



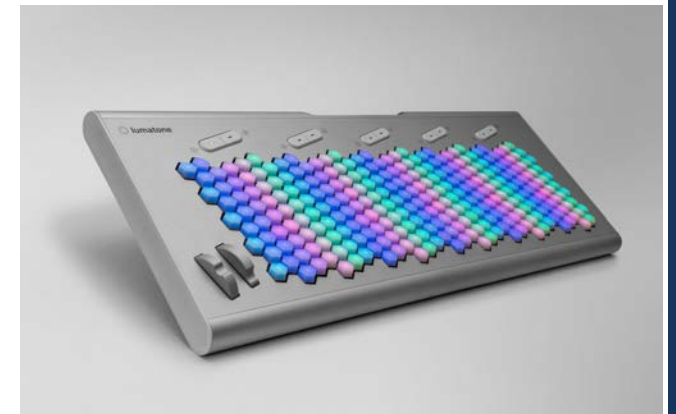
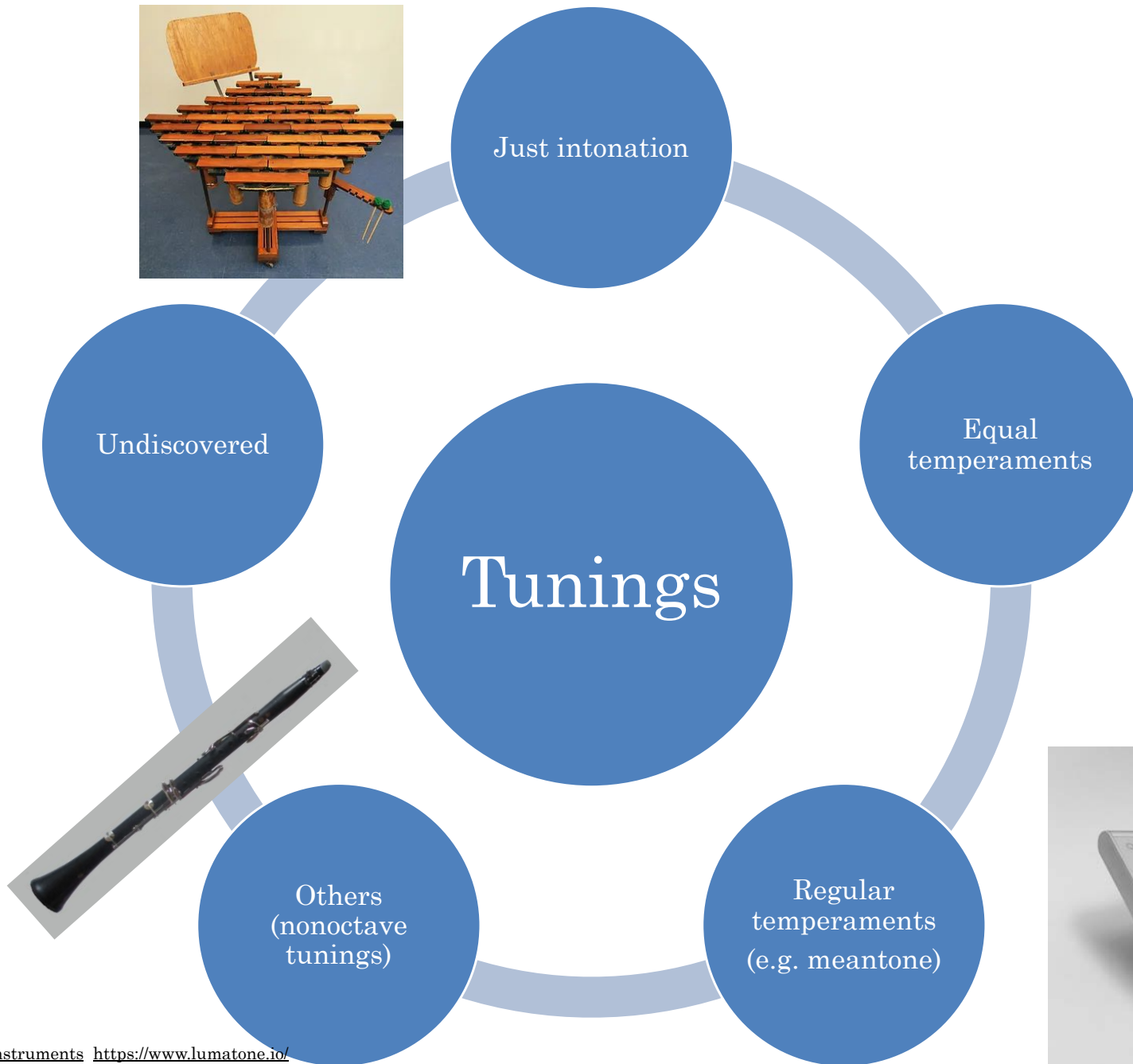
Australian
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Exploring New Soundscapes: A Decatonic Framework for 7-limit Just Intonation

by Nathanael Koh

Supervisors: Prof Frank Millward (ANU), Dr Chloe Hobbs (ANU), Prof Andrew Milne (WSU)

?

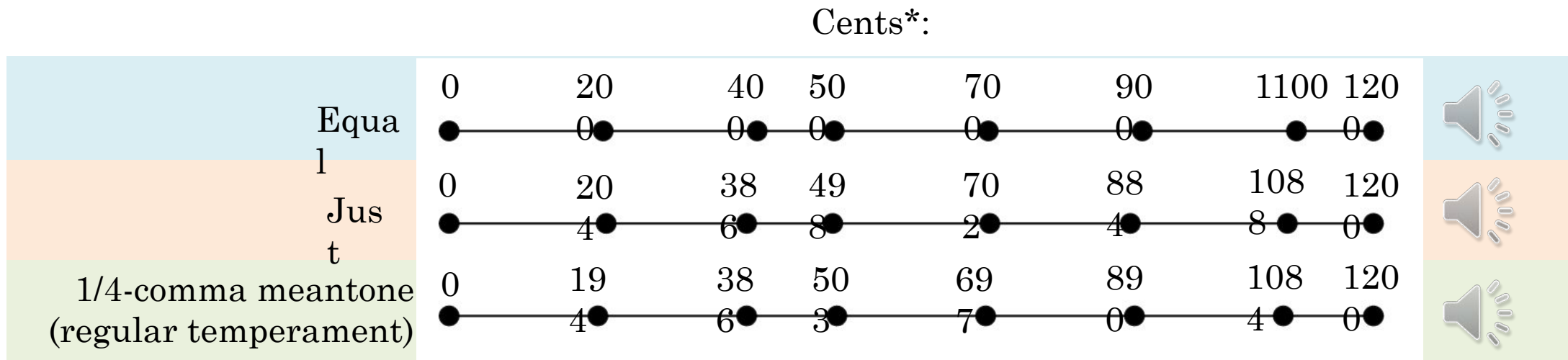


What are regular temperaments?

