* Quiver Hecke algebra * its $\mathbb{Z}$-graded f-d modules
(KLR – algebra)

Quiver Hecke alg is introduced by Khovanov – Lauda and Rouquier independently in 2008. (Generalization of affine Hecke alg of type A. (Brundan – Kleshchev))

I = index set $A = (a_{ij})_{i,j\in I}$: symm'l Cartan matrix

$\mathbb{k}$ = base field.

For $a, b \in I$, set

$$P_{a,b} = \{(x,y) \in \mathbb{Z}^2_{\geq 0} \mid \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \}$$

Take a family of polys $(Q_{i,j})_{i,j\in I}$ in $\mathbb{k}[u,v]$ s.t

$$Q_{i,j}(u,v) = \delta(i \neq j) \sum_{(p,q)\in P_{i,j}} t_{i,j;p,q}\, u^p v^q$$

where $t_{i,j;p,q} \in \mathbb{k}$ s.t $Q_{i,j}(u,v) = Q_{j,i}(v,u)$ and $t_{i,j;-a_{ij},0} \in \mathbb{k}^\times$.

Example [redacted] (1) Let A be of type A_3. Then we have

$$\mathsf{A} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \quad \text{and} \quad \triangle = \underset{1}{②} - \underset{2}{②} - \underset{3}{②}$$

Then we can choose the matrices $(\mathcal{Q}_{i,j}(u,v))_{i,j\in I}$ as follows:

[redacted]

$$P1 = \begin{pmatrix} 0 & (u-v) & 1 \\ (v-u) & 0 & (u-v) \\ 1 & (v-u) & 0 \end{pmatrix}, \qquad P2 = \begin{pmatrix} 0 & (u+v) & 1 \\ (u+v) & 0 & (u-v) \\ 1 & (v-u) & 0 \end{pmatrix}.$$

* There are several choices !!

(2) Let A be of type C_2. Then we have

$$\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad \triangle = \underset{1}{②} - \underset{2}{❹}$$

which implies

$$(\alpha_1,\alpha_1) = 2, \quad (\alpha_1,\alpha_2) = -2 \quad \text{and} \quad (\alpha_2,\alpha_2) = 4.$$

Then we can choose a matrix $(\mathcal{Q}_{i,j}(u,v))_{i,j\in I}$ as follows:

$$\begin{pmatrix} 0 & u^2 - v \\ v^2 - u & 0 \end{pmatrix}. \tag{4.3}$$

$P_{12} = \{(a,b) \in \mathbb{Z}^2_{\geq 0} \mid 2a + 4b = 4\}$
$= \{(2,0), (0,1)\}$

$P_{21} = \{(1,0), (0,2)\}$.

$\mathfrak{S}_n = \langle r_1 \ldots r_{n-1} \rangle$ symmetric group on n-letters, where $r_i = (i, i+1)$ is the simple transposition of i & $i+1$.

$\rightsquigarrow$ $\mathfrak{S}_n$ acts on I^n by place permutation.

($\leftarrow$ height of $\beta = \sum_{i \in I} a_i \alpha_i \rightsquigarrow |\beta| = \sum a_i$)

For $n \in \mathbb{Z}_{\geq 0}$ & $\beta \in \mathsf{R}_A^+$ with $|\beta| = n$, we def

$$I^\beta = \{ \nu = (\nu_1, \ldots \nu_n) \in I^n \mid \alpha_{\nu_1} + \cdots + \alpha_{\nu_n} = \beta \}$$

$\boxed{\text{Def}}$ $\beta \in \mathsf{R}_A^+$ with $|\beta| = n$, the ________________ $R(\beta)$ at $\underline{\beta}$ assoc A & (Q_{ij}) is the $\Bbbk$-alg generated by elts $\{ e(\nu) \mid \nu \in I^\beta \}$, $\{ x_k \mid 1 \leq k \leq n \}$, $\{ \tau_m \mid 1 \leq l < n \}$

satisfying the following relations:

$$e(\nu) e(\nu') = \underline{\delta(\nu = \nu')} e(\nu), \qquad \sum_{\nu \in I^\beta} e(\nu) = 1$$

$$e(\nu) x_k = x_k e(\nu) \qquad x_k x_l = x_l x_k$$

$$\tau_l e(\nu) = e(r_l(\nu)) \tau_l \qquad \tau_k \tau_l = \tau_l \tau_k \text{ if } |k - l| > 1$$

