

A NEW MODEL OF MUSICAL HIERARCHY

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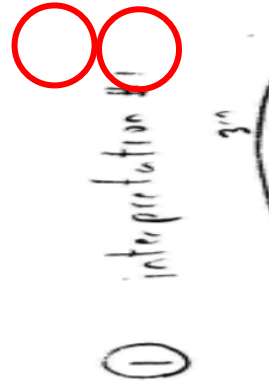
MENUETTO I

Musical score for Menuetto I, measures 1-6. The piece is in 3/4 time with a key signature of one flat (B-flat). The first system consists of six measures. The treble clef part begins with a piano (p) dynamic and features a melodic line with eighth and sixteenth notes. The bass clef part provides a simple accompaniment of quarter notes. Dynamic markings include 'p' at the start of measure 1 and 'f' at the start of measure 5. The piece concludes with a repeat sign at the end of measure 6.

Musical score for Menuetto I, measures 7-12. The second system consists of six measures. The treble clef part continues the melodic line, starting with a piano (p) dynamic in measure 7. The bass clef part continues with quarter notes. Dynamic markings include 'p' at the start of measure 7, 'f' at the start of measure 9, and 'p' at the start of measure 11. The piece concludes with a repeat sign at the end of measure 12.



Mozart K. 282, II



*Poundie
Burstein
(unpublished)*

MENUETTO I



- Analysis is asymmetrical ☹️
 - links first part of first motive with last part of the repeat.

MENUETTO I

The image shows a musical score for 'Menuetto I' in G minor, Op. 9, No. 3 by Franz Schubert. The score is in 3/4 time and features a piano (p) dynamic. The first staff is the treble clef and the second is the bass clef. Three red circles highlight specific notes: the first circle is around the G4 note in the first measure of the treble staff; the second circle is around the G4 note in the fourth measure of the treble staff; the third circle is around the G4 note in the fifth measure of the treble staff. The bass staff has a piano (p) dynamic marking in the first measure.



- Phrase is depicted as descending ☹
 - The motive ascends by scale step.

The image displays a musical score for a piano piece. The top staff is in treble clef, and the bottom staff is in bass clef. The key signature has one flat (B-flat), and the time signature is 3/4. The music is marked with a piano (p) dynamic. A red box highlights a descending phrase in the treble staff, consisting of a half note followed by three quarter notes. A curved arrow above the box points from the first note to the last, indicating the overall downward motion. A second red box highlights a motive within this phrase, consisting of a quarter note followed by three eighth notes, which ascends by a scale step. The bass staff contains a rhythmic accompaniment of eighth notes, also marked with a piano (p) dynamic.



- All of this makes sense if we think of chords as little scales.

scale degree: 3 2 1
(or voice)

scale moves down

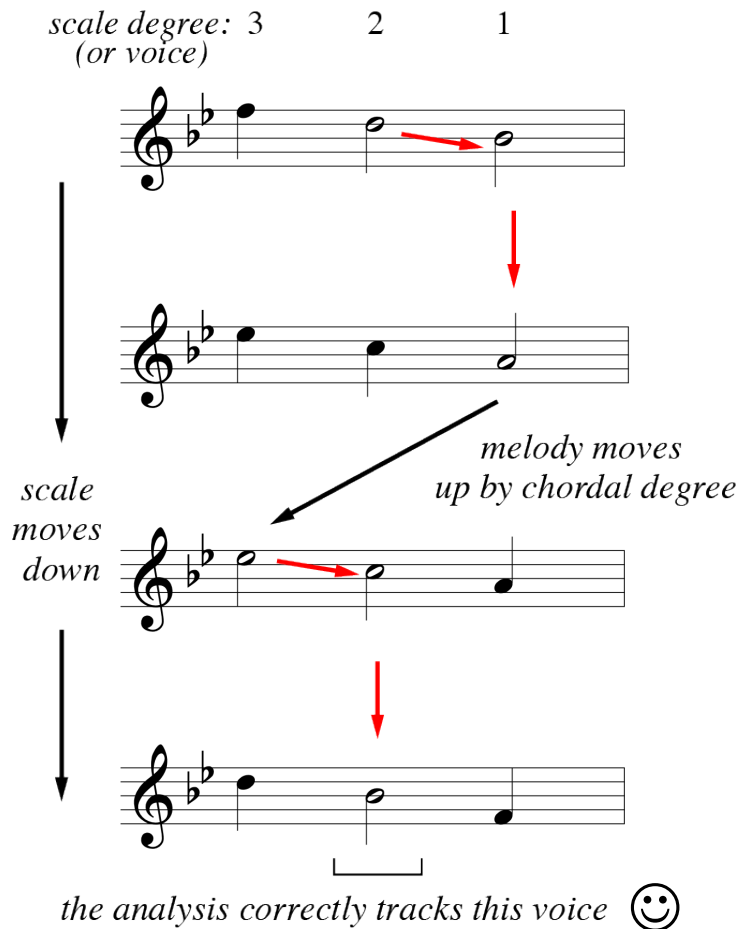
- All of this makes sense if we think of chords as little scales.

scale degree: 3 2 1
(or voice)

scale moves down

melody moves up by chordal degree

- All of this makes sense if we think of chords as little scales.



Once you realize chords can be little scales, you start to find all sorts of manifestations of this pattern.

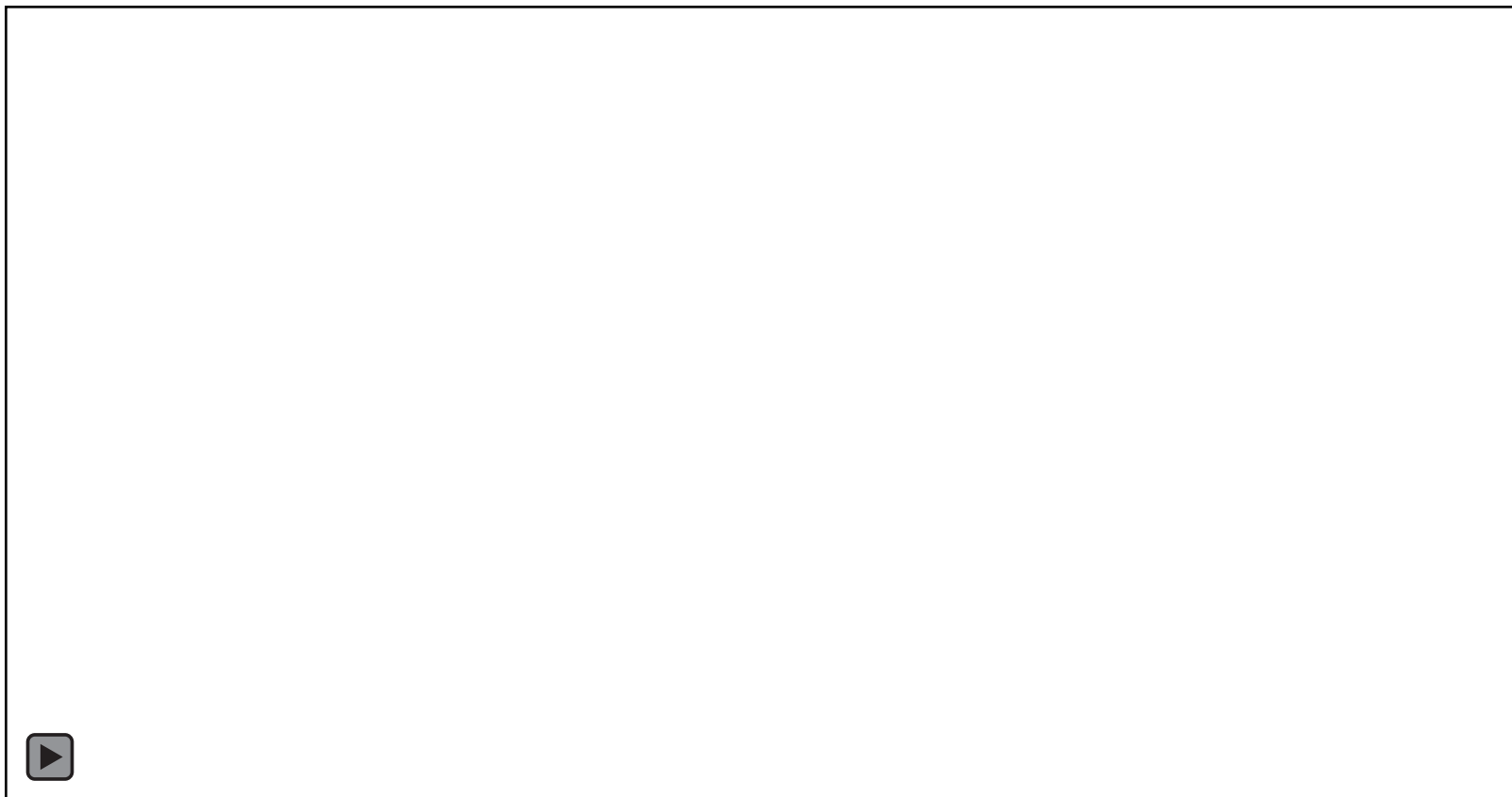
*In particular, you often see voices moving **contrary** to the underlying voice leading.*

EXAMPLE: Waldstein

	+2	+2	+1	+1	+2	+2	+1	+3	0
<i>upper</i>	2	3	1	2	2	3	1	2	3
<i>voices</i>	1	2	3	3	1	2	3	3	2

The musical score consists of three staves. The top staff is the piano part, written in treble clef with a key signature change to B-flat. It includes fingerings: *8va* (octave up) and *8vb* (octave down) for the first two measures, and *15mb* (15th fret, middle bass) for the next two measures. The bottom two staves are for three voices, labeled *voice 3*, *voice 2*, and *voice 1* from top to bottom, all in bass clef.

EXAMPLE: Waldstein



“Eight Days a Week” (The Beatles)



D: I II IV I

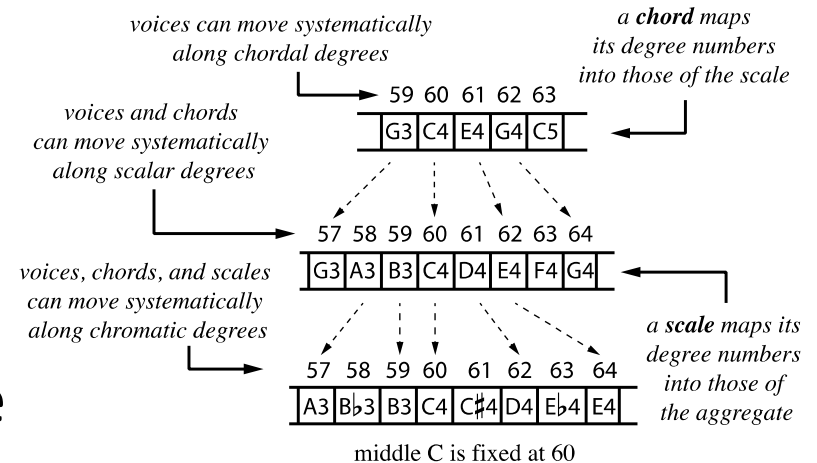
scale degrees:

62 63 64 65 (etc.)

