

Mathematical Morphology in Complete Lattices

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Lattices and Information Processing

Lattices: core mathematical structure in many information processing problems.

Examples:

- soft computing (fuzzy sets, bipolar information),
- knowledge representation,
- logics,
- formal concept analysis,
- automated reasoning,
- decision making,
- image processing and understanding,
- information retrieval,
- etc.

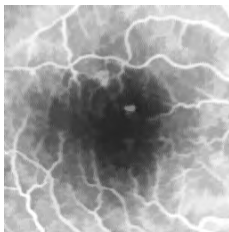
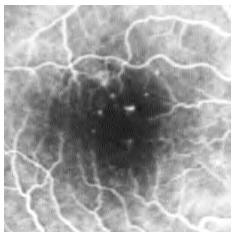
Mathematical morphology on complete lattices.

Mathematical Morphology for Spatial Information

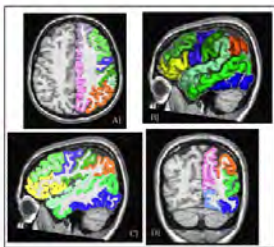
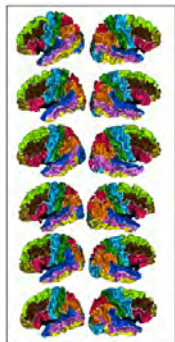
Matheron (mid-1960's), Serra (1982)

- A theory of space.
- Widely used in image processing and interpretation.
- At different levels (local, regional, structural...).
- For different tasks (filtering, enhancement, segmentation, interpretation, spatial knowledge modeling...).

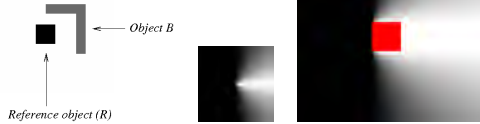
Filtering



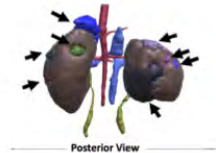
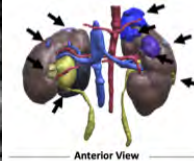
Segmentation



Knowledge modeling What is the region to the right of R ? Is B to the right of R (and to which degree)?



Spatial reasoning and image understanding



Formal Framework: Complete Lattices

- Lattice: $(\mathcal{T}, \leq)$ ($\leq$ partial ordering) such that $\forall (x, y) \in \mathcal{T}, \exists x \vee y$ and $\exists x \wedge y$.
- Complete lattice: every family of elements (finite or not) has a smallest upper bound and a largest lower bound.
- $\Rightarrow$ contains a smallest element 0 and a largest element 1 :

$$0 = \bigwedge \mathcal{T} = \bigvee \emptyset \text{ and } 1 = \bigvee \mathcal{T} = \bigwedge \emptyset$$

- Examples of complete lattices:
 - ▶ $(\mathcal{P}(E), \subseteq)$: complete lattice, Boolean (complemented and distributive)
 - ▶ functions of $\mathbb{R}^n$ in $\overline{\mathbb{R}}$ for the partial ordering $\leq$:
 $f \leq g \Leftrightarrow \forall x \in \mathbb{R}^n, f(x) \leq g(x)$
 - ▶ partitions
 - ▶ logics (propositional logics, modal logics...)
 - ▶ fuzzy sets, bipolar fuzzy sets
 - ▶ rough sets and fuzzy rough sets
 - ▶ formal concepts
 - ▶ graphs and hypergraphs
 - ▶ ...

Algebraic Dilations and Erosions

Heijmans, Ronse (1990)

Complete lattices $(\mathcal{T}, \leq)$, $(\mathcal{T}', \leq')$

Algebraic dilation: $\delta : (\mathcal{T}, \leq) \rightarrow (\mathcal{T}', \leq')$ such that

$$\forall (x_i) \in \mathcal{T}, \delta(\bigvee_i x_i) = \bigvee'_i \delta(x_i)$$

Algebraic erosion: $\varepsilon : (\mathcal{T}', \leq') \rightarrow (\mathcal{T}, \leq)$ such that

$$\forall (x_i) \in \mathcal{T}', \varepsilon(\bigwedge'_i x_i) = \bigwedge_i \varepsilon(x_i)$$

Properties:

- $\delta(0) = 0'$ (in $\mathcal{P}(E)$, $0 = \emptyset$).
- $\varepsilon(I') = I$ (in $\mathcal{P}(E)$, $I = E$).
- δ increasing, ε increasing.
- in $\mathcal{P}(\mathbb{R}^n)$, $\delta(X) = \bigcup_{x \in X} \delta(\{x\})$.