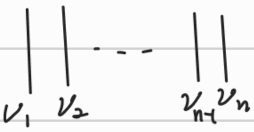
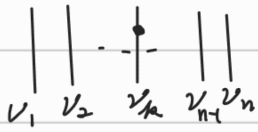
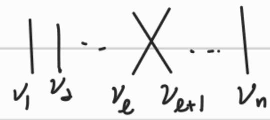
$$\tau_k^2 e(\nu) = Q_{\nu_k \nu_{k+1}}(x_k, x_{k+1}) e(\nu)$$

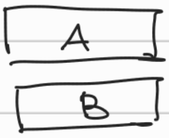
$$(\tau_k x_l - x_{r_k(l)} \tau_k) e(\nu) = \begin{cases} -e(\nu) & \text{if } l = k \text{ \& } \nu_k = \nu_{k+1}, \\ e(\nu) & \text{if } l = k+1 \text{ \& } \nu_k = \nu_{k+1} \\ 0 & \text{o.w} \end{cases}$$

$$(\tau_{k+1} \tau_k \tau_{k+1} - \tau_k \tau_{k+1} \tau_k) e(\nu) = \begin{cases} \overline{Q}_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}, x_{k+2}) & \text{if } \nu_k = \nu_{k+2} \\ 0 & \text{o.w} \end{cases}$$

where $$\overline{Q}_{ij}(u, v, w) = \frac{Q_{ij}(u, v) - Q_{ij}(w, v)}{u - w}$$

Diagrammatic notation

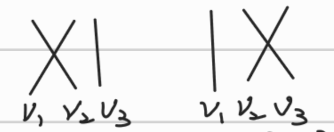
⊛ $e(\nu) =$  $x_k e(\nu) =$  $\tau_\ell e(\nu) =$ 

multiplication $A\,B \Longleftrightarrow$  vertically.

$\boxed{\text{Example}}$ C_2-case with $(Q_{ij}) = \begin{pmatrix} 0 & u^2 - v \\ v^2 - u & 0 \end{pmatrix}$. and $\beta = 2\alpha_1 + \alpha_2$

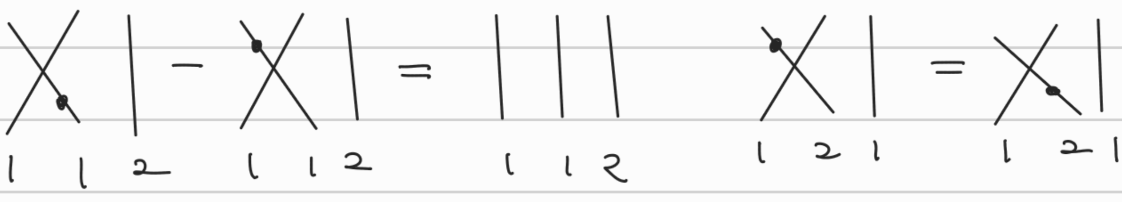
$R(2\alpha_1 + \alpha_2)$ is generated by

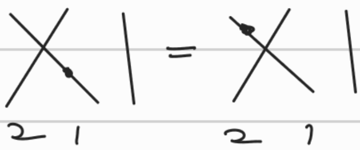
$e(112)$ $e(121)$ $e(211)$ $x_1 e(\nu)$ $x_2 e(\nu)$ $x_3 e(\nu)$