- Generalization of equal temperaments and just intonation
- Reducing the number of notes in just intonation to make it more practical
- Equal temperaments: equal divisions of the octave
 - Example: the current Western tuning (12 equal divisions)
- Just intonation: using the harmonic series to tune intervals
 - Example: Pythagorean tuning

What are regular temperaments?

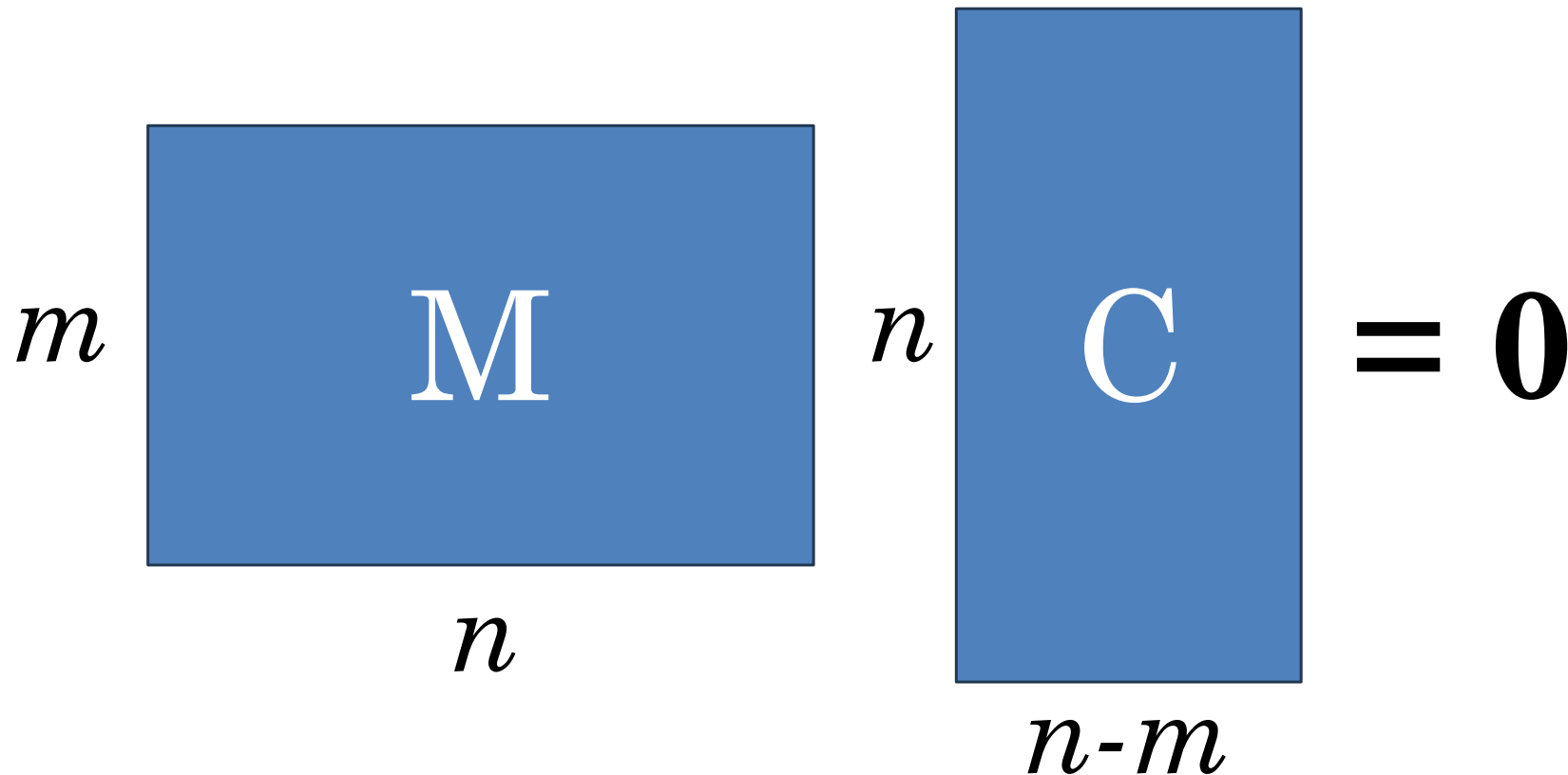
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* 1 cent = 1/1200 of an octave or 1/100 of a semitone

Mathematical motivation

- $MC = 0$ (see Milne 2008, 3)
 - M is the mapping
 - C is the comma space – null space of M



Musical motivation

- Regular temperaments are simplifications of just intonation (JI)
- Allows new interval relationships impossible in JI
- Example: Meantone (Lindley 2001)
 - $\frac{1}{4}$ -comma can be thought of as a chain of 31 fifths (Link Jr 1965).

Musical motivation

4 just fifths



equal a major 3rd and 2 octaves...

but not quite!

4 meantone fifths



equal a major 3rd and 2 octaves!

Why regular temperaments?

- Allow new chord progressions
 - Technically called comma pumps
 - Benedetti's progression:



Why regular temperaments?

