* Quantum affine alg

(untwisted)

$\mathfrak{g}$: f-d simple alg ⤳ $\hat{\mathfrak{g}}$: affine Kac–Moody alg (⤳ $\Delta_{\hat{\mathfrak{g}}} = \underset{0}{\circ}\overset{-a_{10}a_{01}}{\text{——}}\Delta_{\mathfrak{g}}$)

(w/ index set $I_{\mathfrak{g}}$) $\qquad I = \{0\} \sqcup I_{\mathfrak{g}}$

A = affine Cartan matrix asso w/ $\hat{\mathfrak{g}}$.

$U_q(\hat{\mathfrak{g}}) \underset{\neq}{\supseteq} \boxed{U_q'(\hat{\mathfrak{g}})}$ = ________________, subalg of $U_q(\hat{\mathfrak{g}})$ gen'd by

$\qquad \cup\!\!\!\!\!\;|$ $\qquad\qquad e_i, f_i$ and $q_i^{h_i}$ $\forall\, i \in I_0 = \{0\} \sqcup I_{\mathfrak{g}}$.

$U_q(\mathfrak{g})$

(without degree operator) ⟶ ____________________

$\mathbb{K} = \overline{\mathbb{Q}(q)} \subset \bigsqcup_{m \in \mathbb{Z}_{>0}} \mathbb{C}((q^{1/m}))$. base field

Rmk $U_q'(\hat{\mathfrak{g}})$ admits a ________________ f.d integrable module while $U_q(\hat{\mathfrak{g}})$ do not.

Let $\mathscr{C}_{\hat{\mathfrak{g}}}$ be the ____________________________________.

Note · $\mathscr{C}_{\hat{\mathfrak{g}}}$ is not ________________.

· $\mathscr{C}_{\hat{\mathfrak{g}}}$ is a __________ category via $-\otimes-$ induced from the Hopf alg structure of $U_q'(\hat{\mathfrak{g}})$. Here we use the comultiplication

$e_i \mapsto e_i \otimes q_i^{h_i} + 1 \otimes e_i \quad f_i \mapsto f_i \otimes 1 + K_i \otimes f_i$

- For $M, N \in \mathscr{C}_{\hat{\mathfrak{g}}}$ ________________________ in general.
- $\exists$ classification of simple modules of $\mathscr{C}_{\hat{\mathfrak{g}}}$ in terms of Drinfeld polys

$\boxed{\text{Rmk}}$ Roughly speaking

(i) each simple module $M \in \mathscr{C}_{\hat{\mathfrak{g}}}$ has a ________ $I_{\mathfrak{g}}$-highest weight vector ____ of $I_{\mathfrak{g}}$-h.w.

(ii) For each $i \in I_{\mathfrak{g}}$. $\exists!$ f'tal representation $V(\varpi_i)$ which is $\supsetneq V(\Lambda_i)$ as subspace!! $\overline{}$-invariant, admits "simple-crystal", "canonical basis"...., whose ________ $V(\varpi_i)_x$ $(x \in \Bbbk^\times)$ generates $\mathscr{C}_{\mathfrak{g}}$.

In other words, (i) & (ii) tells ________

<u>Recall</u> $M \in R\text{-gmod} \rightsquigarrow M_z := \simeq \Bbbk[z] \otimes M$ affinization of M.
$\rightsquigarrow$ ________

For $M \in \mathscr{C}_{\mathfrak{g}}$, $M^{aff} :=$ ____ $\otimes M$ $U_q'(\hat{\mathfrak{g}})$-module s.t

$$e_i(v_z) = \_\_\_ (e_i v)_z$$
$$f_i(v_z) = \_\_\_ (f_i v)_z$$
$$q_i^{h_i}(v_z) = (q_i^{h_i} \ )_z$$

Here $v_z = 1 \otimes v \in M^{aff}$ for $v \in M$. Then $z_M^{\pm 1}: M^{aff} \to M^{aff}$, $a \otimes v \mapsto za \otimes v$ is an injective $U_q'(\mathfrak{g})$-auto of weight null root δ.
(sometimes we use also M_z for M^{aff}).

[Parameter shift]. For $x \in \Bbbk^\times$, we set
$$M_x := M^{aff} / \underline{(z_M - x) M^{aff}}$$
(Note $M \in \mathscr{C}_{\mathfrak{g}}$ simple $\Rightarrow$ so is M_x) but $\not\cong M$ unless $x = 1$.

