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Extending Borel's Conjecture from Measure to Dimension–Singapore 2024

Borel (1919) defined a subset A of $\mathbb{R}$ to have strong measure zero if for every sequence of positive numbers $(\epsilon_i : i \in \omega)$ there is an open cover of A $(U_i : i \in \omega)$ such that for each i, the diameter of U_i is less than ϵ_i. Besicovitch (1956) showed that A has strong measure zero if and only if A has strong dimension zero, which means that for every gauge function f, A is null for its associated measure H^f. We say that $A \subset \mathbb{R}^N$ has strong dimension f if and only if $H^f(A) > 0$ and for every gauge function g of higher order $H^g(A) = 0$. Here, g has higher order than f when $\lim_{t\to 0^+} g(t)/f(t) = 0$.

Borel conjectured that a set of strong measure zero must be countable. This conjecture naturally extends to the assertion that a set has strong dimension f if and only if it is σ-finite for H^f. Sierpinski (1928) used the continuum hypothesis to give a counterexample to Borel's conjecture and Besicovitch (1963) did the same for its generalization. Laver (1976) showed that Borel's conjecture is relatively consistent with consistent with *ZFC*, the conventional axioms of set theory including the axiom of choice. Recently, we showed that its generalization to strong dimension is also relatively consistent with *ZFC*.

We will give an exposition of these results, along with developing the additional background material needed for their proofs.

Suggested Reading Material

[1] Tomek Bartoszyński and Haim Judah. *Set theory.* A K Peters, Ltd., Wellesley, MA, 1995. On the structure of the real line.

A survey of results on cardinal invariants of the continuum, including Borel's conjecture on sets of strong measure zero.

[2] A. S. Besicovitch. On existence of subsets of finite measure of sets of infinite measure. *Nederl. Akad. Wetensch. Proc. Ser. A.* **55** = *Indagationes Math.*, 14:339–344, 1952.

Shows that a closed set of positive Hausdorff measure has a closed subset of finite positive Hausdorff measure.

[3] A. S. Besicovitch. On the definition of tangents to sets of infinite linear measure. *Proc. Cambridge Philos. Soc.*, 52:20–29, 1956.

Shows that if a closed set has strong dimension f then it is σ-finite for the gauge measure associated with f. Also, shows that a set is null for every gauge measure if and only if it has strong measure zero.

[4] A. S. Besicovitch. A problem on measure. *Proc. Cambridge Philos. Soc.*, 59:251–253, 1963.

Uses the CH to exhibit a set that has strong linear dimension, but is not σ-finite for linear measure.

[5] Lawrence C. Evans and Ronald F. Gariepy. *Measure theory and fine properties of functions.* Textbooks in Mathematics. CRC Press, Boca Raton, FL, revised edition, 2015.

A good introduction to general measure theory, including a chap-ter on Hausdorff measures and Hausdorff dimension.

[6] Richard Laver. On the consistency of Borel's conjecture. *Acta Math.*, 137(3-4):151–169, 1976.

Shows the relative consistency of the Borel Conjecture on sets of strong measure zero with ZFC.

[7] C. A. Rogers. *Hausdorff measures.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1998. Reprint of the 1970 original, With a foreword by K. J. Falconer.

A good introduction to Hausdorff measures and Hausdorff dimen-sion, as developed by the Besicovitch school.

[8] Theodore A. Slaman. On capacitability for co-analytic sets. *New Zealand J. Math.*, 52:865–869, 2021 [2021–2022].

If $V = L$, then the maximal thin Π^1_1-set has full Hausdorff dimen-sion (but no closed subset of positive Hausdorff dimension).

[9] Theodore A. Slaman. Extending Borel's conjecture from measure to dimension. preprint, 2024.

In Laver's model, if a set has strong dimension f then it is σ-finite for H^f.