- Allow new tonal qualities
 - Porcupine



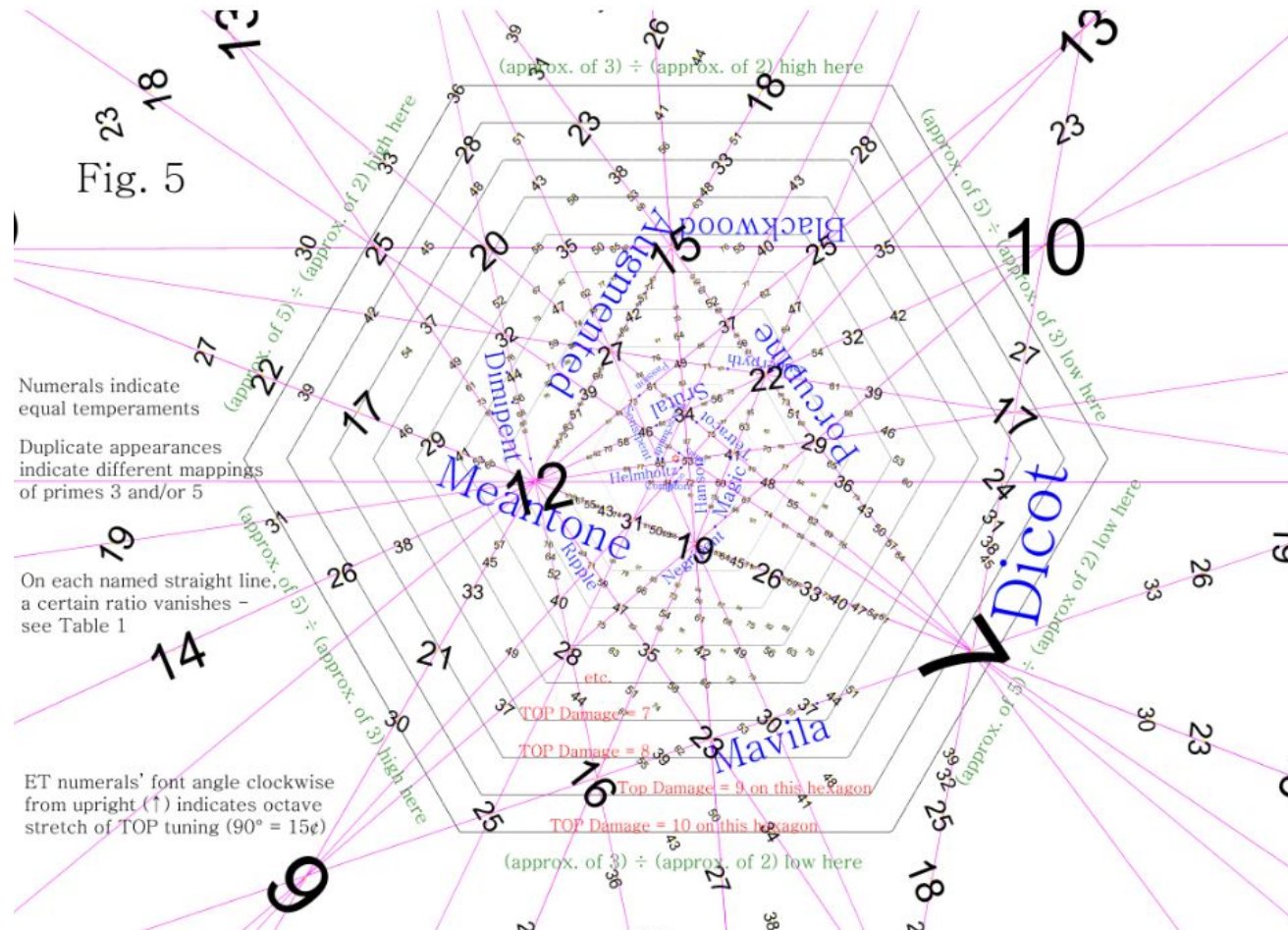
New soundscapes?

- Shower and Biddulph (1931)
 - 6.4 cents as just noticeable difference

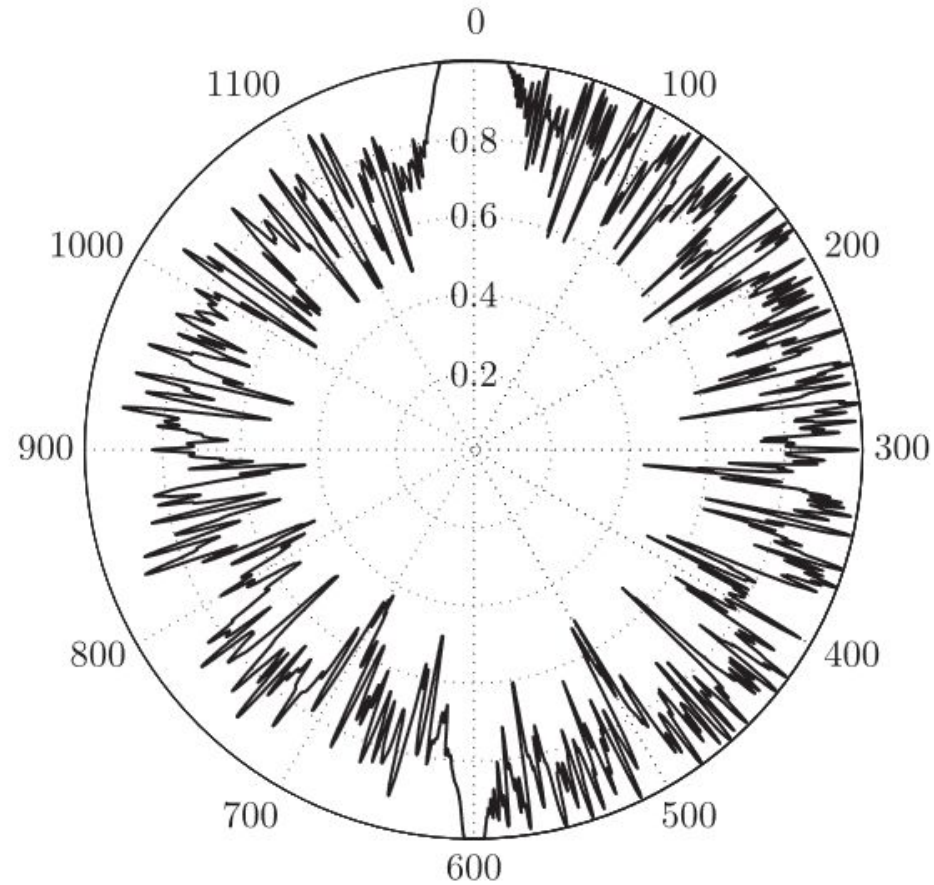
New soundscapes?