Adjunctions

$\delta : (\mathcal{T}, \leq) \rightarrow (\mathcal{T}', \leq'), \varepsilon : (\mathcal{T}', \leq') \rightarrow (\mathcal{T}, \leq), (\varepsilon, \delta)$ **adjunction** if:

$$\forall x \in \mathcal{T}, \forall y \in \mathcal{T}', \delta(x) \leq' y \Leftrightarrow x \leq \varepsilon(y)$$

Properties:

- $\delta(0) = 0'$ and $\varepsilon(1') = 1$.
- (ε, δ) adjunction $\Rightarrow \varepsilon =$ algebraic erosion and $\delta =$ algebraic dilation.
- δ increasing = algebraic dilation iff $\exists \varepsilon$ such that (ε, δ) is an adjunction $\Rightarrow \varepsilon =$ algebraic erosion and $\varepsilon(x) = \bigvee \{y \in \mathcal{T}, \delta(y) \leq' x\}$.
- ε increasing = algebraic erosion iff $\exists \delta$ such that (ε, δ) is an adjunction $\Rightarrow \delta =$ algebraic dilation and $\delta(x) = \bigwedge \{y \in \mathcal{T}', \varepsilon(y) \geq x\}$.
- $\varepsilon\delta \geq Id$ and $\delta\varepsilon \leq Id'$.
- $\varepsilon\delta\varepsilon = \varepsilon$ and $\delta\varepsilon\delta = \delta$.
- $\varepsilon\delta\varepsilon\delta = \varepsilon\delta$ and $\delta\varepsilon\delta\varepsilon = \delta\varepsilon$.
- δ and ε increasing such that $\delta\varepsilon \leq Id'$ and $\varepsilon\delta \geq Id \Rightarrow (\varepsilon, \delta)$ adjunction.

Morphological Dilations and Erosions

- On the lattice of the subsets of $\mathbb{R}^n$ or $\mathbb{Z}^n$, with inclusion:

$$\delta(X) = \cup_{x \in X} \delta(\{x\})$$

- + **invariance under translation** $\Rightarrow \exists B, \delta(X) = D(X, B) = \{x, \check{B}_x \cap X \neq \emptyset\}$
(with $B_x = x + B$).
- B = structuring element (neighborhood, binary relation).
- Same result on the lattice of functions.
- Similar results for erosion: $\exists B, \varepsilon(X) = E(X, B) = \{x, B_x \subseteq X\}$.

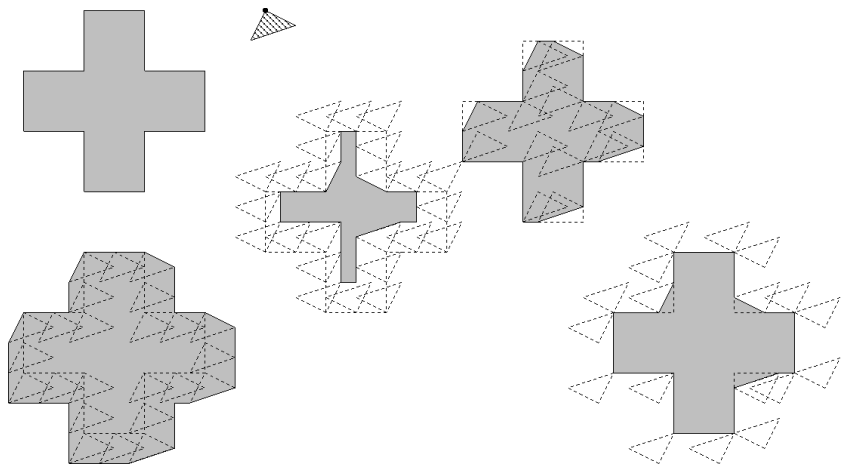
Derived operators: opening, closing, conditional (geodesic) operations, gradient...

Relaxing the assumption on invariance under translation: structuring elements varying in space (ex: projective geometry, omnidirectional images...).