(F#4 is scale degree 65 on returning to D major)

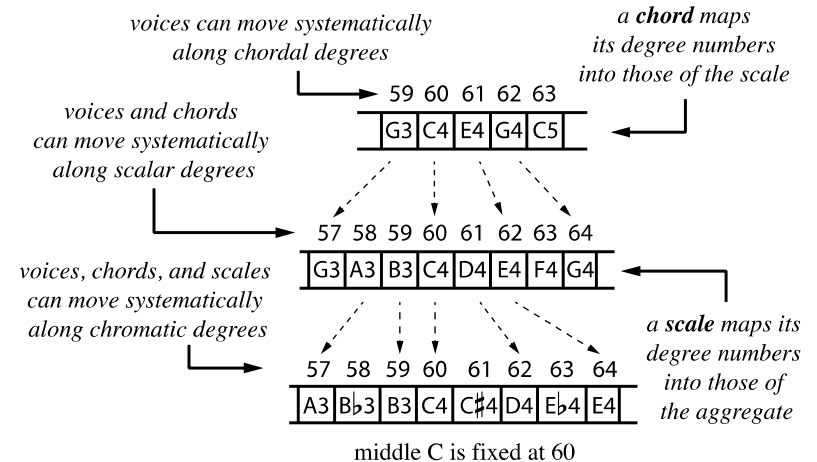
FUNDAMENTAL CLAIM

- Music moves simultaneously along many nested collections:
 - Chord
 - Scale
 - Sometimes chordal or scalar subsets
- This motion is simple when we look at any one level in isolation.
- It becomes quite complex when we look at multiple levels simultaneously.



FUNDAMENTAL CLAIM

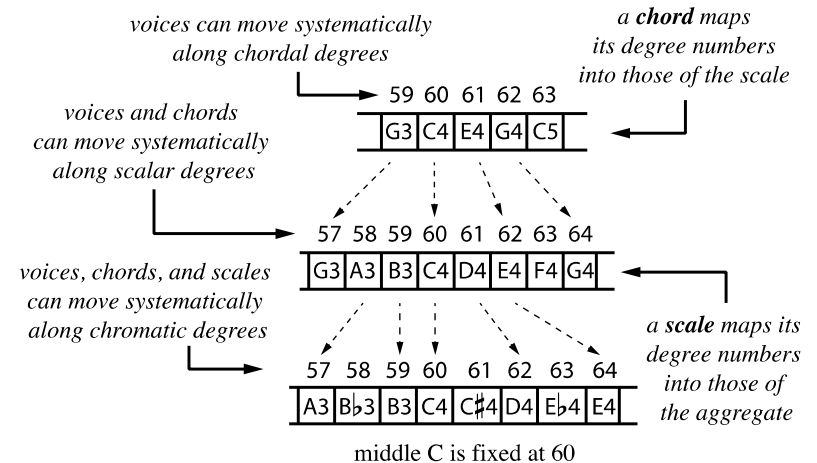
- This picture originates with the cognitive scientist Herbert Simon.
 - It was developed by Deutsch and Feroe.
 - Later Lerdahl
- What I add:
 - Systematic mathematical tools for describing voice leading on each level.
 - Generalize the picture to arbitrary collections.



FUNDAMENTAL CLAIM

- Conjecture: the reason machine-learning doesn't work so well with music is because there isn't enough data, in any one musical style, to do much more than just learn this structure.

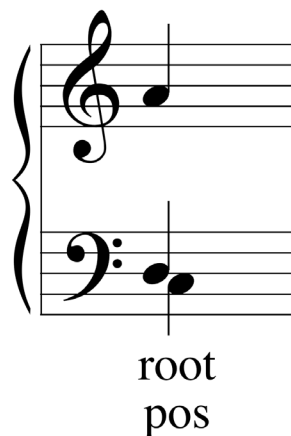
- Ideas about how to fix this.



I. Applications

1. REGISTRAL INVERSION

- Suppose I gave you this chord and told you it was in root position. What is the first-inversion form?



1. REGISTRAL INVERSION

- Suppose I gave you this chord and told you it was in root position. What is the first-inversion form?

The image shows two musical staves, treble and bass clef, with a brace on the left. The first staff (treble clef) has two notes: a quarter note on the second line (G4) and a quarter note on the third space (B4). The second staff (bass clef) has two notes: a quarter note on the second space (D4) and a quarter note on the first space (F4). This represents a D major chord in root position. Below the first staff is the label "root pos". The second staff has two notes: a quarter note on the second space (D4) and a quarter note on the first space (F4). This represents a D major chord in first inversion. Below the second staff is the label "first inv?".

1. REGISTRAL INVERSION

- Suppose I gave you this chord and told you it was in root position. What is the first-inversion form?

The image shows a grand staff with two staves (treble and bass clefs) and a brace on the left. It contains three measures of music, each representing a different voicing of a triad. The notes are: G4 (treble), B4 (treble), and D5 (bass) in the first measure; B4 (treble), D5 (bass), and G4 (bass) in the second measure; and D5 (treble), G4 (treble), and B4 (bass) in the third measure. Below each measure are labels: 'root pos', 'first inv?', and 'second inv?'.

root pos first inv? second inv?

1. REGISTRAL INVERSION

- Suppose I gave you this chord and told you it was in root position. What is the first-inversion form?

The image shows a musical score for a chord in root position and its first, second, and root position inversions. The score is written on a grand staff with a treble clef on the upper staff and a bass clef on the lower staff. The chord is a triad consisting of the notes G, B, and D. The first measure shows the chord in root position (G-B-D). The second measure shows the first inversion (B-D-G). The third measure shows the second inversion (D-G-B). The fourth measure shows the chord in root position (G-B-D) again. The notes are written as quarter notes. Below the staff, the labels 'root pos', 'first inv?', 'second inv?', and 'root pos?' are written under each measure respectively.

root pos first inv? second inv? root pos?

A BETTER ANSWER

- We can form inversions by transposing along the *intrinsic scale* consisting of the chord's own notes.
 - This preserves *voicing*, or spacing in chordal steps.

The diagram illustrates the concept of chord inversions. It features a grand staff with four measures. The first measure is the root position, the second is the first inversion, the third is the second inversion, and the fourth is the root position again. Brackets on the right indicate that the interval between the first and second measures is 'four chordal steps', while the intervals between the second and third, and third and fourth measures are 'one chordal step'. Below the grand staff, a single bass clef staff shows the root notes of each measure, with arrows labeled '+1' indicating the stepwise movement from one root to the next.

four chordal steps

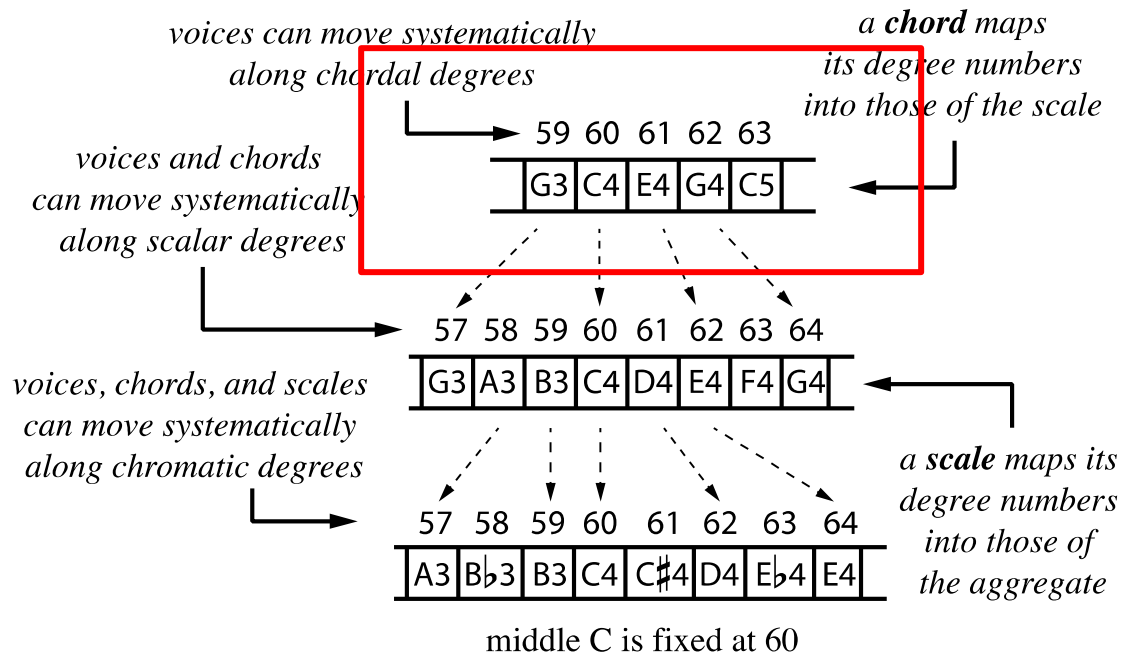
one chordal step

root pos first inv second inv root pos

+1 +1 +1

A BETTER ANSWER

- This is just the top level of my collectional hierarchy.

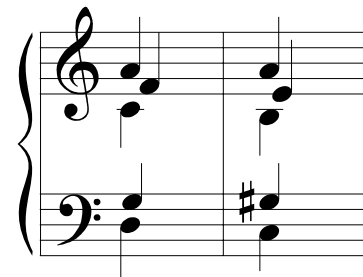


2. VOICING

Every chord presents multiple interval patterns, depending on whether we measure it extrinsically or intrinsically. These give us different ways to transform the chord and different conceptions of chordal identity.

