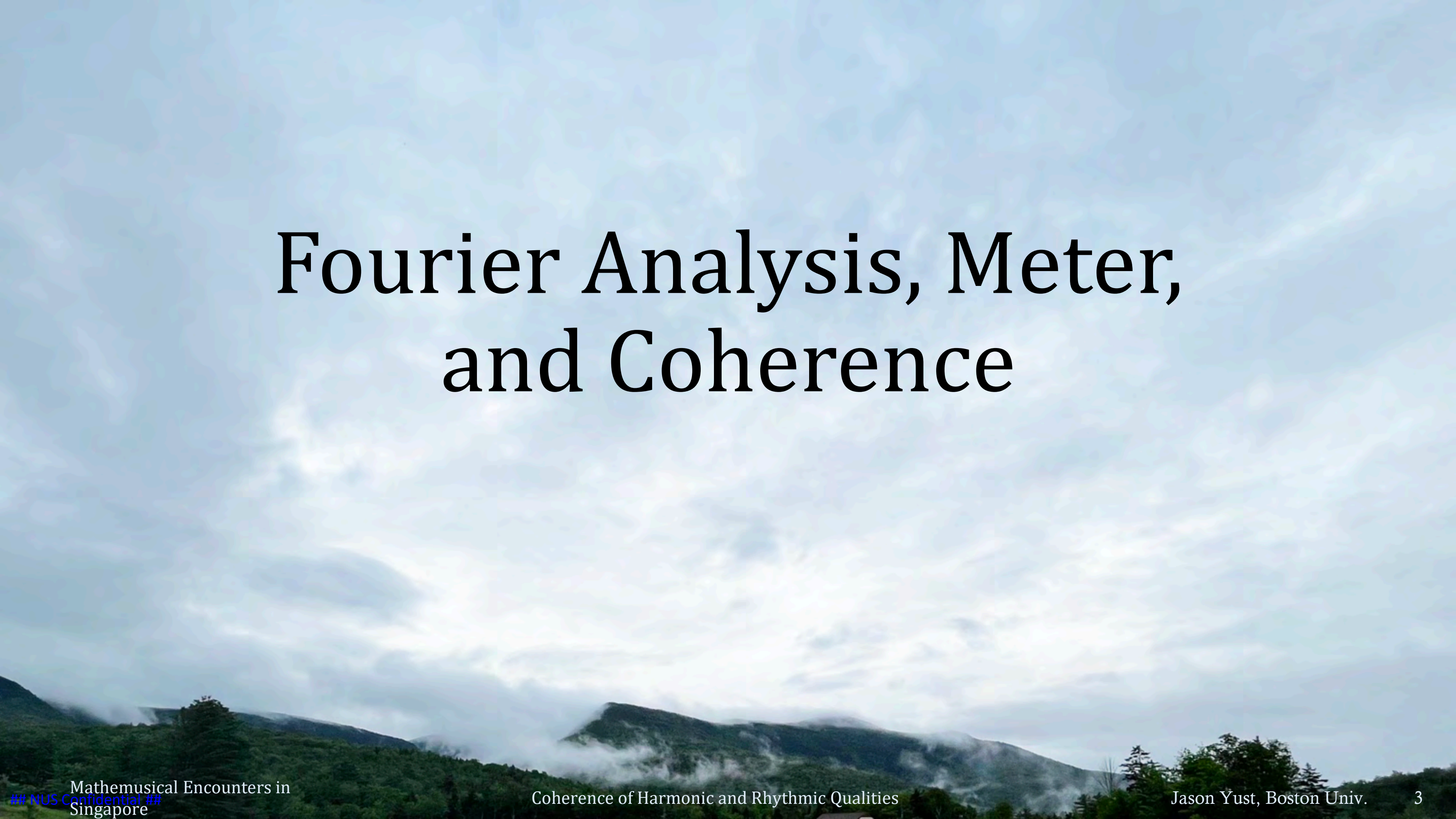
Coherence of Harmonic and Rhythmic Qualities

Jason Yust, BOSTON UNIVERSITY

Mathemusical Encounters in Singpore, Feb. 19, 2024

Outline

1. Fourier Analysis, Meter, and Coherence
   —Introducing coherence and coefficient products via concepts of metrical level and syncopation
2. Ragtime Syncopation
   —Statistical analysis of a corpus of ragtime rhythms
3. Maximal Evenness and Coherence, PC Distributions of Major and Minor Keys
   —Coherence theorem for maximally even sets
   —Balance and major/minor
   —Corpus data: Bach, Scarlatti, Debussy
4. Balinese Pelog
   —Toth data, coefficient products distinguish regional tunings
5. Analysis of an Adowa Performance
   —Coherence in a master drum (atumpan) performance



Fourier Analysis, Meter, and Coherence

Fourier Analysis and Meter

Can we equate **metrical level** with **Fourier coefficient**?

Yes and no . . .

Fourier Analysis and Meter

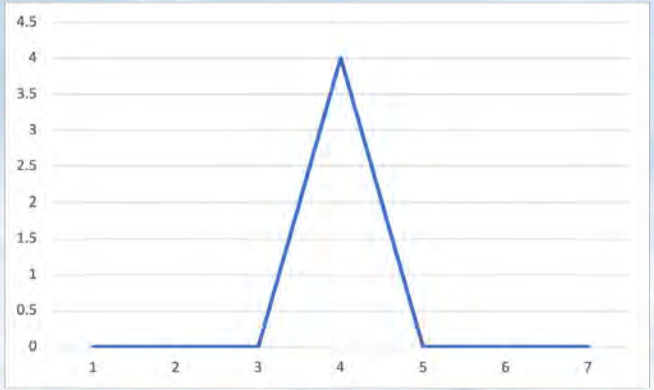
Can we equate **metrical level** with **Fourier coefficient**?

Similarities:

- Fourier prototypes (max. magnitude) are prototypes of metric level

q q q q :||

Spectrum:



Fourier Analysis and Meter

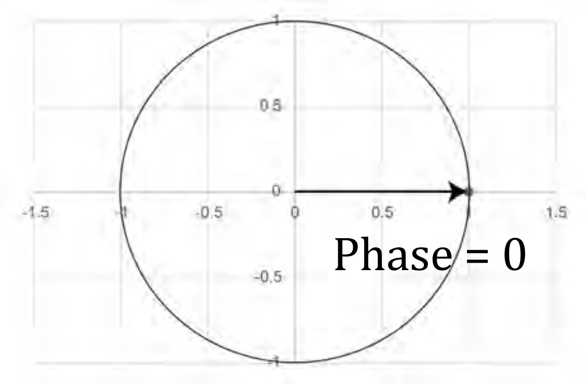
Can we equate **metrical level** with **Fourier coefficient**?

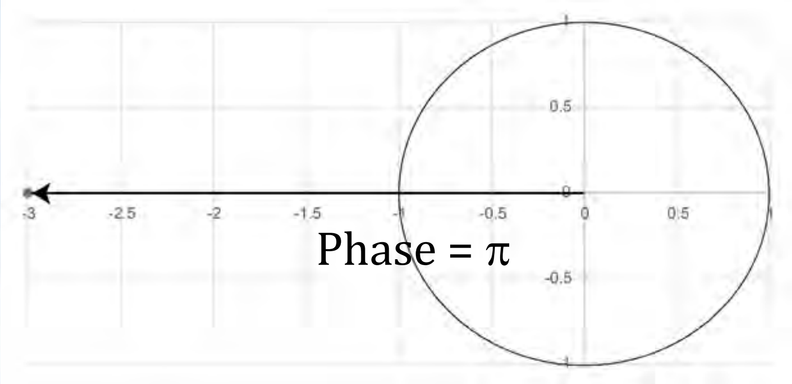
Similarities:

- Phase of Fourier coefficient corresponds to syncopation

f_1: f_4:

eq q q e:||





Fourier Analysis and Meter

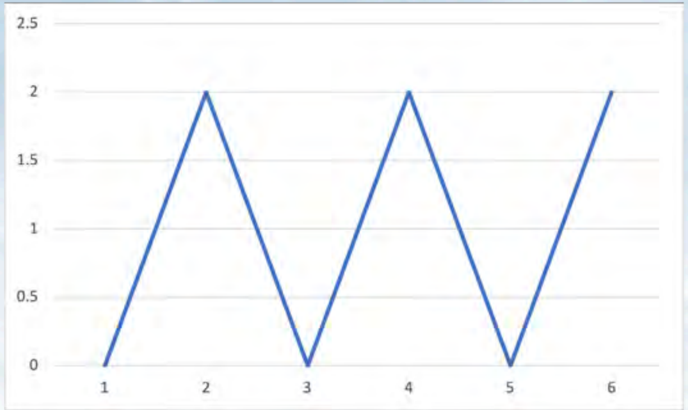
Can we equate **metrical level** with **Fourier coefficient**?

Differences:

- Prototype of metrical level *also* weights possible subdivisions

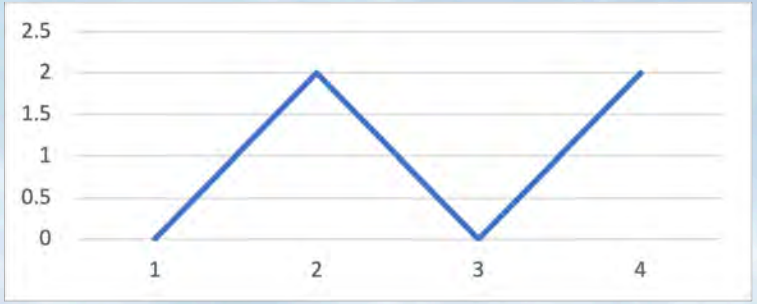
Spectrum:

h. h. :||



Coherence of Metrical Levels

Suppose you're given the following spectrum (for a measure of 8es):



What rhythm did it come from?

You'd probably guess this:

h h :||

But it could also be this:

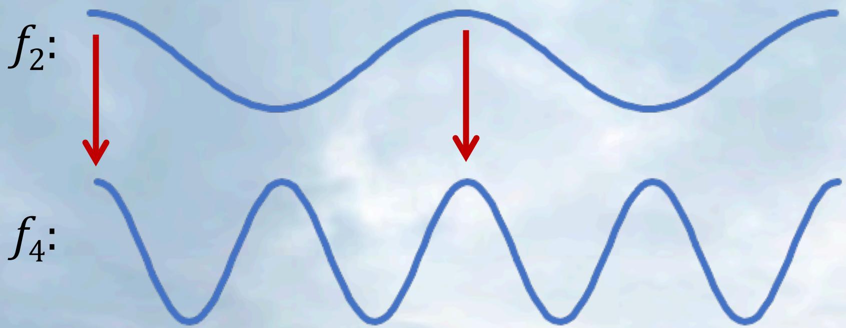
eq eeq e:||

Coherence of Metrical Levels

What's the difference?

Aligned coefficients:

h h :||

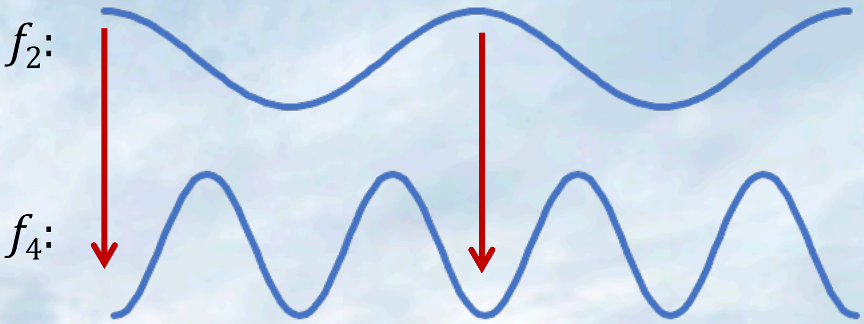


Peaks of f_2 lined up with peaks of f_4

⇒ Coherence

f_4 syncopated against f_2:

e q e e q e :||



Peaks of f_2 lined up with troughs of f_4

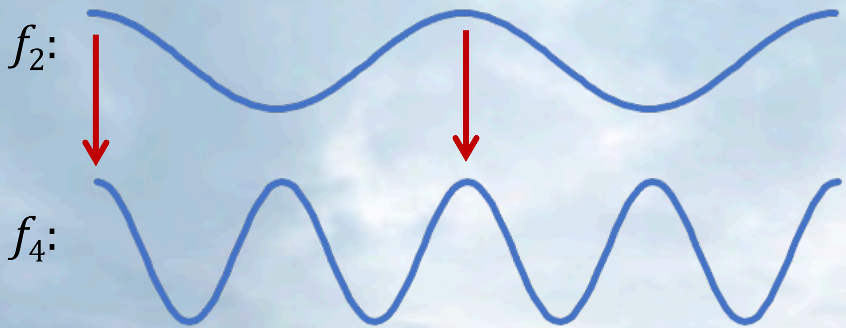
⇒ Anticoherence

Coherence of Metrical Levels

What's the difference?

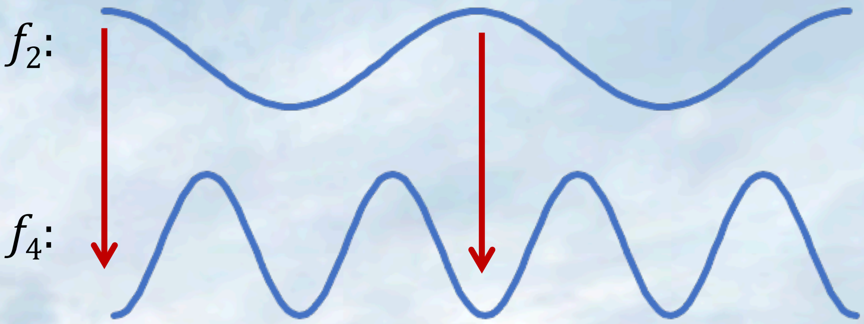
Aligned coefficients:

h h :||



f_4 syncopated against f_2:

e q e e q e :||



Coherence is a sum of phases:

$$2\varphi_2 - \varphi_4 = 0$$

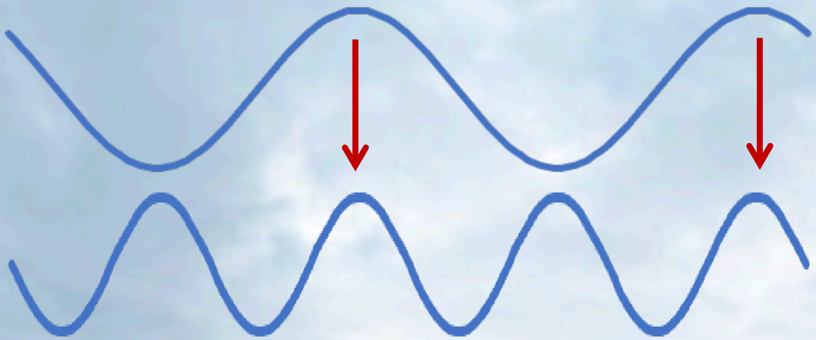
$$2\varphi_2 - \varphi_4 = \pi$$

Coherence of Metrical Levels

Coherence is **invariant w.r.t. translation**!

