First time. Monoidal categorifications, quantum affine alg &
quiver Hecke algebra

* A cluster algebra ^of ______ is $\mathbb{Z}$-subalg of $\mathbb{Z}[x_1^{\pm 1} \dots x_n^{\pm 1}]$ generated by cluster variables, which are groupped into overlapping subsets, called the ______ & clusters are defined inductively by ______ which is controlled by an ______ $\tilde{B}$.

μ_ℓ

cluster

seed $\mathcal{S} = (\underline{\qquad\qquad}, \tilde{B})$ $\xrightarrow{\mu_k}$ $\mu_k(\mathcal{S}) = (\underline{\qquad}, \mu_k(\tilde{B}))$

mutated seed

$\underline{\qquad} = \{x_1', x_2' \dots x_n'\}$ s.t

$x_i = x_i'$ $\forall i \neq k$

$$\mu_k(X)_k = X_k' = \frac{\prod_{b_{ik}>0} X_i^{b_{ik}} + \prod_{b_{ik}<0} X_i^{-b_{ik}}}{X_k}$$

______ : a monomial of cluster variables in a cluster.

ex) $x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$

$x_1'^{b_1} x_2'^{b_2} \dots x_n'^{b_n}$

c.ex) $x_k^{a_k} x_k'^{b_k}$

x_i : cluster variable

One of important motivation of cluster algebra is to observe ______ of elements in ______ basis $\mathcal{B}$ of ______

- More precisely for $\mathbb{b}, \mathbb{b}' \in \mathbb{B}$, when do they q-commute?
  "q-commute" means $\underline{\mathbb{b}\,\mathbb{b}' = q^{*}\,\mathbb{b}'\,\mathbb{b}}$.

- If they do q-commute, $\underline{\qquad\qquad \in q^{\mathbb{Z}}\,\mathbb{B}}$?

We say $\mathbb{b} \in \mathbb{B}$ is ______ if $\underline{\qquad\qquad \in q^{\mathbb{Z}}\mathbb{B}}$.

Dream? ____

____

Two famous conjecture : ① Linear independency : the set of all cluster monomials is $\mathbb{Z}$-linear independent. (Irelli-Keller-Labardini-Fragoso Plamondon)

② Laurent positivity : every cluster variable is in $\mathbb{Z}_{\geq 0}[X_1^{\pm 1} \ldots X_n^{\pm 1}]$ for any cluster $\{X_1 \ldots X_n\}$. (Lee-Schiffler. Gross-Hacking-Keel-Kontsevich)

"Quantum cluster alg" of ______________ is a q-deformation of cluster alg. "Big differce" compared with "cluster alg" is that it is not ______________

"Quantum cluster alg" is a $\mathbb{Z}[q^{\pm 1/2}]$-subal of quantum torus $\underline{\mathbb{Z}[q^{\pm 1/2}]\langle \tilde{X}_1^{\pm 1}, \ldots \tilde{X}_n^{\pm 1}\rangle}$ where $\hat{X}_i \tilde{X}_i^{-1} = \tilde{X}_i^{-1}\tilde{X}_i = 1$

= ______________

↳ Quantum Laurent phenomena

↳ Quantum torus

The ______________ can be encoded a matrix $\Lambda = (\Lambda_{ij})$: i.e

$$\tilde{X}_i \tilde{X}_j = \underline{q^{\Lambda_{ij}}}\, \tilde{X}_j \tilde{X}_i$$

$\xrightarrow{\mu_k}$ quantum seed $(\underline{\mu_k(\mathcal{X})}, \mu_k(\Lambda), \mu_k(\tilde{B}))$

μ_ℓ

quantum seed $(\underline{\mathcal{X} = \{\tilde{x}_1, \ldots \tilde{x}_n\}}, \Lambda, \tilde{B})$

↑ quantum cluster

* variables in a quantum cluster are ______________ !!

and their "q"-commutativity encoded $\mu(\Lambda)$.

Note when we have two elts in a quantum torus, determining whether ______________.

⇝ ①' quantum linear independency

②' quantum Laurent positivity

follow.

$\mathbb{Z}[q^{\pm 1/2}]$ - independency!

$\mathbb{Z}_{\geq 0}[q^{\pm 1/2}]\langle \tilde{X}_1^{\pm 1} \ldots \tilde{X}_n^{\pm 1}\rangle$

(Davison : skew-symmetric)

* Monoidal categorification of "quantum" cluster algebra.
for ①', ②' [this notion is introduced by Hernandez–Leclerc]

$\mathbb{k}$: base field A: $\mathbb{k}$-algebra.
$\mathcal{C}$: monoidal category of A-modules, with auto funtor q
w/ $\exists \otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$. $q : \mathcal{C} \longrightarrow \mathcal{C}$ (Ex: A is a Hopf alg —)

A simple $M \in \mathcal{C}$ is "______" if $\nexists$ non-trivial simples $N_1, N_2 \in \mathcal{C}$ s.t ______

A simple M in $\mathcal{C}$ is said to be "______" if $M \otimes M$ is still ______ in $\mathcal{C}$

For simples M & N, we say M and N ______ if ______

⊛ [Notion of monoidal categorification] Let $\mathcal{A}_{q^{1/2}}$ be a quantum cluster alg

Assume ☆0 $\mathcal{A}_{q^{1/2}} \simeq \mathbb{Z}[q^{\pm 1/2}] \otimes_{\mathbb{Z}[q^{\pm 1}]} K(\mathcal{C})$ ← Grothendieck ring of $\mathcal{C}$ induced by $-\otimes-$.

☆1 q. cluster monomials in $\mathcal{A}_{q^{1/2}} \longleftrightarrow$ classes of ______ of $\mathcal{C}$

☆2 " variables in $\mathcal{A}_{q^{1/2}} \longleftrightarrow$ " of / ______ of $\mathcal{C}$

$\Rightarrow$ ①', ②' follows ! ①' is obvious !!

②' $$X_i' = \frac{\sum_t f_t(q) X_1^{b_{1t}} \cdots X_n^{b_{nt}}}{X_1^{a_1} \cdots X_n^{a_n}}$$

$\Leftrightarrow$

$f_t(q)$ is a coefficient of $[M_1^{b_{1t}} \otimes \cdots \otimes M_n^{b_{nt}}]$ w/ $f_t(q) \in \mathbb{Z}_{\geq 0}[q^{\pm 1/2}]$
in $[M_i'] \cdot [M_1^{a_1} \otimes \cdots \otimes M_n^{a_n}]$

If ☆0, ☆1, ☆2 are satified, we say that $\mathcal{C}$ is a <u>monoidal categorification</u> of $A_{q^{1/2}}$.

However,

Let us first find "known" "good"-circumstance for m. categorification.