$\rightsquigarrow$ $e_0 v_x = \_\_\_ \cdot (e_0 v)_x$ for $v \in M$

____

<u>One more feature of $\mathcal{C}_{\hat{\mathfrak{g}}}$</u> For $M \in \mathcal{C}_{\hat{\mathfrak{g}}}$, $\exists$ $\mathcal{D}M$ & $\mathcal{D}^{-1}M \in \mathcal{C}_{\hat{\mathfrak{g}}}$, called right dual & left dual of M s.t $\exists$ homos

$$M \otimes \mathcal{D}M \longrightarrow \mathbb{1} \qquad \& \qquad \mathcal{D}^{-1}M \otimes M \longrightarrow \mathbb{1}$$

In particular, the duals of $V(\varpi_i)_x$ $(x \in \mathbb{k}^\times)$ are given as follows:

$$\mathcal{D}(V(\varpi_i)_x) = V(\varpi_{i^*})_{p^* x} \qquad \mathcal{D}^{-1}(V(\varpi_i)_x) = V(\varpi_{i^*})_{p^{*-1} x}$$

where $p^* = (-1)^{h_{\mathfrak{g}}} q^{h^\vee_{\mathfrak{g}}}$. Here $h_{\mathfrak{g}}$: Coxeter # of $\mathfrak{g}$ ($\underline{\qquad\qquad\qquad\qquad}$)

$h^\vee_{\mathfrak{g}}$: dual Coxeter # of $\mathfrak{g}$.

($\underline{\qquad\qquad\qquad\qquad}$)

* Dynkin pair $D_{\hat{\mathfrak{g}}}$ of $\hat{\mathfrak{g}}$

For each $\hat{\mathfrak{g}}$, we assign the Dynkin pair $D_{\hat{\mathfrak{g}}} = (\Delta, \sigma)$ of $\hat{\mathfrak{g}}$ as:

$$A_n^{(1)} = (\triangle_{A_n}, \mathrm{Id}), \quad B_n^{(1)} = (\triangle_{A_{2n-1}}, \vee), \quad C_n^{(1)} = (\triangle_{D_{n+1}}, \vee), \quad D_n^{(1)} = (\triangle_{D_n}, \vee),$$
$$E_n^{(1)} = (\triangle_{E_n}, \mathrm{Id}) \ (n = 6, 7, 8), \quad F_4^{(1)} = (\triangle_{E_6}, \vee), \quad G_2^{(1)} = (\triangle_{D_4}, \widetilde{\vee}).$$

Note ______________________________ !

Recall The notions of $S_Q(\beta)$ & $S_Q(i,p)$ in R-gmod.

For $\hat{\mathfrak{g}}$ and $(i,p) \in \hat{\Delta}_0$ of $D_{\hat{\mathfrak{g}}}$, we set

$$V(i,p) := \begin{cases} V(\varpi_{i^\sigma})_{(-q_s^{1/|\sigma|})^p} & A_n^{(1)}, C_n^{(1)}, D_n^{(1)}, E_n^{(1)}, G_2^{(1)} \\ V(\varpi_{i^\sigma})_{(-1)^{d(i,n)}(q_s^{1/|\sigma|})^p} & B_n^{(1)} \\ V(\varpi_{i^\sigma})_{(-1)^{d(i,2)}(q_s^{1/|\sigma|})^p} & F_4^{(1)} \end{cases}$$

For $Q = (\Delta, \sigma, \xi)$ on $D_{\hat{\mathfrak{g}}}$ & $\beta \in \Phi_\Delta^+$, we set

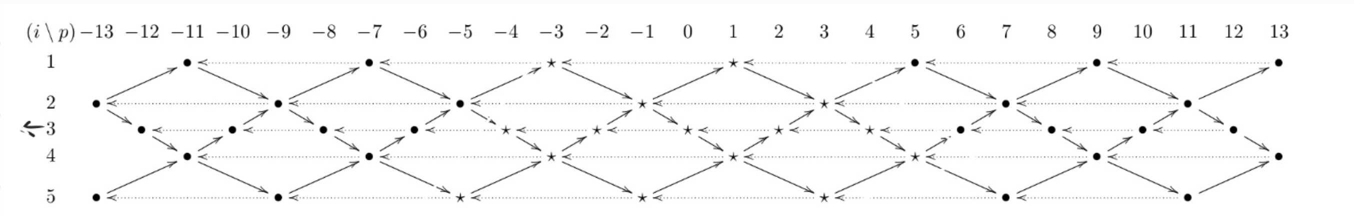
$V_Q(\beta) :=$ __________ where $\phi_Q(\beta, 0) =$ ____________________