(q.) e(q. e(:||

f_2:



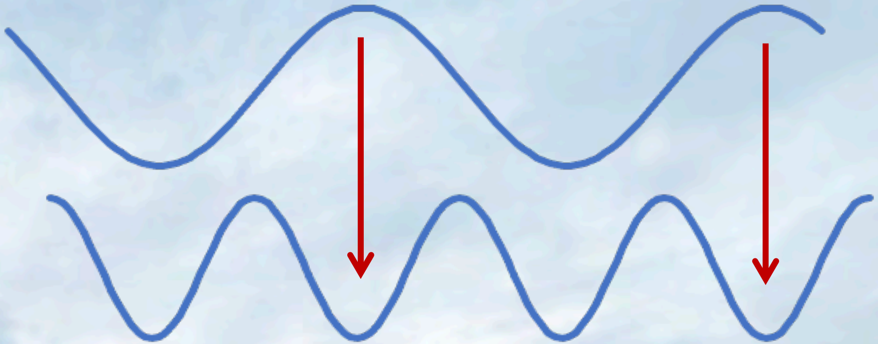
f_4:

Both coefficients syncopated by e

$$2\varphi_2 - \varphi_4 = 0$$

q e e q e e :||

f_2:



f_4:

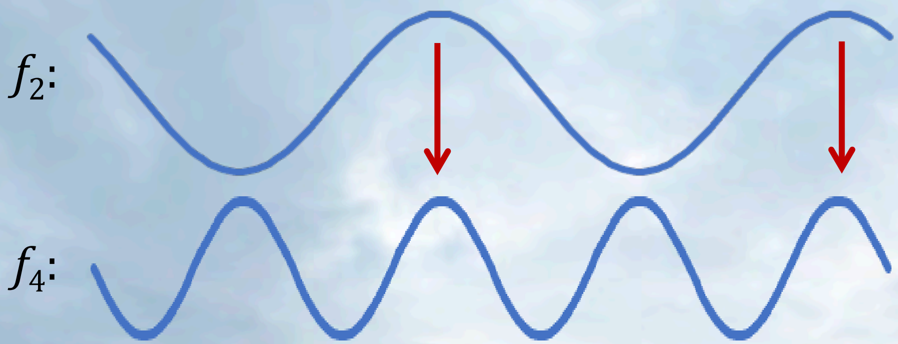
f_2 syncopated by e, f_4 unsyncopated

$$2\varphi_2 - \varphi_4 = \pi$$

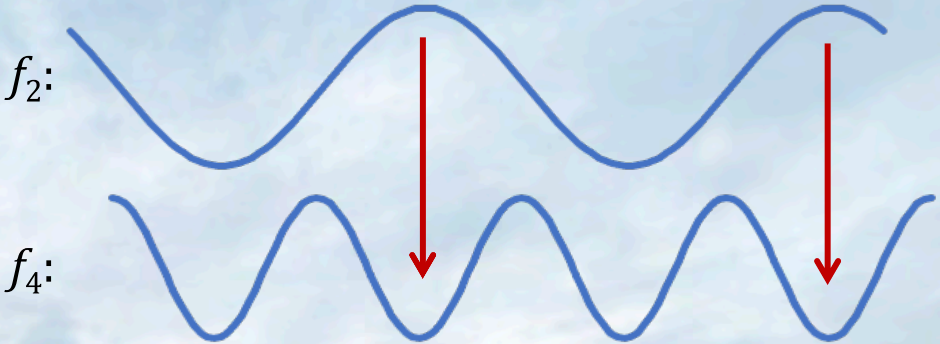
Coherence of Metrical Levels

Coherence is **invariant w.r.t. translation**!

(q.) e(q. e(:||



q e e q e e :||



Notice also that the first rhythm is **2-in-8 maximally even** . . .

. . . while the second is **6-in-8 maximally even**

Coherence

Coherence is the **phase of a product of Fourier coefficients**,

$$f_a \times f_b \times \overline{f_c} \ ,$$

$\varphi_{a \cdot b/c}$, where $a + b = c$.

Properties:

- $\varphi_{a \cdot b/c}$ is **transposition invariant** but depends only on the phases, not the magnitudes of the Fourier coefficients (independent of spectrum)
- **Inversion** of a pcset conjugates the coefficient product, so:
  $\varphi_{a \cdot b/c}(\mathrm{I}(A)) = -\varphi_{a \cdot b/c}(A)$
- **Complementation** of a pcset rotates the product 180°:
  $\varphi_{a \cdot b/c}(\mathrm{Complement}(A)) = \varphi_{a \cdot b/c}(A) + \pi$

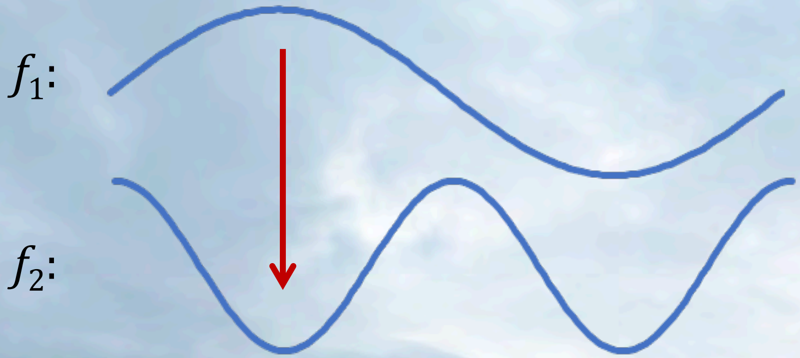
See Yust & Amiot 2022

Homometry

Coherence can distinguish homometric sets.

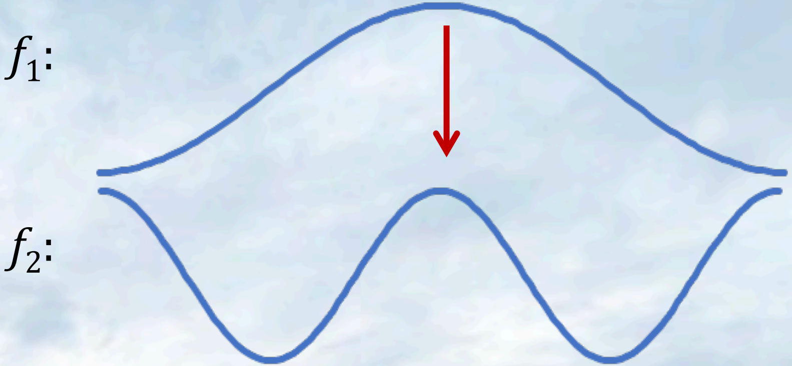
Spectrum, $(|f_1|, |f_2|, |f_3|, |f_4|) = (1.41, 2, 1.41, 0)$

e q   e h   :||



Anticoherent: $\varphi_{1\cdot 1/2} = \pi$

q.   e e q.   :||



Coherent: $\varphi_{1\cdot 1/2} = 0$

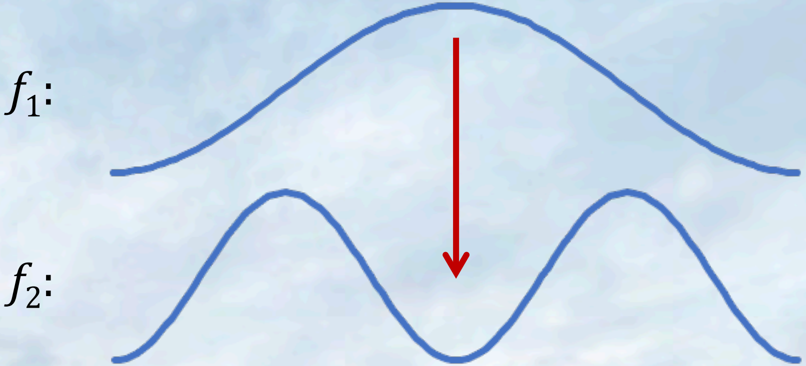
Homometry

Homometric multisets (> = ×2), Spectrum = (1.41, 2, 1.41, 4)

q e e e e q :||
\> \>

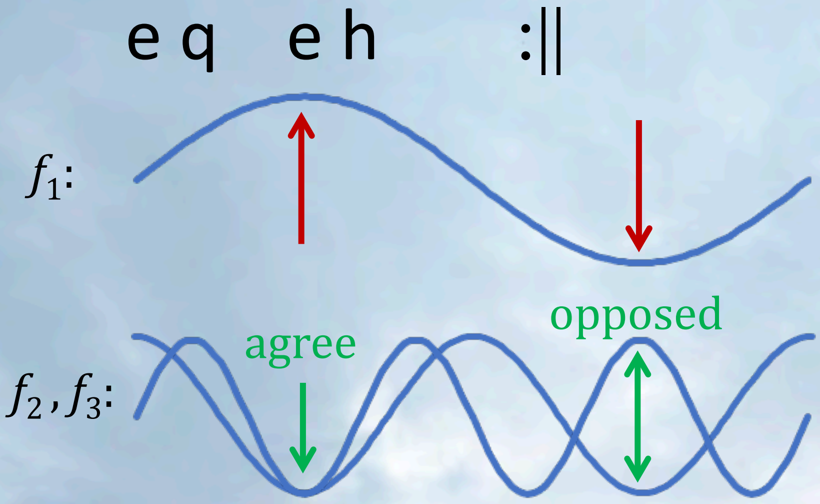
Coherent: $\varphi_{1\cdot 1/2} = 0$

q e e e e q :||
\> \>



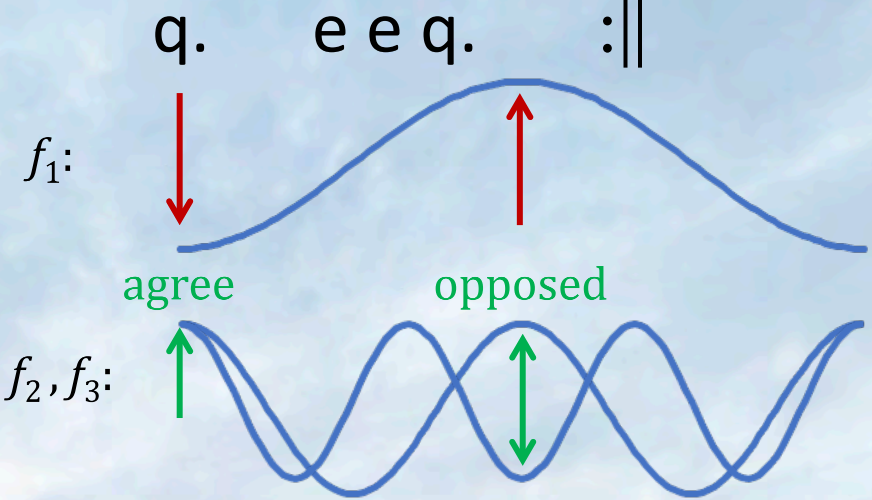
Anticoherent: $\varphi_{1\cdot 1/2} = \pi$

Products of three coefficients



Coherent: $\varphi_{1 \cdot 2/3} = 0$

f_1 peaks where f_2 and f_3 agree,
f_1 troughs where f_2 and f_3 opposed.



Anticoherent: $\varphi_{1 \cdot 2/3} = \pi$

f_1 peaks where f_2 and f_3 opposed,
f_1 troughs where f_2 and f_3 agree.

Products of three coefficients

e q e h :||

Subset of *cinquillo* (5-in-8 ME):

e q e q q :||

(c = 8 – 5)

Coherent: $\varphi_{1\cdot 2/3} = 0$

q. e e q. :||

Superset of *tresillo* (3-in-8 ME):

q. q q. :||

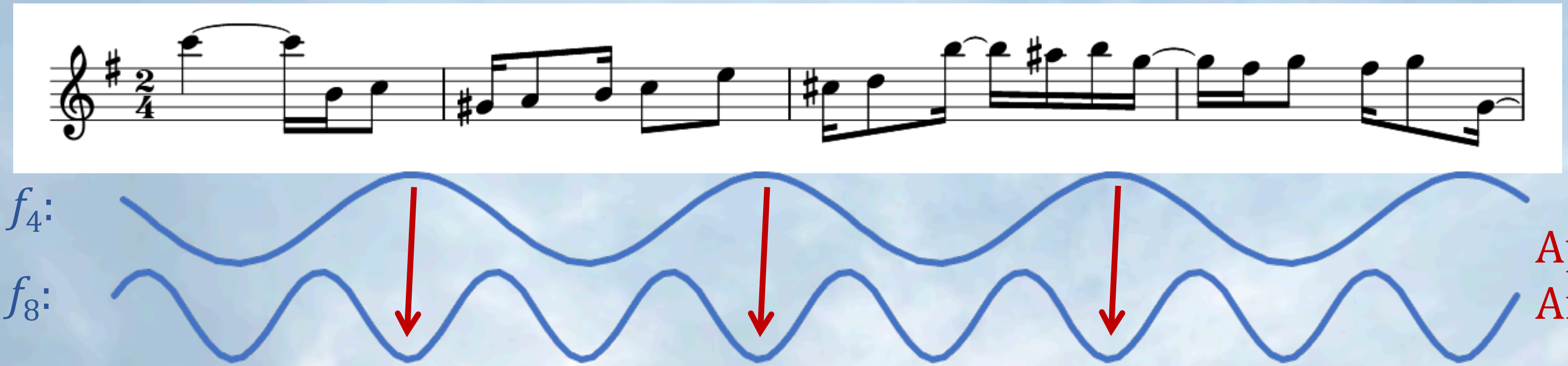
(c = 3)

Anticoherent: $\varphi_{1\cdot 2/3} = \pi$

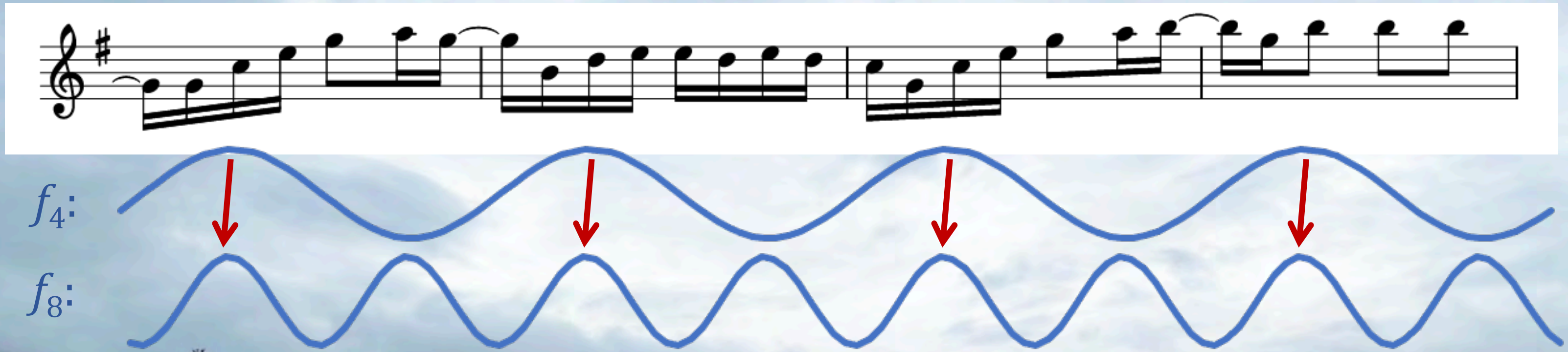
Ragtime Syncopation

Ragtime Syncopation

Second strain from Joplin's "Weeping Willow Rag," phrases 1 and 2



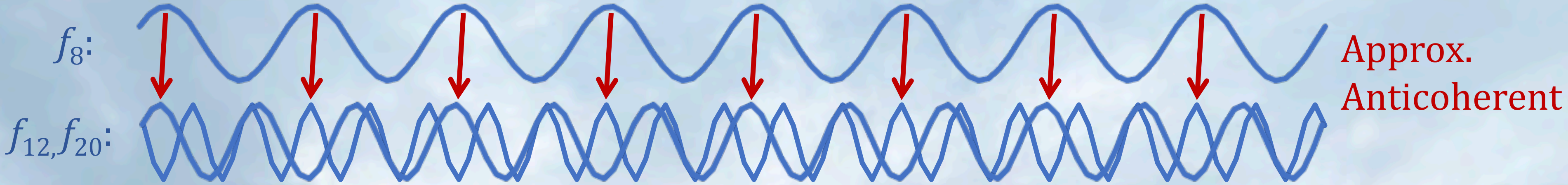
Approx.
Anticoherent



Approx.
Coherent

Ragtime Syncopation

Second strain from Joplin's "Weeping Willow Rag," phrases 1 and 2

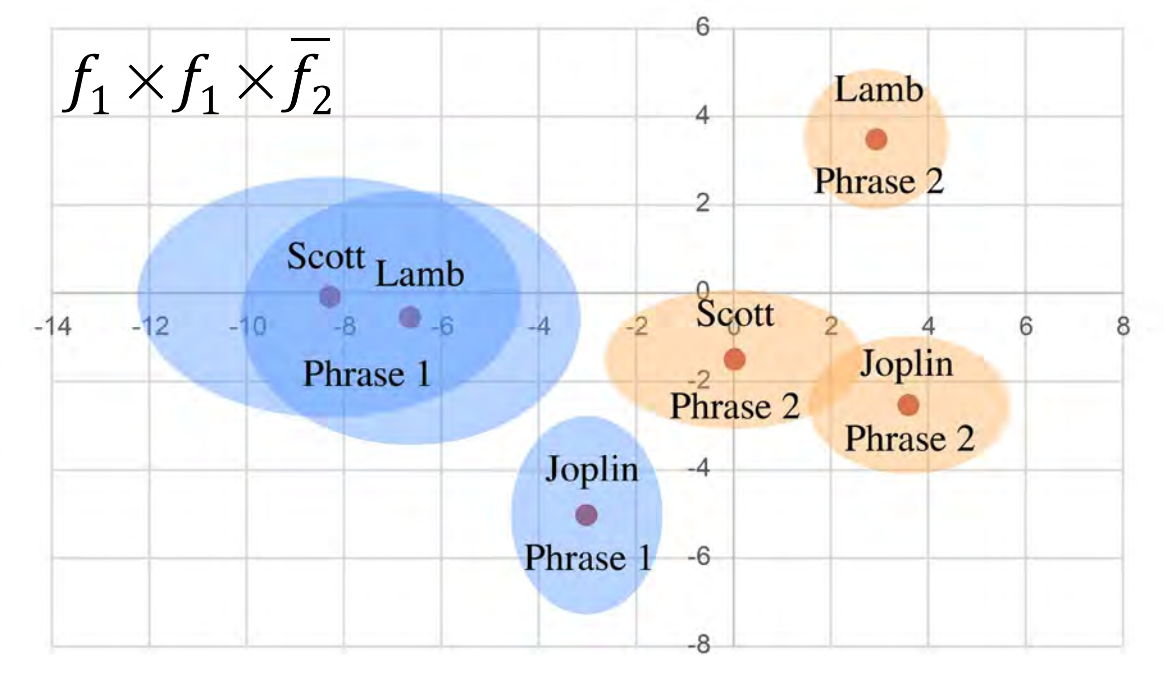


Anticoherence of $f_2 \times f_3 \times \overline{f_5}$ is a feature of the cinquillo (5-in-8 ME) rhythm that distinguishes it from the spectrally indistinguishable tresillo (3-in-5 ME) rhythm.