In this course, we will mainly consider f-d simple Lie alg $\mathfrak{g}$

my $\mathfrak{g}$: $A_n, B_n, C_n, D_n, E_{6,7,8}, F_4, G_2$

<u>Known</u> • Restricted dual $A_q(\mathfrak{n})$ of $U_q(\mathfrak{g})$ has "quantum cluster algebra" structure

Here the restricted dual means

$$A_q(\mathfrak{n}) = \bigoplus_{\beta} A_q(\mathfrak{n})_\beta \quad \text{where} \quad A_q(\mathfrak{n})_\beta := \mathrm{Hom}_{\mathbb{Q}(q)}(U_q^+(\beta)_{-\beta}, \mathbb{Q}(q))$$

(<u>unipotent quantum coordinate ring</u>)

Then $\dim A_q(\mathfrak{n})_\beta < \infty$ and $A_q(\mathfrak{n})$ has also algebra structure

(<u>non-commutative!</u>)

<u>Note</u>

$$\begin{tikzcd} A_q(\mathfrak{n}) \supset & \overset{\text{integral form}}{A_{\mathbb{Z}[q^{\pm 1/2}]}(\mathfrak{n})} \arrow[d, "q=1"] & \text{has the } \mathbb{B} \,!! \\ & \mathbb{C}[N] & \text{commutative} \end{tikzcd}$$

$\boxed{\text{Convention}}$ For a statement P, $\delta(P) = 1, 0$ according to whether P is true or not.

* "Known" "good-circumstance" for m. Cat

Thm [Geiß-Leclerc-Schröer, Goodearl-Yakimov]
$A_q(\mathfrak{n})$ has a quantum cluster alg structre. → $\mathcal{A}_{qk}$

Thm [Khovanov-Laura, Rouquier (independently)] [Categorification]
The $\mathbb{Z}$-graded finite dimensional category $R^{\mathfrak{g}}$-gmod of quiver Hecke alg $R^{\mathfrak{g}}$ (KLR) corresponding to $\mathfrak{g}$ categorifies $A_q(\mathfrak{n})$.

$$K(R^{\mathfrak{g}}\text{-gmod}) \simeq A_{\mathbb{Z}[q^{\pm 1}]}(\mathfrak{n}) \qquad \rightsquigarrow ☆0.$$

↳ Grothendieck ring of $R^{\mathfrak{g}}$-gmod.

[Categorification] Fujita-H-O-Oya, Scrimshaw-O

Thm [Hernandez-Leclerc, Nakajima, Varagnolo-Vasserot, Kashiwara-O.]
The quantum Grothendieck ring $K_q(\mathcal{C}^{\mathcal{Q}}_{\hat{\mathfrak{g}}})$ of heart subcategory $\mathcal{C}^{\mathcal{Q}}_{\hat{\mathfrak{g}}}$ of f-d integrable modules over quantum affine alg $U_q'(\hat{\mathfrak{g}})$ → $\mathcal{C}$ is iso to $A_q(\mathfrak{n})$ for simply-laced $\mathfrak{g}$ (ADE), while $\hat{\mathfrak{g}}$ is any affine type!

(Ex) $\hat{\mathfrak{g}} = B_n^{(1)} \rightsquigarrow \mathfrak{g} = A_{2n-1}$.

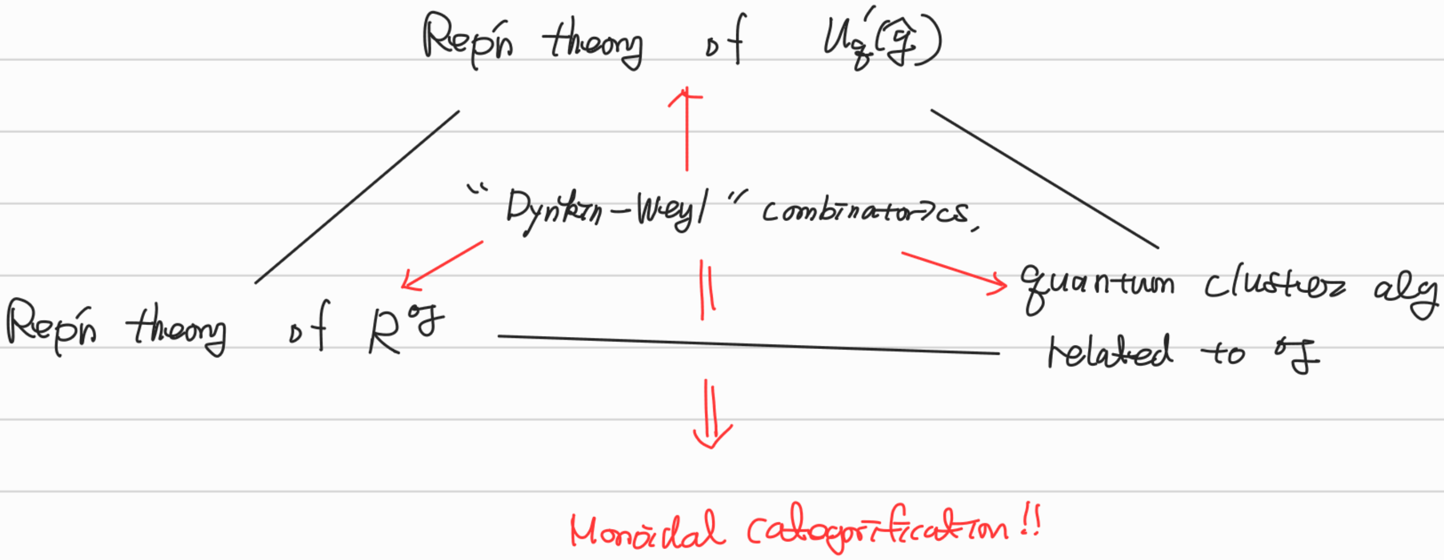
$$K_q(\mathcal{C}^{\mathcal{Q}}_{\hat{\mathfrak{g}}}) \simeq A_q(\mathfrak{n})$$

Note $\mathcal{C}^{\mathcal{Q}}_{\hat{\mathfrak{g}}}$ is "quite small" when we compare it with $\mathcal{C}^0_{\mathfrak{g}}$
the "skeleton" category of the f-d integrable modules over $U_q'(\hat{\mathfrak{g}})$
which is not semi-simple * not braided.

The one of goals of this course is to introduce

"Dynkin-Weyl" combinatorics,

which is located in the center (?) of



"Combinatorics" provides us tools for practical computing!

* preparation

$\mathcal{C}$: monoidal category with $\otimes$ such $\otimes$ is bi-exact.
of A-modules A: $\mathbb{k}$-algebra.

Def $M \in \mathcal{C}$ ① $\mathrm{hd}(M) := $ the largest completely reducible quotient of M
(head of M)

② $\mathrm{soc}(M) := $ the largest completely reducible submodule of M.
(socle of M)

③ For $M, N \in \mathcal{C}$, we denote by $M \nabla N$: the head of $M \otimes N$
$M \Delta N$: the socle of $M \otimes N$.

Def $\mathcal{C}$ is said to be R-category if . . .