	CHR	DIA	CHORD
F–A	4	2	2
C–F	5	3	2
G–C	5	3	2
D–G	5	3	2

	CHR	DIA	CHORD
E–A	5	3	2
B–E	5	3	2
G \sharp –B	3	2	2
C–G \sharp	8	4	2



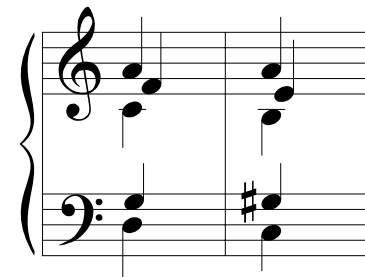
2. VOICING

For a chord of a given size, we can relate voicings to chord quality.

i.e. for a five-note chord, the (2, 2, 2, 2) voicing will be most “quartal,” while the (2, 1, 1, 2) voicing will be most “tertian.”

	CHR	DIA	CHORD
F–A	4	2	2
C–F	5	3	2
G–C	5	3	2
D–G	5	3	2

	CHR	DIA	CHORD
E–A	5	3	2
B–E	5	3	2
G#–B	3	2	2
C–G#	8	4	2



2. VOICING

In a recent paper I argue that *voicing* provides a set of approximate chord categories that simplify the world of musical possibilities. These correspond to the intuitive terms *clustered*, *tertian*, and *quartal*.

Approximate Set Theory

Chord Categories, Voicings, and Interval Cycles

Dmitri Tymoczko

Abstract This article describes an approximate set theory modeling intuitions shared by musicians such as Cowell, Schoenberg, Messiaen, and Persichetti. The author considers five approximation strategies, showing that in each case the result resembles an exact seven-tone set theory. Since most seven-tone sets are interval cycles, approximate twelve-tone sets are approximately cyclic as well. The theory explains how to highlight this cyclic structure using *voicings*, modeled by intervals in the *intrinsic scale* formed from a chord's own notes. This connection to voicing is what gives approximate chord categories much of their significance. The approach is most useful for chords with five or fewer notes and works tolerably for hexachords, but it breaks down with larger collections. This is not a failure of the model but a reflection of the fact that quality space contracts as cardinality increases.

Keywords set theory, voicing, intrinsic scale, analysis

APPROXIMATE INTERVAL CATEGORIES are a staple of informal musical discourse; musicians of all stripes speak of *steps* and *thirds*, or *clustered* and *quartal* harmonies. Approximate terminology has the cognitive advantage of reducing our harmonic taxonomy and the perceptual advantage of reflecting the often imprecise nature of musical experience: rather than requiring (or postulating, or hoping) that listeners maintain an exact tally of all the intervals they hear, generic qualia (e.g., *tertian*, *quartal*) allow for a degree of listener imperfection. Analysts frequently encounter passages saturated with a single type of generic interval, for example, an abundance of major and minor thirds or perfect and augmented fourths.¹ And as we will see, approximate categories highlight compositional affordances that might otherwise go unnoticed.

Yet academic music theory tends to valorize exact relationships. This is most obviously true of musical set theory, which categorizes chords by exact interval content, measured along the diatonic, chromatic, or some other scale. It is also a feature of twelve-tone music, which emphasizes rigid transformations of ordered

¹ The distinction between "generic" and "specific" intervals originates with Clough and Myerson 1985, though that work emphasizes scale membership rather than direct categorization of chromatic intervals.

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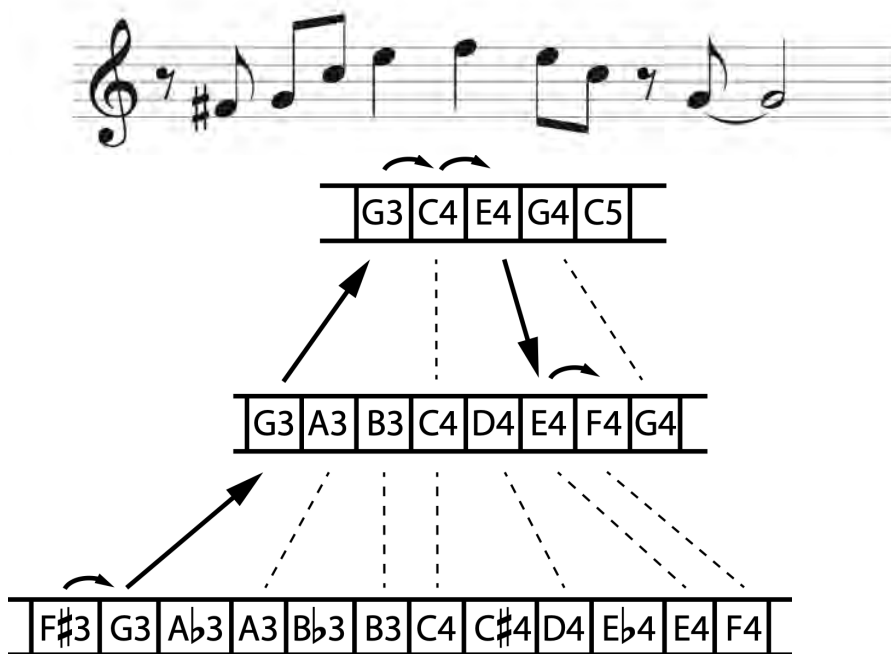
2. VOICING

This is a useful compositional tool.

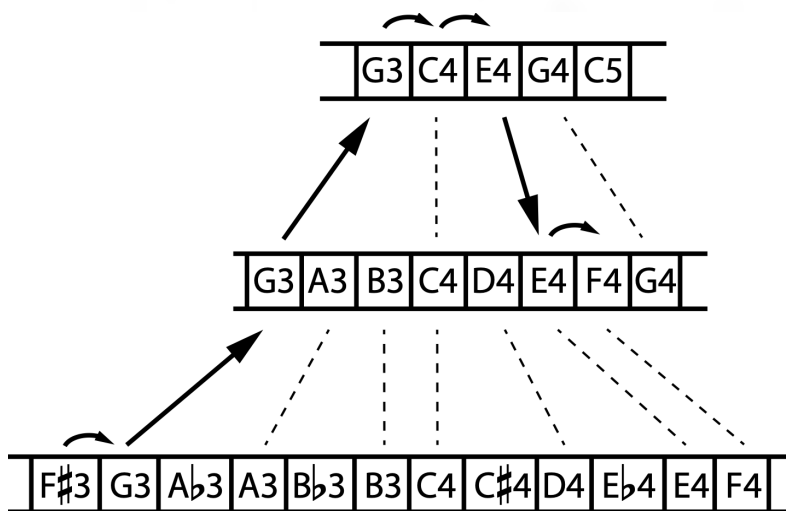
<http://www.madmusicalsscience.com/nr.html>

3. MOTIVES

- Motives are voicings augmented with
 - Rhythmic structure
 - Hierarchical pitch structure



3. MOTIVES



ARCA

- Because this gets so complex I wrote a syntax for manipulating motives:
 - Website
 - Free python libraries
 - DAW (patent applied for!)

4. VOICE LEADING

- The most efficient voice leading between two chords preserves voicing.
 - Suppose we want to move from this chord to a minor triad.



4. VOICE LEADING

- The most efficient voice leading between two chords preserves voicing.
 - Suppose we want to move from this chord to a minor triad.
 - There are 3 basic possibilities:

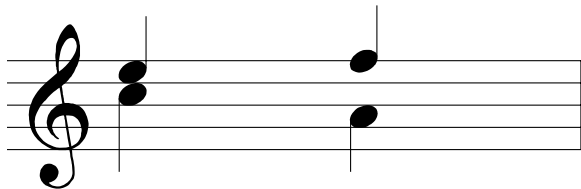
C → root C → third C → fifth

4. VOICE LEADING

- This means that motion-along-a-chord can be explanatorily useful *even if composers are not explicitly thinking about it.*

5. REPETITION

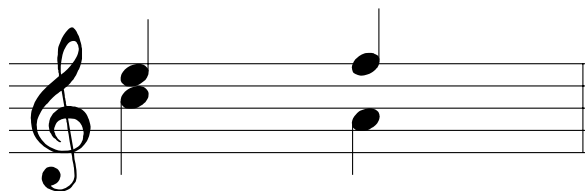
- When we describe a voice leading using a combination of intrinsic and extrinsic transposition, we provide a recipe for *repeating* it.



5. REPETITION

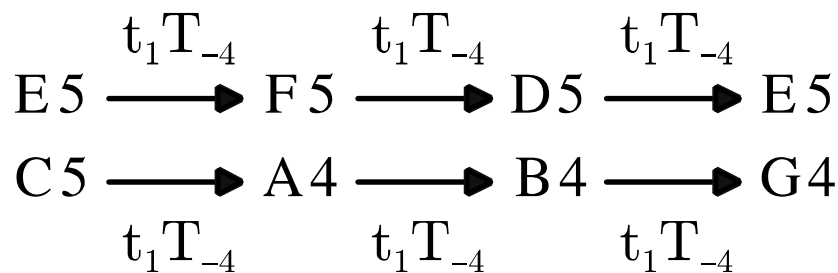
- When we describe a voice leading using a combination of intrinsic and extrinsic transposition, we provide a recipe for *repeating* it.

$$\begin{array}{ccc} & t_1 T_{-4} & \\ E5 & \longrightarrow & F5 \\ C5 & \longrightarrow & A4 \\ & t_1 T_{-4} & \end{array}$$

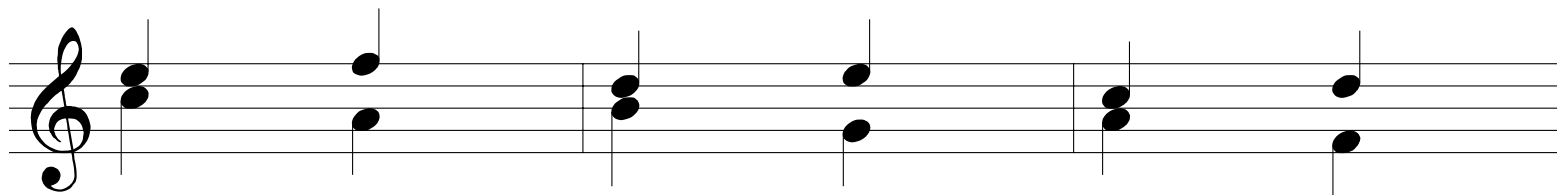


5. REPETITION

- When we describe a voice leading using a combination of intrinsic and extrinsic transposition, we provide a recipe for *repeating* it.