The tresillo has a coherent $f_2 \times f_3 \times \overline{f_5}$.

Ragtime Syncopation

Descriptive stats. for 71 pieces (4–5 strains each) by Scott Joplin, James Scott, Joseph Lamb

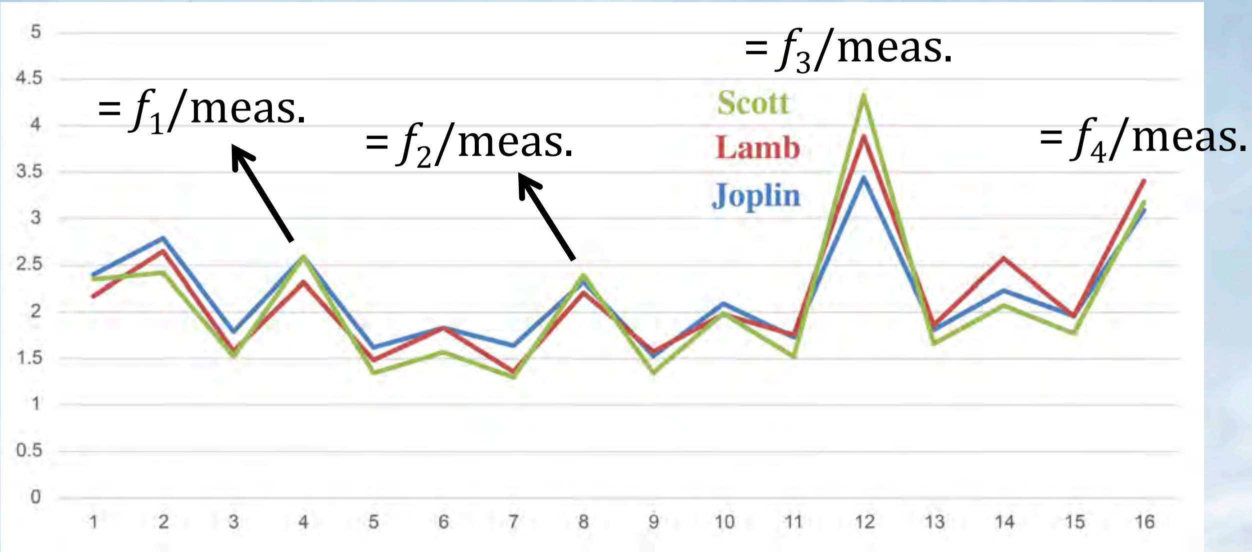


All of the composers tend to anticoherent w and h levels in phrase 1, and coherent in phrase 2.

Ovals show standard errors

Ragtime Syncopation

Averaged spectra over 4-measure phrases

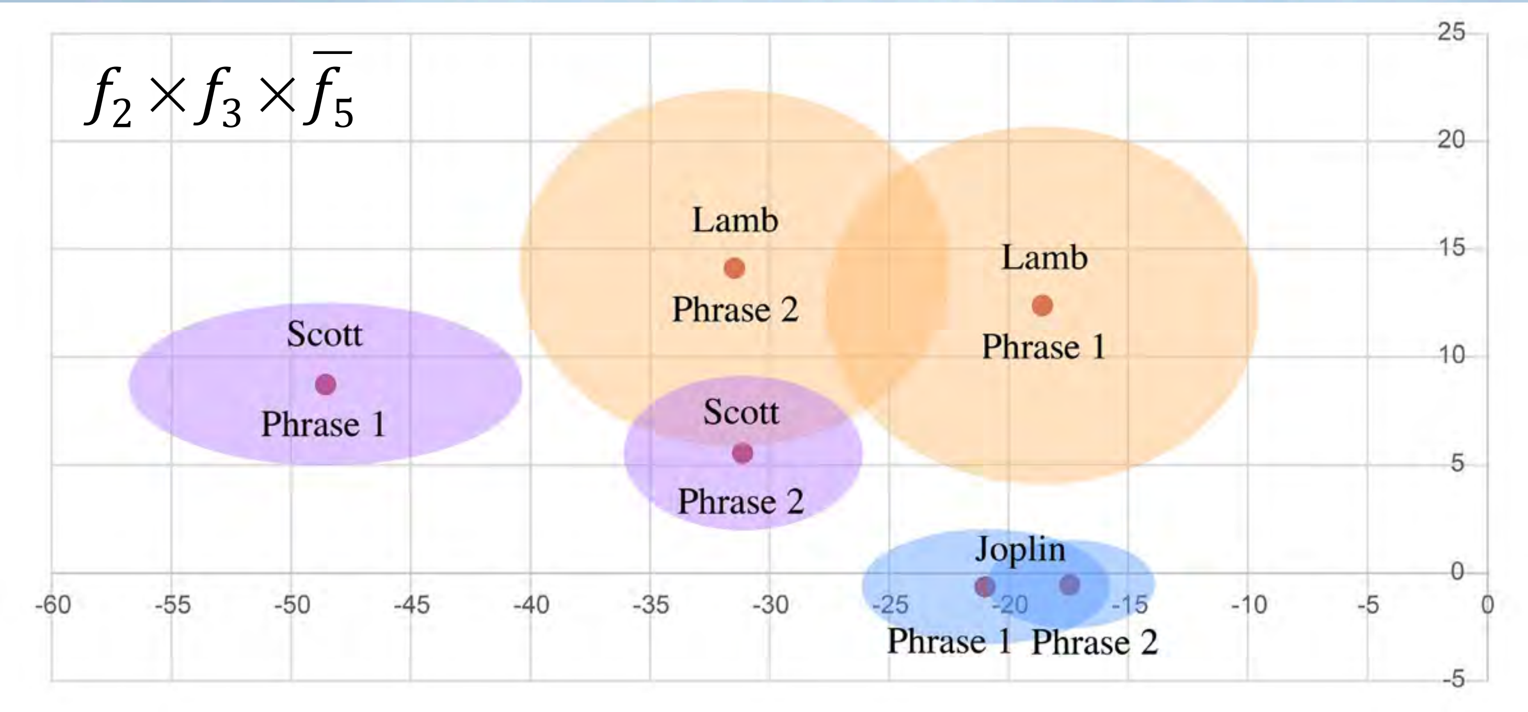


Ragtime melodies consistently emphasize f_3 (f_{12} across 4-meas. phrase)
Tresillo and *cinquillo* rhythms are prototypes of this quality.

from Yust & Kirlin 2021

Ragtime Syncopation

Descriptive stats. for 71 pieces (4–5 strains each) by Scott Joplin, James Scott, Joseph Lamb



All phrases are characterized by strong anticoherence of $f_2 \times f_3 \times \overline{f_5}$ (more like cinquillo than tresillo)

Ovals show standard errors

Maximal Evenness and Coherence

Maximal Evenness and Coherence

Theorem: Let A be c-in-d maximally even (c,d coprime).

Then $\varphi_{a\cdot b/c}(A) = \pi$. —proven recently by E. Amiot

Note, this implies that, for the (d – c)-in-d ME complement,

$\varphi_{a\cdot b/c}(\text{compl}(A)) = 0$.

Example: Cinquillo (5-in-8 ME): $\varphi_{1\cdot 2/3}(A) = 0$, $\varphi_{2\cdot 3/5}(A) = \pi$.
Tresillo (3-in-8 ME): $\varphi_{1\cdot 2/3}(A) = \pi$, $\varphi_{2\cdot 3/5}(A) = 0$.

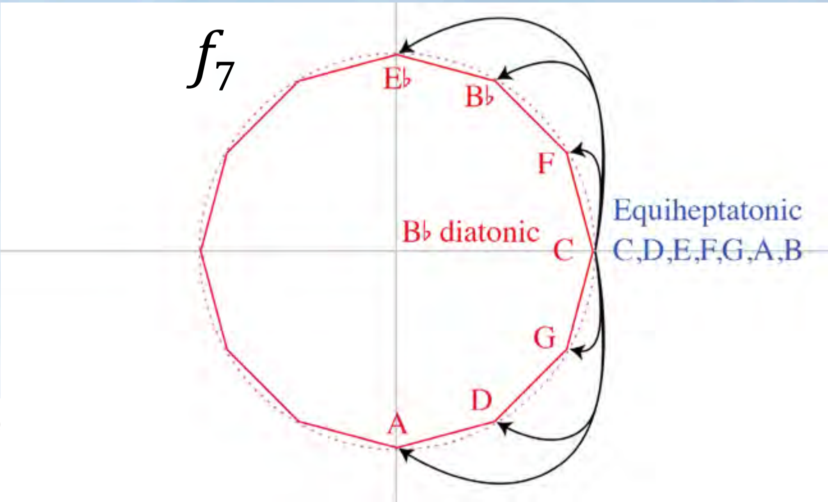
Example: Diatonic (7-in-12 ME): $\varphi_{2\cdot 3/5}(A) = 0$, $\varphi_{3\cdot 4/7}(A) = \pi$.
Pentatonic (5-in-12 ME): $\varphi_{2\cdot 3/5}(A) = \pi$, $\varphi_{3\cdot 4/7}(A) = 0$.

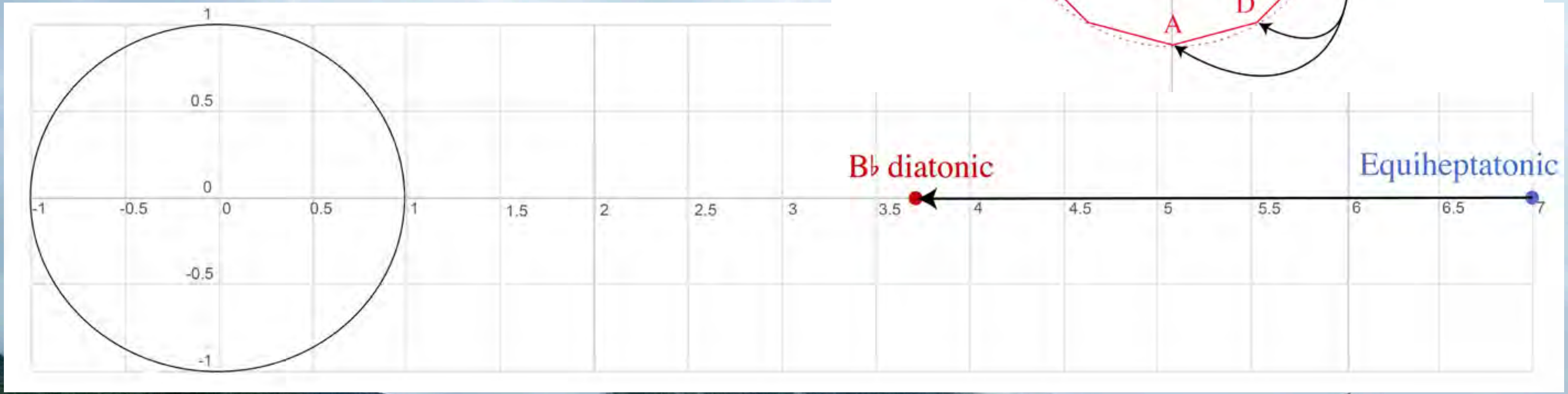
Maximal Evenness and Coherence

A diatonic scale is a symmetrical displacement of an equiheptatonic one.

All equiheptatonic pcs are at $\varphi_7 = 0$.

The displacement is not large enough to change the $\varphi_7 = 0$ value.





Maximal Evenness and Coherence

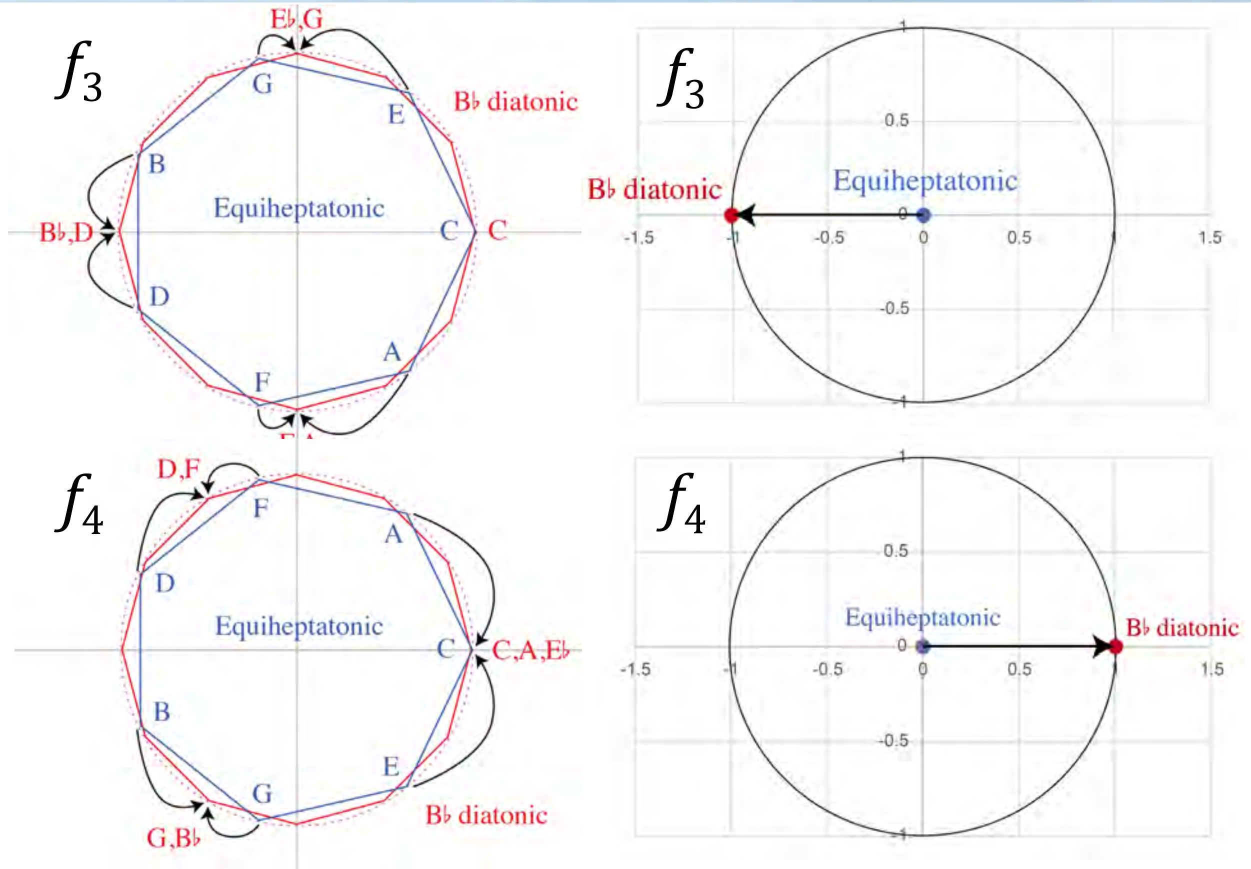
For coefficients $a + b = 7$, positions of equiheptatonic pcs are reflected over real line.

The displacement has opposite effects, so that one of φ_a, φ_b is 0 and the other is π.