(1) $\forall$ $L, M, N \in \mathcal{C}$, if $\underset{\text{submodule}}{X \subset L \otimes M}$ & $Y \subset M \otimes N$ s.t $X \otimes N \subset L \otimes Y$,
then $\exists$ ________ s.t ________ & ________

(2) $\forall$ $L, M, N \in \mathcal{C}$ if $X \subset M \otimes N$ & $Y \subset L \otimes M$ s.t $L \otimes X \subset Y \otimes N$,
then $\exists$ ________ s.t ________ & ________

(3) For simples M & N in $\mathcal{C}$ s.t one of them is real,

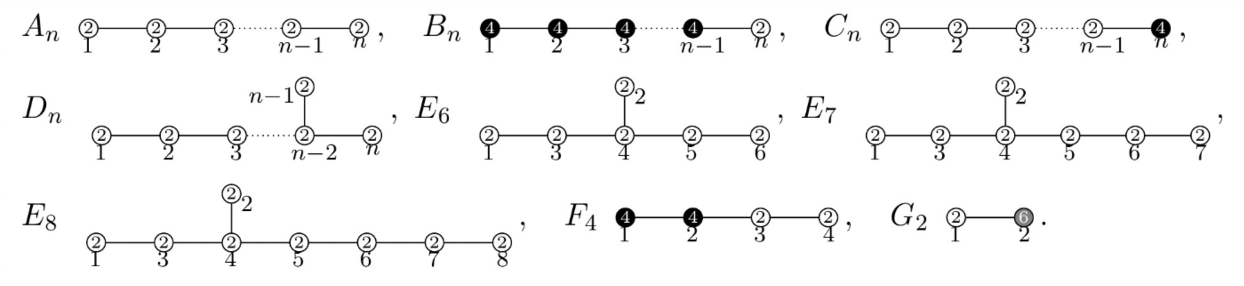
(i) $\mathrm{Hom}_A(M \otimes N, N \otimes M) = \mathbb{k}\, \mathbf{r}_{M,N}$ ← an non-zero elt in the 1 dim'l space.

(ii) $\mathrm{Im}(\mathbf{r}_{M,N}) \simeq M \nabla N$, $\mathrm{Im}(\mathbf{r}_{N,M}) \simeq M \Delta N$ ← (up to constant)

(iii) $M \otimes N$ is simple $\Leftrightarrow$ $M \otimes N \simeq N \otimes M$ $\Leftrightarrow$ $M \nabla N \simeq N \nabla M$
$\Leftrightarrow$ $\mathbf{r}_{M,N}$ & $\mathbf{r}_{N,M}$ are inverses to each other.

* Notations for f-d simple Lie alg $\mathfrak{g}$ and its $U_q(\mathfrak{g})$.

Let $\Delta = (\overset{\text{vertices}}{\Delta_0}, \overset{\text{edges}}{\Delta_1})$ be a Dynkin diagram of f.d simple Lie alg $\mathfrak{g}$



Here $\textcircled{t}_k :\Leftrightarrow (\alpha_k, \alpha_k) = t$

$d(i,j)$ = # edges between i and $j \in \Delta_0$ (Example)

W_Δ : Weyl group of Δ w_0 : the longest elt of W_Δ.

$W_\Delta = \langle s_i \mid i \in \Delta_0 \rangle /$

square rel ______

comm. rel ______

braid rel ______

$A_\Delta = (a_{ij})_{i,j \in \Delta_0}$: Cartan matrix of Δ

$\Pi^\vee = \{h_i\}$ simple coroots. $\Pi = \{\alpha_i\}$ simple roots

Φ^+_Δ : set of positive roots, P_Δ : weight lattice of Δ

R_Δ : root lattice of $\Delta = \bigoplus_{i \in \Delta_0} \mathbb{Z}\alpha_i$

$(\ , \ )$: symmetric bilinear form on weight lattice

$U_q(\mathfrak{g})$: quantum group of $\mathfrak{g}$ (Recall quantum Serre's relation)

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix} e_i^{1-a_{ij}-r} e_j e_i^r = \sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix} f_i^{1-a_{ij}-r} f_j f_i^r = 0 \quad \text{if } i \neq j.$$

Weyl - combinatorics

$\underline{w_0} = s_{i_1} \cdots s_{i_\ell}$ reduced expression of w_0

$\Rightarrow \{ \beta_k^{\underline{w_0}} = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \mid 1 \le k \le \ell \} =$ ______ s.t ______.

(Ex) A_2. $\underline{w_0} = s_1 s_2 s_1$

Define

$\beta_k^{\underline{w_0}} <_{\underline{w_0}} \beta_s^{\underline{w_0}}$ if and only if ______.

$\Rightarrow$ $<_{\underline{w_0}}$ is ______ in the following sense: If $\alpha, \beta \in \Phi^+$ s.t $\alpha+\beta = \gamma \in \Phi^+$

we have either

______ $<_{\underline{w_0}}$ ______ $<_{\underline{w_0}}$ ______ or ______ $<_{\underline{w_0}}$ ______ $<_{\underline{w_0}}$ ______

① Commutation class

For $\underline{w_0}', \underline{w_0}$ of w_0, we write $\underline{w_0}' \sim \underline{w_0}$ if they are connected by commutation relation: $s_i s_j = s_j s_i$ for $i, j \in \Delta_0$ with $d(i,j) > 1$.

(Ex) Δ of A_3 $\quad \underline{w_0} = s_1 s_2 \underline{s_3 s_1} s_2 s_1 \sim \underline{w_0}' = s_1 \underline{s_2 s_1} \underline{s_3} s_2 s_1$

$W_\Delta \simeq \mathfrak{S}_4$

$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad f$

$\underline{w_0}'' = \underline{s_2 s_1 s_2} s_3 s_2 s_1$

The relation $\sim$ is equivalent & we denote $[\underline{w_0}]$ the equ. class of $\underline{w_0}$ and called it ______ of $\underline{w_0}$.

Note w_0 induces an auto $\ast$ of $\Delta_0^{\circlearrowleft}$, $i \longmapsto i^*$, as follows:

$w_0(\alpha_i) = -\alpha_{i^*}$ $\qquad$ (Ex) A_3 $\quad \underline{w_0} = s_1 s_2 s_3 s_1 s_2 s_1$

② r-cluster point

For a red ex $\underline{w_0} = s_{i_1} s_{i_2} \cdots s_{i_\ell}$, $\underline{w_0}' = s_{i_2} \cdots s_{i_\ell} s_{i_1^*}$ is also reduced.

We write $\underline{w_0} \overset{r}{\sim} \underline{w_0}'$. $\rightsquigarrow$ $[\underline{w_0}] \neq [\underline{w_0}']$ $\rightsquigarrow$ $[[\underline{w_0}]]$ is defined ← a r-cluster point

induces $[\underline{w_0}] \overset{r}{\sim} [\underline{w_0}']$

Dynkin-combinatorics.