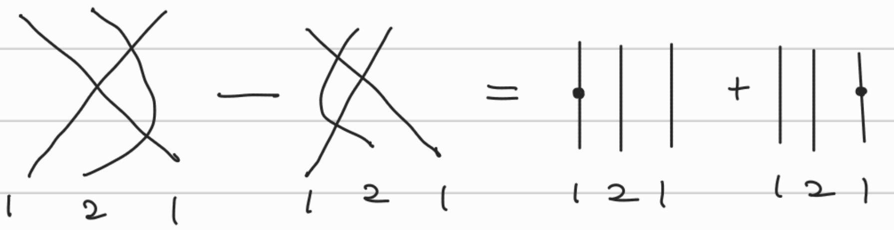


$\tau_1 e(\nu)$ $\tau_2 e(\nu)$

w/ relations

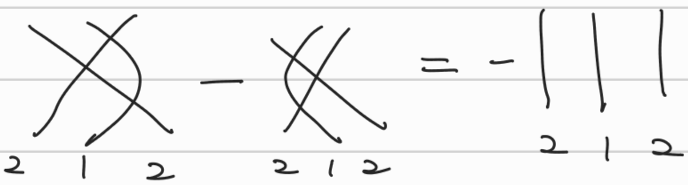






$\left(\because\ \bar{Q}_{1,2}(u,v,w) = u+w\right)$

Note in $R(2\alpha_2+\alpha_1)$



$\left(\because\ \bar{Q}_{2,1}(u,v,w) = -1\right)$

Note $R(\beta)$ is $\mathbb{Z}$-graded by assigning

$\deg e(\nu) = \underline{\qquad}$ $\quad \deg x_k e(\nu) = \underline{\qquad\qquad}$ $\quad \deg \tau_\ell e(\nu) = \underline{\qquad\qquad}$

$\star$ $R(\beta)$-gmod: the category of $\underline{\qquad\qquad\qquad\qquad\qquad\qquad}$, with $\underline{\qquad\qquad}$

$\underline{\qquad\qquad\qquad\qquad\qquad\qquad}$.

- For $M \in R(\beta)$-gmod, we set

$$\mathrm{wt}(M) = -\beta \in \mathsf{Q}^-.$$

- We define the grading shift functor q by

$$(qM)_k = \underline{\qquad\qquad\qquad} \quad \text{for } M = \bigoplus_{k\in\mathbb{Z}} M_k \in R(\beta)\text{-gmod}.$$

* Faithful rep'n of $R(\beta)$

Let $P_\beta := \bigoplus_{\nu \in I^\beta} \mathbb{K}[x_1, \dots x_n] e(\nu) \subset R(\beta)$

Note that $\mathfrak{S}_n = \langle r_1 \dots r_{n-1} \rangle$ acts on P_β in the following way: $\sigma \in \mathfrak{S}_n$

$$\left(f(x_1 \dots x_n)\, e(\nu_1, \dots \nu_n) \right) \cdot \sigma$$
$$= f(x_{\sigma(1)}, x_{\sigma(2)}, \dots x_{\sigma(n)})\, e(\nu_{\sigma^{-1}(1)}, \nu_{\sigma^{-1}(2)}, \dots, \nu_{\sigma^{-1}(n)})$$

Ex) $\quad r_1 \mid x_1 x_2\, e((2))) :=$ ______

Thm [Khovanov-Lauda, Rouquier]

(1) P_β is a ______ of $R(\beta)$

i.e $\quad \underset{\text{subalg}}{\leq}$

______

(2) For each $\sigma \in \mathfrak{S}_n$, let us choose its red exp $\tilde{\sigma} = r_{i_1} \cdots r_{i_t}$ and set $\tau_\sigma :=$ ______. Then R_n is a ______ with a basis $\{\tau_\sigma \mid \sigma \in \mathfrak{S}_n\}$

(3) The center $Z(R_n)$ of R_n coincides with ______.

$\boxed{\text{Example}}$ For $i \in I$ & $\beta \in R^+$ with $|\beta| = n$, set

$$z_{i,\beta} = \sum_{\nu \in I^\beta} \Big(\prod_{\substack{a \in [1 \dots n] \\ \nu_a = i}} x_a \Big) e(\nu)$$

$\Rightarrow \quad z_{i,\beta} \in Z(R(\beta))$

For $\beta = 2\alpha_1 + \alpha_2 \in R^+_{A_3}$ $\quad z_{1,\beta} = x_1 x_2\, e(1,1,2) + x_1 x_3\, e(1,2,1) + x_2 x_3\, e(2,1,1)$

"Def" We say that a quiver Hecke alg $R(\beta)$ is ______ if $Q_{i,j}(u,v)$ is a poly in ______ $\forall i,j \in \mathrm{supp}(\beta)$. When $R(\beta)$ is symmetric, we write $R(\beta)$ ______ instead of $R(\beta)$

Here for $\beta = \sum_{i \in I} a_i \alpha_i \in Q^+$, $\mathrm{supp}(\beta) = \{ i \in I \mid a_i > 0 \}$.