$\Rightarrow \varphi_{a \cdot b/7} = 0 + \pi + 0 = \pi$

e.g.:

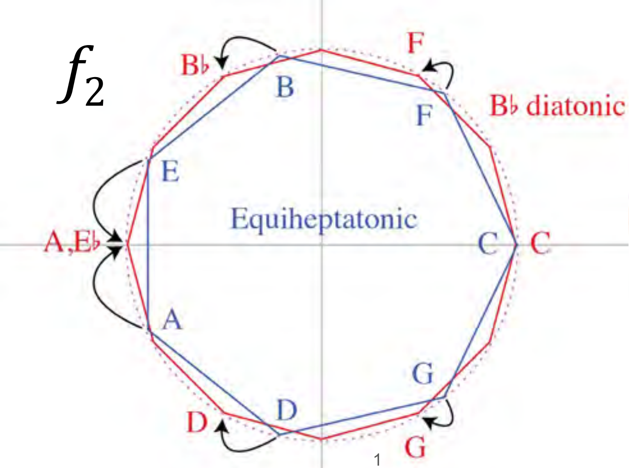
$\varphi_{3 \cdot 4/7} = \pi$

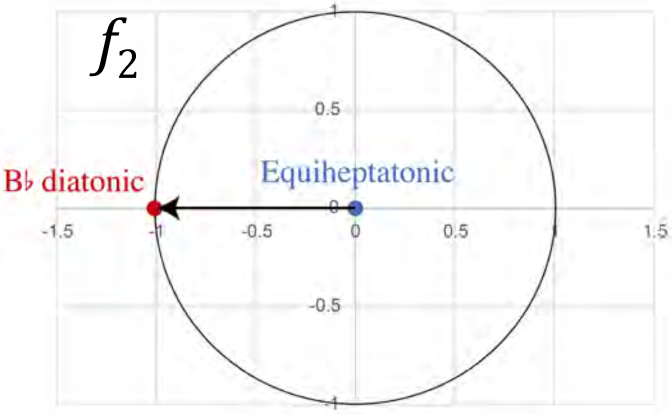


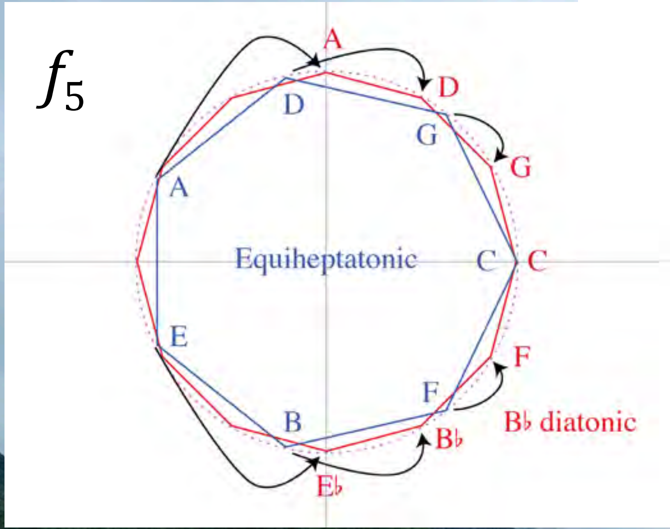
Maximal Evenness and Coherence

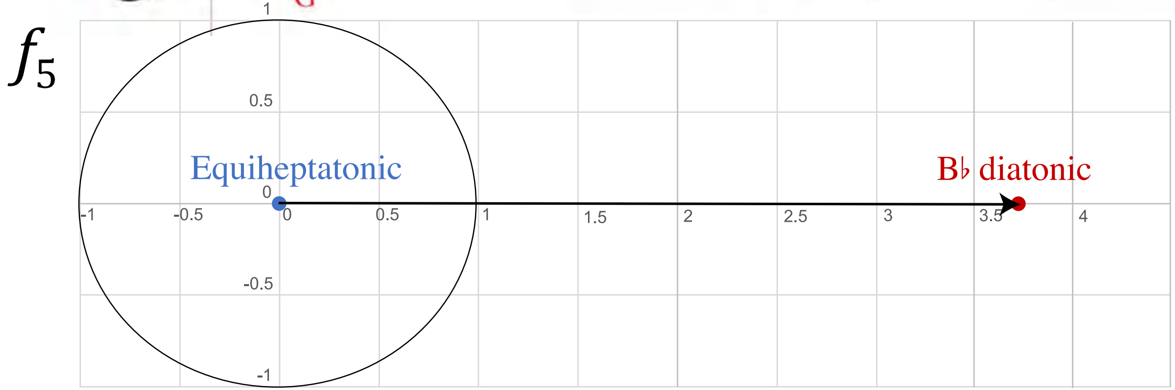
Similarly

$\varphi_{2\cdot5/7} = \pi$









Diatonic vs. Pentatonic

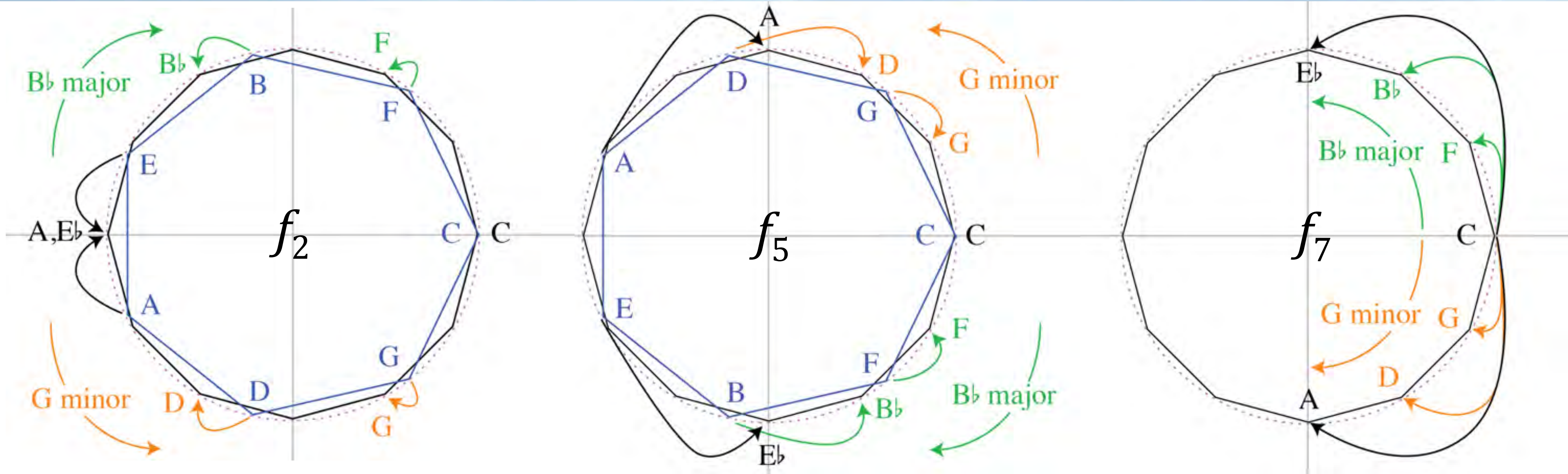
Diatonic and pentatonic collections have equivalent spectra, emphasizing f_5/f_7.

They are distinguishable by coherence:

Diatonic:	Pentatonic:
$\varphi_{1\cdot4/5} = 0$	$\varphi_{1\cdot4/5} = \pi$
$\varphi_{2\cdot3/5} = 0$	$\varphi_{2\cdot3/5} = \pi$
$\varphi_{1\cdot6/7} = \pi$	$\varphi_{1\cdot6/7} = 0$
$\varphi_{2\cdot5/7} = \pi$	$\varphi_{2\cdot5/7} = 0$
$\varphi_{3\cdot4/7} = \pi$	$\varphi_{3\cdot4/7} = 0$

Major vs. Minor

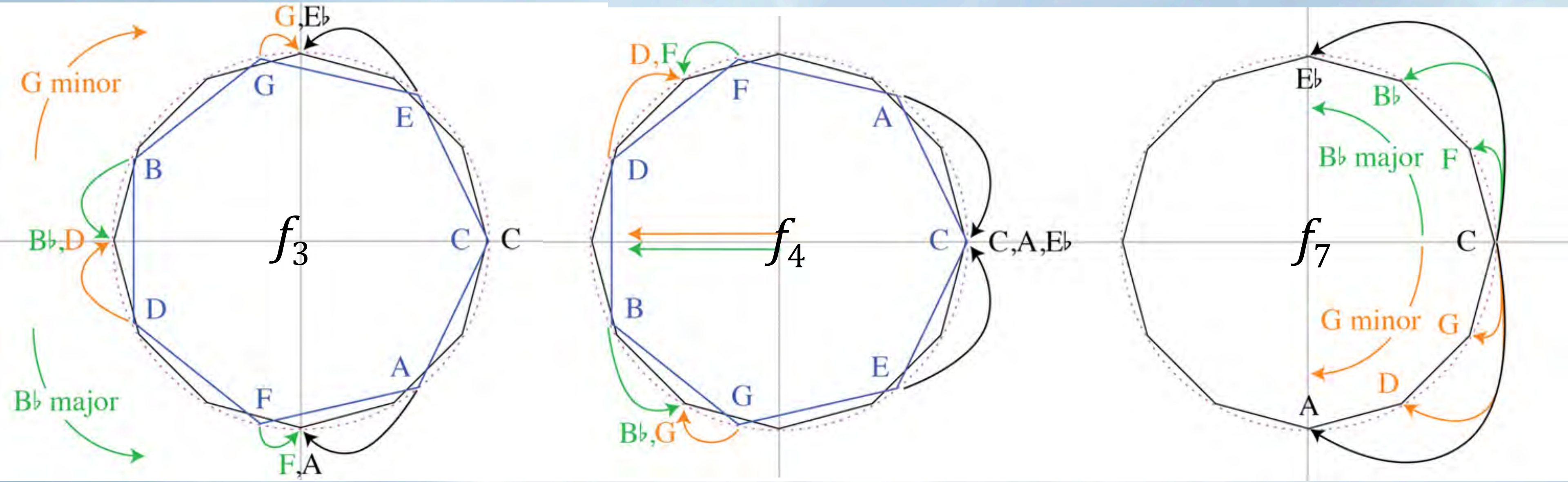
The major and minor tonic–fifth are precisely opposite the balance point of the diatonic.



For $\varphi_{2\cdot5/7}$ the effects reinforce one another (lower φ_2, φ_5, higher φ_7 for major).

Major vs. Minor

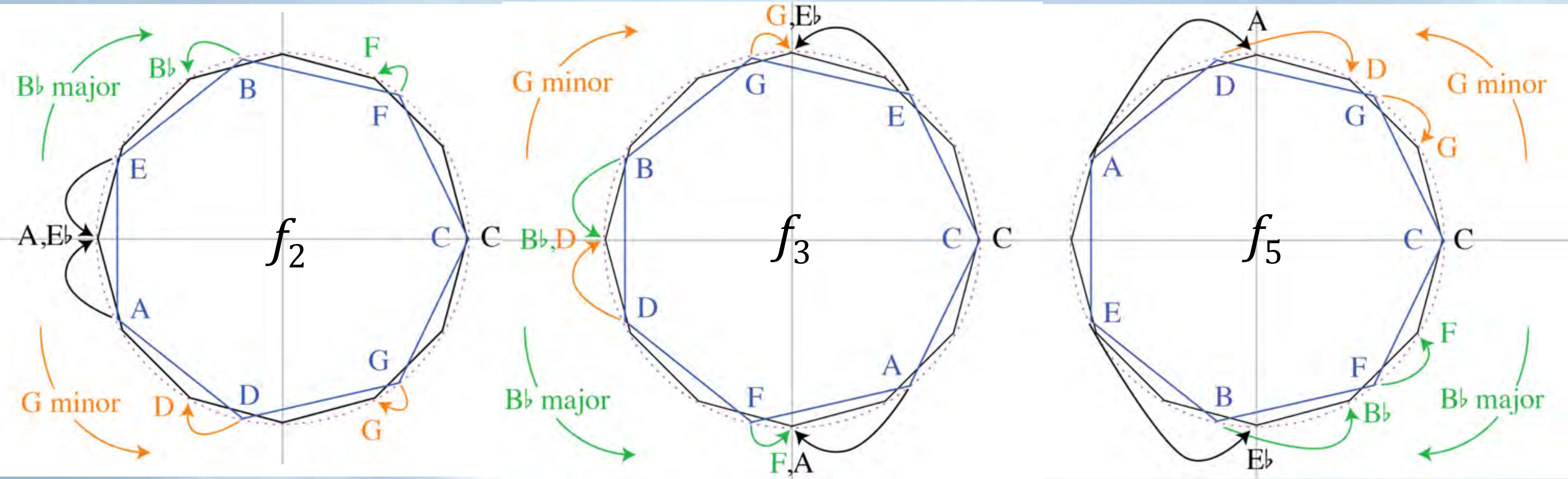
The major and minor tonic–fifth are precisely opposite the balance point of the diatonic.



Mode effects are less pronounced for $\varphi_{3\cdot4/7}$ (higher φ_3 for major counteracts higher φ_7).

Major vs. Minor

The major and minor tonic–fifth are precisely opposite the balance point of the diatonic.



For $\varphi_{2\cdot3/5}$, effects of φ_2 partially counteract φ_3 and φ_5.

Corpus data: Bach partitas

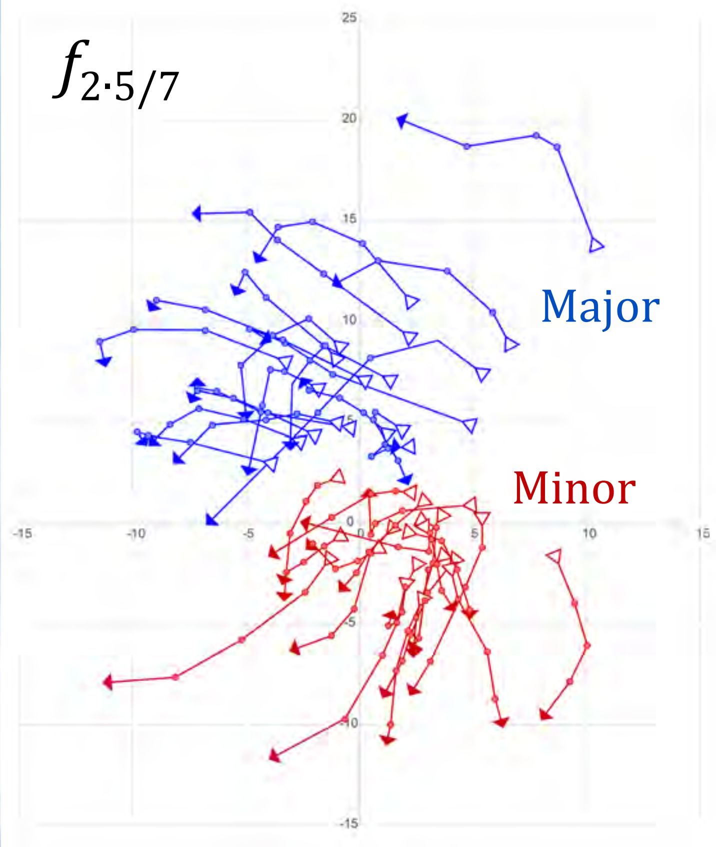
All movements of Bach's 6 Partitas (3 major, 3 minor) on the complex plane for $f_{2\cdot5/7}$ (normalized for share of power)

Modes are clearly distinct.

Arrows point from smaller (white arrow) to larger (Filled arrow) window sizes.

With increasing window size in both modes:

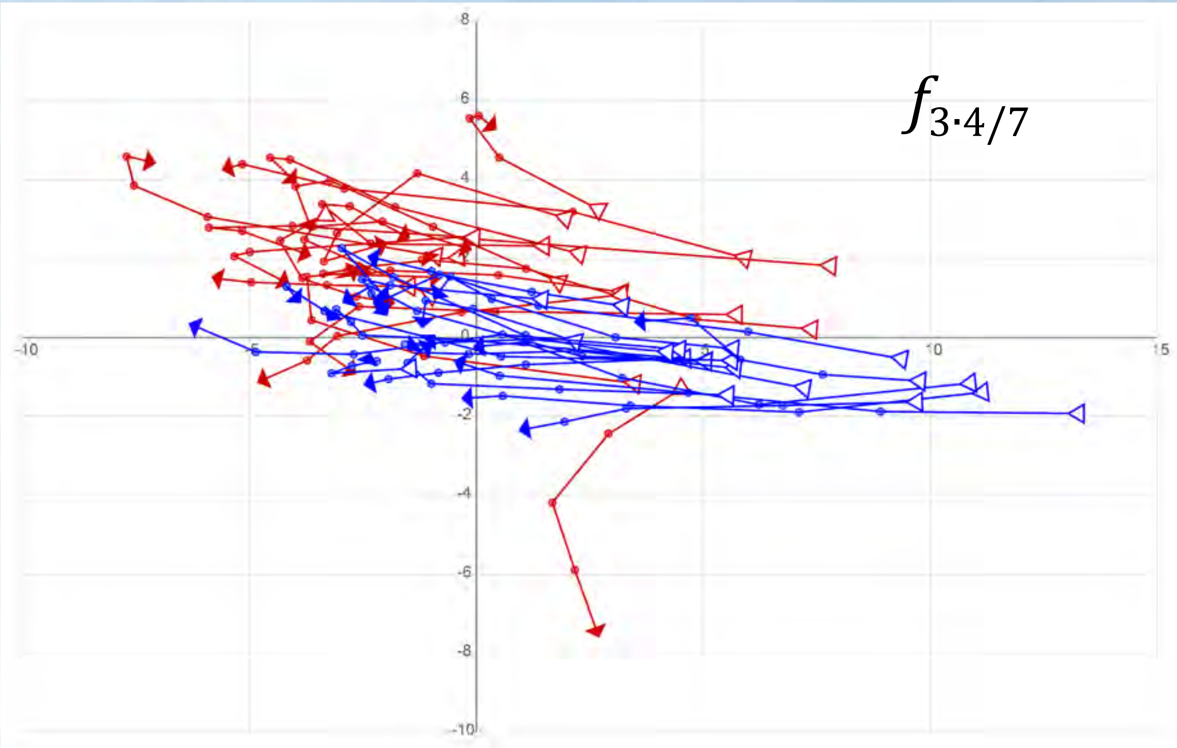
- Magnitudes get larger,
- Phase moves towards π.