$\Rightarrow \mathcal{D}^{\pm} V_Q(\beta) =$ ____________________

$\mathscr{C}^0_{\mathfrak{g}} :=$ the ______ of $\mathscr{C}_{\hat{\mathfrak{g}}}$ containing $\{V(i,p) \mid$ ______ $\}$
and stable under taking ______, ______ & ______

For a convex subset S of $\hat{\Delta}_0$, $\mathscr{C}^S_{\mathfrak{g}}$: the smallest containing $\{$ ______ $\}$.
In particular, ______ for a Q-datum Q on $D_{\mathfrak{g}} = (\Delta, \sigma)$.

Example $U_q'(B_3^{(1)})$ $\quad q_s = q^{1/2}$



Thm (Chari-Moura, Chari-Pressley, HL, KO, FO, Scrimshaw-O).
① The Category $\mathscr{C}^0_{\mathfrak{g}}$ is the skeleton subcategory of $\mathscr{C}_{\hat{\mathfrak{g}}}$: i.e
For every ______ module M, $\exists N \in \mathscr{C}^0_{\mathfrak{g}}$ and $x \in \mathbb{k}^\times$ s.t it is ______ &
______

② For each $Q = (\Delta, \sigma, \xi)$ of $D_{\mathfrak{g}}$, $\mathscr{C}^Q_{\mathfrak{g}}$ is a heart subcat of $\mathscr{C}^0_{\mathfrak{g}}$; i.e
for every f'tal module $V(i,p)$ in $\mathscr{C}^0_{\mathfrak{g}}$, ______ $\in \mathscr{C}^Q_{\mathfrak{g}}$ and ______ $\in \mathbb{Z}$ s.t
______ $\simeq V(i,p)$.

* R-matrix M, N simples in $\mathcal{C}^0_{\hat{\mathfrak{g}}}$.

Known $\exists!$ $\Bbbk(z) \otimes U_q'(\mathfrak{g})$-module ______

$$R^{norm}_{M,N_z} : \Bbbk(z) \underset{\Bbbk[z^{\pm1}]}{\otimes} (M \otimes N_z) \longrightarrow \Bbbk(z) \underset{\Bbbk[z^{\pm1}]}{\otimes} (N_z \otimes M)$$

sending

$$u_M \otimes (u_N)_z \longrightarrow (u_N)_z \otimes u_M.$$

and ______ $\in \Bbbk[z]$ s.t

______ $: M \otimes N_z \longrightarrow N_z \otimes M$

s.t $d_{M,N}(z) R^{norm}_{M,N_z}\big|_{z=k}$ ______ $(k \in \Bbbk^\times)$ (but not iso !!)

Note that R^{norm} satisfies YB-equation!

We denote by $\underline{\mathbf{r}_{M,N}} := d_{M,N}(z) R^{norm}_{M,N_z}\big|_{z=1}$ and call it ______

$d_{M,N}(z) =$ ______ (M,N)

Note, even though M, N are non-simple, there are pairs of modules (M,N) s.t $\mathbf{r}_{M,N}$ is defined. We call such pairs <u>rrp</u>

<u>Note</u> <u>computing $d_{M,N}(z)$ looks not easy.</u>

Example Consider $B_3^{(1)}$ case. Then the $U_q'(B_3^{(1)})$-f'tal module $V(\varpi_1)$ has the following "crystal graph"

$$\otimes \qquad \begin{tikzcd} \boxed{1} \arrow[r, "1"] & \boxed{2} \arrow[r, "2"] & \boxed{3} \arrow[r, "3"] & \boxed{0} \arrow[r, "3"] & \boxed{\overline{3}} \arrow[r, "2"] & \boxed{\overline{2}} \arrow[r, "1"] \arrow[lllll, bend right, "0"'] & \boxed{\overline{1}} \arrow[lllll, bend left, "0"] \end{tikzcd} \qquad (D_{B_3^{(1)}} = (\Delta_{A_5}, V))$$

$U_q'(B_3^{(1)})$-action is given as usual. In particular, $f_3 \boxed{0} = [2]_3 \boxed{\overline{3}}$ $\quad f_0 \boxed{\overline{1}} = \boxed{2}$

$e_3 \boxed{0} = [2]_3 \boxed{3}$ $\quad f_0 \boxed{\overline{2}} = \boxed{1}$

In this case, $V(\varpi_1) \simeq V(\Lambda_1)$ as $U_q(B_3)$-modules.