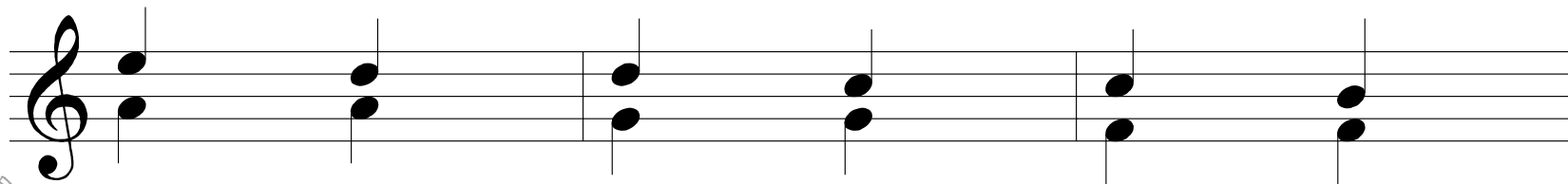
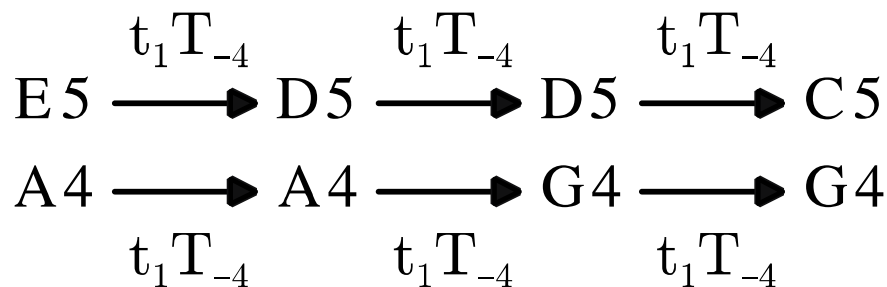


*Little t creates
canons!*



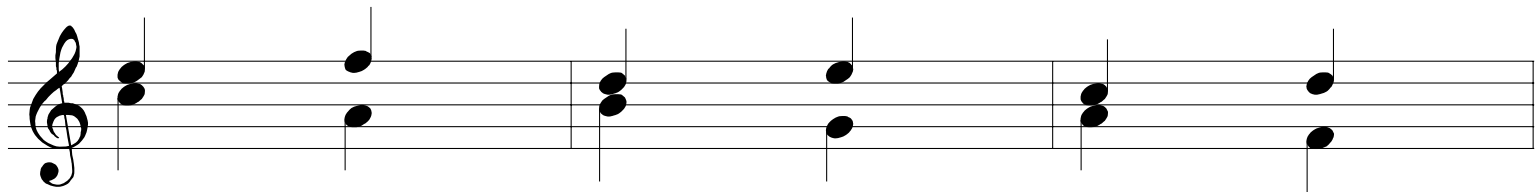
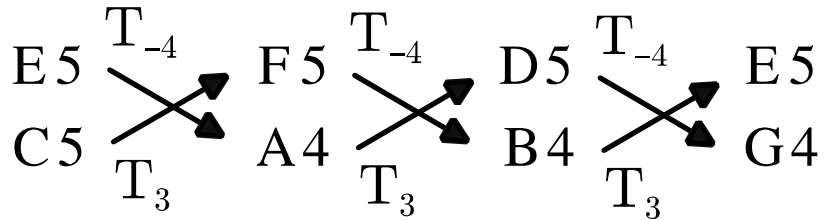
5. REPETITION

- This description is *structure neutral*, applying to all chords of that size.



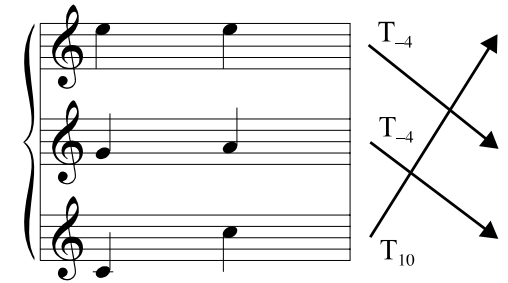
5. REPETITION

- This description is *equivalent to*, but different from, a description in terms of symmetries.

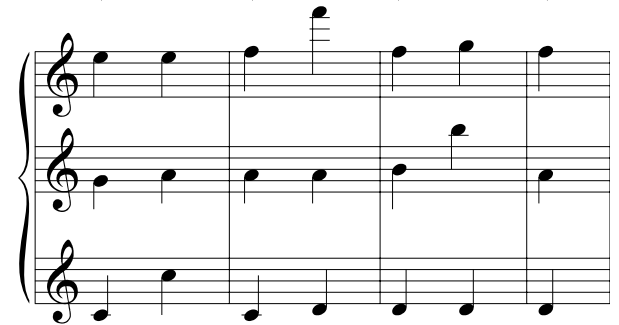


5. SEQUENCES

- The *repetition group* is isomorphic to the *voice leading group*.
 - Every repetition is a voice leading.
 - This gives us two different languages for describing repetition.
 - It is possibly the origin of the concept of a chord.



$T_{-4}t_2$ relates each initial chord to the next



chords are related by $c_{it}c_{it}c_{it}T_{-4}t_2$

II. The spiral diagrams

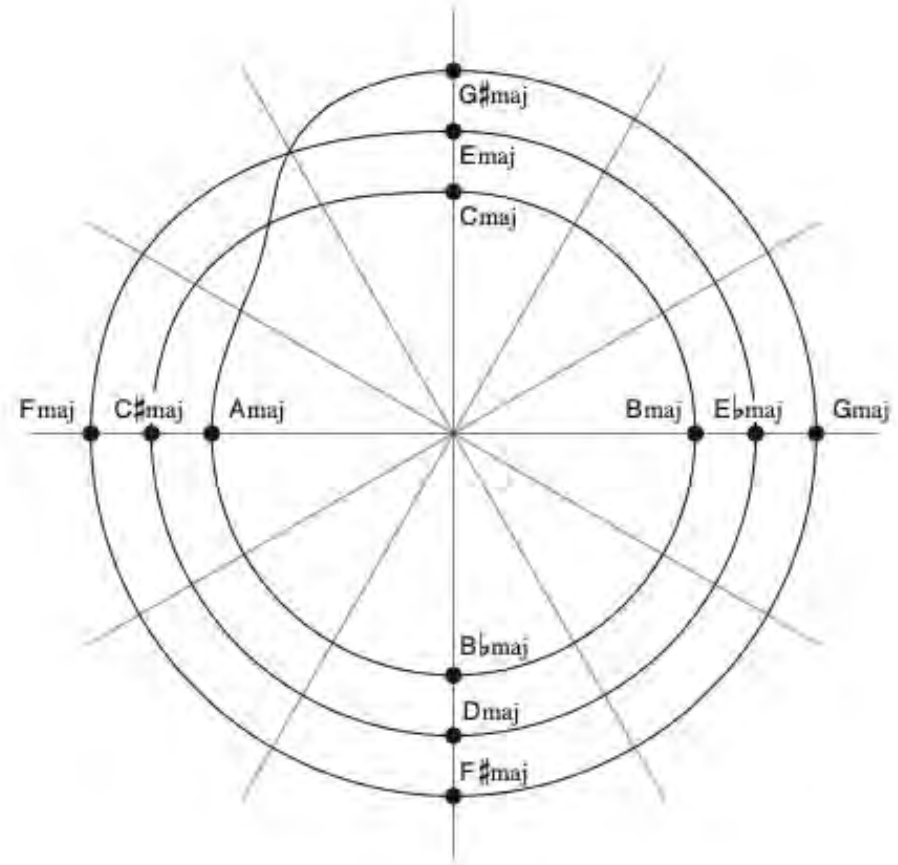
There is a simple two-dimensional geometry of hierarchically nested transposition, intrinsic and extrinsic, for any pair of objects whatsoever.

*They represent all the (n-voice) voice leadings linking the transpositions of any chord in any scale, except for the **voice exchanges***

(which we can add to the diagrams if we want)

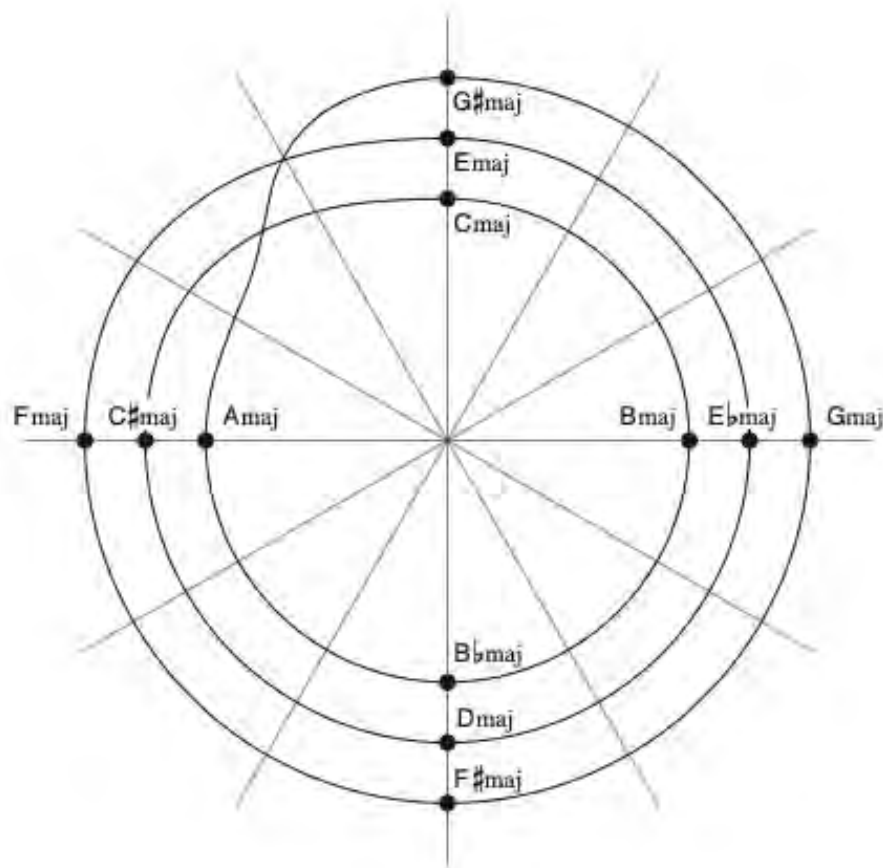
SPIRAL DIAGRAMS

- The combined operation of “transposition along a chord” (intrinsic transposition) and “transposition along a scale” (extrinsic transposition) can be represented, for any chord and any scale, by a spiral.