Corpus data: Bach partitas

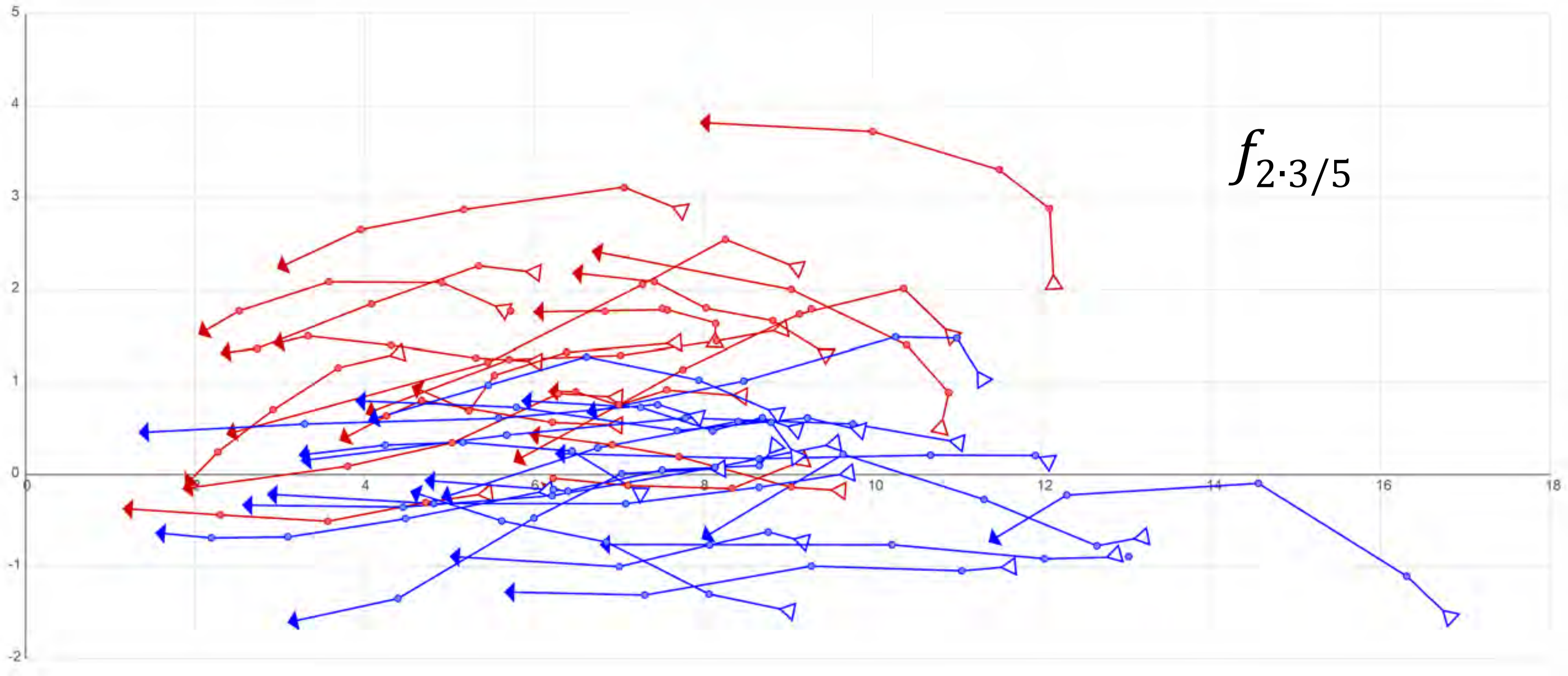
$f_{3\cdot 4/7}$ also differentiates modes, but less consistently.

Most pieces go from coherent at smaller windows to anticoherent at larger windows.



Corpus data: Bach partitas

$f_{2\cdot3/5}$ also differentiates modes, and is coherent at all window sizes.



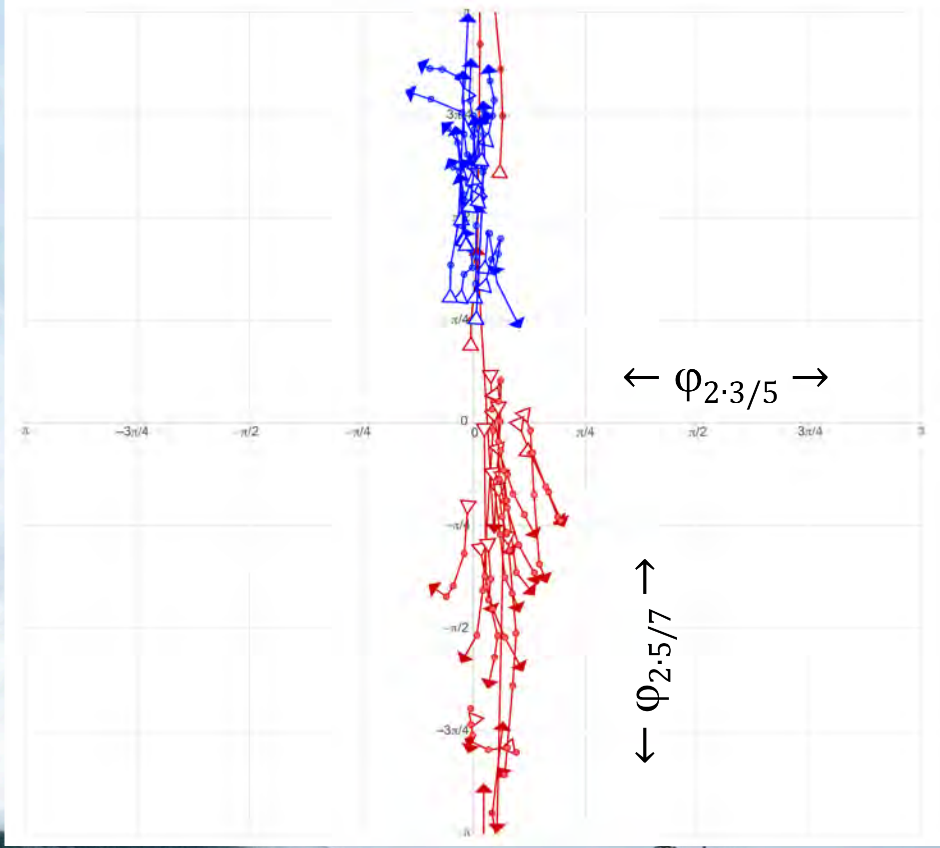
Corpus data: Bach partitas

A phase space on $\varphi_{2\cdot 3/5} \times \varphi_{2\cdot 5/7}$ shows modal and diatonic/pentatonic contrasts.

The highly restricted range of $\varphi_{2\cdot 3/5}$ shows that the repertoire is exclusively diatonic.

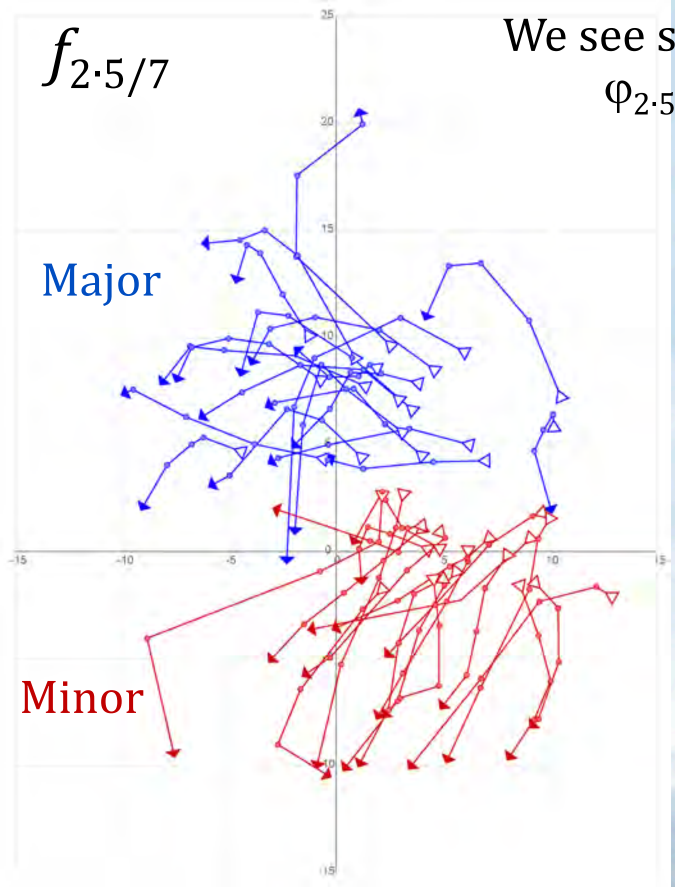
Mode differences are evident in the positive and negative values of $\varphi_{2\cdot 5/7}$.

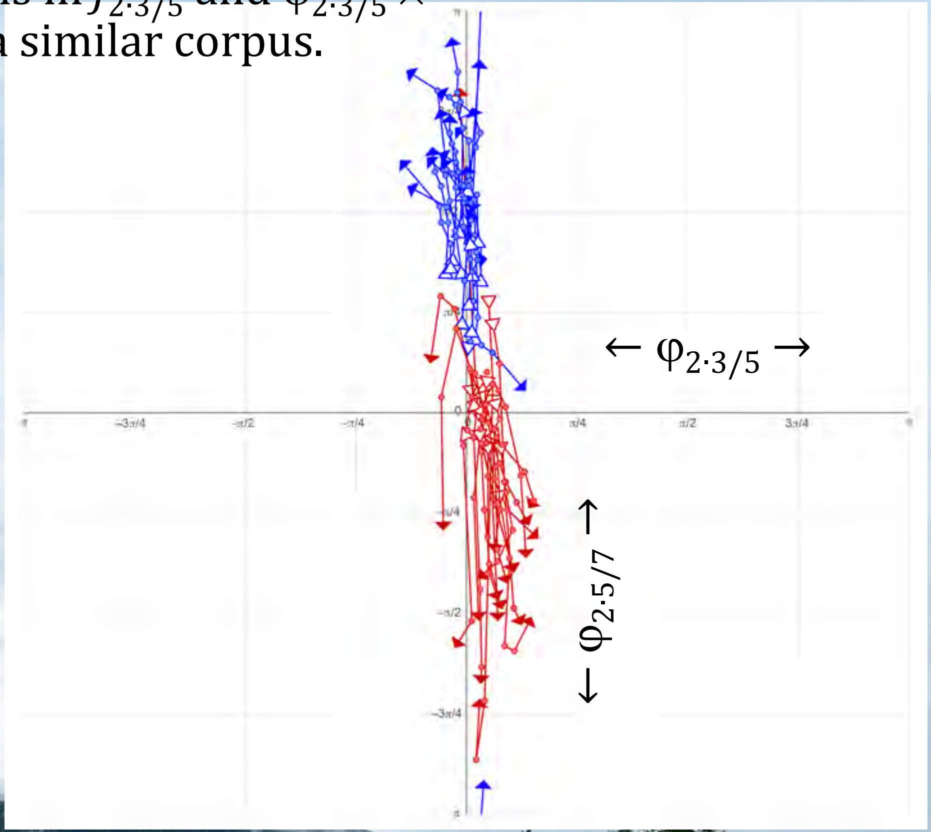
The tendency for larger windows to be less coherent is also evident in $\varphi_{2\cdot 5/7}$.



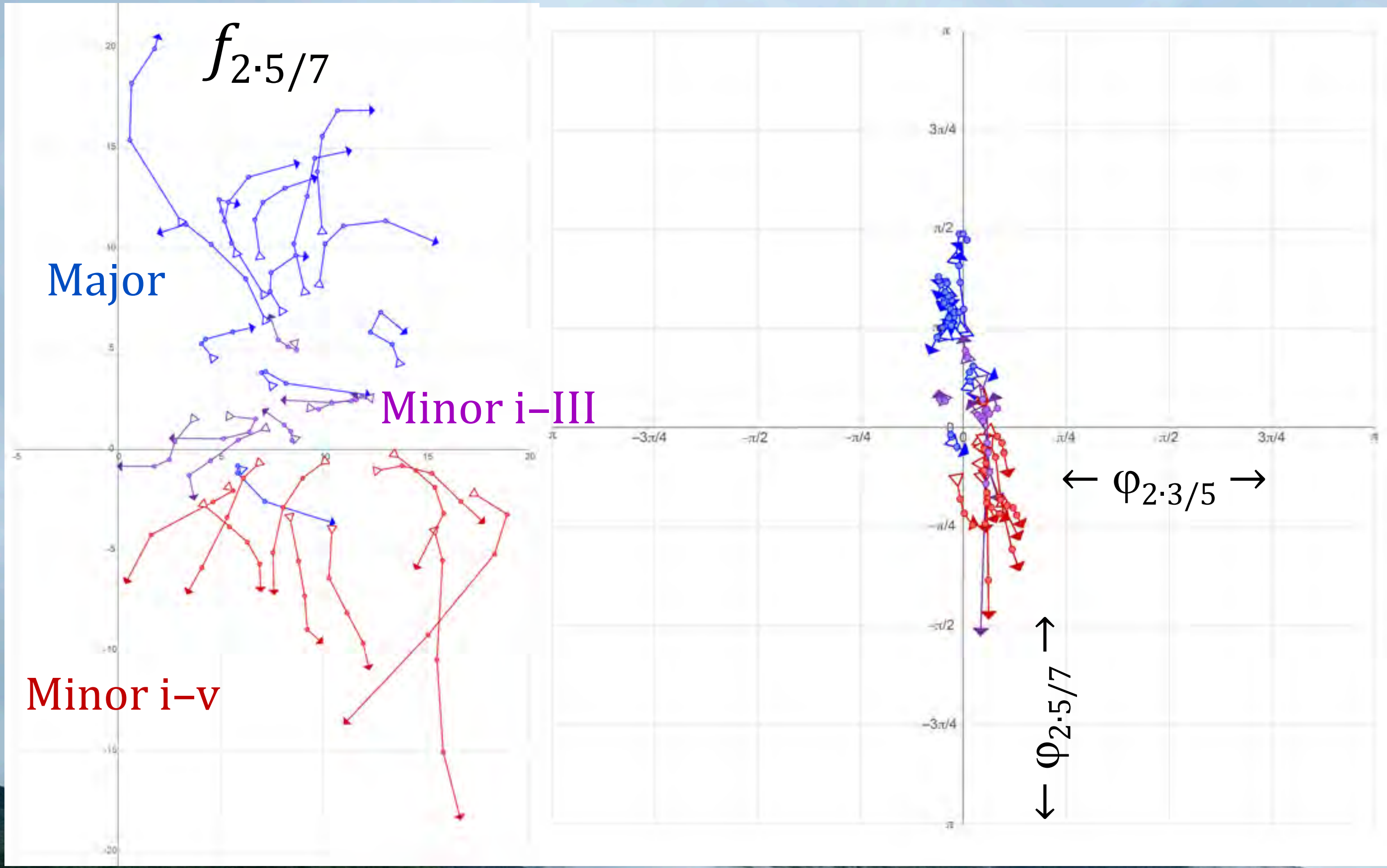
Corpus data: Bach French Suites

We see similar patterns in $f_{2\cdot3/5}$ and $\varphi_{2\cdot3/5} \times \varphi_{2\cdot5/7}$ spaces for a similar corpus.