- Shower and Biddulph (1931)
 - 6.4 cents as just noticeable difference
- 12 to 25 cents would be audible for most musical purposes

Searching for temperaments



Searching for temperaments



Taken from Milne (2011,
16)

Gaps in research

- Discovering new regular temperaments using a systematic mathematical and musical search
 - Computer program (Python)
 - Mathematical
 - Musical properties
- **Musical compositions**

Compositional process

- Find a suitable temperament (using a program, mathematically and musically)
- Listen to the sounds
- Notation
- Create a tonal and harmonic system based on it
- **COMPOSE MUSIC!!!**
 - Functional harmony or an extension thereof (?)

```
#Mapping
mapping = []
for i in range(len(supported)):
    val = [supported[i], int(round(supported[i]*math.log2(3), 0)
    if i == 0:
        val0 = val
        mapping.append(val0)
    if i >= 1:
        if len(mapping) == 1:
            if math.gcd(val0[0], val[0]) == g:
                mapping.append(val)

mapping = mapping[0:2]
badcount = 0

for i in range(0, 4):
    if math.gcd(mapping[0][i], mapping[1][i]) == mapping[0][i]:
        badcount = badcount + 1

if badcount != 4:

    for i in range(0, 4):
        mapping[0][i] = mapping[0][i]/g
        mapping[1][i] = mapping[1][i]/g

def solve(a, b):
    if b > a:
        return solve(b, a)
    else:
        q, r = divmod(a, b)
        if r == 0:
            return [0, 1/b]
        else:
            s = solve(b, r)
            x = s[0]
            y = s[1]
            return [y, x-q*y]

sol = solve(mapping[0][0], mapping[1][0])
newmap = [[], []]
for i in range(0, 4):
    c = mapping[1][i]*sol[0]+mapping[0][i]*sol[1]
    newmap[0].append(c)
for i in range(0, 4):
```

A new theoretical framework?

	Pentatonic	Diatonic
Notes	5	7
Interval	Semifourth	Semififth
Missing interval	Semififth	Semifourth
Number of notes in doubled scale	10	14
Mapping	(10 16 23 28)	(14 22 32 39)