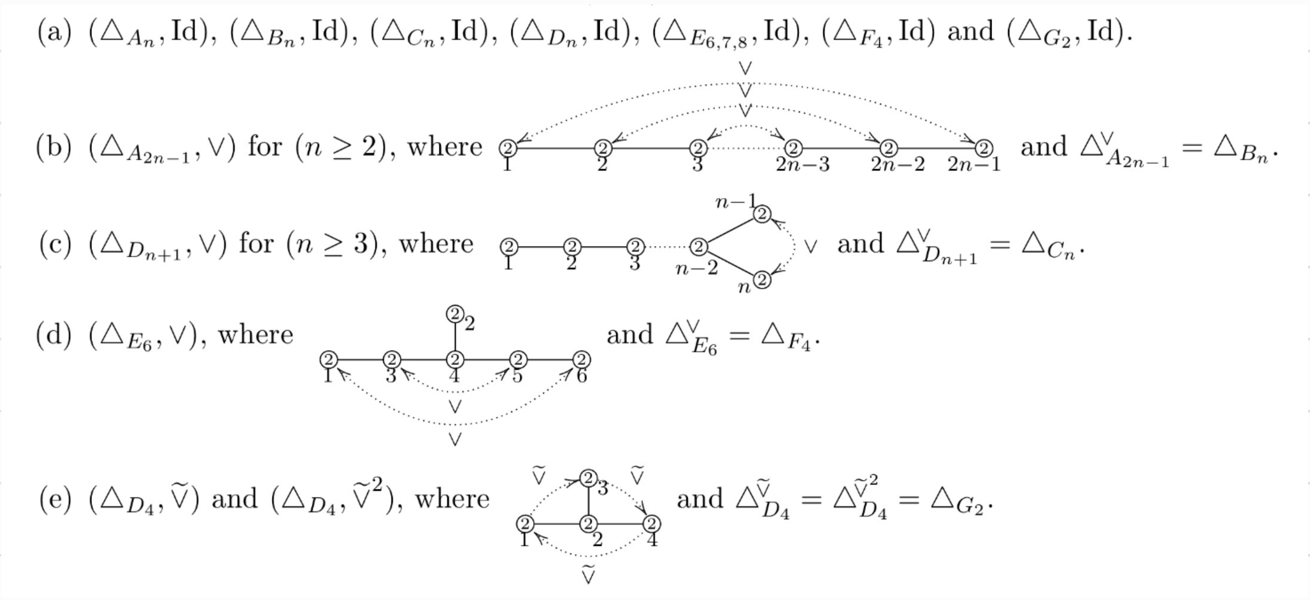
σ : Dynkin diagram automorphism on $\Delta_0 \circlearrowleft$ i.e

$a_{ij}^{\Delta} = \underline{\hspace{3cm}}$ ${}^{\forall} i, j \in \Delta_0$ when $A^{\Delta} = (a_{ij}^{\Delta})_{i,j\in\Delta_0}$ Cartan matrix.

In this course we assume that

$\underline{\hspace{3cm}} \geq 0$ ${}^{\forall} i \in \Delta_0$ —— (*)! (only A_{2n}-case!)

We call (Δ, σ) a $\underline{\hspace{3cm}}$, sometimes denoted by Δ^{σ}



(a) $(\triangle_{A_n}, \mathrm{Id})$, $(\triangle_{B_n}, \mathrm{Id})$, $(\triangle_{C_n}, \mathrm{Id})$, $(\triangle_{D_n}, \mathrm{Id})$, $(\triangle_{E_{6,7,8}}, \mathrm{Id})$, $(\triangle_{F_4}, \mathrm{Id})$ and $(\triangle_{G_2}, \mathrm{Id})$.

(b) $(\triangle_{A_{2n-1}}, \vee)$ for $(n \geq 2)$, where [diagram] and $\triangle^{\vee}_{A_{2n-1}} = \triangle_{B_n}$.

(c) $(\triangle_{D_{n+1}}, \vee)$ for $(n \geq 3)$, where [diagram] and $\triangle^{\vee}_{D_{n+1}} = \triangle_{C_n}$.

(d) $(\triangle_{E_6}, \vee)$, where [diagram] and $\triangle^{\vee}_{E_6} = \triangle_{F_4}$.

(e) $(\triangle_{D_4}, \widetilde{\vee})$ and $(\triangle_{D_4}, \widetilde{\vee}^2)$, where [diagram] and $\triangle^{\widetilde{\vee}}_{D_4} = \triangle^{\widetilde{\vee}^2}_{D_4} = \triangle_{G_2}$.

Here Δ^{σ} : Dynkin diagram obtained from the orbit of σ.

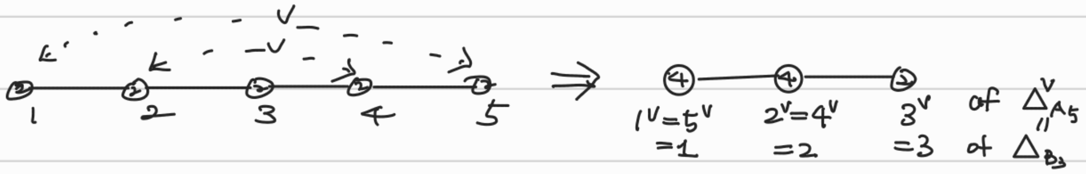
Convention

For $i \in \Delta_0$ & σ, i^{σ}: ____________ $\subset \Delta^{\sigma}$

$\#(i^{\sigma})$ = ________________________ $(=1$ or $r)$

We also "understand" i^{σ} a ____________ of Δ^{σ}.

(EX) $(A_5, \vee)$:



$1^\vee = \{1, 5\}$, $\#(1^\vee) = \#(5^\vee) = 2 = r$, $\#(3^\vee) = 1$

$1^\vee = 5^\vee = 1$ as a vertex of $\Delta_{B_3} = \Delta^\vee_{A_5}$

Def (Δ, σ) A function $\xi : \Delta_0 \longrightarrow \mathbb{Z}$, $i \longmapsto \xi_i$ is a ____________ on (Δ, σ)

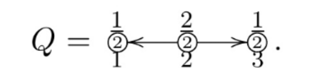
if (1) $^{\forall}$ $i, j \in \Delta_0$ s.t $d(i,j) = 1$ & $\#(i^{\sigma}) = \#(j^{\sigma})$, $|\xi_i - \xi_j| =$ ________

(2) // & $\#(i^{\sigma}) = 1 < \#(j^{\sigma}) = r$, $^{!\exists}$ $\jmath \in j^{\sigma} \subset \Delta_0$ s.t

$|\xi_i - \xi_{\jmath}| =$ ___ and $\xi_{\sigma^{\ell}(\jmath)} = \xi_{\jmath}$ ______ $^{\forall}$ $1 \le \ell < r$.

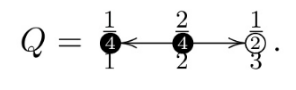
We refer a triple $Q = (\Delta, \sigma, \xi)$ as a ____________ of (Δ, σ).

Example (1) $(\Delta_{A_3}, \mathrm{Id}) \rightsquigarrow$



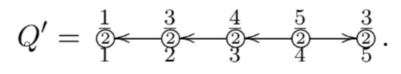
$\#(i^{\mathrm{Id}}) = 1$ $^{\forall}$ $i \in \Delta_0$

(2) $(\Delta_{B_3}, \mathrm{Id}) \rightsquigarrow$



$\#(i^{\mathrm{Id}}) = 1$ $^{\forall}$ $i \in \Delta_0$

(3) $(\Delta_{A_5}, \vee) \rightsquigarrow$



$\#(3^\vee) = 1$

$\#(1^\vee) = \#(2^\vee) = \#(4^\vee) = \#(5^\vee)$

$= 2$

50분 ⊥

For a Q-datum $\mathcal{Q}=(\Delta,\sigma,\xi)$, we say $i\in\Delta_0$ ________ if

$\exists\forall$ $j\in\Delta_0$ s.t $d(i,j)=1$.