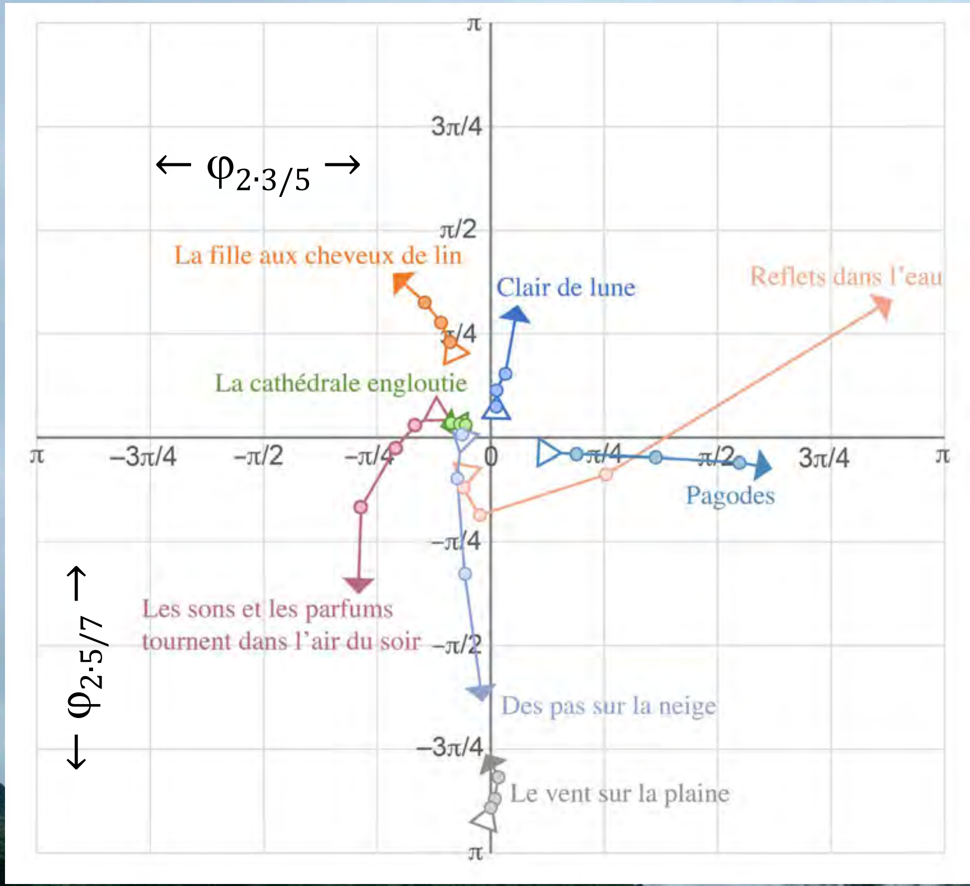
Corpus data: Scarlatti sonatas



In a corpus of Scarlatti sonatas the use of minor dominant vs. relative major secondary keys has an influence (also parallel minor in secondary themes).

The range of variation of $\varphi_{2\cdot5/7}$ is lower than for Bach, but still much more than $\varphi_{2\cdot3/5}$.

Corpus data: Debussy



Debussy's music uses all parts of the $\varphi_{2\cdot 3/5} \times \varphi_{2\cdot 5/7}$ space. Some pieces reflect diatonic organization and traditional major–minor distinctions, while it reveals pentatonic organization in pieces like "Reflets dans l'eau" and "Pagodes".

Balinese Pelog

Gamelan Kebyar Tunings: Toth data

Andrew Toth's Data

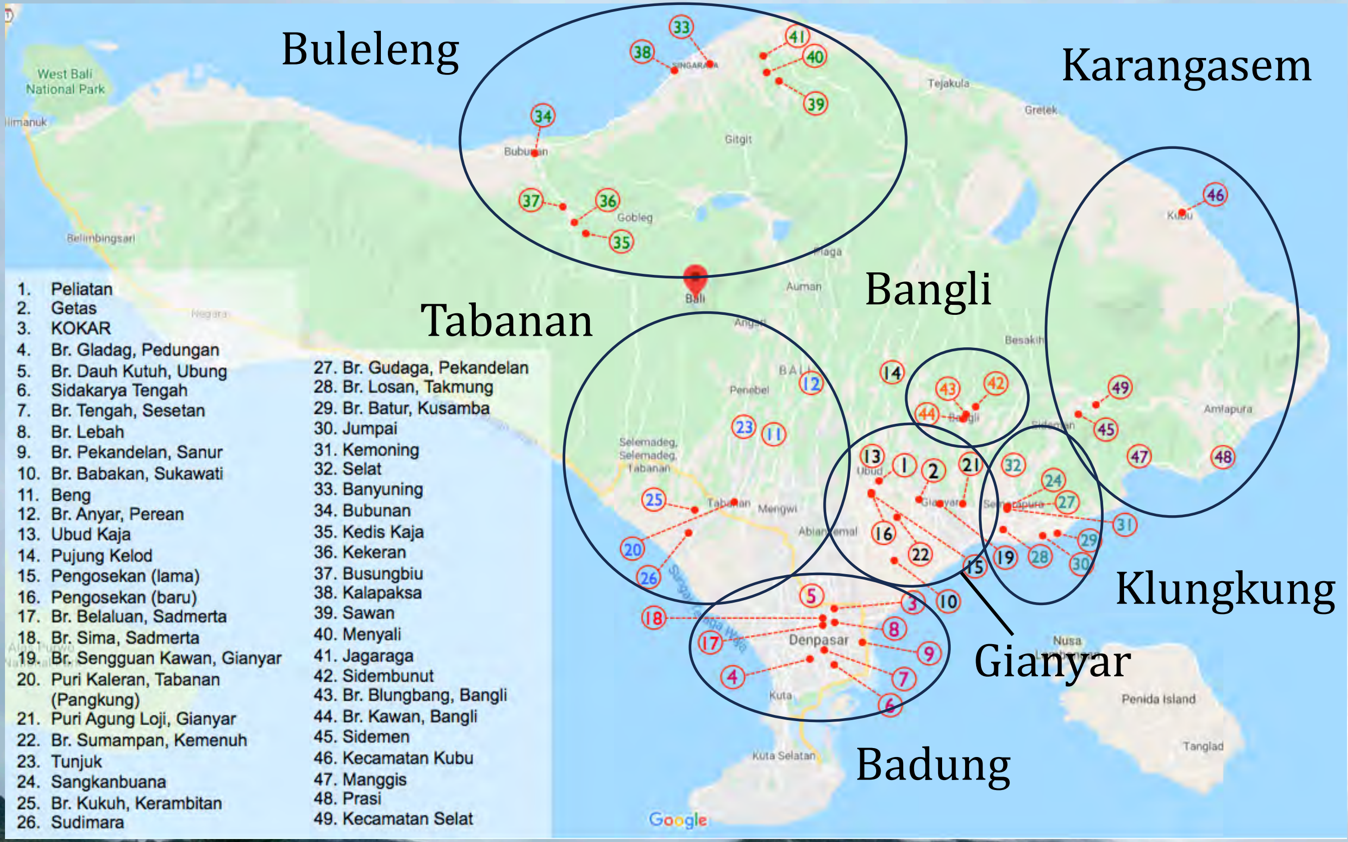
Toth measured 50 gamelans across all regions of Bali

Thanks to Wayne Vitale and Bill Sethares for data. ("Balinese Gamelan Tuning: The Toth Archives" *Analytical Approaches to World Music*, 2021)

Processing:

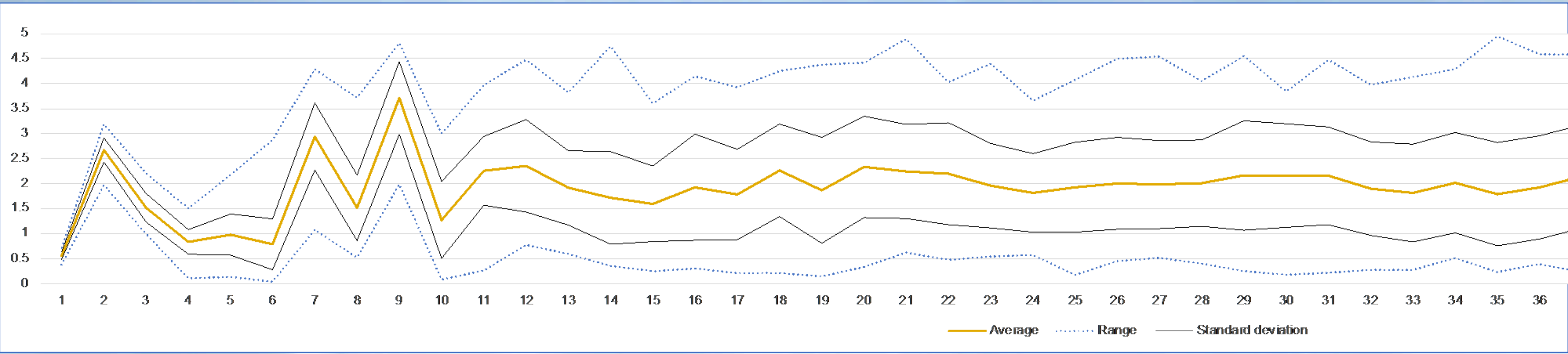
- Average across instruments.
- Average step sizes between second and third octave.
- Stretch/compress to a 1200¢ octave.

Regions of Bali (Toth data)



From Vitale and Sethares 2021

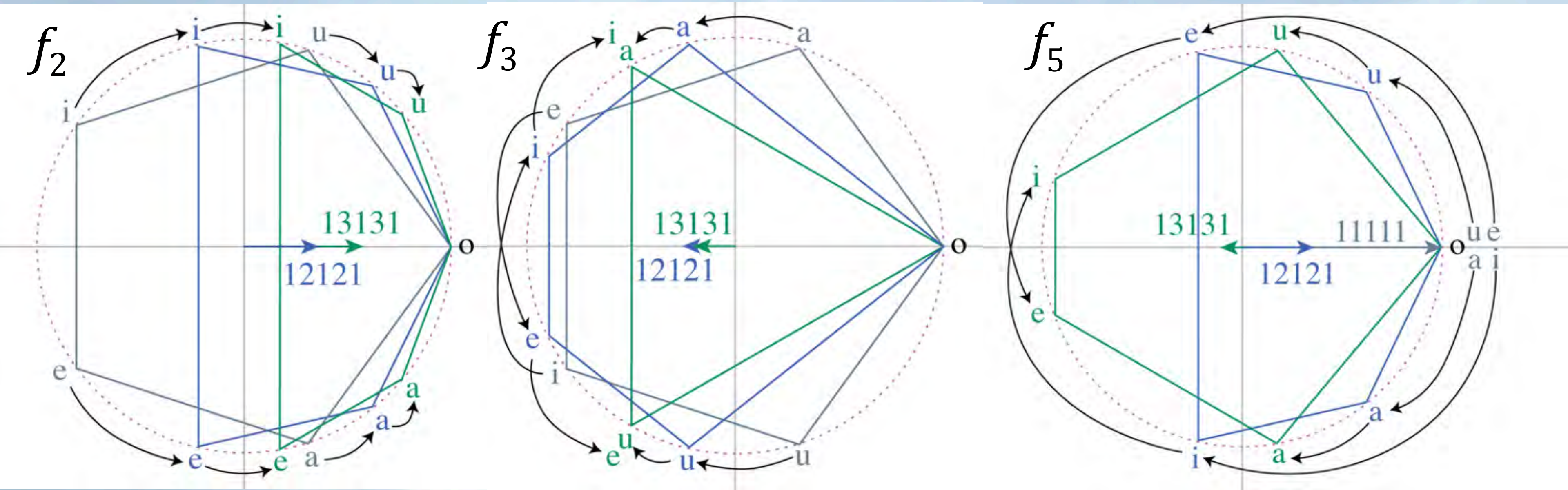
Pelog Spectra



- Peaks at f_2, f_7, and f_9 and troughs in between are consistent.
- Above f_9, little discernable consistency.

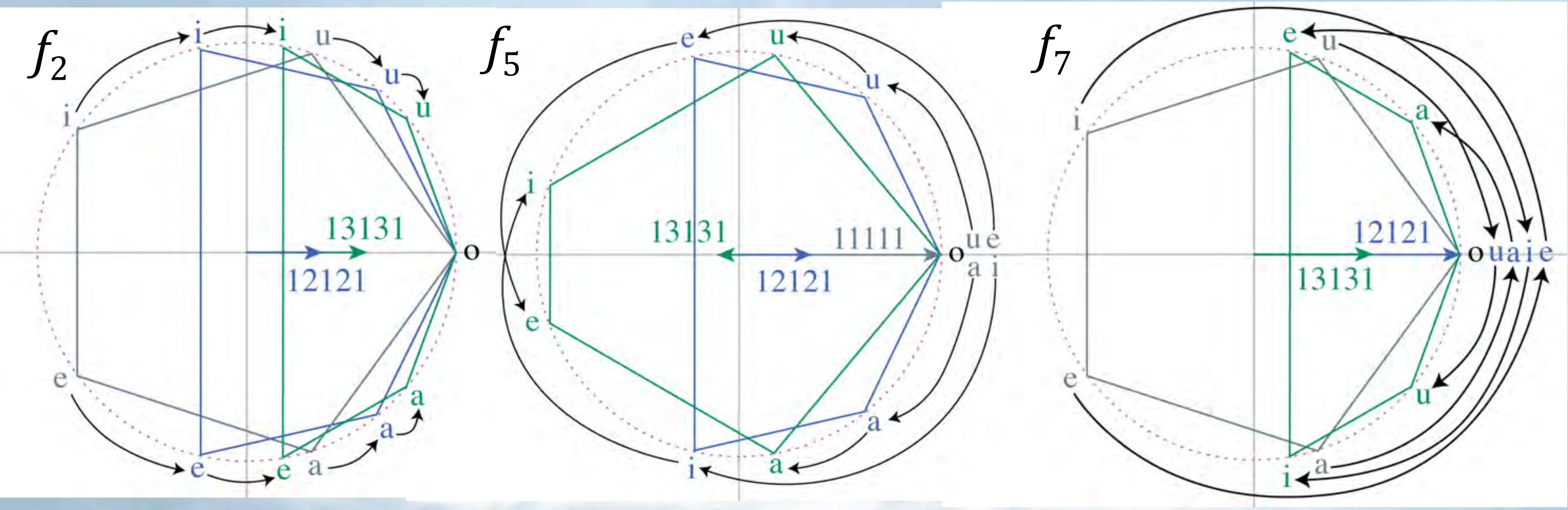
Generated Pentatonics

The maximally even 5-in-7 pentatonic (12121) has the expected $\varphi_{2\cdot3/5} = \pi$,
but as the generator gets closer to ½-octave, φ_5 flips so that $\varphi_{2\cdot3/5} = 0$.