Algebraic Opening and Closing

- **Algebraic opening:** γ increasing, idempotent and anti-extensive.
- **Algebraic closing:** φ increasing, idempotent and extensive.
- Examples: $\gamma = \delta\varepsilon$ and $\varphi = \varepsilon\delta$ with (ε, δ) adjunction.
- Invariance domain: $Inv(\varphi) = \{x \in \mathcal{T}, \varphi(x) = x\}$.
- γ opening $\Rightarrow \gamma(x) = \bigvee \{y \in Inv(\gamma), y \leq x\}$.
- φ closing $\Rightarrow \varphi(x) = \bigwedge \{y \in Inv(\varphi), x \leq y\}$.
- (γ_i) openings $\Rightarrow \bigvee_i \gamma_i$ opening.
- (φ_i) closings $\Rightarrow \bigwedge_i \varphi_i$ closing.
- γ_1 and γ_2 openings $\Rightarrow$ equivalence between:
 - 1 $\gamma_1 \leq \gamma_2$
 - 2 $\gamma_1\gamma_2 = \gamma_2\gamma_1 = \gamma_1$
 - 3 $Inv(\gamma_1) \subseteq Inv(\gamma_2)$
- Similar result on closings.

A simple example on sets



(Illustration: C. Ronse)

Lattice of Fuzzy Sets and Fuzzy Morphology

- Space $\mathcal{S}$ (e.g. $\mathbb{Z}^n$ or $\mathbb{R}^n$)
- $\mathcal{F}$: set of fuzzy sets on $\mathcal{S}$ – $\mu \in \mathcal{F}$, $\mu : \mathcal{S} \rightarrow [0, 1]$ (or $\mu : \mathcal{S} \rightarrow L$, L lattice).
- Partial ordering:

$$\forall (\mu_1, \mu_2) \in \mathcal{F}^2, \mu_1 \leq \mu_2 \Leftrightarrow \forall x \in \mathcal{S}, \mu_1(x) \leq \mu_2(x)$$

- $(\mathcal{F}, \leq)$ = complete lattice
- $\wedge = \min$
- $\vee = \max$
- Algebraic dilation and erosion: as in any complete lattice
- Operators: t-norm t , t-conorm T , complementation c , implication I derived from T and c , residual implication I_R derived from t .
- **Structure = residuated lattice**

Morphological Operations in the Fuzzy Case

Fuzzy dilation of μ by ν :

$$\delta_\nu(\mu)(x) = \sup_{y \in \mathcal{S}} t[\nu(x - y), \mu(y)]$$

Fuzzy erosion of μ by ν :

- by duality:

$$\varepsilon_\nu(\mu)(x) = \inf_{y \in \mathcal{S}} T[c(\nu(y - x)), \mu(y)] = \inf_{y \in \mathcal{S}} I[\nu(y - x), \mu(y)]$$

- by adjunction:

$$\varepsilon_\nu(\mu)(x) = \inf_{y \in \mathcal{S}} I_R[\nu(y - x), \mu(y)]$$

Equivalence for Lukasiewicz operators (up to a bijective permutation on $[0, 1]$).

Properties: as in classical morphology.

Examples of application: modeling spatial relations, spatial reasoning under imperfect information and knowledge.

Morpho-Logics: First Steps

With Jérôme Lang, IRIT (now at LAMSADE):
mathematical morphology on logical formulas, via the models, in propositional logic [IPMU-2000, TCIS-2002, JANCL-2023].

- Set of all models of a formula φ : $\llbracket \varphi \rrbracket = \{\omega \in \Omega \mid \omega \models \varphi\}$
- Lattice structure on the set of all models $\Leftrightarrow$ lattice structure on the set of formulas (up to an equivalence relation)
- $\llbracket \varphi \vee \psi \rrbracket = \llbracket \varphi \rrbracket \cup \llbracket \psi \rrbracket$
- $\llbracket \varphi \wedge \psi \rrbracket = \llbracket \varphi \rrbracket \cap \llbracket \psi \rrbracket$
- $\llbracket \varphi \rrbracket \subseteq \llbracket \psi \rrbracket$ iff $\varphi \models \psi$, and φ is consistent iff $\llbracket \varphi \rrbracket \neq \emptyset$
- Algebraic dilations and erosions as in any complete lattice