Doubled pentatonic: V V#/Wb W W#/Xb X X#/Yb Y Y#/Zb Z Z#/Vb V

Doubled diatonic: D D#/Eb E E#/Fb F F#/Gb G G#/Ab A A#/Bb B B#/Cb C C#/Db D

Mapping matrix of the decimal temperament

$$\begin{pmatrix} 10 & 16 & 23 & 28 \\ 14 & 22 & 32 & 39 \end{pmatrix}$$

Hermite normal
form

$$\begin{pmatrix} 2 & 0 & 3 & 4 \\ 0 & 2 & 1 & 1 \end{pmatrix}$$

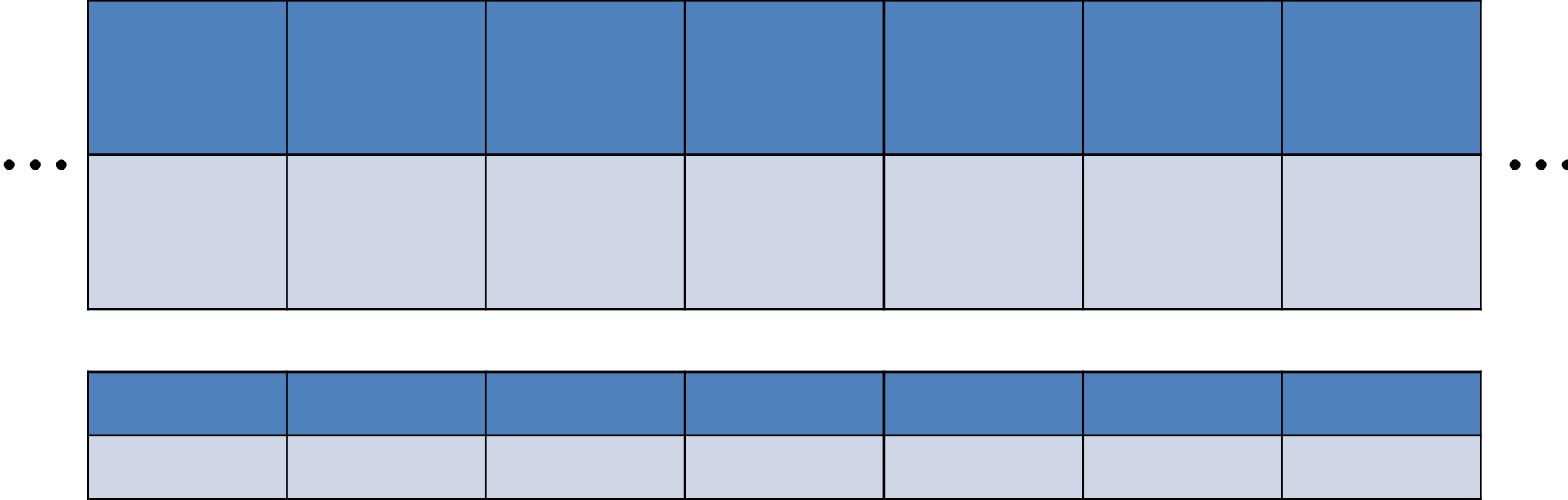
‘Semiquartal’ form

$$\begin{pmatrix} 2 & 4 & 5 & 6 \\ 0 & -2 & -1 & -1 \end{pmatrix}$$

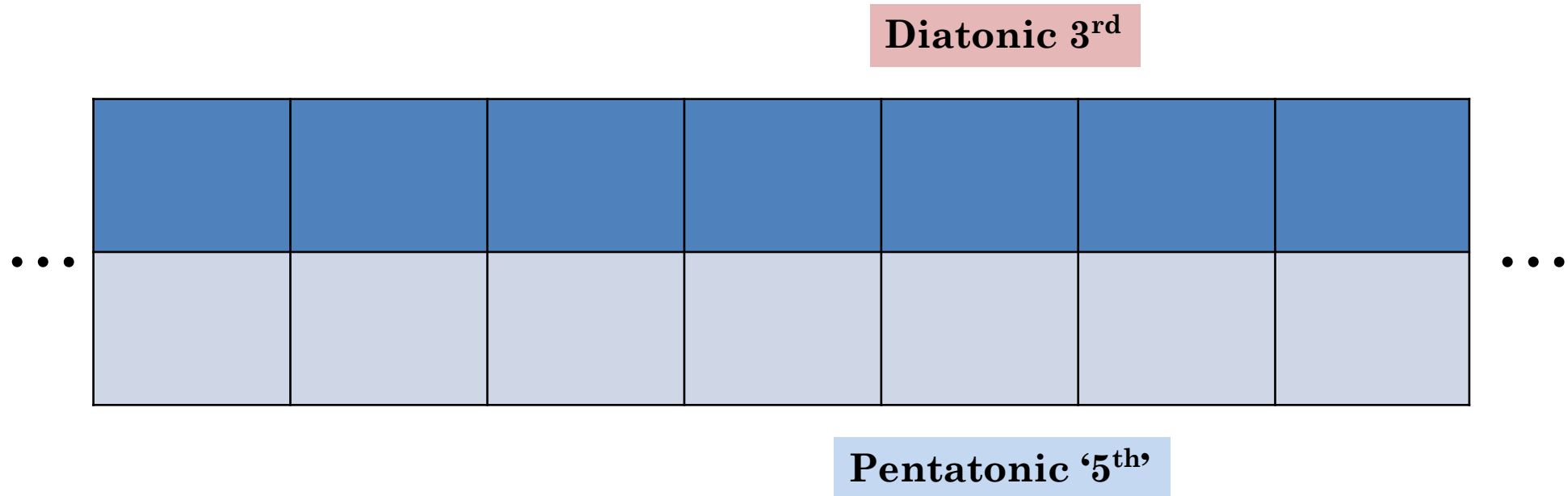
‘Semiquintal’ form

$$\begin{pmatrix} 2 & 2 & 4 & 5 \\ 0 & 2 & 1 & 1 \end{pmatrix}$$

Relationship with Semi-Pythagorean intervals

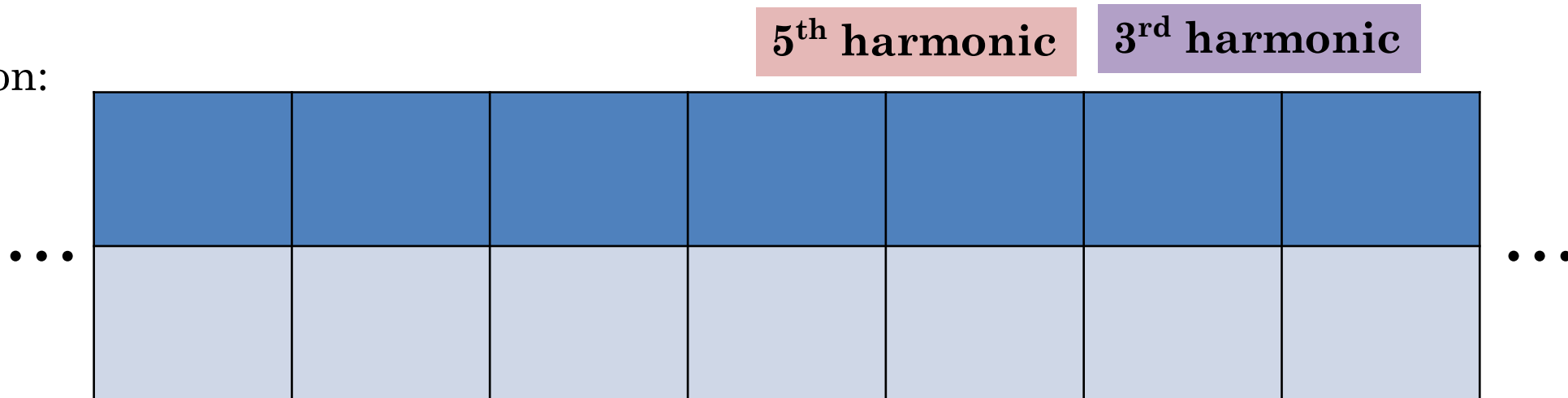


Relationship with Semi-Pythagorean intervals



Relationship with Semi-Pythagorean intervals

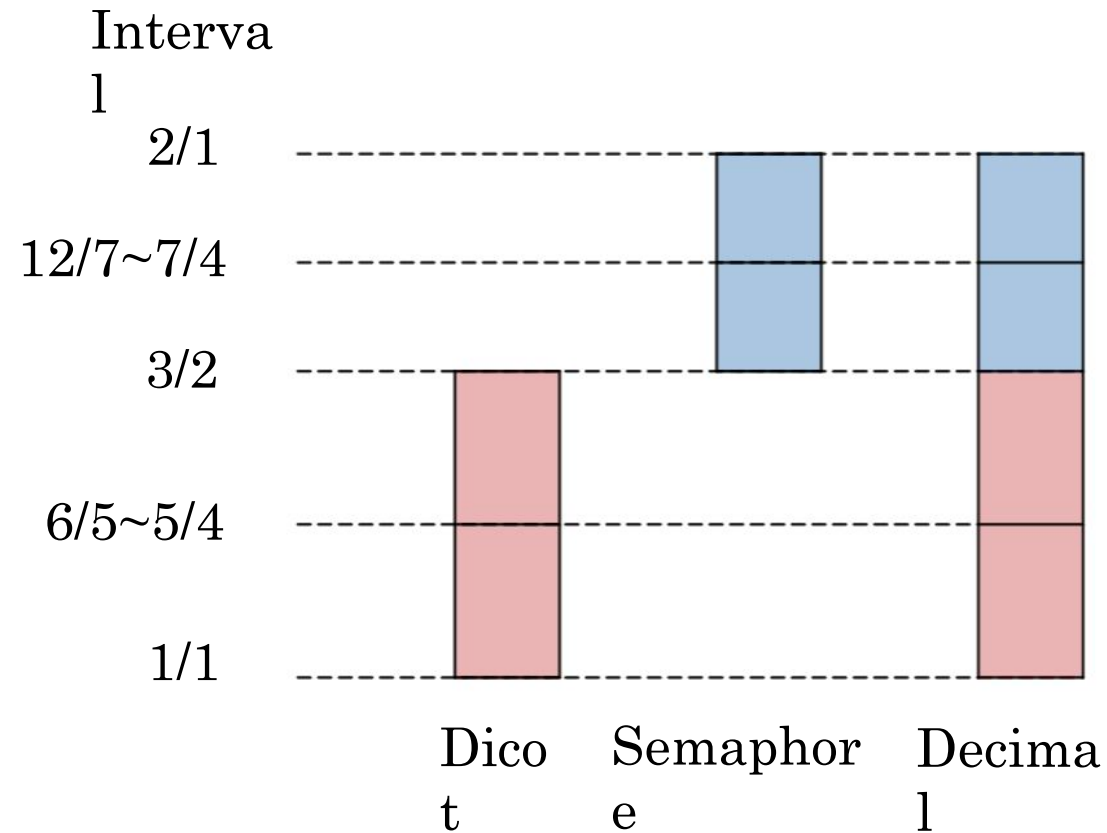
In Just
Intonation:



In Just
Intonation:

7th harmonic

Relationships



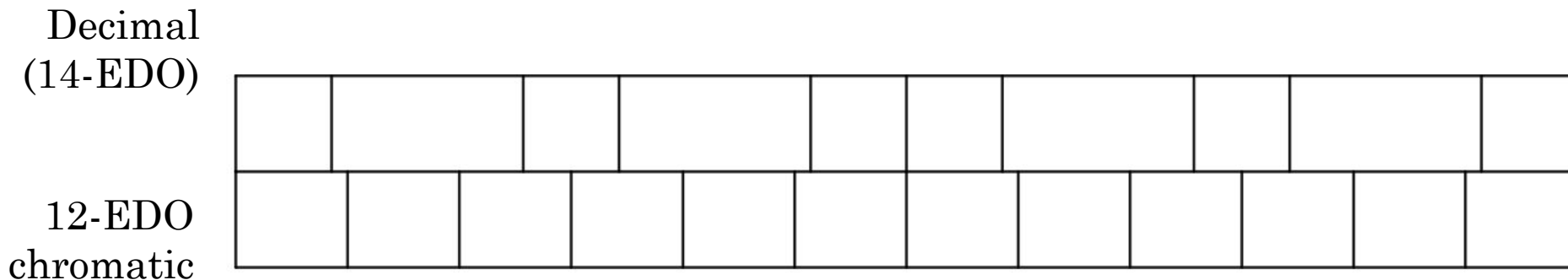
Combination of dicot and semaphore intervals
to form the decimal temperament

Decimal temperament

- Tempers out $25/24$ and $49/48$ (Xenharmonic Wiki, n.d., a.)
- 14-EDO (Equal Divisions of the Octave)

Decimal temperament

- Has a 10-note (4+6) MOS (Moment-of-Symmetry; see Milne 2007) scale (referred to as the *decimal scale*)
- Minimum symmetrical set which contains all prime harmonics above tonic
 - Contains the whole 7-limit Tonality Diamond (tempered in the decimal temperament; see Partch 1974)