(However $V(\varpi_2) \simeq V(\Lambda_2) \oplus V(0)$) $\quad \boxed{}\boxed{} \to$ domino (Branching rule!)

Note $V(\Lambda_1) \otimes V(\Lambda_1) \simeq V(2\Lambda_1) \oplus V(\Lambda_2) \oplus V(0)$ as $U_q(B_3)$-module. $\quad$ $\underline{q_s = q^{1/2}}$ $\quad q_3 = q_s \quad q_1 = q_2 = q_s^2 = q_0$

$$u_{2\Lambda_1} = \boxed{1} \otimes \boxed{1} \qquad u_{\Lambda_2} = \boxed{1} \otimes \boxed{2} - q_s^2 \boxed{2} \otimes \boxed{1}$$

$$u_0 = \boxed{1} \otimes \boxed{\overline{1}} - q_s^2 \boxed{2} \otimes \boxed{\overline{2}} + q_s^4 \boxed{3} \otimes \boxed{\overline{3}} - [2]_0^{-1} q_s^4 \boxed{0} \otimes \boxed{0}$$
$$+ q_s^6 \boxed{\overline{3}} \otimes \boxed{3} - q_s^8 \boxed{\overline{2}} \otimes \boxed{2} + q_s^{10} \boxed{\overline{1}} \otimes \boxed{1}.$$

For generic $x, y \in \mathbb{K}^\times$, $R := d_{V(\varpi_1)_x, V(\varpi_1)_y}(y/x) R^{\mathrm{norm}}_{V(\varpi_1)_x, V(\varpi_1)_y}$ gives an iso

$$V(\varpi_1)_x \otimes V(\varpi_1)_y \xrightarrow{\sim} V(\varpi_1)_y \otimes V(\varpi_1)_x$$

as $U_q'(B_3^{(1)})$-module (and hence $U_q(B_3)$-module).

So R sends h.t-vectors / $U_q(B_3)$ to themselves.

i.e $R(u_{2\Lambda_1}) = a^{2\Lambda_1} u_{2\Lambda_1}$ $\quad R(u_{\Lambda_2}) = a^{\Lambda_2} u_{\Lambda_2}$ $\quad R(u_0) = a^0 u_0$

Then, using $\circledast$, we have the following by direct computation + wt consideration. $\overline{\alpha_0} = -\overline{\alpha_1} - 2\overline{\alpha_2} - 2\overline{\alpha_3}$

① $f_0 u_0 = q_s^{-2}(y^{-1} - q_s^{10} x^{-1}) u_{\Lambda_2}$

② $f_0 f_2 f_3^{(2)} f_2 u_{\Lambda_2} = q_s^{-2}(y^{-1} - q_s^{4} x^{-1}) u_{2\Lambda_1}$.

Thus

①′ $R(q_s^{-2}(y^{-1} - q_s^{10} x^{-1}) u_{\Lambda_2}) = R(f_0 u_0) = f_0 R(u_0) = f_0 a^0 u_0$

By taking $x^{-1}y = z$, $\Rightarrow$ $\dfrac{a^0}{(1 - q_s^{10} z)} = \dfrac{a^{\Lambda_2}}{(z - q_s^{10})}$

②′ $R(q_s^{-2}(y^{-1} - q_s^{4} x^{-1}) u_{2\Lambda_1}) = R(f_0 f_2 f_3^{(2)} f_2 u_{\Lambda_2}) = f_0 f_2 f_3^{(2)} f_2 R(u_{\Lambda_2})$

$= f_0 f_2 f_3^{(2)} f_2 a^{\Lambda_2} u_{\Lambda_2}$

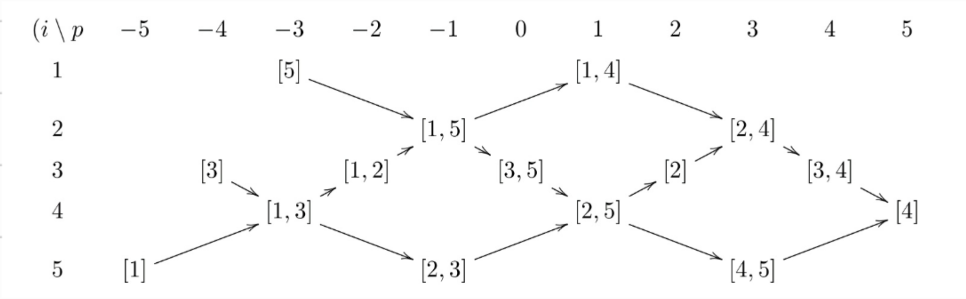
$\Rightarrow \dfrac{a^{2\Lambda_1}}{(z - q^4)} = \dfrac{a^{\Lambda_2}}{(q^4 - z)}$