____

$*$ Q-datum $\mathcal{Q}=(\Delta,\sigma,\xi)$, i : source of $\mathcal{Q}$, we define $s_i\xi:\Delta_0\longrightarrow\mathbb{Z}$

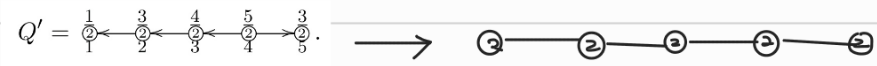
s.t $(s_i\xi)_j = \xi_j -$ ________________

$\Rightarrow$ $\underline{s_i\mathcal{Q} = (\qquad\qquad\qquad)}$ is another Q-datum of (Δ,σ).

(EX) (1)' 2 is

(2)' 2 is

(3)' 4 is



Weyl-Dynkin Combinatorics!!

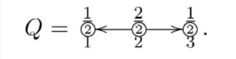
$\mathcal{Q}:(\Delta,\sigma,\xi)$ Q-datum $\underline{w_0}= s_{i_1}s_{i_2}\cdots s_{i_\ell}$ red exp of $w_0\in W_\Delta$

$\underline{w_0}$ is said to be adapted to $\mathcal{Q}$ if

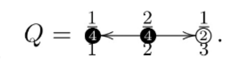
i_k is a source of $s_{i_{k-1}}\cdots s_{i_2}s_{i_1}\mathcal{Q}$ $\exists\forall$ $1\le k\le\ell$.

(Exercise)

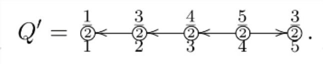
(1)'' $s_2s_1s_3s_2s_1s_3$



(2)'' $s_2s_1s_3s_2s_1s_3s_2s_1s_3$



(3)'' $s_4s_3s_2s_5s_3s_1s_4s_3s_2s_5s_3s_1s_4s_3s_5$



Thm (Bedard + . . . , Fujita-O, Kashiwara-O) .

$\mathcal{Q}=(\Delta,\sigma,\xi)$ Q-datum

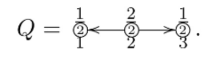
(1) $\exists$ $\underline{w_0}$ of $w_0\in W_\Delta$ s.t $\underline{w_0}$ is adapted to $\mathcal{Q}$

(2) All of Q-adapted red. exps forms a comm. class ______ of $w_0\in W_\Delta$

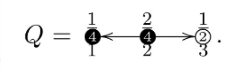
(3) For $\mathcal{Q}'=(\Delta,\sigma,\xi')$, $[\mathcal{Q}]\neq[\mathcal{Q}']$ unless $\exists k\in\mathbb{Z}$ s.t ______ $\forall i\in\Delta_0$. But $[\mathcal{Q}]\overset{r}{\sim}[\mathcal{Q}']$. Furthermore, all $[\mathcal{Q}]$'s that shares Δ & σ forms a r-clust pt $\underline{[\![\Delta]\!]}$.

For each $\mathcal{Q}$, $!\exists$ $\tau_\mathcal{Q}\in W_\Delta\rtimes\langle\sigma\rangle$ satisfying certain property !

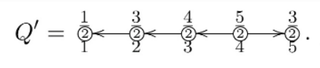
(EX) (1)''' $\tau_\mathcal{Q}=s_2s_1s_3$



(2)''' $\tau_\mathcal{Q}=s_2s_1s_3$



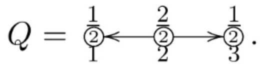
(3)''' $\tau_{\mathcal{Q}'}=s_4s_3s_2\,V$



We call $\tau_\mathcal{Q}$ the ______________________ .

For Q-datum $Q=(\Delta,\sigma,\xi)$, $\underline{w_0}=s_{i_1}\cdots s_{i_\ell}\in[Q]$ and $1\le k\le \ell$,
we assign a pair $\phi^{-1}_{\underline{w_0},\xi}(\beta^{\underline{w_0}}_k,0):=(i,p)\in\Delta_0\times\mathbb{Z}$ as follows:

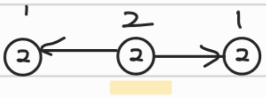
$i=$ ____ and $p=$ ________________

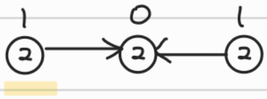
Example (1) $Q=$ 

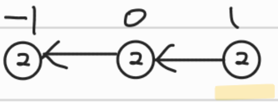
$\underline{w_0}=\overset{1}{s_2}\,\overset{2}{s_1}\,\overset{3}{s_3}\,\overset{4}{s_2}\,\overset{5}{s_1}\,\overset{6}{s_3}$

$\beta^{\underline{w_0}}_1=\alpha_2$ $\beta^{\underline{w_0}}_2=\alpha_1+\alpha_2$ $\beta^{\underline{w_0}}_3=\alpha_2+\alpha_3$

(2, 2) (1, 1) (3, 1)



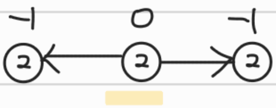


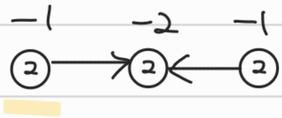


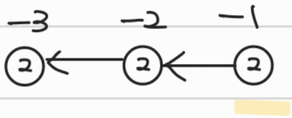
i/p	-1	0	1	2
1				
2				
3				

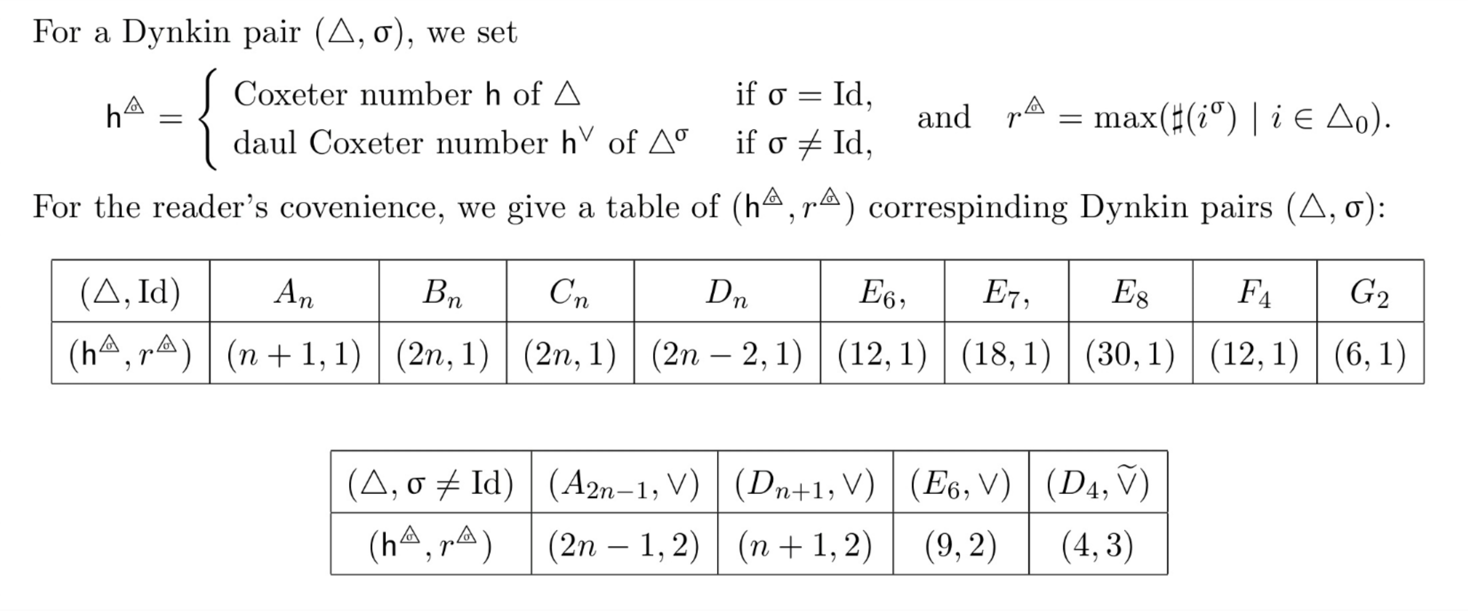
$\beta^{\underline{w_0}}_4=\alpha_1+\alpha_2+\alpha_3$ $\beta^{\underline{w_0}}_5=\alpha_3$ $\beta^{\underline{w_0}}_6=\alpha_1$