 - Repeats every half an octave



Semi-Pythagorean interval (mathematical calculation)	$\sqrt{\frac{1}{1}} = \frac{1}{1}$	$\sqrt{\frac{9}{8}}$	$\sqrt{\frac{4}{3}}$	$\sqrt{\frac{3}{2}}$	$\sqrt{\frac{16}{9}} = \frac{4}{3}$	$\sqrt{\frac{2}{1}}$	$\sqrt{\frac{9}{4}} = \frac{3}{2}$	$\sqrt{\frac{8}{3}}$	$\sqrt{\frac{3}{1}}$	$\sqrt{\frac{32}{9}}$	$\sqrt{\frac{4}{1}} = \frac{2}{1}$
Semi-Pythagorean interval (cents/¢)	0¢	101.95¢	249.02¢	350.98¢	498.04¢	600¢	701.95¢	849.02¢	950.98¢	1098.04¢	1200¢
Steps in 14-tone equal temperament (musical result)	0	1	3	4	6	7	8	10	11	13	14
Step size		<i>Small</i>	<i>Large</i>	<i>Small</i>	<i>Large</i>	<i>Small</i>	<i>Small</i>	<i>Large</i>	<i>Small</i>	<i>Large</i>	<i>Small</i>
14-tone equal temperament interval in cents	0¢	85.71¢	257.14¢	342.86¢	514.29¢	600¢	685.71¢	857.14¢	942.86¢	1114.29¢	1200¢
Just interval(s)	1/1	21/20, 15/14	8/7, 7/6	6/5, 5/4	4/3	7/5, 10/7	3/2	8/5, 5/3	12/7, 7/4	28/15, 40/21	2/1
Error of semi-Pythagorean interval from Just Intonation (cents/¢)	0¢	±17.49¢	±17.85¢	±35.34¢	0¢	±17.49¢	0¢	±35.34¢	±17.85¢	±17.49¢	0¢
Note name	<i>0</i>	<i>1</i>	<i>2</i>	<i>3</i>	<i>4</i>	<i>5</i>	<i>6</i>	<i>7</i>	<i>8</i>	<i>9</i>	<i>0</i>

Clef and staff

This is the notation of the decimal scale in 14-tone equal temperament.



Clef and staff

This is the notation of the decimal scale in 14-tone equal temperament.