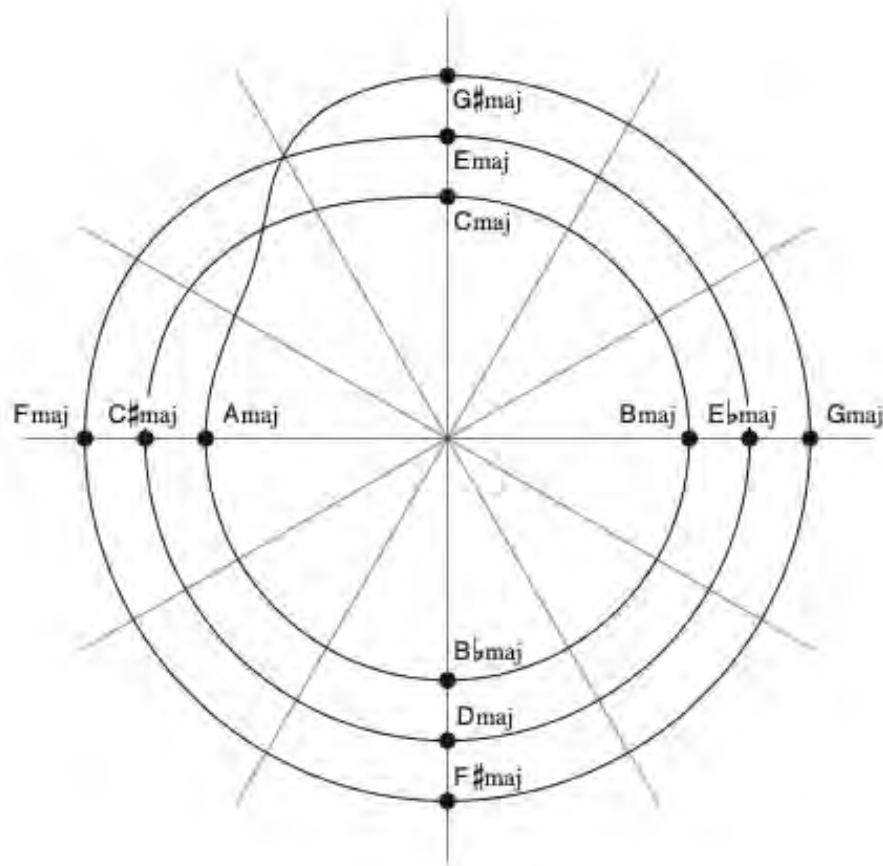
SPIRAL DIAGRAMS

- These diagrams derive from the more complicated orbifolds representing *all* chords of a given size.



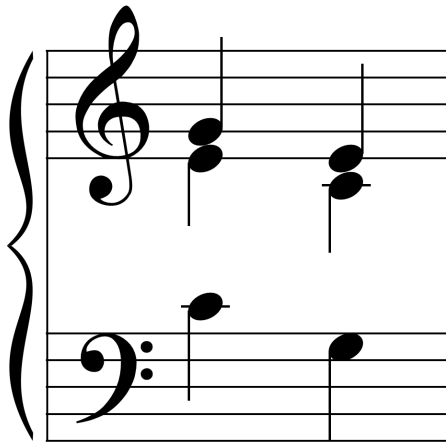
SPIRAL DIAGRAMS

- Clockwise motion descends, counterclockwise motion ascends.

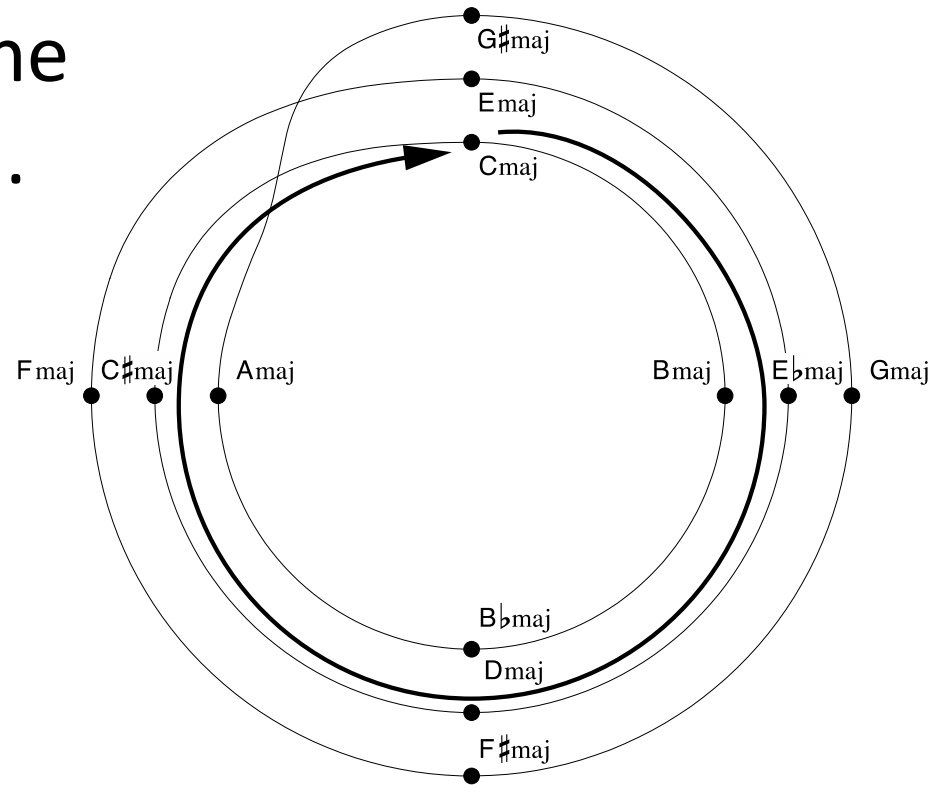


SPIRAL DIAGRAMS

- *Loops* represent transposition along the intrinsic scale (chord).



t_{-1}

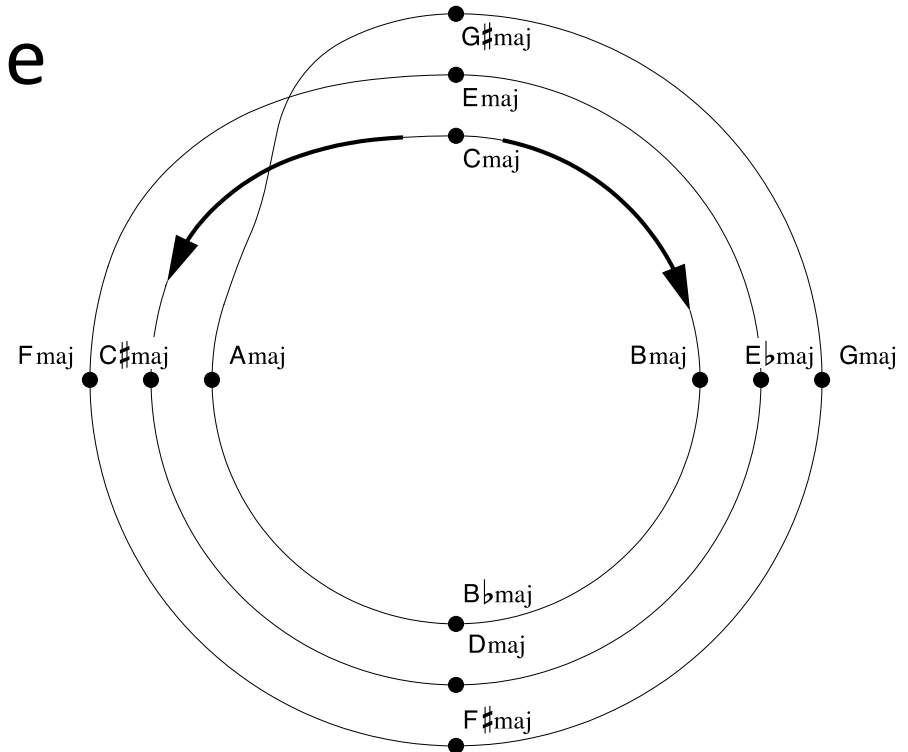


SPIRAL DIAGRAMS

- *Slides* represent transposition along the extrinsic or enclosing collection (scale).

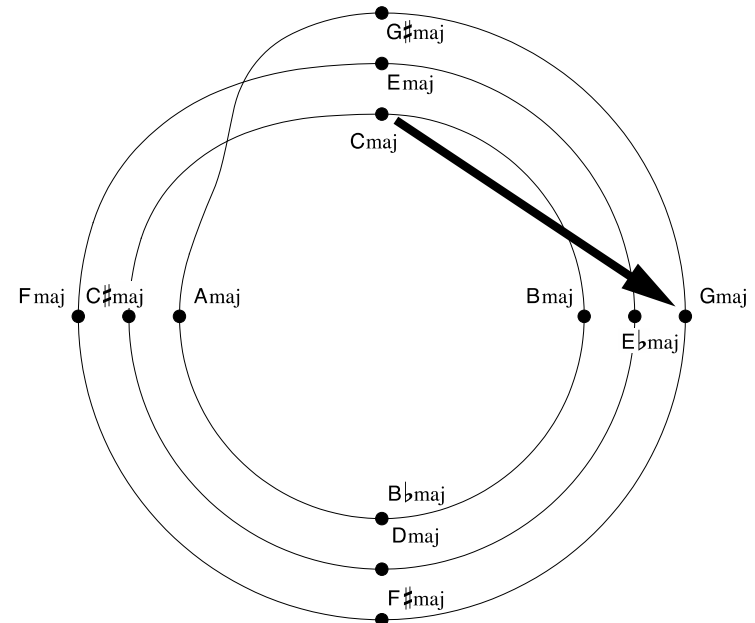
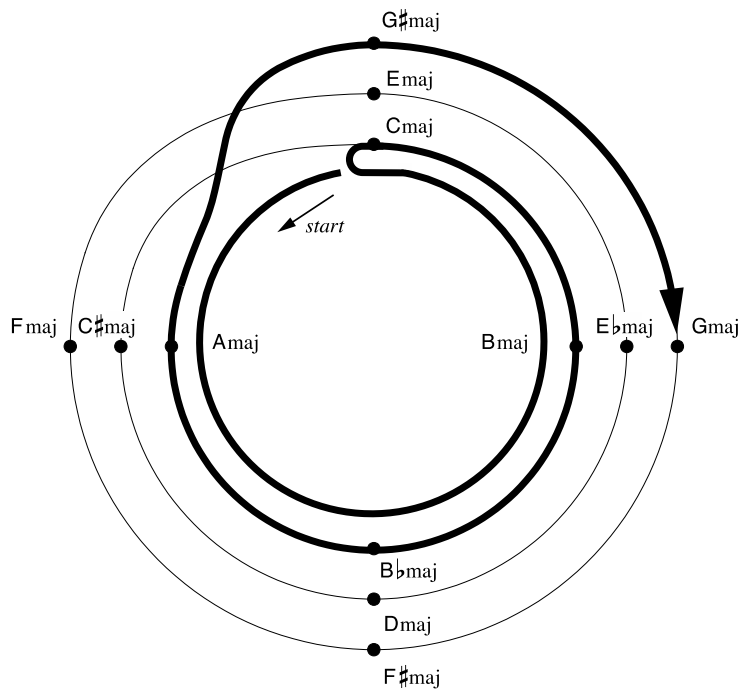