(2, 0) (1, -1) (3, -1)









For a Dynkin pair $(\triangle,\sigma)$, we set

$$\mathsf{h}^{\sigma\!\!\!\!\!\triangle}=\begin{cases}\text{Coxeter number } \mathsf{h} \text{ of } \triangle & \text{if } \sigma=\mathrm{Id},\\ \text{daul Coxeter number } \mathsf{h}^\vee \text{ of } \triangle^\sigma & \text{if } \sigma\neq\mathrm{Id},\end{cases}\quad\text{and}\quad r^{\sigma\!\!\!\!\!\triangle}=\max(\sharp(i^\sigma)\mid i\in\triangle_0).$$

For the reader's covenience, we give a table of $(\mathsf{h}^{\sigma\!\!\!\!\!\triangle},r^{\sigma\!\!\!\!\!\triangle})$ correspinding Dynkin pairs $(\triangle,\sigma)$:

$(\triangle,\mathrm{Id})$	A_n	B_n	C_n	D_n	$E_6,$	$E_7,$	E_8	F_4	G_2
$(\mathsf{h}^{\sigma\!\!\!\!\!\triangle},r^{\sigma\!\!\!\!\!\triangle})$	$(n+1,1)$	$(2n,1)$	$(2n,1)$	$(2n-2,1)$	$(12,1)$	$(18,1)$	$(30,1)$	$(12,1)$	$(6,1)$

$(\triangle,\sigma\neq\mathrm{Id})$	$(A_{2n-1},\vee)$	$(D_{n+1},\vee)$	$(E_6,\vee)$	$(D_4,\widetilde{\vee})$
$(\mathsf{h}^{\sigma\!\!\!\!\!\triangle},r^{\sigma\!\!\!\!\!\triangle})$	$(2n-1,2)$	$(n+1,2)$	$(9,2)$	$(4,3)$

Lemma ξ : a ht fn on (Δ,σ). $Q=(\Delta,\sigma,\xi)$ Q-datum, $\underline{w_0}\in[Q]$.

(a) $\phi^{-1}_{\underline{w_0},\xi}(\beta,0) \neq \phi^{-1}_{\underline{w_0},\xi}(\alpha,0)$ ____________

(b) For $\underline{w_0}\sim\underline{w_0}'\in[Q]$ and $\beta\in\Phi^+$, ________________________

(c) Let $\widehat{I}_Q=\{(i,k)\in\Delta_0\times\mathbb{Z} \mid$ ________________ $\}$. For $k\in\mathbb{Z}$, we set

$$\phi_Q^{-1}(\beta,k) := (\underline{\qquad\qquad\qquad})^{*\overset{k\text{-times}}{}}$$

Then $\phi_Q : \widehat{I}_Q \xrightarrow[\text{onto}]{1-1} \Phi^+_\Delta\times\mathbb{Z}$. Here $i^{*k} := (i^*)^{*\cdots *}$.

Note that, for Q-data Q, Q' on (Δ,σ), if

$$\xi_i-\xi'_i\equiv 0 \mod 2\#(i^\sigma) \quad \text{for all } i\in\Delta_0$$

then $\widehat{I}_Q=\widehat{I}_{Q'}$. Thus we can use ________________ by fixing the parity of height fns on (Δ,σ).

Def (Δ,σ) : Dynkin pair. The ____________ $\widehat{\Delta}=(\widehat{\Delta}_0,\widehat{\Delta}_1)$ is defined as

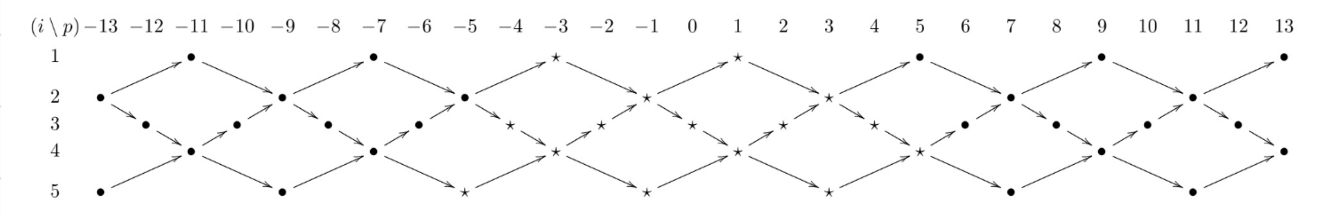
$\widehat{\Delta}_0=\widehat{I}_{\Delta}$

$\widehat{\Delta}_1=\{(i,p)\xrightarrow{-a_{ij}}(j,s) \mid (i,p),(j,s)\in\widehat{\Delta}_0,$ ________________ $\}$

Here $(i,p)\xrightarrow{-a_{ij}}(j,s)$ denotes that we assign $(-a_{ij})$-many arrow from (i,p) to (j,s)

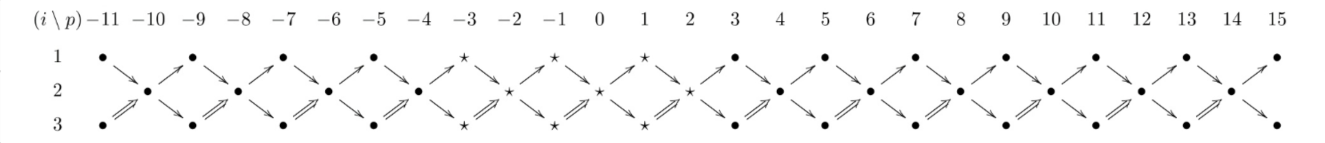
(Example) (3) $(A_5,\vee)$ $Q=\overset{1}{②}\leftarrow\overset{3}{②}\leftarrow\overset{4}{②}\leftarrow\overset{5}{②}\rightarrow\overset{3}{②}$

$\widehat{\Delta}=$



(2) (B_3, Id) $Q=\overset{1}{④}\leftarrow\overset{2}{④}\rightarrow\overset{1}{②}$

$\widehat{\Delta}=$

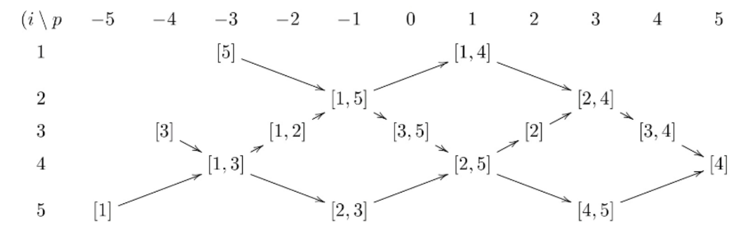


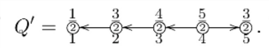
- For Q-datum $\mathcal{Q}$ on (Δ, σ), We denote by $\Gamma^{\mathcal{Q}} = (\Gamma^{\mathcal{Q}}_0, \Gamma^{\mathcal{Q}}_1)$ the full subquiver of $\widehat{\triangle}$ whose set of vertices is ______________ and call it (______________)

Convention For $\Gamma^{\mathcal{Q}}$, we use positive roots for their vertices.