The image shows a musical staff with a five-line structure and a '5' clef. Above the staff, the digits 4, 5, 6, 7, 8, 9, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, 1, 2, 3, 4, 5, 6 are aligned with the notes. The notes are quarter notes, and the scale is ascending. A red bracket is drawn below the staff, starting under the 0th note and ending under the 6th note. Below the bracket, the text 'One decatave' is written.

Clef and staff

This is the notation of the decimal scale in 14-tone equal temperament.



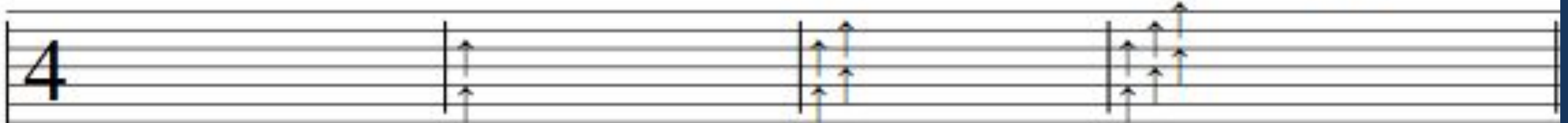
Key Signatures

0 or 5 decimal
No accidentals

2 or 7 decimal
6 \uparrow , 1 \uparrow

4 or 9 decimal
6 \uparrow , 1 \uparrow , 8 \uparrow , 3 \uparrow

6 \uparrow or 1 \uparrow decimal
6 \uparrow , 1 \uparrow , 8 \uparrow , 3 \uparrow , 0 \uparrow , 5 \uparrow

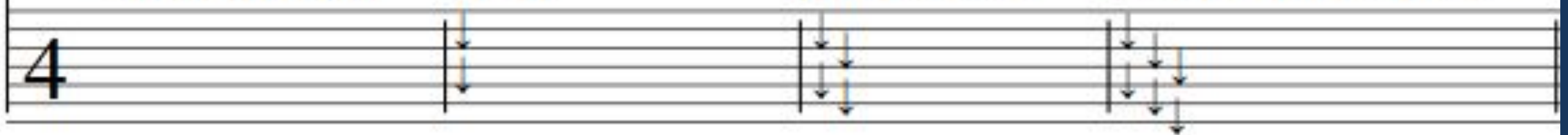


0 or 5 decimal
No accidentals

8 or 3 decimal
4 \downarrow , 9 \downarrow

6 or 1 decimal
4 \downarrow , 9 \downarrow , 2 \downarrow , 7 \downarrow

4 \downarrow or 9 \downarrow decimal
4 \downarrow , 9 \downarrow , 2 \downarrow , 7 \downarrow , 0 \downarrow , 5 \downarrow



Modes

Names derived from Horn (1958)

The image displays five musical staves, each representing a different mode. Each staff begins with a treble clef and a 3/4 time signature. The notes are written in a sequence that represents the scale of the mode, starting from a central pitch (the tonic) and moving up and then down. The modes shown are:

- Phrygian:** The first staff, with a green vertical line marking the start of the scale. The notes are: G, A, Bb, C, D, E, F, G.
- Aeolian:** The second staff. The notes are: G, A, B, C, D, E, F, G.
- Dorian:** The third staff. The notes are: G, A, B, C, D, E, F, G.
- Mixolydian:** The fourth staff. The notes are: G, A, B, C, D, E, F, G.
- Ionian:** The fifth staff. The notes are: G, A, B, C, D, E, F, G.

Intervals

The image displays three staves of musical notation, each in 4/4 time, illustrating various intervals. A green vertical line marks the beginning of the first staff.

- Staff 1:** Contains four measures of intervals:
 - Small 1sts: A sequence of eighth notes ascending from G4 to E5.
 - Large 1sts: A sequence of eighth notes ascending from G4 to E5, with a larger interval between notes.
 - Small 2nds: A sequence of pairs of eighth notes ascending from G4 to E5.
 - Large 2nds: A sequence of pairs of eighth notes ascending from G4 to E5, with a larger interval between pairs.
- Staff 2:** Contains four measures of intervals:
 - Small 3rds: A sequence of pairs of eighth notes ascending from G4 to E5.
 - Large 3rds: A sequence of pairs of eighth notes ascending from G4 to E5, with a larger interval between pairs.
 - Small 4ths: A sequence of pairs of eighth notes ascending from G4 to E5.
 - Small 4ths: A sequence of pairs of eighth notes ascending from G4 to E5.
- Staff 3:** Contains two measures of intervals:
 - Large 4ths: A sequence of pairs of eighth notes ascending from G4 to E5, with a larger interval between pairs.
 - 5ths: A sequence of pairs of eighth notes ascending from G4 to E5.

Chords

0 semisextal 0 semiquartal 0 semisextal symmetric 0 semiquartal symmetric 0 semisextal quadratic 0 semiquartal quadratic