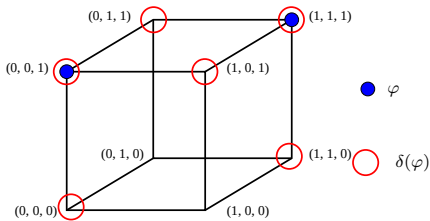
Morphological dilation of a formula φ with a structuring element B :

$$\llbracket \delta_B(\varphi) \rrbracket = \delta_B(\llbracket \varphi \rrbracket) = \{\omega \in \Omega \mid \check{B}_\omega \wedge \varphi \text{ consistent}\}$$

Morphological erosion:

$$\llbracket \varepsilon_B(\varphi) \rrbracket = \varepsilon_B(\llbracket \varphi \rrbracket) = \{\omega \in \Omega \mid B_\omega \models \varphi\}$$

B : a relation between worlds, e.g. neighborhood, distance



Example of a dilation of size 1 (Hamming distance):

$$\varphi = (a \wedge b \wedge c) \vee (\neg a \wedge \neg b \wedge c) \text{ and } \delta(\varphi) = (\neg a \vee b \vee c) \wedge (a \vee \neg b \vee c)$$

Reasoning with morpho-logics (in PL)

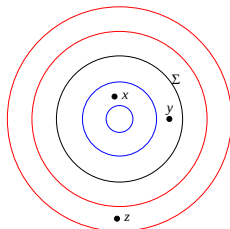
With Ramón Pino Pérez and Carlos Uzcátegui, Los Andes, Merida, Venezuela
(now at CRIL, Univ. Artois, and Santander Univ., Columbia, respectively)

[ECSQARU-2001, KR-2004, ECAI-2006, JANCL-2023]:

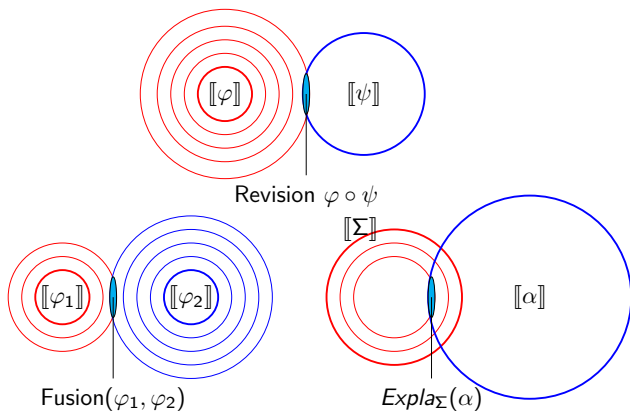
- Revision
- Merging (fusion)
- Abductive (explanatory) reasoning
- Mediation

Using dilations and erosions

Morphological partial ordering: stratification of the models from successive dilations and erosions



Example: $x \preceq_f y \preceq_f z$



Extensions to other Logics

Modal logics [JANCL-2002]

Accessibility relation	Structuring element
$\square$	ε
$\diamond$	δ

Description logics (joint work with Jamal Atif, LAMSADE, and Céline Hudelot, MICS) [FSS-2008, IEEE SMC-2014]: δ and ε as binary predicates, ontological reasoning.

Satisfaction systems, institutions, topos (joint work with Marc Aiguier, MICS) [AI-2018, IJAR-2018, JANCL-2019, IJAR-2023]:

- General framework for many logics.
- Revision based on relaxations.
- Abduction based on cuttings and retractions.
- Dual operators from dilations and erosions.
- Towards spatial reasoning.

Structural Information: Spatial Relations

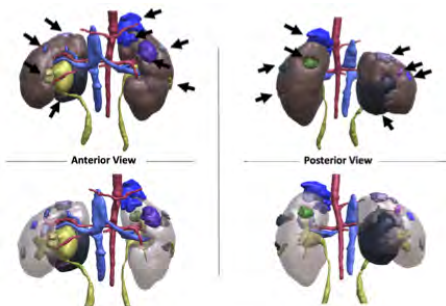
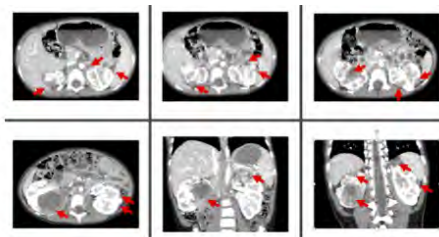
Expression of several spatial relations in terms of morphological operators:

- adjacency,
- distance (nearest point distance, Hausdorff distance),
- relative direction,
- more complex relations (between, along...).