Thus R :

$$
\begin{array}{lcll}
V(2\Lambda_1) & \xrightarrow{\sim} & V(2\Lambda_1) & u_{2\Lambda_1} \longrightarrow \underline{(z - q_s^4)(z - q_s^{10})} u_{2\Lambda_1} \quad \square\!\square \\
\oplus & & & \\
V(\Lambda_2) & \xrightarrow{\sim} & V(\Lambda_2) & u_{\Lambda_2} \longrightarrow \underline{(z q_s^4 - 1)(z - q_s^{10})} u_{\Lambda_2} \quad \boxminus \\
\oplus & & & \\
V(0) & \longrightarrow & V(0) & u_0 \longrightarrow \underline{(z q_s^4 - 1)(z q_s^{10} - 1)} u_0 \quad \phi
\end{array}
$$

$\Rightarrow d_{1,1}(z) := d_{V(\varpi_1), V(\varpi_1)}(z) = \underline{(z - q_s^4)(z - q_s^{10})}$

$$\mathrm{Ir}_{\underset{= x}{V(\varpi_1)_{(-q_s)^{-3}}},\ \underset{= y}{V(\varpi_1)_{(-q_s)^{1}}}}\left(V(\varpi_1)_{(-q_s)^{-3}} \otimes V(\varpi_1)_{(-q_s)^{1}}\right) \simeq \underline{\qquad\qquad}$$



· **Def**

For a simple module $M, N \in \mathcal{C}_{\hat{\mathfrak{g}}}$, we set

$\partial(M, N) =$ the order of zero at $z=1$ for $d_{M,N}(z)\, d_{N,M}(z^{-1})$.

$$:= \underline{\mathrm{Zero}_{z=1}\left(d_{M,N}(z)\, d_{N,M}(z^{-1})\right)}$$

(Example) $\partial\left(V(\varpi_1)_{(-q_s)^{-2}},\ V(\varpi_1)_{(-q_s)^{2}}\right)$ (over $B_3^{(1)}$)

$$= \mathrm{Zero}_{z=1}\left((q_s^4 z - q_s^4)(q_s^4 z - q_s^{10}) \cdot (q_s^{-4} z^{-1} - q_s^4)(q_s^{-4} z^{-1} - q_s^{10}) \right)$$

$= \underline{\qquad}$

<u>Note</u> for $k \gg 0$, $d(D^{\pm k} M, N) = d(M, D^{\mp k} N) = $ ___ .

Thus $\Lambda(M,N) := \sum_{k\in\mathbb{Z}} (-1)^{k+\delta(k<0)}$ __________ are well-defined!

$\Lambda^\infty(M,N) := \sum_{k\in\mathbb{Z}} (-1)^k$ __________

<u>Rmk</u> the defs of Λ, Λ^∞ are ________________________

<u>Thm</u> For each $\hat{\mathfrak{g}}$,

$\mathscr{C}_{\hat{\mathfrak{g}}}$ is a R-category proposition M, N : grrp (1) $M \nabla N$ is simple.

(2) $M \otimes N$ is simple $\Leftrightarrow$ $\mathfrak{d}(M, N) = 0$

(3) $\mathfrak{d}(M, N) \leq 1 \Rightarrow$ $M \nabla N$ is real simple

(4) If $\mathfrak{d}(M, N) = 1$, $[M \otimes N]$ is of composition length 2, whose factors are $[M \nabla N] \neq [M \Delta N]$. & $\{M, M \nabla N, M \Delta N\}$ $\{N, M \nabla N, M \Delta N\}$ commuting families!

(5) $\mathrm{Hom}(M \otimes N, N \otimes M) = \mathbb{k}\, \mathrm{r}_{M,N}$

(6) $M \otimes N$ is simple $\Leftrightarrow M \otimes N \simeq N \otimes M \Leftrightarrow M \nabla N \simeq N \nabla M$
$\Leftrightarrow \mathrm{r}_{M,N}$ & $\mathrm{r}_{N,M}$ are inverses to each other

* Categorification & Monoidal categorification.

For (Δ, σ), let $\mathcal{X}^{\hat{\Delta}} :=$ ______ .