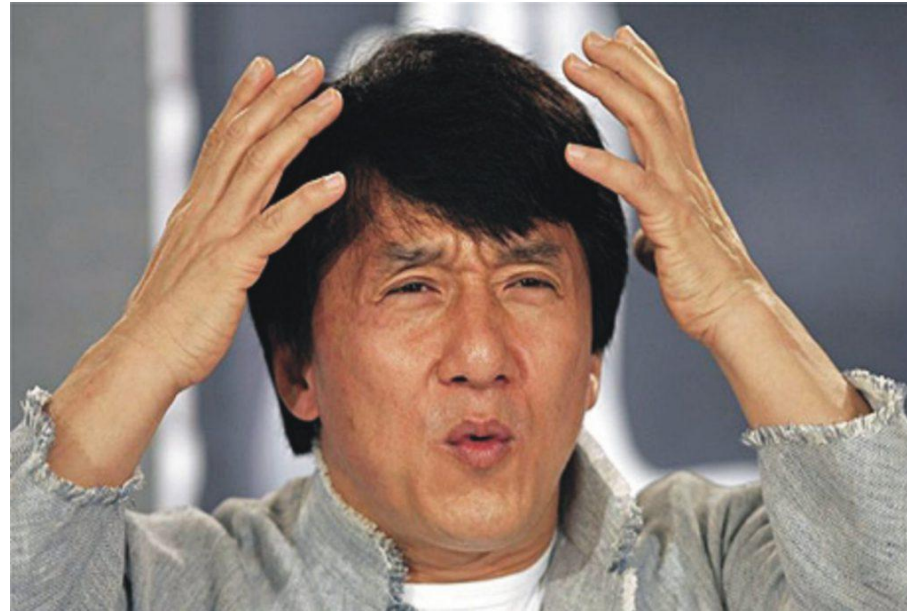
4

The image displays a musical staff with six chords. A green vertical line is positioned at the first chord, with the number '4' written to its left. Each chord is represented by a vertical line with a wavy, scribbled pattern. The labels above the chords are: '0 semisextal', '0 semiquartal', '0 semisextal symmetric', '0 semiquartal symmetric', '0 semisextal quadratic', and '0 semiquartal quadratic'. The staff is flanked by black bars at the top and bottom.

Decimal temperament

Tonic (one scale can have two tonics)

Intervals/notes (one note can have two different interpretations depending on scale)



Chords (one chord can have four different harmonic functions)

Decimal temperament

Tonic (one scale can have two tonics)

Intervals/notes (one note can have two different interpretations depending on scale)



Chords (one chord can have four different harmonic functions)

Decimal temperament

- Harmonic series

$$\begin{pmatrix} 2 & 0 & 3 & 4 & 7 & 9 & 5 & 10 & 9 \\ 0 & 2 & 1 & 1 & 0 & -1 & 2 & -1 & 0 \end{pmatrix}$$

Hermite normal
form

Harmonic numbers	Note (θ as fundamental)
1, 2, 4, 8, 16	0
17	1
9, 18	2↓
19	2
5, 10, 20	3
21	4↓
11, 22	5
23	5
3, 6, 12, 24	6
13	7
7, 14	8
15	9↓

Decimal temperament

- Musical composition
 - Sibelius 8
 - NotePerformer
 - Organ manual
 - Organ pedal
 - French horn
 - Vibraphone
 - Strings (group)
 - Solo viola
 - Offtonic N-TET using 14-TET preset
- ‘Eternity’
 - Movement 1 – of eternal time (approx. 7’)
 - Movement 2 – Life and Knowledge (approx. 6’)
 - Movement 3 – void (approx. 7’)

Hear the difference!

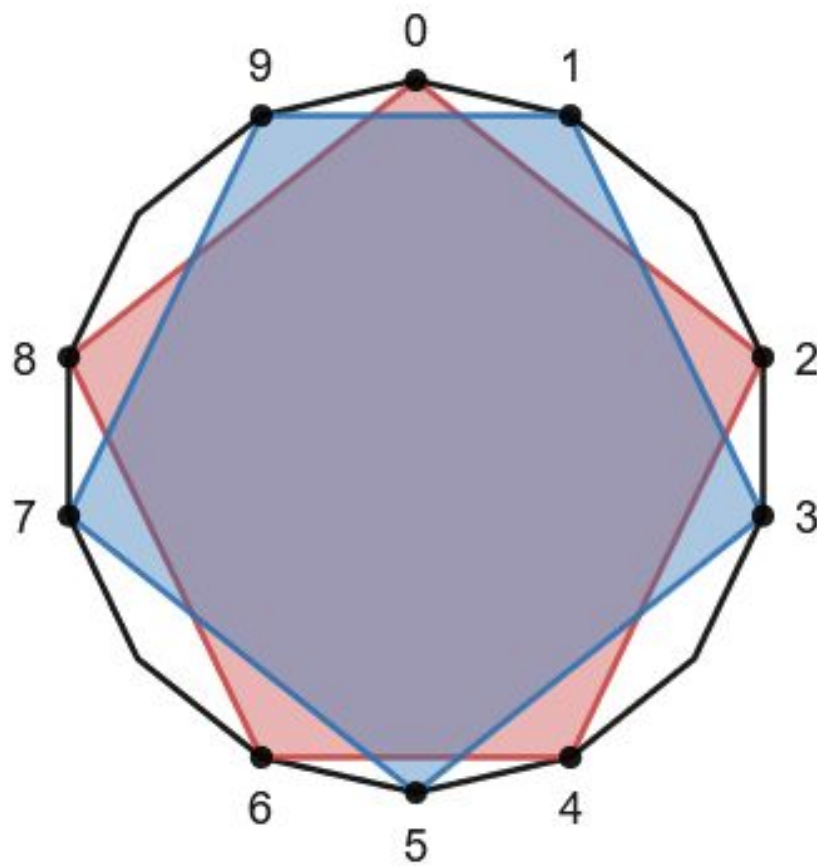
119 a tempo ♩ = 132

Musical score for piano, measures 119-124. The score is in 4/4 time and marked "a tempo" with a quarter note equal to 132 beats per minute. The piece is in 5/4 time signature. The score is written for four staves, numbered 2, 3, 4, and 5 from bottom to top. The bottom staff (2) is marked *pp* and *col 10mb*. The third staff (3) is marked *pp* and *p*. The fourth staff (4) is marked *pp*. The fifth staff (5) is marked *pp*. The score shows a series of notes and rests across the measures, with some notes beamed together and some notes marked with accents.

Significance of my research

- Moving ahead of mathematics
- Making an alternative framework for analysis and composition
- Using this as a framework to devise new systems
 - Rhythm
 - Timbre
 - Other musical parameters
- Using the sounds practically
- Extensions to higher limits?

Di-pentatonic scales?



Acknowledgements

- Organizers of Mathematical Encounters in Singapore: A Diderot Legacy (Prof Moreno Andreatta, Prof Carlos Agon, Prof Gérard Assayag, Prof Elaine Chew)
- Australian National University
- My supervisors: Prof Frank Millward (ANU), Dr Chloe Hobbs (ANU) and Prof Andrew Milne (WSU)
- Mr Adrian Mann
- My family and my cat

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Thank you!

‘Eternity’

Scan this QR code for the score-reader video of ‘Eternity’:



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