Unifying formalization, applies in various settings

Applications:

- spatial reasoning,
- object recognition,
- image understanding.



Spatial Reasoning using Modal Logics

Examples in mereotopology (with $\Box \equiv \varepsilon$ and $\Diamond \equiv \delta$):

- tangential part: $\varphi \rightarrow \psi$ and $\Diamond\varphi \wedge \neg\psi$ consistent, or $\varphi \rightarrow \psi$ and $\varphi \wedge \neg\Box\psi$ consistent
- non tangential part: $\Diamond\varphi \rightarrow \psi$, or, $\varphi \rightarrow \Box\psi$
- external connection:
 $\varphi \wedge \psi$ inconsistent and $\Diamond\varphi \wedge \psi$ consistent (or $\varphi \wedge \Diamond\psi$ consistent)
- tangential proper part: tangential part and $\neg\varphi \wedge \psi$ consistent
($TPP(X, Y) = P(X, Y) \wedge \neg P(Y, X) \wedge \neg P(\delta(X), Y)$)

Hypergraphs

With Alain Bretto (GREYC, Caen)

- Underlying space: $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V}$ the set of vertices and $\mathcal{E}$ the set of hyperedges.
- Hypergraph: $H = (V, E)$ with $V \subseteq \mathcal{V}$ and $E \subseteq \mathcal{E}$, $E = ((e_i)_{i \in I})$ ($e_i \subseteq V$).
- $v(e)$ = set of vertices forming the hyperedge e (equivalence).
- Isolated vertex: $x \in V \setminus \cup_{i \in I} v(e_i) = V \setminus E$.
- Dual hypergraph: $H^* = (V^* \simeq E, E^* \simeq (H(x))_{x \in V})$
with $H(x) = \text{star}(x) = \{e \mid x \in e\}$.

Partial Ordering and Complete Lattices

- On vertices: $\mathcal{T}_1 = (\mathcal{P}(\mathcal{V}), \subseteq)$.
- On hyperedges: $\mathcal{T}_2 = (\mathcal{P}(\mathcal{E}), \subseteq)$.
- On $\mathcal{H}$: $\mathcal{T}_3 = (\{H\}, \preceq)$, with $\{H = (V, E)\} =$ set of hypergraphs defined on $(\mathcal{V}, \mathcal{E})$ such that $\forall e \in E, v(e) \subseteq \mathcal{V}$.
 - ▶ Partial ordering: $\forall (H_1 = (V_1, E_1), H_2 = (V_2, E_2)) \in \mathcal{T}_3^2$,

$$H_1 \preceq H_2 \Leftrightarrow V_1 \subseteq V_2 \text{ and } E_1 \subseteq E_2$$

- ▶ Inf and Sup:

$$H_1 \wedge H_2 = (V_1 \cap V_2, E_1 \cap E_2)$$

$$H_1 \vee H_2 = (V_1 \cup V_2, E_1 \cup E_2)$$

and extensions to any family.

- ▶ Smallest element: $H_\emptyset = (\emptyset, \emptyset)$, largest element: $\mathcal{H} = (\mathcal{V}, \mathcal{E})$.

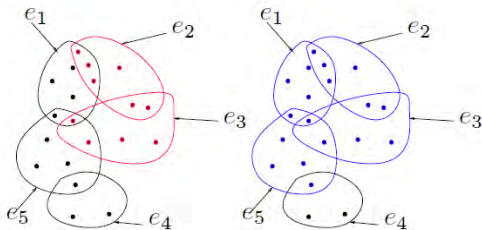
Algebraic erosions and dilations as in any complete lattice.

Structuring element B : binary relation between two elements.