Then $\mathcal{X}^{\hat{\Delta}}$ is ______ & $\simeq$ ______

Thm [Frenkel – Reshetikhin]

$D_{\mathfrak{g}} = (\Delta, \sigma)$, $\exists$ ______

$\mathcal{X} : K(\mathscr{C}^{0}_{\mathfrak{g}}) \hookrightarrow \mathcal{X}^{\hat{\Delta}}$

We call $\mathcal{X}$ t-character homo. For $M \in \mathscr{C}^{0}_{\mathfrak{g}}$, $\mathcal{X}([M])$ encodes l-weight space decomposition of M

$\Rightarrow \mathcal{X}([M]) \in$ ______ $[X_{i,p}^{\pm 1}]_{(i,p) \in \hat{\Delta}_0}$

We denote by ______ $= \mathcal{X}(K(\mathscr{C}^{0}_{\mathfrak{g}}))$

Note For $Q=(\Delta,\sigma,\xi)$ and a mono m in $\mathcal{X}^{Q}$, we can define

$\underline{wt_Q(m)} \in \mathsf{R}_\Delta$

(Example)

$T^Q =$

$(i \setminus p$	-5	-4	-3	-2	-1	0	1	2	3	4	5
1			[5]				[1,4]				
2					[1,5]				[2,4]		
3		[3]		[1,2]		[3,5]		[2]		[3,4]	
4			[1,3]				[2,5]				[4]
5	[1]				[2,3]				[4,5]		

$Q = (\Delta_{A_5}, V, \xi)$ of $D_{B_3^{(1)}}$

$wt_Q(X_{1,-3}) = \alpha_5$ $\quad wt_Q(X_{2,3}) = \alpha_2+\alpha_3+\alpha_4$

$wt_Q(X_{2,-1}X_{4,5}X_{5,3}^{-1}X_{1,1}^{-1}) = 0$

we can define $\underline{\text{homo elts in } \mathcal{X}^{Q}}$.

For a Q-datum $Q=(\Delta,\sigma,\xi)$ of $D_{\mathfrak{g}}$, we have

$$K(\mathcal{C}^0_{\mathfrak{g}}) = \bigoplus_{\beta\in \mathsf{R}_+} K(\mathcal{C}^0_{\mathfrak{g}})[\beta] \quad \text{—— ☆}$$

where $K(\mathcal{C}^0_{\mathfrak{g}})[\beta] := \{x \in K(\mathcal{C}^0_{\mathfrak{g}}) \mid x \text{ is homo } \& \ wt_Q(x)=\beta\}$

Thm (Chari-Moura, KKOP, FO)

$\mathcal{C}^0_{\mathfrak{g}}$ has a ____________ according to ☆.

i.e every module M decomposes into $\underline{M=\bigoplus M_\beta}$ s.t $M_\beta \in \mathcal{C}^0_{\mathfrak{g}}[\beta]$

and $\underline{Hom_{\mathcal{C}^0_{\mathfrak{g}}}(M,N)=0}$ if $M\in\mathcal{C}^0_{\mathfrak{g}}[\beta]$, $N\in\mathcal{C}^0_{\mathfrak{g}}[\gamma]$ s.t $\beta\neq\gamma$.

When $\mathfrak{g}$ is simply-laced, Nakajima, Varagnolo–Vasserot constructed a q-deformation of $\mathcal{K}_q(\mathcal{C}_{\mathfrak{g}}^0)$ of $\mathcal{K}(\mathcal{C}_{\mathfrak{g}}^0)$ by using geometrical method.
By Hernandez, $\mathcal{K}_q(\mathcal{C}_{\mathfrak{g}}^0)$ for any $\mathfrak{g}$ constructed in a uniform way.

$$\begin{tikzcd}
K(\mathcal{C}_{\mathfrak{g}}^0) \arrow[r, hook, "\chi"] & \mathcal{K}(\mathcal{C}_{\mathfrak{g}}^0) \arrow[r, hook] \arrow[d, dotted, "\text{quantization}"'] & \mathcal{X}^{\blacktriangle} \arrow[d, dotted] \\
& \mathcal{K}_q(\mathcal{C}_{\mathfrak{g}}^0) \arrow[r, hook] \arrow[u, bend right, "q\to 1"'] & \mathcal{X}_q^{\blacktriangle} \arrow[u, bend right, "q\to 1"']
\end{tikzcd}$$

