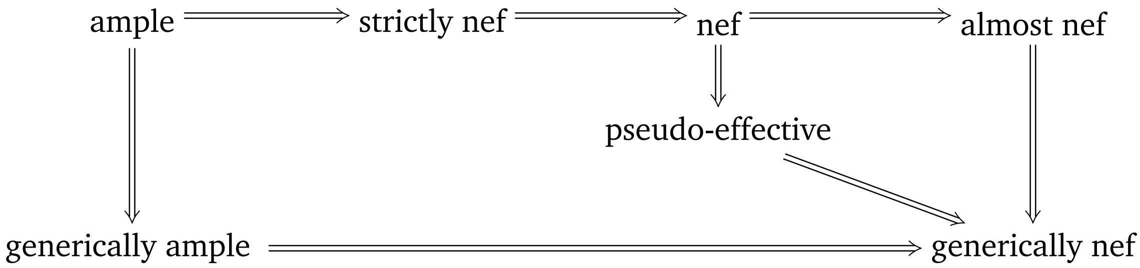
(2) *$\mathcal{E}$ is said to be almost nef if there exist countably many proper subvarieties $Z_i \subseteq X$ ensuring that the sheaf $\mathcal{E}|_C := \mathcal{E} \otimes \mathcal{O}_C$ is nef for any curve $C \not\subseteq \cup_i Z_i$ (see* [DPS01, Definition 6.4] *and* [LOY20, Definition 3.6]*). The notation $\mathbb{S}(\mathcal{E})$ denotes the smallest countable union $\cup_i Z_i$ with the above property.*

Henceforth, we further assume that $\mathcal{E}$ is torsion-free.

(3) *$\mathcal{E}$ is said to be pseudo-effective if for any $a \in \mathbb{Z}_+$ and any ample Cartier divisor A on X, there exists $b \in \mathbb{Z}_+$ such that* Sym$^{[ab]}(\mathcal{E}) \otimes A^b$ *is globally generated at a general point of X (see* [Nak04, Definition 3.20] *and* [BDPP13, Definition 7.1]*; cf.* [Vie83]*).*

(4) *$\mathcal{E}$ is said to be generically ample (resp. generically nef) if $\mu^{min}_{H_1\cdots H_{n-1}}(\mathcal{E}) > 0$ (resp. ≥ 0) holds for any ample Cartier divisors $H_1, \ldots, H_{n-1}$ on X (see* [Miy87]*).*

Our definition of the pseudo-effectivity of $\mathcal{E}$ differs from the pseudo-effectivity of $\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(1)$ even when $\mathcal{E}$ is locally free. For torsion-free sheaves, we can summarize relations among the above-mentioned notions by the following table:



We refer to [IMZ23, Section 2] for more information and the proofs; moreover, other implications do not hold.

Now, we will state our main results discussed in this talk. We note that in our joint paper [IMZ23], we also establish another structure theorem for projective klt varieties with positively curved tangent sheaf [IMZ23, Theorem 1.1], while this theorem was not contained in this talk due to the organization.

Theorem 0.2 (=[IMZ23, Theorem 1.2]). *Let X be a projective klt variety with almost nef tangent sheaf. Then, there exists a fibration $\alpha\colon X \to Y$ satisfying the following properties*:

(1) *The fibration $\alpha\colon X \to Y$ is flat, and every irreducible component of the singular locus of X dominates Y.*
(2) *The base variety Y is an étale quotient of an abelian variety (i.e., there exists a finite étale cover $A \to Y$ from an abelian variety A).*
(3) *Any fiber of $\alpha\colon X \to Y$ is an irreducible and reduced rationally connected klt variety, and a very general fiber has almost nef tangent sheaf.*

The following are some applications to the above structure theorem.

Theorem 0.3 (=[IMZ23, Corollary 1.3]). *Let X be a projective klt variety with almost nef tangent sheaf. If the canonical divisor K_X is numerically trivial, then X is an étale quotient of an abelian variety (in particular, it is smooth).*

Theorem 0.4 (cf. [IMZ23, Theorem 1.4]). *Let X be a projective klt variety with almost nef tangent sheaf. After we replace X with an appropriate quasi-étale cover, there exists a fibration $\alpha\colon X \to A$ onto an abelian variety A of dimension $\widehat{q}(X)$ satisfying the following properties*:

(1) *The pullback $\alpha^*\mathcal{T}_A$ of the tangent bundle $\mathcal{T}_A$ coincides with the reflexive hull of the flat part of the Fujita decomposition of $\mathcal{T}_X$. In particular, the augmented irregularity $\widehat{q}(X)$ is equal to the rank of the flat part of $\mathcal{T}_X$.*
(2) *Any fiber of $\alpha\colon X \to A$ is maximally quasi-étale, i.e., any quasi-étale cover of a fibre has to be étale, and a very general fiber F has vanishing augmented irregularity $\widehat{q}(F) = 0$.*

Theorem 0.5 (cf. [IMZ23, Corollary 1.5]). *Let X be a projective klt variety with almost nef tangent sheaf. Then, the following conditions are equivalent.*

(1) *The tangent sheaf $\mathcal{T}_X$ is generically ample.*
(2) *The augmented irregularity $\widehat{q}(X)$ is zero.*
(3) *Any finite quasi-étale cover of X is rationally connected.*

We propose the question below at the end of the talk.

Question 0.6 (=[IMZ23, Question 7.4]; cf. [HIM22, Problem 3.12]). *Let X be a rationally connected projective klt variety with almost nef tangent sheaf. Is the augmented irregularity of X zero? Is X even of Fano type?*

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