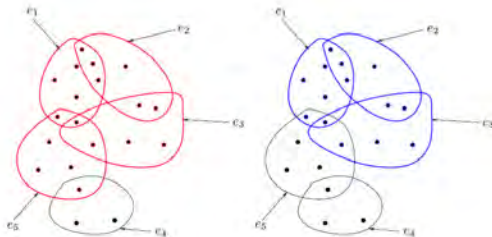
$$B_x = \delta(\{x\})$$

Example: morphological opening on $\mathcal{T}_3 = (\{H\}, \preceq)$

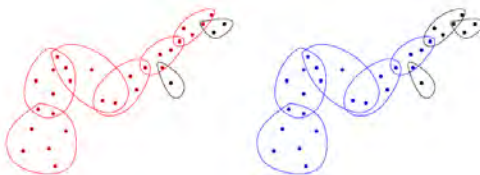
- Canonical decomposition of H : $H = (\vee_{e \in E} (v(e), \{e\})) \vee (\vee_{x \in V_{\setminus E}} (\{x\}, \emptyset))$.
- **Dilation:** $\forall x \in V_{\setminus E}, \delta(\{x\}, \emptyset) = (\{x\}, \emptyset), \forall e \in E, \delta(v(e), \{e\}) = (\cup \{v(e') \mid v(e') \cap v(e) \neq \emptyset\}, \{e' \in \mathcal{E} \mid v(e') \cap v(e) \neq \emptyset\})$.
- If no isolated vertices: $\delta(\{e\}) = B_e = \{e' \in \mathcal{E} \mid v(e) \cap v(e') \neq \emptyset\}$, and $\delta(E) = \cup_{e \in E} \delta(\{e\}) = \{e' \in E \mid \exists e \in E, v(e') \cap v(e) \neq \emptyset\}$.



Adjoint erosion: $\varepsilon(E) = \cup \{E' \in \mathcal{P}(\mathcal{E}) \mid \delta(E') \subseteq E\} = \{e \in \mathcal{E} \mid B_e \subseteq E\}$ and $\varepsilon(H) = (\cup_{e \in \varepsilon(E)} v(e), \varepsilon(E))$.



Opening $\gamma_2 = \delta\varepsilon$: $\gamma_2(E) = \cup_{B_e \subseteq E} B_e$, and $\gamma_2(H) = (\cup_{e \in \gamma_2(E)} v(e), \gamma_2(E))$.



Formal Concept Analysis: Adjunction and Galois Connection

Equivalent concepts by reversing the order on one space.

$$\delta : A \rightarrow B, \varepsilon : B \rightarrow A$$
$$\delta(a) \leq_B b \Leftrightarrow a \leq_A \varepsilon(b)$$

increasing operators

$$\varepsilon\delta\varepsilon = \varepsilon, \delta\varepsilon\delta = \delta$$

$\varepsilon\delta$ = closing, $\delta\varepsilon$ = opening

$$\text{Inv}(\varepsilon\delta) = \varepsilon(B), \text{Inv}(\delta\varepsilon) = \delta(A)$$

$$\varepsilon(B) = \text{Moore family}$$

$$\delta(A) = \text{dual Moore family}$$

$$\delta = \text{dilation: } \delta(\bigvee_A a_i) = \bigvee_B (\delta(a_i))$$

$$\varepsilon = \text{erosion: } \varepsilon(\bigwedge_B b_i) = \bigwedge_A (\varepsilon(b_i))$$

$$\alpha : B \rightarrow A, \beta : A \rightarrow B$$

$$a \leq_A \alpha(b) \Leftrightarrow b \leq_B \beta(a)$$

$$(\Leftrightarrow \beta(a) \leq'_B b \text{ with } \leq'_B \equiv \geq_B)$$

decreasing operators

$$\alpha\beta\alpha = \alpha, \beta\alpha\beta = \beta$$

$\alpha\beta$ and $\beta\alpha$ = closings

$$\text{Inv}(\alpha\beta) = \alpha(B), \text{Inv}(\beta\alpha) = \beta(A)$$

$$\alpha(B) \text{ and } \beta(A) = \text{Moore families}$$

$$\alpha(\bigvee_B b_i) = \bigwedge_A \alpha(b_i)$$

$$\beta(\bigvee_A a_i) = \bigwedge_B \beta(a_i) \text{ (anti-dilation)}$$

($M \subseteq \mathcal{L}$ is a Moore family if any element of $\mathcal{L}$ has a smallest upper bound in M)

Homotopic Transformations

- Non-increasing operators.
- High importance in images, among others (discrete, structured data).
- Simple elements: voxels, cellular complexes, tetrahedral meshes representations.
- Local characterizations based on results on topology, homology and relative homology groups.

Teaser

Mathematical morphology and symbolic representations of music:

- Talk by Paul Lascabettes: Using Mathematical Morphology to Discover Repeated Patterns in a Multidimensional Representation of Music.
- Talk by Gonzalo Romero: Mathematical Morphology as a tool for analyzing symbolic music.