Note that $\exists$ a canonical basis $\mathcal{L}_q$ of $\mathcal{K}_q(\mathcal{C}_{\mathfrak{g}}^0)$ parametrized by simple modules in $\mathcal{C}_{\mathfrak{g}}^0$. For a simple $M \in \mathcal{C}_{\mathfrak{g}}^0$, we write $[M]_q$ the corresponding elt in $\mathcal{L}_q$. We call $[M]_q$ the (q,t)-simple character of M

Then we can naturally expect

(a) $[M]_q \in \mathbb{Z}_{\geq 0}[q^{\pm 1/2}][\widetilde{X}_{i,p}]$

☆ (b) $[M]_q|_{q=1} = \chi([M])$

Thm (Nakajima) If $\mathfrak{g}$ is of type ABDE, ☆ holds.
ADE
Fujita–Hernandez
–O–OYa, B

For a convex subset S of $\hat{\triangle}$ asso w/ $D_{\mathfrak{g}}$, we set

$$\mathcal{K}_q(\mathscr{C}_{\mathfrak{g}}^S) := \bigoplus_{[M]_q \in L_q^S} \mathbb{Z}[q^{\pm 1/2}][M]_q \quad \text{where } \underline{L_q^S} := \{[M_q] \mid M \text{ simple in } \mathscr{C}_{\mathfrak{g}}^S\}$$

(Categorification)

<u>Thm</u> [GLS, HL, KKK, KO, KKOP, FHOO]

$\mathcal{Q} = (\Delta, 6, \xi)$ of $D_{\mathfrak{g}}$. $A_q(\mathfrak{n})$: uni quantum coord ring asso w/ Δ

(note Δ is of simply-laced type)

$\Rightarrow$ $\exists$ $\underline{\mathbb{Z}[q^{\pm 1/2}]\text{-alg iso}}$

$$\Phi_{\mathcal{Q}} : \mathbb{Z}[q^{\pm 1/2}] \otimes A_{\mathbb{Z}[q^{\pm 1}]}(\mathfrak{n}) \simeq \mathcal{K}_q(\mathscr{C}_{\mathfrak{g}}^{\mathcal{Q}})$$

as $\underline{\text{quantum cluster algs of skew-symmetric type.}}$

Moreover, the isomorphism sends

(i) $\underline{\mathbb{B} \text{ to } L_q^{\mathcal{Q}}}$ (up to $\mathbb{Z}[q^{\pm 1/2}]^{\times}$).

(ii) $\underline{F_{\mathcal{Q}}^{up}(\alpha)}$ to $\underline{[V_{\mathcal{Q}}(\alpha)]_q}$ for $\alpha \in \Phi_{\Delta}^{+}$

$\rightsquigarrow$ dual PBW basis of $A_q(\mathfrak{n})$ assoc / $[\mathcal{Q}]$

$\longleftrightarrow$ monomial basis of ordered products $[V_{\mathcal{Q}}(\beta)]$'s.

Thm [HL, FHOO]

$\hat{\mathfrak{g}}_{(1)}, \hat{\mathfrak{g}}_{(2)}$ s.t $D_{\hat{\mathfrak{g}}_{(1)}}$ & $D_{\hat{\mathfrak{g}}_{(2)}}$ shares the same $\triangle$ (Ex $B_3^{(1)}$, $A_5^{(2)}$!!)

Then the presentation of $\mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1/2}]} K_q(\mathscr{C}^0_{\hat{\mathfrak{g}}_{(t)}})$ $(t=1,2)$
has the same presentation as follows: It is gen'd by $\{Y_{i,m} \mid (i,m) \in \Delta_0 \times \mathbb{Z}\}$

$$\sum_{r=0}^{1-a_{ij}^{\triangle}} (-1)^r \begin{bmatrix} 1-a_{ij}^{\triangle} \\ r \end{bmatrix} Y_{i,m}^{1-a_{ij}^{\triangle}-r} Y_{j,m} Y_{i,m}^r = 0$$

$$Y_{i,m} Y_{j,m+1} = q^{-a_{ij}^{\triangle}} Y_{j,m+1} Y_{i,m} + (1-q^{-2})\delta_{ij}$$

$$Y_{i,m} Y_{j,\ell} = q^{(-1)^{k+\ell} a_{ij}^{\triangle}} Y_{j,\ell} Y_{i,m} \quad (\ell > m+1)$$

Here $Y_{i,m} \longleftrightarrow [\mathcal{D}^m(V_Q(\alpha_i))]$

Thm (KKOP, FHOO22⁺, HL, ...)