T₁



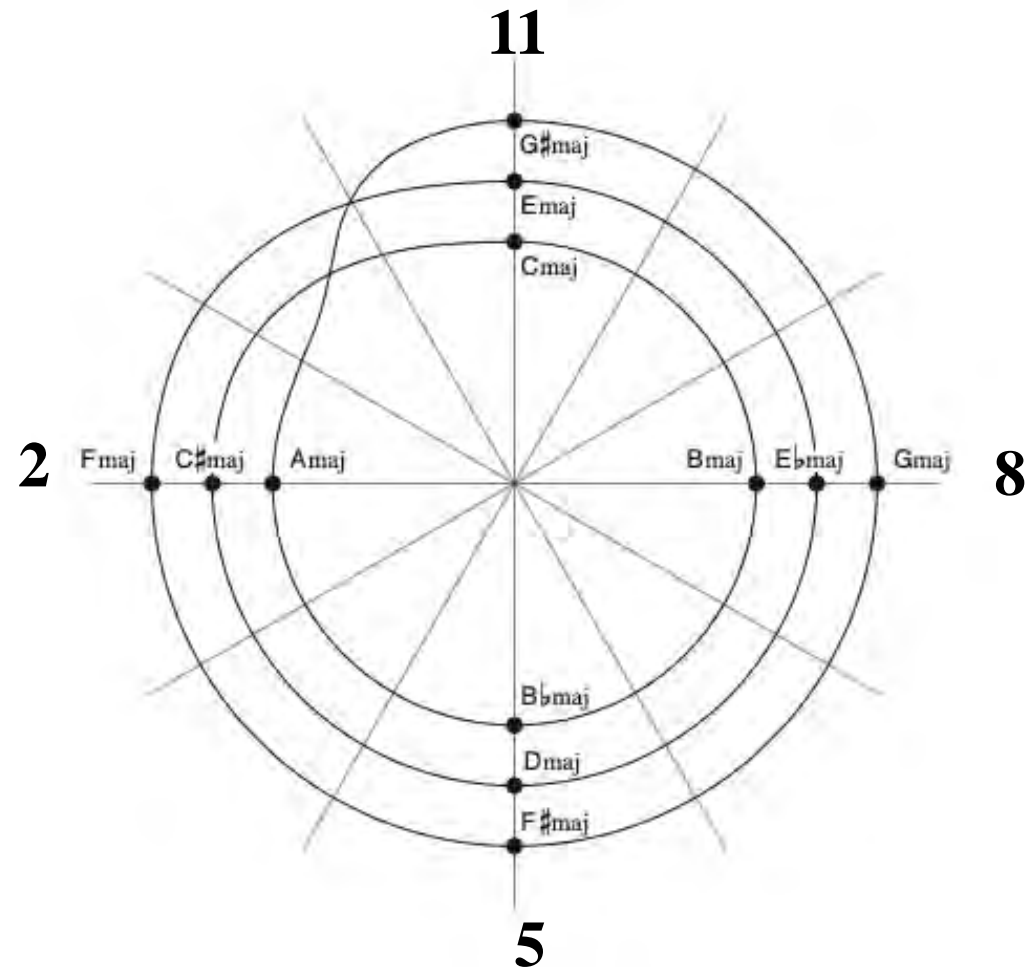
SPIRAL DIAGRAMS

- Two paths represent the same voice leading if they start and end at the same points and involve the same total amount of angular motion.



SPIRAL DIAGRAMS

- Chords' angular position is determined by the sum of their pitch classes (mod 12).
- The sum of the melodic intervals in a voice leading corresponds to the total amount of angular motion in the corresponding path.



SPIRAL DIAGRAMS

<http://www.madmusicalscience.com/cs.html>

<http://www.madmusicalscience.com/multichord.html>

Rock, Beethoven, NR-voice leadings, etc. ...

III. The quadruple hierarchy

THE QUADRUPLE HIERARCHY

1. *Voices* move inside
2. (Possibly scale-like) *chords* that move inside
3. *Scales*

THE QUADRUPLE HIERARCHY

1. *Voices* move inside
2. (Possibly scale-like) *chords* that move inside
3. *Scales* that move inside
4. A chromatic collection.

(which could be 12-tone equal temperament,
some unequal but fixed temperament,
or the continuous space of just intonation)

EXAMPLE: Beethoven's Ninth



EXAMPLE: McCoy Tyner

voice p

chord $T_{-3}t_1$ T_{-1} T_1t_{-1} quantizes T_{-1} T_{-1} quantizes

scale F mixolydian E \flat minor pentatonic

Detailed description: This system contains three staves. The top staff is labeled 'voice' and shows a melodic line in F major with a piano dynamic marking 'p'. The middle staff is labeled 'chord' and shows a sequence of chords: F major (T₋₃t₁), E \flat major (T₋₁), G \flat major (T₁t₋₁), and F major (T₋₁). The word 'quantizes' is written above the first and last two chords, with an equals sign between the first and second chords, and between the last two chords. The bottom staff is labeled 'scale' and shows the F mixolydian scale (F, G, A, B \flat , C, D, E \flat) and the E \flat minor pentatonic scale (E \flat , G \flat , A \flat , C \flat , D \flat).

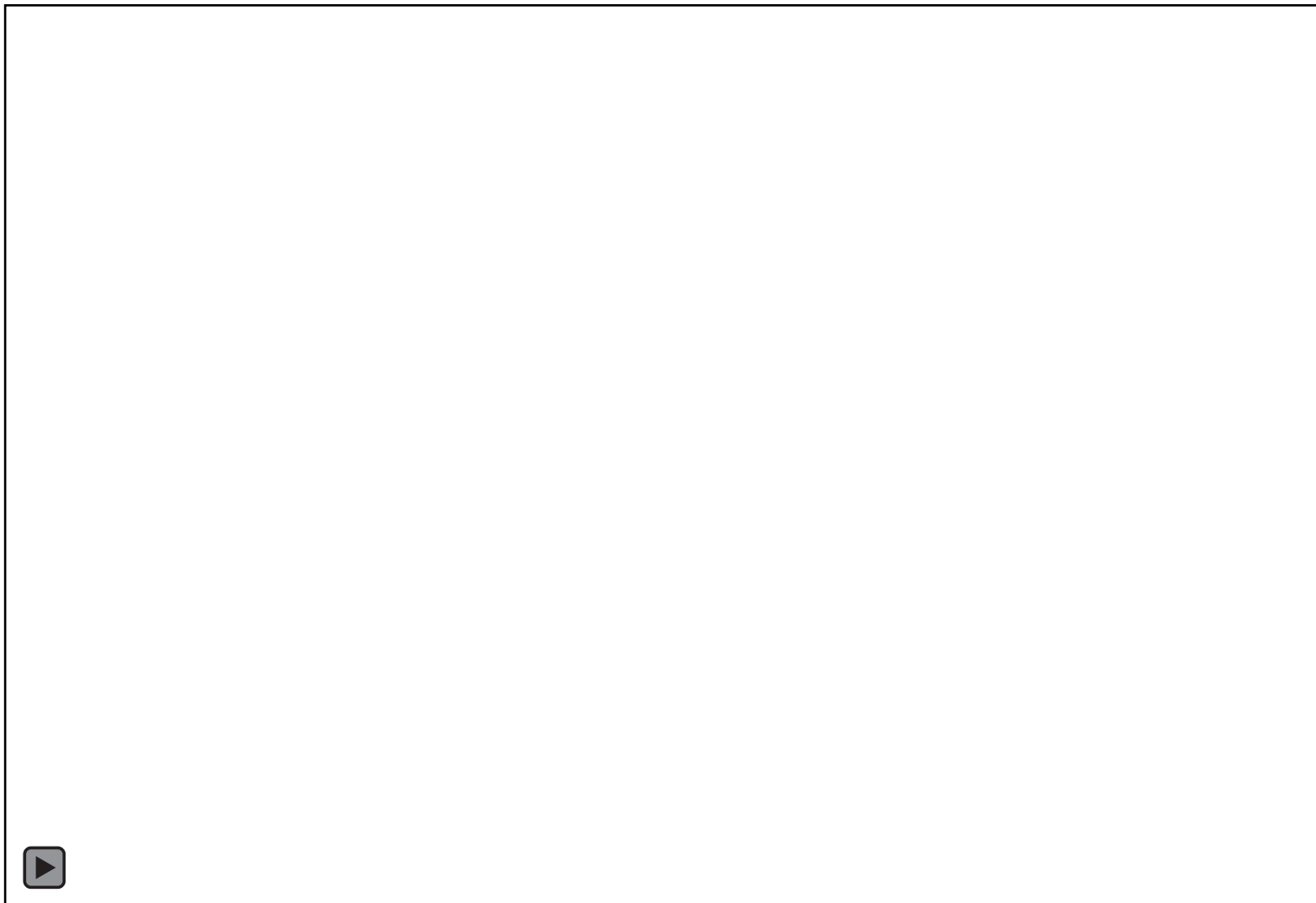
T_1 T_{-1} T_1t_{-1} $T_{-3}t_1$ $T_{-4}t_1$ T_{-1}

F mixolydian

Detailed description: This system contains three staves. The top staff shows a melodic line in F major. The middle staff is labeled 'chord' and shows a sequence of chords: F major (T₁), E \flat major (T₋₁), G \flat major (T₁t₋₁), F major (T₋₃t₁), E \flat major (T₋₄t₁), and F major (T₋₁). The bottom staff is labeled 'scale' and shows the F mixolydian scale (F, G, A, B \flat , C, D, E \flat).