(Ex) $(3)'$

$\Gamma_{\mathcal{Q}} =$



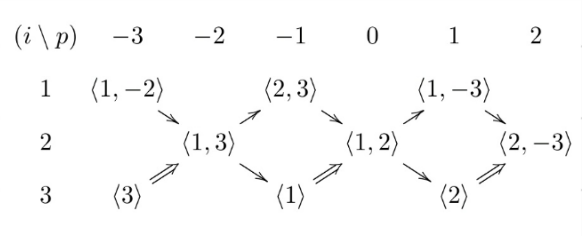


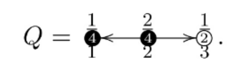
$[a,b] = \sum_{k=a}^{b} \alpha_k$

$[c] = \alpha_c$

$(2)'$

$\Gamma_{\mathcal{Q}} =$





$\langle a, \pm b \rangle = \varepsilon_a \pm \varepsilon_b$

$\langle c \rangle = \varepsilon_c$

$\varepsilon_k = \sqrt{2}\, \epsilon_k$

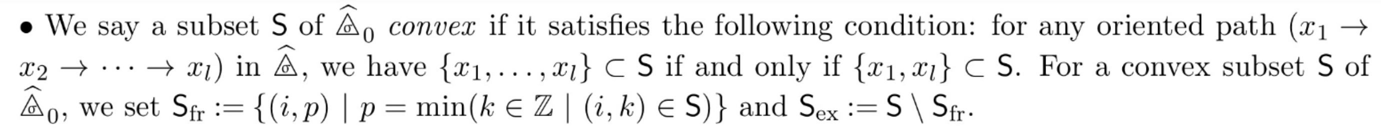
<u>Note</u> for $a_{ij} < 0$, we have $|a_{ij}| = 1$ or $|a_{ji}| = 1$.

<u>Thm</u> $\Gamma^{\mathcal{Q}}$ with labelling via Φ^+ realizes the convex partial order $\prec_{[\mathcal{Q}]}$ on Φ^+ defined as follows:

$\alpha \prec_{[\mathcal{Q}]} \beta \iff \alpha \underline{\prec}_{w} \beta$ $^{\forall}\underline{w}' \in [\mathcal{Q}]$

Here "realization" means $\alpha \prec_{[\mathcal{Q}]} \beta \iff {}^{\exists}$a path from β to α in $\Gamma^{\mathcal{Q}}$.

- Convex subset of $\widehat{\triangle}_0$



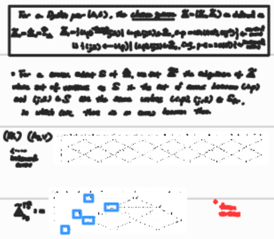
• We say a subset S of $\widehat{\triangle}_0$ *convex* if it satisfies the following condition: for any oriented path $(x_1 \to x_2 \to \cdots \to x_l)$ in $\widehat{\triangle}$, we have $\{x_1, \ldots, x_l\} \subset \mathsf{S}$ if and only if $\{x_1, x_l\} \subset \mathsf{S}$. For a convex subset S of $\widehat{\triangle}_0$, we set $\mathsf{S}_{\mathrm{fr}} := \{(i,p) \mid p = \min(k \in \mathbb{Z} \mid (i,k) \in \mathsf{S})\}$ and $\mathsf{S}_{\mathrm{ex}} := \mathsf{S} \setminus \mathsf{S}_{\mathrm{fr}}$.

↳ frozen vertices

↳ exchange vertices.

* Weyl–Dynkin Combinatoric ⇝ Quantum cluster algebra.

☒ Exchange matrix $\widetilde{B}$



∘ For a convex subset S of $\widehat{\triangle_{6}}$, we associate an exchange matrix $\widetilde{B}_S = (b_{(i,p),(j,s)})_{(i,p)\in S,\ (j,s)\in S_{ex}}$ as follows :

$$b_{(i,p),(j,s)} = \begin{cases} (-1)^{\delta(s>p)}\, a_{ij}^{\triangle} & \text{if } |p-s| = \underline{\hspace{6cm}} \\ (-1)^{\delta(s>p)} & \text{if } |p-s| = \underline{\hspace{4cm}} \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$\widetilde{B}_S$ is $\underline{\hspace{5cm}}$ with $D_S := \mathrm{diag}(\, d_{i,p} \mid d_{i,p} = \underline{\hspace{3cm}}\,)$

i.e., $(D_S \widetilde{B}_S)_{(i,p),(j,s)} = \underline{\hspace{10cm}}$

In particular, we set $\widetilde{B}_Q := \underline{\hspace{1.5cm}}$

☒ Quantum torus

Let $\mathcal{Q} = (\Delta, \sigma, \xi)$ be a $\mathcal{Q}$-datum on (Δ, σ). q = indeterminate

<u>Def</u> $\mathcal{X}_q(\mathcal{Q})$: $\mathbb{Z}[q^{\pm 1/2}]$-alg generated by $\{\widetilde{X}_{i,p}^{\pm 1} \mid (i,p) \in \widehat{\Delta}_0\}$
subject to the following relation:

(a) $\widetilde{X}_{i,p} \widetilde{X}_{i,p}^{-1} = \widetilde{X}_{i,p}^{-1} \widetilde{X}_{i,p} = 1$

(b) $\widetilde{X}_{i,p} \widetilde{X}_{j,s} = q^{(i,p\,;\,j,s)_{\mathcal{Q}}} \widetilde{X}_{j,s} \widetilde{X}_{i,p}$ where

$$(i,p\,;\,j,s)_{\mathcal{Q}} = \underline{\hspace{8cm}}$$

where $\underline{\hspace{4cm}}$ and $\underline{\hspace{4cm}} \in \Phi_\Delta^+ \times \mathbb{Z}$

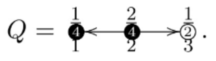
(c) We set $\mathrm{wt}_{\mathcal{Q}}(\widetilde{X}_{i,p}) = \underline{\hspace{2cm} \in \Phi_\Delta}$

Then for a monomial $m = q^{s/2} \widetilde{X}_{i_1,p_1}^{a_1} \cdots \widetilde{X}_{i_r,p_r}^{a_r}$.

$$\mathrm{wt}_{\mathcal{Q}}(m) := \sum_{k=1}^{r} \underline{\hspace{4cm}}$$

$\rightsquigarrow$ $\mathcal{X}_q(\mathcal{Q})$ is R_Δ-graded.