Example For any symmetrizable $\mathfrak{g}$ and $i \in I$,

$R(\alpha_i)$ is symmetric even though $\mathfrak{g}$ is ______

Note $R(\alpha_i) \simeq$ ______ $e(i) \longmapsto 1 \quad x_1 \longmapsto x$ (No tau!)

For each $i \in I$, $\exists$ a 1-dim'l simple $R(\alpha_i)$-module

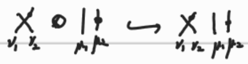
$L(i) := \mathbb{K}u(i) \simeq R(\alpha_i)/x_1 R(\alpha_i)$ $(\Rightarrow \mathbb{K} \cdot \frac{!}{i})$

For an $R(\beta)$-module M & an $R(\gamma)$-module N,

____ denotes the $R(\beta+\gamma)$-module defined as follows

$$\underline{M \circ N} = R(\beta+\gamma)\, e(\beta,\gamma) \underset{R(\beta)\otimes R(\gamma)}{\otimes} (M \otimes N)$$ ($\circ$: convolution product)

where $R(\beta) \otimes R(\gamma) \hookrightarrow R(\beta+\gamma)$



(horizontal product of diagrams!)

and $e(\beta,\gamma) = \sum_{\substack{\nu \in I^\beta \\ \mu \in I^\gamma}} e(\nu * \mu)$

<u>Example</u> $i, j \in I$ s.t. $a_{ij} < 0$

$\Rightarrow$ $L(i) \circ L(j)$ is 2-dim'l $\simeq \mathbb{K}$ ____ $\oplus$ $\mathbb{K}$ ____

In $R(\alpha_i+\alpha_j)$, $\nexists$ negative degree elts !!

$\leadsto$ $\mathbb{K}$ ____ is the ____ . (ie ____)

$\Rightarrow$ $\mathbb{K}$ ____ $\simeq$ ____ . ( )

Note that $R(\alpha_i+\alpha_j)$ is ____ in general.

* Now R-gmod is a monoidal category s.t. $- \circ -$ is bi-exact.

⊙ Intertwiners. $\beta \in Q^+$ with $|\beta| = n$. $1 \le a < n$.

Define

$$\varphi_a \in R(\beta) \text{ by } \varphi_a e(\nu) = \begin{cases} (\tau_a x_a - x_a \tau_a) e(\nu) & \text{if } \nu_a = \nu_{a+1} \\ \tau_a e(\nu) & \text{if } \nu_a \neq \nu_{a+1} \end{cases}$$

We call $\{\varphi_a\}$ the intertwiners.

(Example) $\beta = 2\alpha_1 + \alpha_2 \in Q^+_{A_3}$

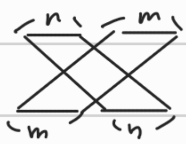
$$\varphi_1 = (\tau_1 x_1 - x_1 \tau_1) e(1\,1\,2) + \tau_1 e(1\,2\,1) + \tau_1 e(2,1,1) \in R(2\alpha_1 + \alpha_2)$$

______________ ! but $\varphi_a e(\nu)$ is ________ !

Lemma (Important lemma) $\{\varphi_a\}_{1 \le a < n}$ satisfies the ____________.