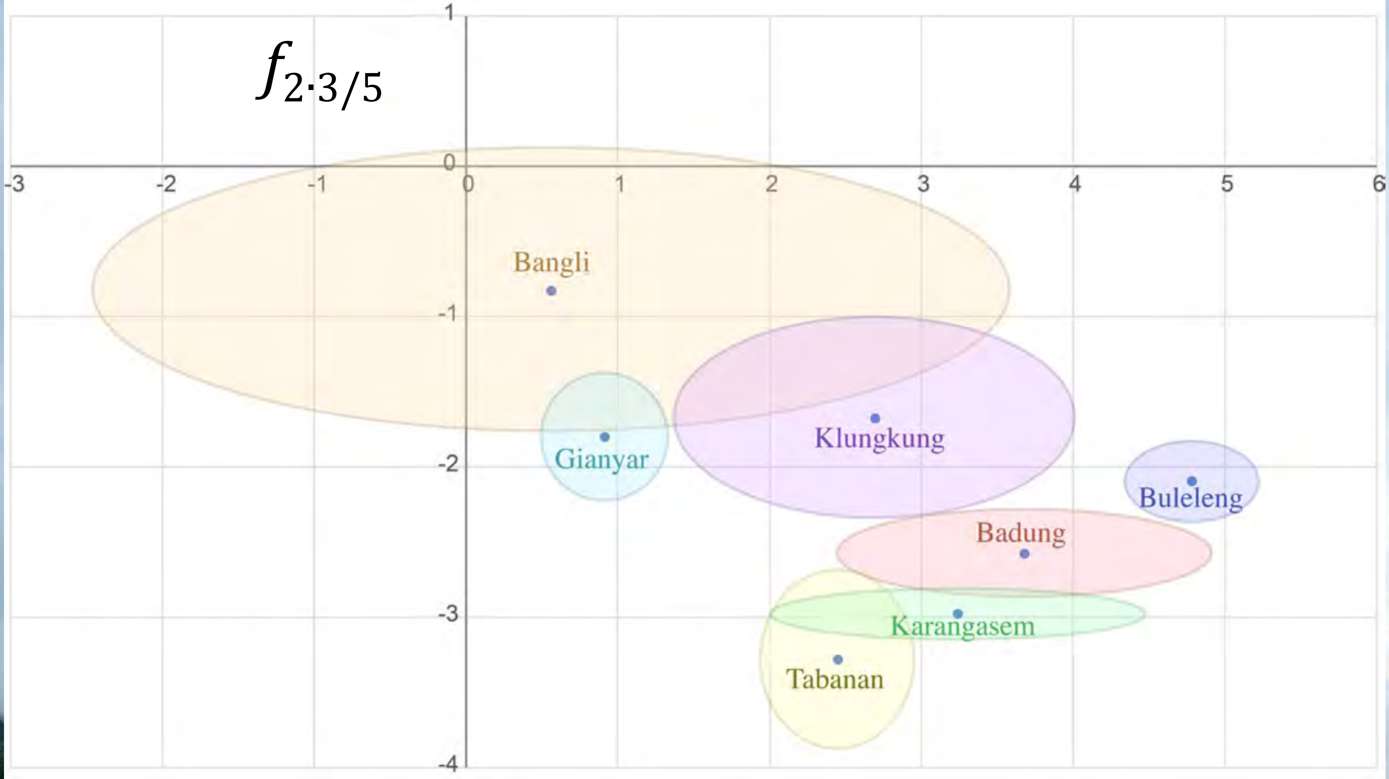
Generated Pentatonics

The φ_5 flip has the opposite effect on $\varphi_{2\cdot 5/7}$,
which is 0 for 12121 (pentatonic mod 7), but π for 13131 (subset of 7-in-9 ME)



Pelogs in $f_2 \times f_3 \times \overline{f_5}$ space

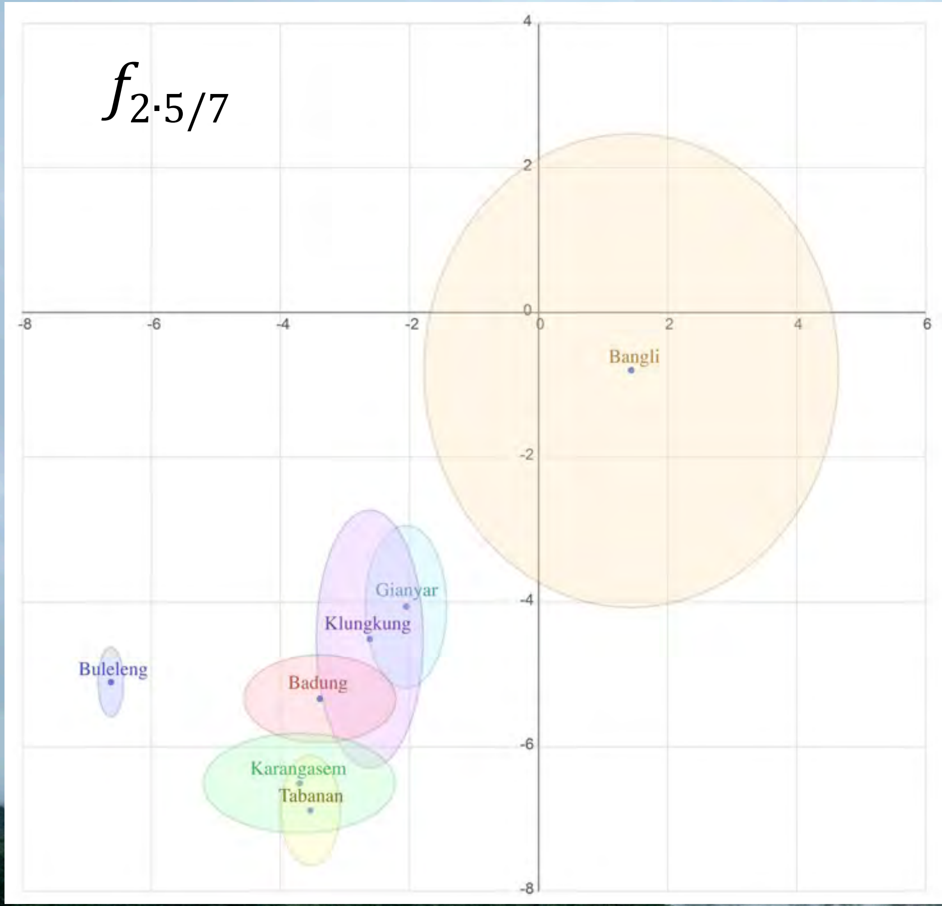
Average and standard error for Toth data by region



All the major regions (Gianyar, Buleleng, Badung, Klungkung, Tabanan) are distinguishable.

	n
Gianyar:	11
Buleleng:	9
Badung:	9
Klungkung:	7
Tabanan:	6
Karangasem:	5
Bangli:	3

Pelogs in $f_2 \times f_5 \times \overline{f_7}$ space



Anticoherence of $\varphi_{2\cdot5/7}$ indicates that pelog is more like a gapped 7-note ME scale than a 5-note ME scale.

All the major regions except Klungkung are distinguishable.

	n
Gianyar:	11
Buleleng:	9
Badung:	9
Klungkung:	7
Tabanan:	6
Karangasem:	5
Bangli:	3

Imaginary part of $f_{2\cdot 3/5}$ and $f_{2\cdot 5/7}$

The data also show that a negative imaginary part of $f_{2\cdot 3/5}$ and $f_{2\cdot 5/7}$ is a feature of pelog.

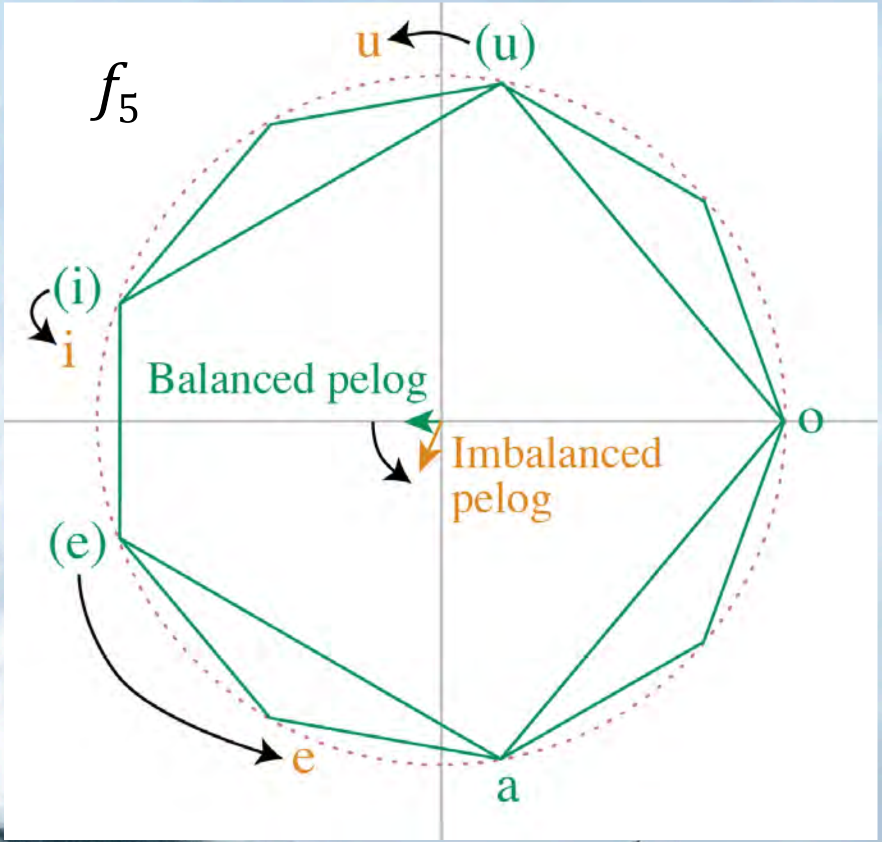
This reflects an asymmetry:
dong–deng > ding-dong
dong–dung > dang-dong,

a — i - o - e — u

$<$ (a — i — o) $>$ (o — e — u)

$\Rightarrow \varphi_{2\cdot 3/5} > 0, \varphi_{2\cdot 5/7} < \pi.$

Imbalance helps distinguish regions:
Larger in Gianyar, smaller in Buleleng

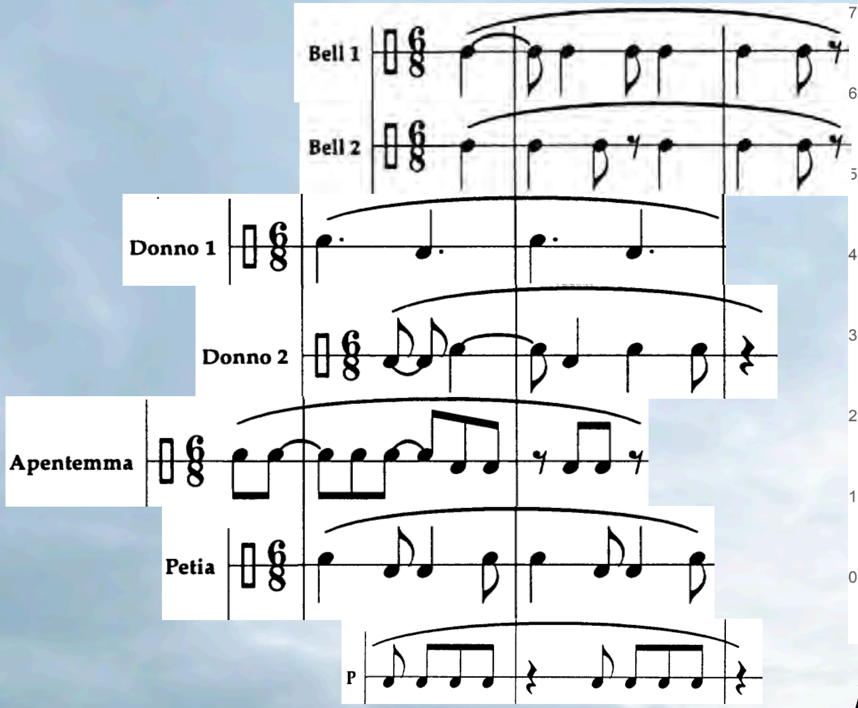


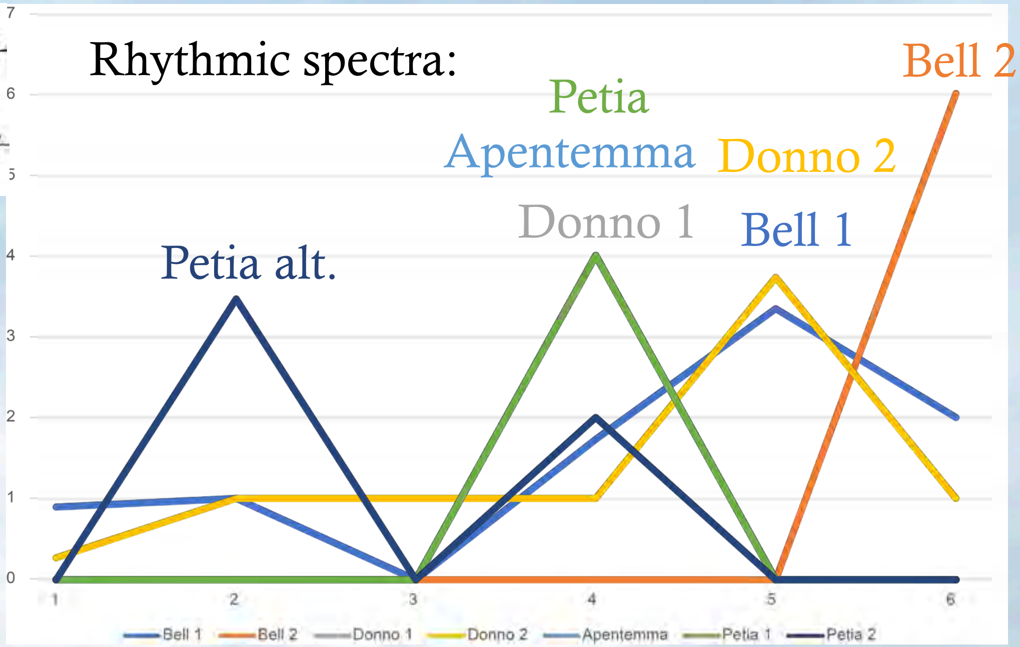
Rhythmic Qualities and Coherence in Adowa

Analysis of a transcription by Willi Anku

Adowa

The Adowa ensemble (in Anku's transcription)

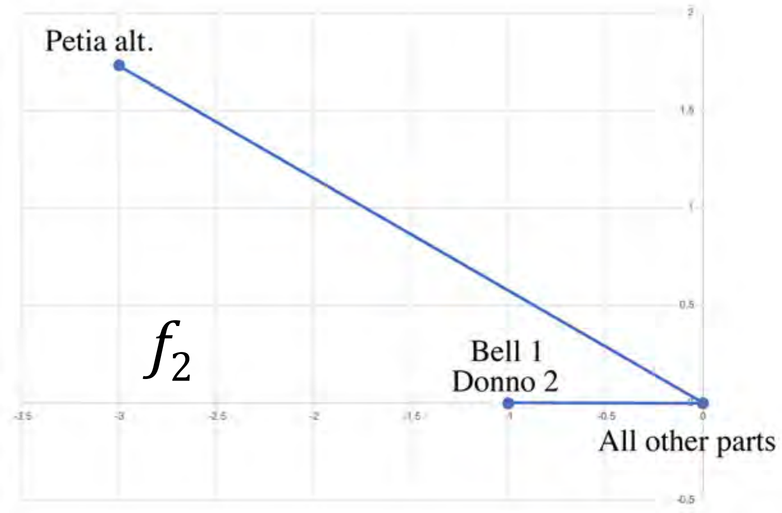




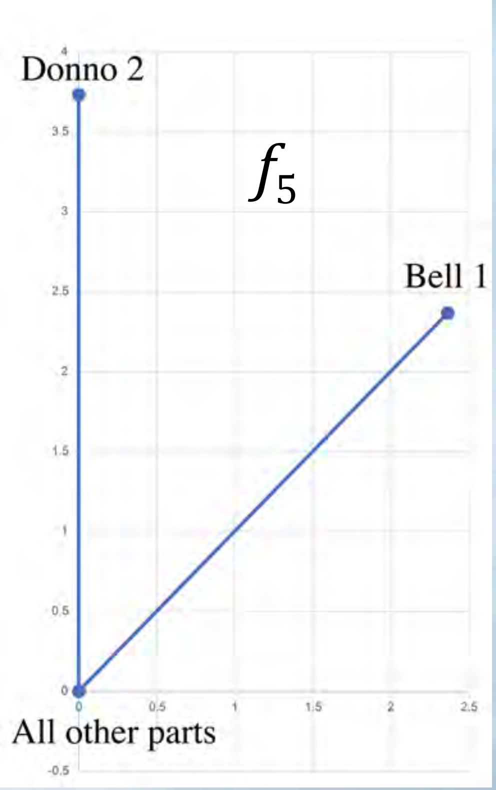
f_4 and f_5 prominent, represented by multiple parts

Adowa

The Adowa ensemble on 2-d. planes

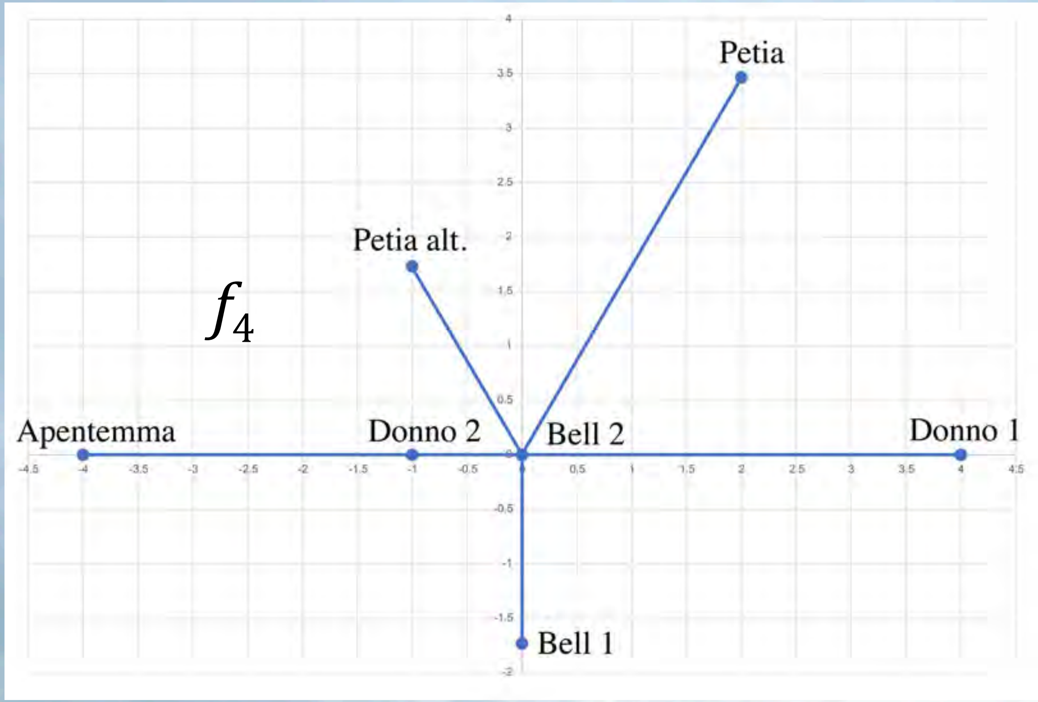


f_2 and f_5 (periodicities h. and ⅘q.) are both represented by just 2–3 parts, similar in phase.