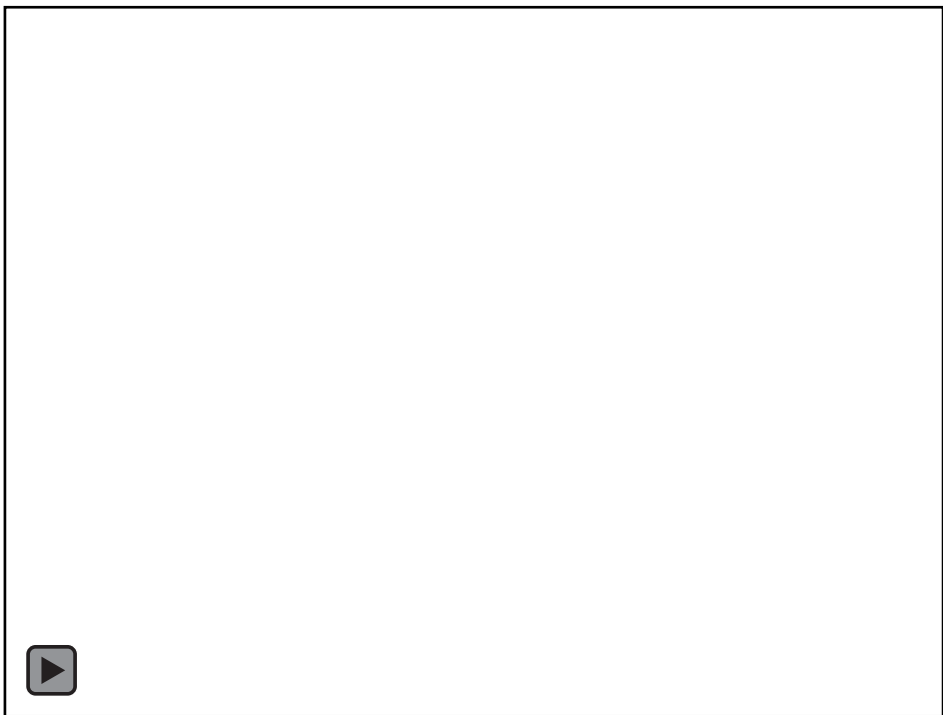
EXAMPLE: McCoy Tyner



EXAMPLE: McCoy Tyner

phrasing on the beat *phrasing moves one eighth note later*

chromatic parallelism G diatonic? chromatic parallelism? pentatonic parallelism



GEOMETRY AND HIERARCHY

- This perspective links the flat spaces of voice-leading geometry with a more hierarchical perspective reminiscent of Schenker.
 - Allows us to describe contrapuntal processes occurring in the background, decorated on the surface in non-obvious ways.
 - *Chords* can be little *scales*.
 - The spiral models provide unified and intuitible representations of each of three different levels.

IV. Making music

let's generalize the Waldstein opening by writing a progression where voices move contrary to abstract harmonies.

COMPOSITION

- We will start with a progression of diatonic fourth chords.



COMPOSITION

- We repeat it, permuting the voices to form a round.

The image shows a musical score for a round, consisting of two staves (treble and bass clef) with a brace on the left. The music is written in a key with one flat (B-flat). The notation consists of four measures. Above the first two measures, there are two arrows labeled t_{-1} pointing to the second and third measures, indicating a transposition of one step down. The third measure contains a B-flat note in the bass clef, with the text "(B \flat for variety)" written below it. The fourth measure continues the sequence. The text "etc." is written to the right of the second arrow.



COMPOSITION

- This gives us a sequence of descending scales.

Diagram illustrating a sequence of descending scales. The notation shows a grand staff with a treble and bass clef. A bracket labeled t_{-1} spans the first two measures, with arrows pointing down to the first and second notes of the first measure.

read scale degrees down in columns:

	1	2	3	4	5	6	7	8
A								
B								
C								
A'								

(back to the top, but adding 1 to each scale degree:
middle C is now scale degree 2!!)

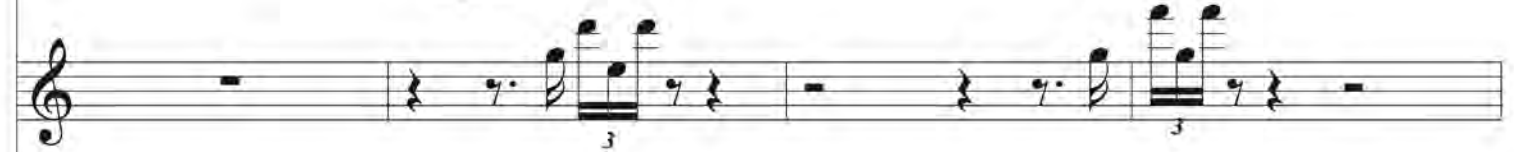
COMPOSITION

- We can create melodic patterns along these scales.

pattern: +1
degrees: 1 2 3 4 5 (= old 4) etc.



pattern: ||: +3, -3, +3, +3 :||
degrees: 5 8 5 8

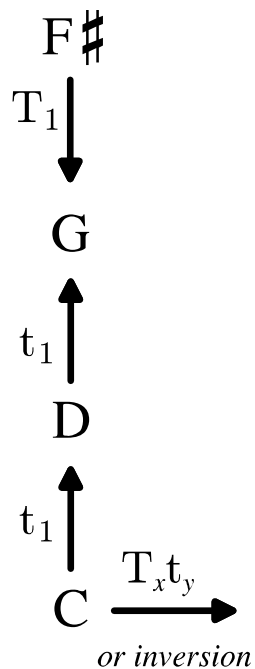


11 14 11 14



MOTIVES

fourth chords with one neighbor note



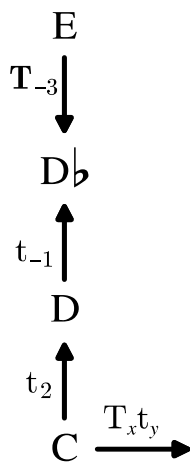
MOTIVES

top voice: 012 chromatic clusters, with a "nonharmonic" tone a minor third above the last note
bottom voice: free

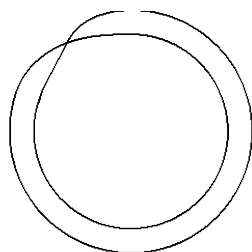
atonal

t_1 t_1T_{-2} $t_{-1}T_1$ t_{-1} $t_{-1}T_6$

"n" "n" "n" (etc.)



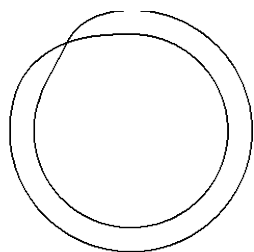
WHEELS WITHIN WHEELS



voices in chord

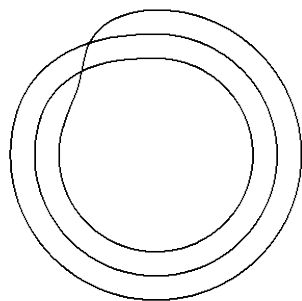
1. *Voices* move inside
2. (Possibly scale-like)
chords

WHEELS WITHIN WHEELS



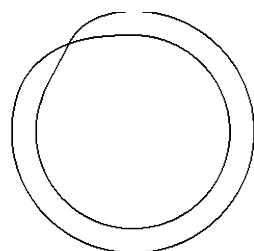
voices in chord

1. *Voices* move inside
2. (Possibly scale-like) *chords* that move inside
3. *Scales*

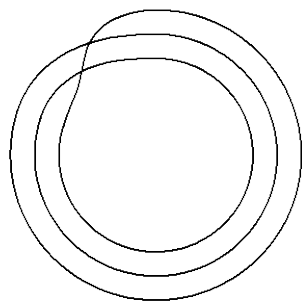


chord in scale

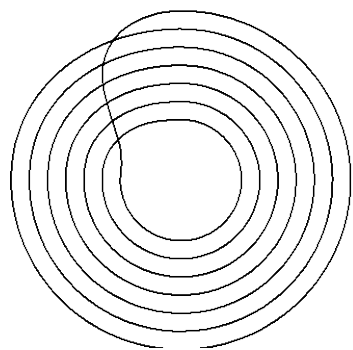
WHEELS WITHIN WHEELS



voices in chord



chord in scale

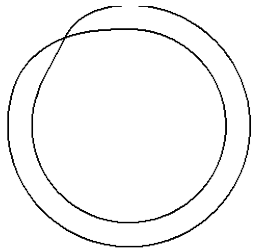


scale in aggregate

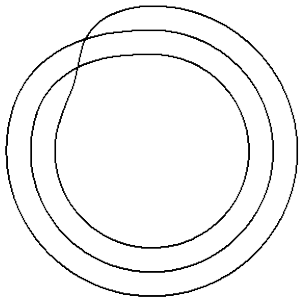
1. *Voices* move inside
2. (Possibly scale-like) *chords* that move inside
3. *Scales* that move inside
4. A chromatic aggregate.

This is a very simple organizational strategy that generates music of extraordinary psychological complexity.

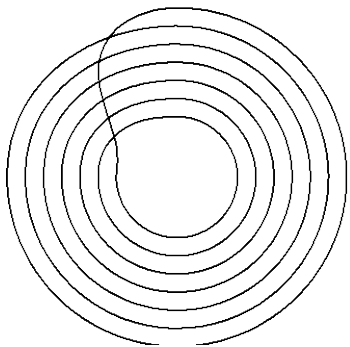
WHEELS WITHIN WHEELS



voices in chord



chord in scale

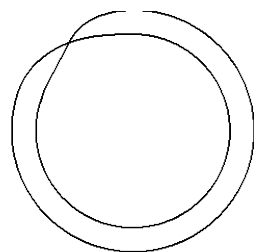


scale in aggregate

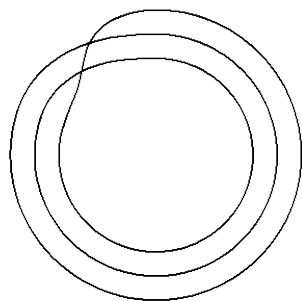
1. *Voices* move inside
2. (Possibly scale-like) *chords* that move inside
3. *Scales* that move inside
4. A chromatic aggregate.