$S \subset \hat{\triangle}_0$ (S can be $\hat{\triangle}$) Then
Any convex subset

$\exists$ $\mathcal{S}_S$ s.t $\mathscr{C}^S_{\mathfrak{g}}$ is a

of a quantum cluster alg $\mathcal{A}_{q^{1/2}}(\mathcal{S}_S) \simeq$

and hence

- any cluster monomial in $K_q(\mathscr{C}^S_{\hat{\mathfrak{g}}})$ corresponds to an elt in $\mathcal{L}^S_q$ (up to $\mathbb{Z}[q^{\pm 1/2}]^\times$)

In particular, we have the followings:

(a) For a Q-datum $Q = (\Delta, \sigma, \xi)$ on $D_{\hat{\mathfrak{g}}}$, $\mathscr{C}^Q_{\hat{\mathfrak{g}}}$ has an admissible quantum monoidal seed

$$\mathcal{S}_Q = (\quad , \quad , \quad) \text{ in } \mathscr{C}^Q_{\hat{\mathfrak{g}}}.$$

In this case, in $K_q(\mathscr{C}^Q_{\hat{\mathfrak{g}}})$ corr to (up to $\mathbb{Z}[q^{\pm 1/2}]^\times$)

(b) For a height fn ξ on $D_{\hat{\mathfrak{g}}} = (\Delta, \sigma)$, and its $\hat{\triangle}_{\le \xi} \underset{\text{subset}}{\subset} \hat{\triangle}_0$, $\mathscr{C}^{\hat{\triangle}_{\le\xi}}_{\hat{\mathfrak{g}}}$ has an admissible quantum monoidal seed

$$\mathcal{S}_{\le \xi} := \left(\{M_\xi(i,p)\}_{(i,p)\in \hat{\triangle}_{\le\xi}}, \ L_{\hat{\triangle}_{\le\xi}}, \ \widetilde{B}_{\hat{\triangle}_{\le\xi}} \right) \text{ in } \mathscr{C}^{\hat{\triangle}_{\le\xi}}_{\hat{\mathfrak{g}}}$$

where

$$(L_{\hat{\triangle}_{\le\xi}})_{(i,p),(j,s)} = -\Lambda(M_\xi(i,p), M_\xi(j,s)).$$
$$= (\quad)_{\hat{\triangle}}$$

$$M_\xi(i,p) := \mathrm{hd}\left(V(i,\xi_i) \otimes \cdots \otimes V(i, p+2\#(i^\sigma)) \otimes V(i,p) \right)$$

$*$ Quantum affine Schur-Weyl duality functor.

Let us fix $D_{\hat{\mathfrak{g}}} = (\Delta, \sigma)$ and Q-datum on (Δ, σ).

$\rightsquigarrow$ $SW_Q = \{ V_Q(\alpha_i) \mid i \in I \}$

<u>Thm</u> [KKKO]

$\exists$ exact monoidal functor

$$\mathcal{F}_Q : \bigoplus R(\beta)\text{-gmod} \longrightarrow \mathcal{C}_{\hat{\mathfrak{g}}}^{Q}$$

s.t ① $\mathcal{F}_Q(L(i)) \simeq \underline{V_Q(\alpha_i)}$

② $\mathcal{F}_Q(M \circ N) \simeq \underline{\mathcal{F}_Q(M) \otimes \mathcal{F}_Q(N)}$

③ sends simples to <u>simples bijectively</u>

<u>Naive idea</u>

For each $\beta \in \Phi^+_\Delta$,

$$\widehat{V}(\nu) := \underset{\text{extension } *}{\text{some}} \otimes \left(V_Q(\alpha_{\nu_1}) \otimes \cdots \otimes V_Q(\alpha_{\nu_n}) \right) \quad \text{for } \nu \in I^\beta$$

$$\widehat{V}(\beta) := \bigoplus_{\nu \in I^\beta} V(\nu)$$

$\rightsquigarrow$ $\widehat{V}(\beta)$ has $(U_q'(\mathfrak{g}), R(\beta))$-bimodule structure derived from $d(V_Q(\alpha_i), V_Q(\alpha_j))$

Then for $M \in R(\beta)\text{-gmod}$

$$\mathcal{F}_Q(M) := \widehat{V}(\beta) \otimes_{*} M \in \mathcal{C}_{\hat{\mathfrak{g}}}^{Q}$$

Moreover $\mathcal{F}_Q$ preserve R-matrices $\mathbf{r}_{M,N}$ in R-gmod and hence invariants Λ, d !!.