(Example)

$Q =$ 

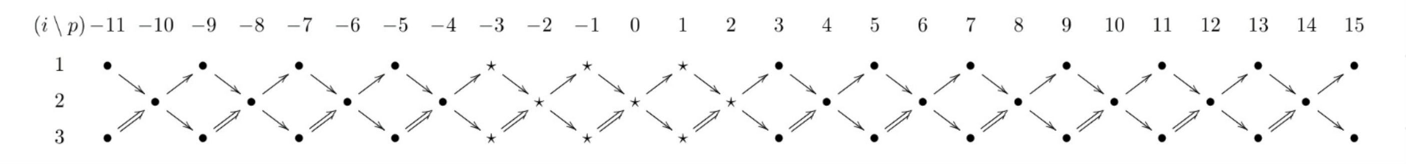
$$\Gamma^{Q} = \begin{tikzcd} (i \setminus p) & -3 & -2 & -1 & 0 & 1 & 2 \\ 1 & \langle 1,-2\rangle \arrow[dr] & & \langle 2,3\rangle \arrow[dr] & & \langle 1,-3\rangle \arrow[dr] & \\ 2 & & \langle 1,3\rangle \arrow[ur] \arrow[dr] & & \langle 1,2\rangle \arrow[ur] \arrow[dr] & & \langle 2,-3\rangle \\ 3 & \langle 3\rangle \arrow[ur, Rightarrow] & & \langle 1\rangle \arrow[ur, Rightarrow] & & \langle 2\rangle \arrow[ur, Rightarrow] & \end{tikzcd}$$

of $Q = (\Delta_{B_3}, \mathrm{id}, \xi)$

$$\widetilde{X}_{2,2}\widetilde{X}_{2,-2} = \underline{\qquad\qquad\qquad\qquad} \widetilde{X}_{2,-2}\widetilde{X}_{2,2} \quad \Big(\because \ \phi_Q(2,2) = (\langle 2,-3\rangle, 0)$$
$$= \underline{\qquad} \widetilde{X}_{2,-2}\widetilde{X}_{2,2} \qquad \phi_Q(2,-2) = (\langle 1,3\rangle, 0) \ \Big)$$

$\mathrm{wt}_Q(\widetilde{X}_{1,-3}) = \underline{\qquad\qquad\qquad\qquad}$

$\mathrm{wt}_Q(\widetilde{X}_{2,0}\widetilde{X}_{2,2}\widetilde{X}_{1,1}^{-1}\widetilde{X}_{3,1}^{-2}) = \underline{\qquad}$



$$f := q\widetilde{X}_{1,1} + q\widetilde{X}_{2,2}\widetilde{X}_{1,3}^{-1} + q^2\widetilde{X}_{3,3}^{2}\widetilde{X}_{2,4}^{-1} + (q+q^{-1})\widetilde{X}_{3,3}\widetilde{X}_{3,5}^{-1}$$
$$+ q^2\widetilde{X}_{2,4}\widetilde{X}_{3,5}^{-2} + q\widetilde{X}_{1,5}\widetilde{X}_{2,6}^{-1} + q^{-1}\widetilde{X}_{1,7}^{-1}$$

$\Rightarrow$ f is homogeneous of $\underline{\qquad\qquad\qquad\qquad}$.

$\boxed{\text{Lemma}}$ (HL, FO, KO)

For any Q-data Q, Q' on (Δ,σ),

$\underline{\qquad\qquad\qquad\qquad\qquad\qquad}$ for any $(i,p), (j,s) \in \widehat{\Delta}_0$

$\Rightarrow$ $\chi_q^{\widehat{\Delta}}$ is $\underline{\qquad\qquad\qquad}$ for each Dynkin pair (Δ,σ).

Now let us denote $\underline{\qquad\qquad\qquad\qquad}$ instead of $(i,p\,;j,s)_Q$.

For monomials $m = \overrightarrow{\prod}\widetilde{X}_{i_k,p_k}^{a_k}$, $m' = \overrightarrow{\prod}\widetilde{X}_{j_t,s_t}^{b_t}$, we define

$$(m,m')_{\widehat{\Delta}} := \sum_{k,s} \underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad} \quad \text{for any } Q \text{ on } \widehat{\Delta}.$$

Thm (HL, GY-FO, KO, FHOO, Jang-Lee-O)

$\mathcal{Q} = (\Delta, \sigma, \xi)$ Q-datum $\mathcal{A}_q(\mathfrak{n})$: unip. quantum cor. alg asso / Δ

(i) $\exists$ ________ $\mathbb{Q}(q^{1/2})$-alg homo

$$\bar{\iota}_{\mathcal{Q}} : \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1}]} \mathcal{A}_{\mathbb{Z}[q^{\pm 1}]}(\mathfrak{n}) \hookrightarrow \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1/2}]} \underline{\qquad}$$

(ii) $\exists$ a quantum seed

$$\mathcal{S}_{\mathcal{Q}} = \left(\{ f^{\xi}_{i,p} \in \mathcal{X}^{\mathcal{Q}}_q \}_{(i,p) \in \Gamma^{\mathcal{Q}}_0},\ L_{\mathcal{Q}} = (\lambda_{(i,p),(j,s)}),\ \widetilde{B}_{\mathcal{Q}} \right) \text{ in } \mathcal{X}^{\Delta}_q$$

s.t (a) $f^{\xi}_{i,p}$ is ________, $f^{\xi}_{i,p} f^{\xi}_{j,s} = q^{\lambda_{(i,p),(j,s)}} f^{\xi}_{j,s} f^{\xi}_{i,p}$

(b) $\widetilde{B}_{\mathcal{Q}} =$ ________

and

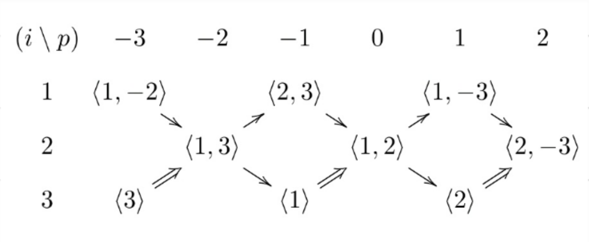
$$\bar{\Phi}_{\mathcal{Q}} : \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1}]} \mathcal{A}_{\mathbb{Z}[q^{\pm 1}]}(\mathfrak{n}) \xrightarrow{\sim} \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1/2}]} \underline{\qquad\qquad} \hookrightarrow \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1/2}]} \mathcal{X}^{\Delta}_q$$

Here

$$\lambda_{(i,p),(j,s)} = (m_{\xi}(i,p), m_{\xi}(j,s))_{\Delta}$$

where $m_{\xi}(i,p) =$ ________

$m_{\xi}(j,s) =$ ________



$(i \setminus p)$	-3	-2	-1	0	1	2
1	$\langle 1, -2 \rangle$		$\langle 2, 3 \rangle$		$\langle 1, -3 \rangle$	
2		$\langle 1, 3 \rangle$		$\langle 1, 2 \rangle$		$\langle 2, -3 \rangle$
3	$\langle 3 \rangle$		$\langle 1 \rangle$		$\langle 2 \rangle$	