Adowa

The adowa ensemble on 2-d. planes



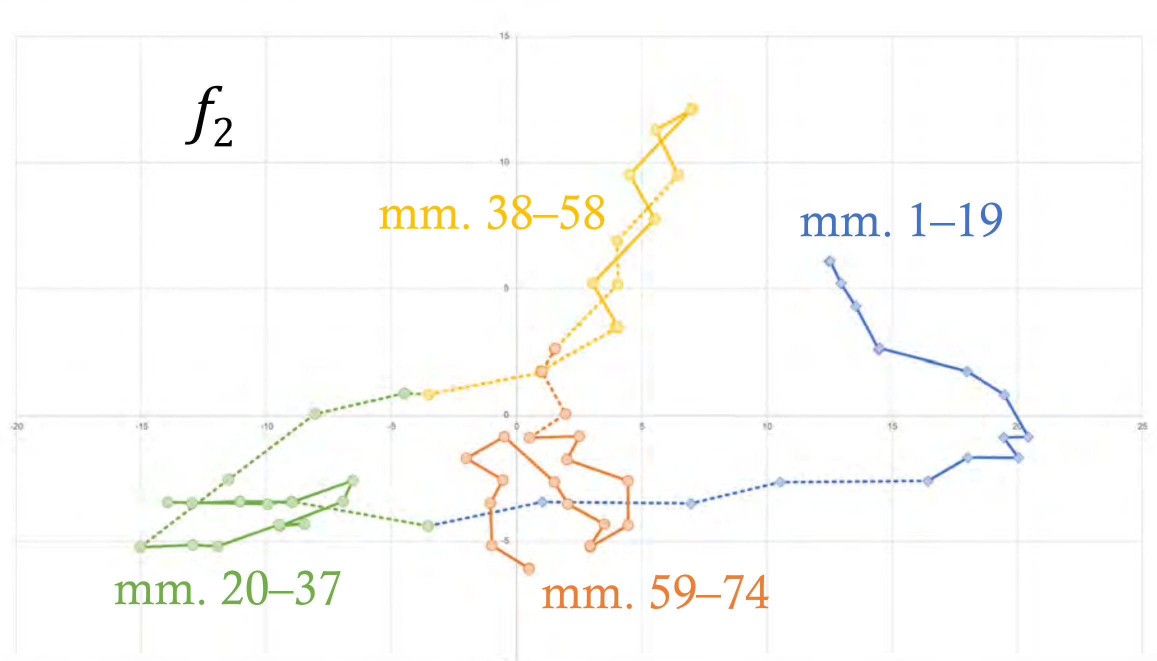
f_4 represents the periodicity of the beat (q.) and here multiple parts cover different regions of the space (on-beat, off-beat, ahead of beat, behind beat).

Adowa: Atumpan (lead drum)

Anku's transcription of a performance by Solomon Amonquandoh divided into 4 sections

Amonquandoh begins centered on the main two beats.

Then he explores each region of the space in turn: off-beats, ahead of the beat, and behind the beat.



(Values summed over 3½ cycles)

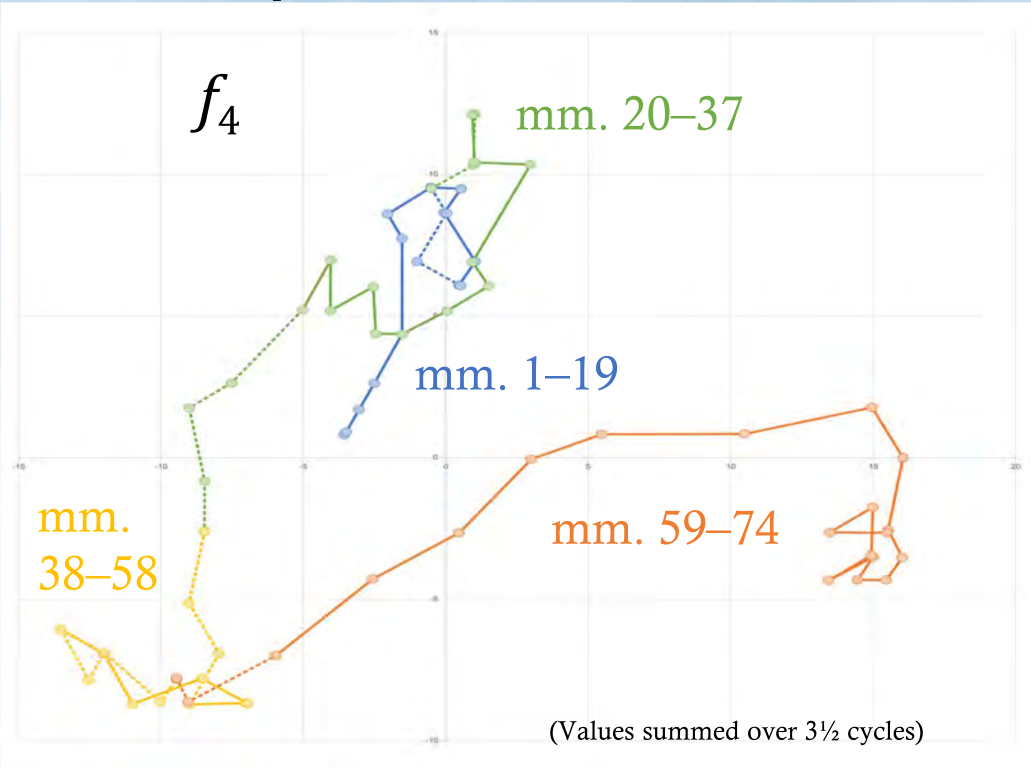
Adowa: Atumpan (lead drum)

Anku's transcription of a performance by Solomon Amonquandoh divided into 4 sections

Frequency 4 measures orientation with respect to the main 4 q. beats.

The performance also explores all the regions of this space in sequence.

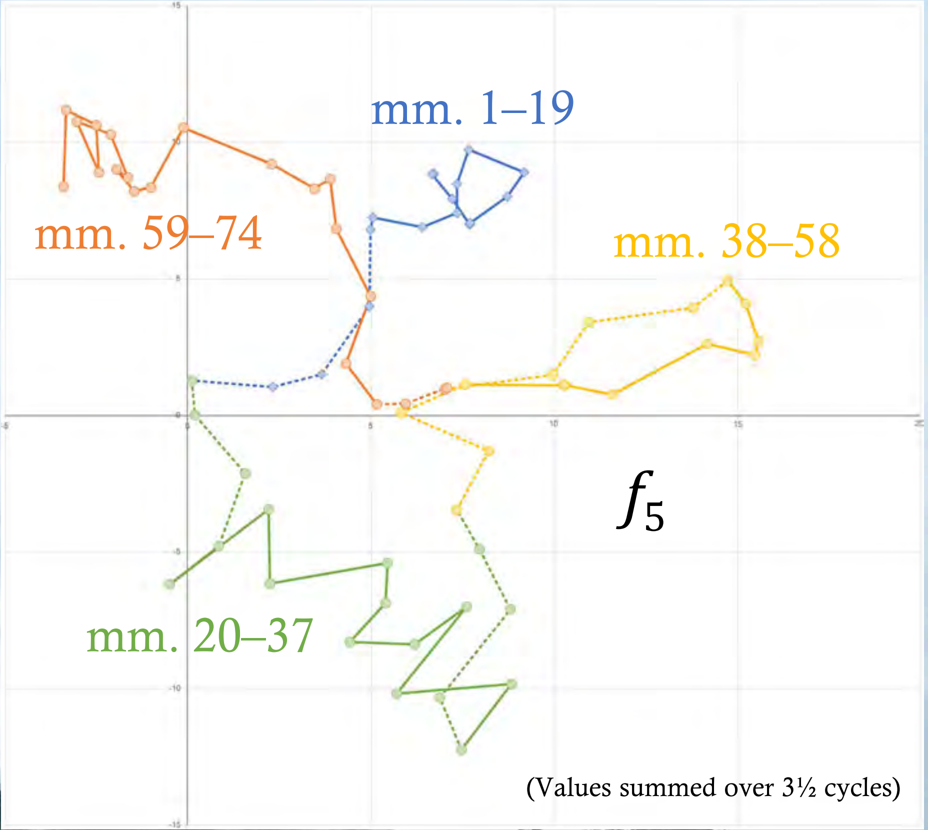
The first two sections are consistently ahead of the beat, the third section behind, and the last section on the beat.



Adowa: Atumpan (lead drum)

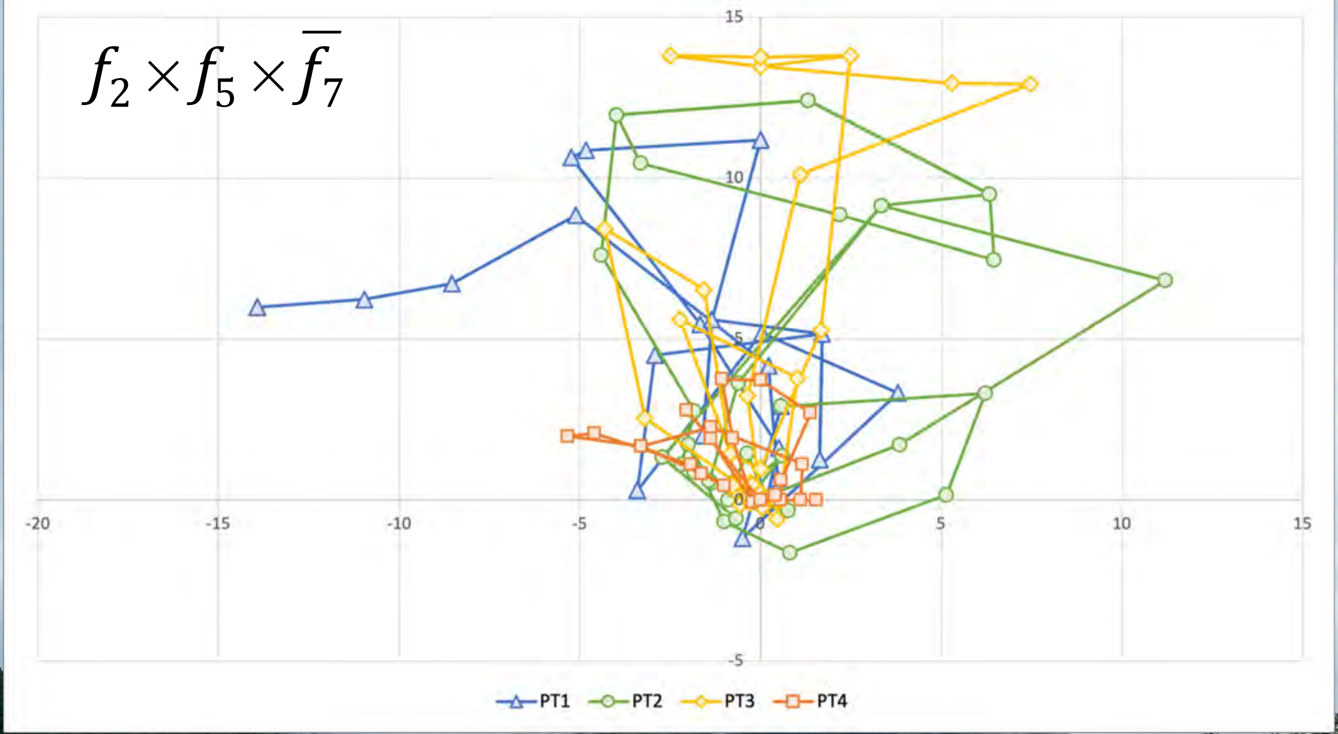
Anku's transcription of a performance by Solomon Amonquandoh divided into 4 sections

This is frequency articulated by the timeline rhythm of the bell, which is in the upper right quadrant of the space. Amonquandoh starts in the vicinity of the bell and explores all the adjacent regions, avoiding the half of the space across from the bell.



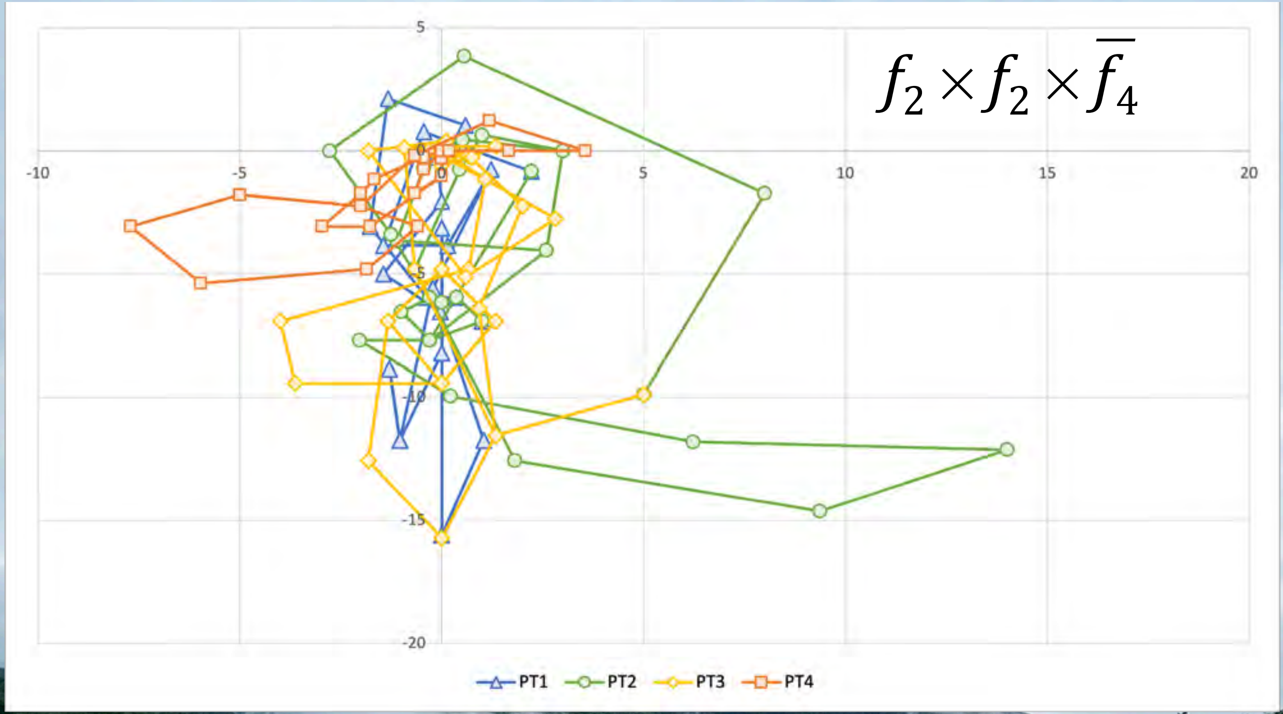
Amonquandoh's Atumpan in $f_2 \times f_5 \times \overline{f_7}$ space

While exploring different regions of f_2 and f_5 spaces,
Amonquandoh maintains a consistent $\varphi_{2 \cdot 5/7}$ of about $3\pi/2$.



Amonquandoh's Atumpan in $f_2 \times f_2 \times \overline{f_4}$ space

Similarly while exploring different regions of f_2 and f_4 spaces,
Amonquandoh maintains a consistent $\varphi_{2\cdot 2/4}$ of about $\pi/2$.



Thanks!

Powerpoint and more here:

sites.bu.edu/jyust