For $m, n \in \mathbb{Z}_{\ge 0}$, $w[m,n] \in \mathfrak{S}_{m+n}$ defined by

$$w[m,n](k) = \begin{cases} k+n, & \text{if } 1 \le k \le m, \\ k-m & \text{if } m < k \le m+n. \end{cases}$$

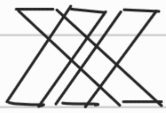
($\Leftrightarrow$ )

$\exists$ $R(\beta+\gamma)$ — module homo $\qquad |\gamma| = m \quad |\beta| = n$

$$R_{M,N} : M \circ N \longrightarrow q^{-*}\, N \circ M \qquad \text{☆ : degree of } R_{M,N}$$
$$x\,(u \otimes v) \longrightarrow \underline{x\, \varphi_{w[m,n]}}\,(v \otimes u) \qquad \text{(computable!!)}$$

Note

$$\begin{tikzcd}
L \circ M \circ N \arrow[r, "R_{L,M}"] \arrow[dr, "R_{L,\, M\circ N}"'] & M \circ L \circ N \arrow[d, "R_{L,N}"] \\
 & M \circ N \circ L
\end{tikzcd}$$



$$\begin{tikzcd}
 & L \circ M \circ N \arrow[dl, "R_{L,M}"'] \arrow[dr, "R_{M,N}"] & \\
M \circ L \circ N \arrow[d, "R_{L,N}"'] & & L \circ N \circ M \arrow[d, "R_{L,N}"] \\
M \circ N \circ L \arrow[dr, "R_{M,N}"'] & & N \circ L \circ M \arrow[dl, "R_{L,N}"] \\
 & N \circ M \circ L &
\end{tikzcd}$$

<u>Warning</u> $R_{M,N}$ can !

(Ex) A_3 $\Rightarrow$

* Affinization From now on, we usually assume • R or $R(\beta)$ (symmetric one!)

• $(\alpha_i, \alpha_i) = 2$ ($\leadsto \deg(x_k e(\nu)) = 2$)

Let z be an indeterminate with degree ___

$\Rightarrow$ $\exists$ $\mathbb{Z}$-grading preserving homo

$$\psi_z : R(\beta) \longrightarrow \Bbbk[z] \otimes R(\beta)$$

$$x_k \longmapsto \_\_\_$$

$$\tau_k \longmapsto \_\_\_$$

$$e(\nu) \longmapsto \_\_\_$$

If $R(\beta)$ is ___ symmetric, ψ_z can be homo!

$(\because \psi_z(\tau_k^2 e(\nu)) \neq \psi_z(Q_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}) e(\nu))$ in general)

For an R-module M, $\Bbbk[z] \otimes_{\Bbbk} M$ has a R-module structure via ψ_z

$$x_k(a \otimes u) = za \otimes u + a \otimes x_k u \quad \forall\, a \in \Bbbk[z],\ u \in M.$$

Then we can understand z of $M_z := \Bbbk[z] \otimes_{\Bbbk} M$ as R-module endo of M_z injective

s.t $M \simeq M_z / z M_z$. $\boxed{}$ ___

(symmetric)

Example ① $L(i)_z \simeq \Bbbk[z] \simeq R(\alpha_i)$ s.t $x \cdot u(i)_z = z\, u(i)_z$

Here $u(i)_z := 1 \otimes u(i)$.

② (non-sym) $\mathfrak{g}$: C_2-type $L(12)$: 1-dim'l $R(\alpha_1 + \alpha_2)$-module

Define $\tilde{L}(12) := \Bbbk[z]\, u(12)_z$ s.t

$\tau_1 u(12)_z =$ ___ $\quad x_1 u(12)_z =$ ___ $\quad x_2 u(12)_z =$ ___

$\Rightarrow$ $L(12) \simeq$ ___

* R-matrices $M: R(\beta)$-module, $N: R(\gamma)$-module

Let s be the order of zero of

$$R_{M_z,N} : M \circ_z N \longrightarrow N \circ M_z$$

that is, the ______________________________ s.t the image of $R_{M_z,N}$ is contained in $\square\,(N \circ M_z)$. (Note $R_{M_z,N}$ __________________ !!)

($\Rightarrow$ If $s>0$, __________)

We set $R^{ren}_{M_z,N} :=$ __________ $\quad \mathbf{r}_{M,z} :=$ __________

Then $R^{ren}_{M_z,N}$ and $\mathbf{r}_{M,N}$ does not vanish !! we say $\mathbf{r}_{M,N}$ the R-matrix

$\boxed{\circledast \text{ Rmk}}$ $\mathbf{r}$ satisfies YB-equations.