*It is an opportunity for theory!
Algorithmic composition,
compositional assistants.*

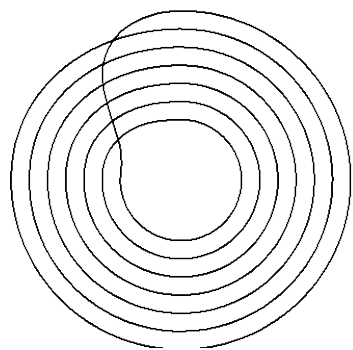
WHEELS WITHIN WHEELS



voices in chord



chord in scale

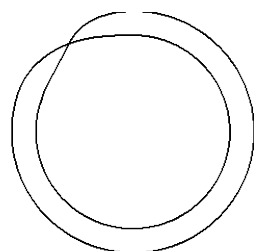


scale in aggregate

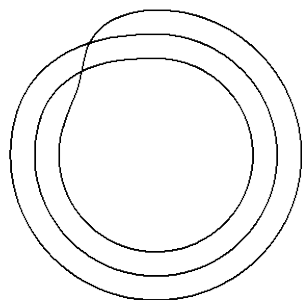
1. *Voices* move inside
2. (Possibly scale-like) *chords* that move inside
3. *Scales* that move inside
4. A chromatic aggregate.

*It is **really hard** to manage three separate levels of motion-along-a-collection, particularly with unfamiliar chords and scales.*

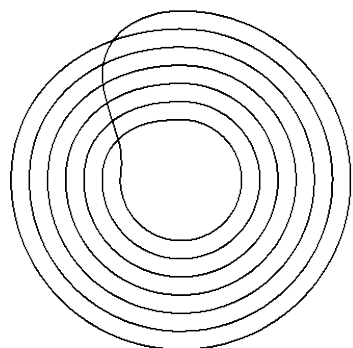
WHEELS WITHIN WHEELS



voices in chord



chord in scale

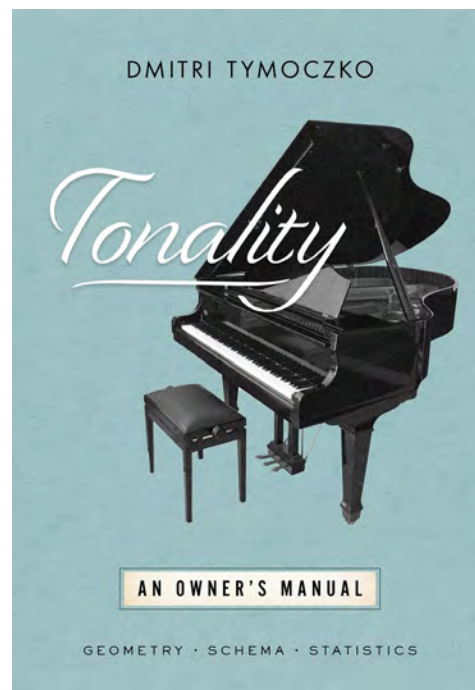
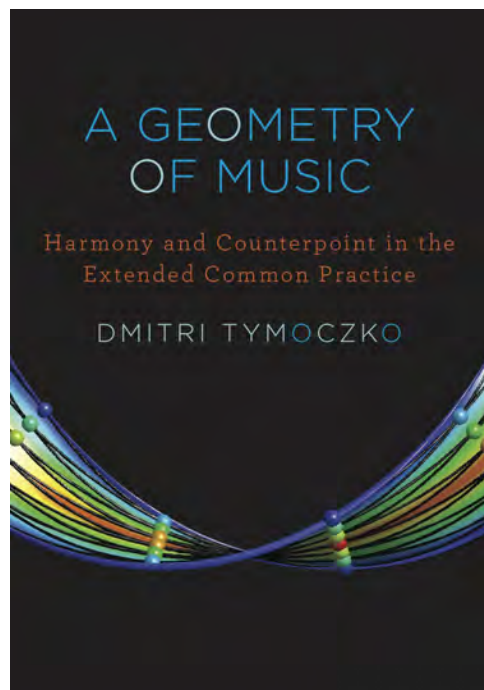


scale in aggregate

1. *Voices* move inside
2. (Possibly scale-like) *chords* that move inside
3. *Scales* that move inside
4. A chromatic aggregate.

*For analysts, it can be **really hard** to recognize these structures in pre-existing music!*

Thank you!

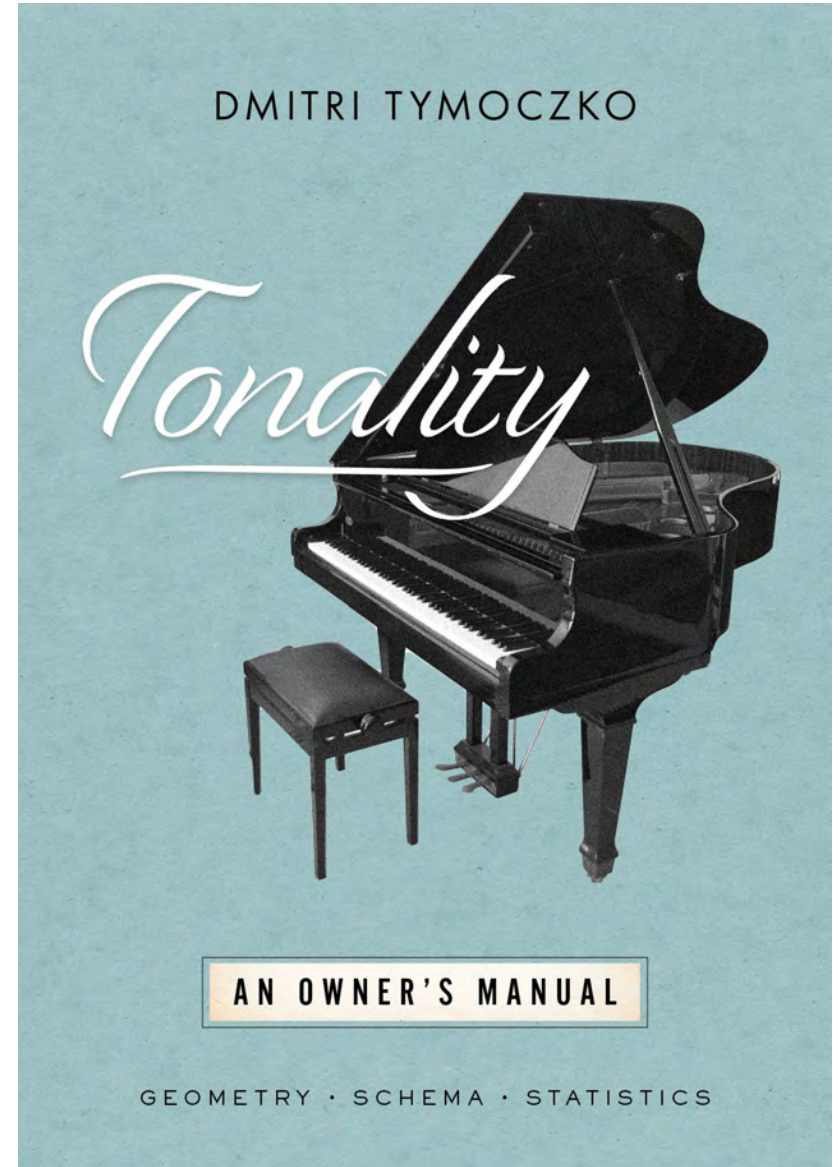
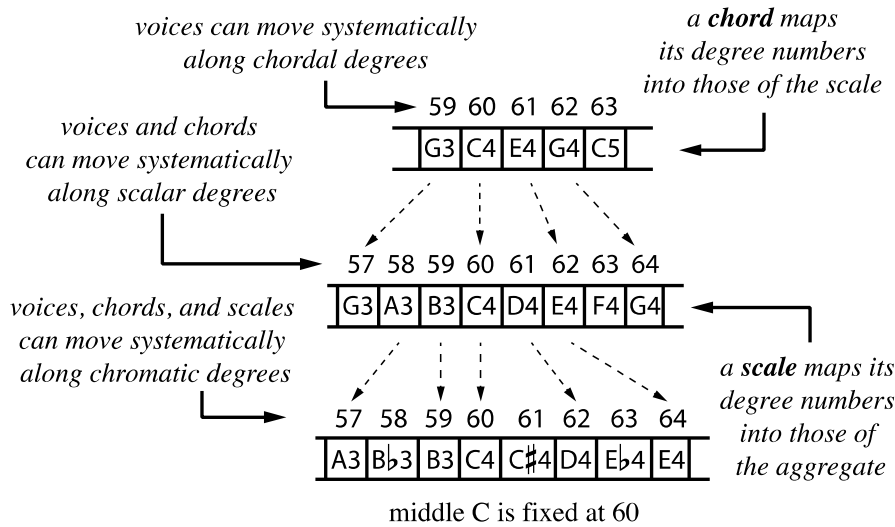


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METHODOLOGY

- I am working in the tradition called “American Set Theory.”
 - I believe transposition is a fundamentally important *symmetry* of musical structure.
 - Inversion is also important though less so.
- My version of set theory is different from the one you know:
 - Hierarchical
 - Collection relative
 - Approximate
- Describes traditional and vernacular practice (folk, rock, jazz, etc.) as well as modern music.



A CHORD IS AN ABSTRACT OBJECT

- Beethoven moves the upper voice systematically along a three-note chord that is itself moving.
 - Chords neither localized in pitch or time.
 - They are essentially scales.

$||: +1 +1 +1 (n) -2 +1 -1 :||$

Beethoven,
Op. 54, II,
mm. 38–41.



⇐ *standard notation
is misleading*

A CHORD IS AN ABSTRACT OBJECT

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||: +1 +1 +1 (n) -2 +1 -1 :||

The musical score illustrates the concept of a chord as an abstract object. The piano part shows a sequence of chords: C major, Eb major, F#m, and G major. The violin part shows a melodic line that moves systematically through the notes of these chords, demonstrating how the chord's structure is preserved even as its pitch and time change.

better:

*octave-free
notation:*

C	E \flat	F \sharp
↓	↓	↓ ⁺¹
C	E \flat	G
↓	↓	↓ ⁺²
C	E \flat	A

EXTRINSIC AND INTRINSIC

- Every musical object, including chords, sets, set classes, motives, etc., defines two scalar contexts:
 1. The *extrinsic* or enclosing scale that contains that object as a proper subset.
 - E.g. quarter-tone, chromatic, diatonic, pentatonic, etc. Even continuous unquantized space.
 2. The *intrinsic* scale consisting of that object's own notes.

EXTRINSIC AND INTRINSIC

- These two contexts define two separate notions of transposition.



BIG T little t

- T_x = transposition by x scale steps.
- **T** for chromatic transposition
- Regular T for any other scale (e.g. diatonic)
- t_y = transposition by y chordal steps.
- $T_x t_y$ = simultaneous transposition along chord and scale.



BIG T little t

These turn out to have a simple and beautiful geometry which we will discuss later.