To define $\mathrm{r}_{M,N}$, we need the following ingradient

② One of M an N admits an affinization in the following sense:

① $\exists$ M_z : R-module with an inj endo z s.t

$$M_z / z M_z \simeq M$$

(+ more conditions...).

As we see $M = L(12)$ over $R_{C_2}(\alpha_1+\alpha_2)$, some simple R-module (R: non-symmetric) admits an affinization and hence we can construct non-zero R-module homo $\underline{\underline{\mathrm{r}_{M,N}}}$, R-matrix. In this case, we say $L(12)$ admits an ______________

<u>Thm</u> For a symmetric quiver Hecke algebra R, R-gmod is a R-category

* Consequences of R-category (R-gmod !!)

<u>Def</u> $\Lambda(M,N)$ = the <u>homogeneous degree</u> of $\mathbf{r}_{M,N}$, when $\mathbf{r}_{M,N}$ is defined. In this case, we say (M,N) as "<u>rationally renormalizable pair</u>" (<u>rrp</u>)

$$\mathbf{r}_{M,N} : M \circ N \longrightarrow N \circ M .$$

<u>Lemma</u> [Kang-Kashiwara-Kim-O] for rrp $M,N \in R$-gmod,

$$d(M,N) := \frac{1}{2}\big(\Lambda(M,N) + \Lambda(N,M)\big) \in \mathbb{Z}_{\geq 0}$$

If $M, N \in R$-gmod s.t they are ______ $\ast$ rrp $\ast$ ______________ ______ , we will say (M,N) ______________ (grrp).

Thm (M,N) grrp $M \nabla N$ and $N \nabla M$ ______________

<u>proposition</u> M, N : grrp

(1) $M \circ N$ is simple $\Leftrightarrow$ $\underline{\mathfrak{d}(M,N) = 0}$

(2) $\mathfrak{d}(M,N) \leq 1 \Rightarrow$ $\underline{M \nabla N \text{ is real simple}}$

(3) If $\mathfrak{d}(M,N) = 1$, $[M \circ N]$ is of composition length 2. whose factors are $[M \nabla N] \neq [M \Delta N]$. & $\{M, M \nabla N, M \Delta N\}$
$\{N, M \nabla N, M \Delta N\}$
commuting families!

(4) $\mathrm{Hom}(M \circ N, N \circ M) = \mathbb{k}\, \mathrm{r}_{M,N}$

(5) $M \circ N$ is simple $\Leftrightarrow M \circ N \simeq N \circ M \Leftrightarrow M \nabla N \simeq N \nabla M$
$\Leftrightarrow \mathrm{r}_{M,N}$ & $\mathrm{r}_{N,M}$ are inverses to each other

* Categorification of Bases

Dual canonical / upper global basis

Thm ☆ [Varagnolo-Vasserot, Rouquier]

R: symmetric quiver Hecke alg over $\mathbb{k}$ of characteristic zero.

Then Φ sends " ______________________________ "

to the ____________ ! (up to $q^{\mathbb{Z}}$)

* Dual PBW basis. asso w/ w_0

Note $A_q(\mathfrak{n})$ for f.d-simple $\mathfrak{g}$ has a dual PBW-basis: Let $w_0 = s_{i_1}\cdots s_{i_\ell}$ be a reduced expression of $w_0 \in W_{\mathfrak{g}}$.

$\Rightarrow \quad \beta_k^{w_0} = s_{i_1}\cdots s_{i_{k-1}}(\alpha_{i_k}) \in \Phi_{\mathfrak{g}}^+$

$\Rightarrow ! \exists \ \underline{F_{w_0}^{up}(\beta_k^{w_0})} \in \underline{\mathcal{B}} \subset A_q(\mathfrak{n})$ s.t. $\mathrm{wt}(F_{w_0}^{up}(\beta_k^{w_0})) = -\beta_k^{w_0}$ (dual PBW-vector asso w/ w_0)

and $\{ F_{w_0}^{up}(\beta_1^{w_0})^{a_1} \cdots F_{w_0}^{up}(\beta_\ell^{w_0})^{a_\ell} \mid a_i \in \mathbb{Z}_{\geq 0} \}$ a PBW basis asso w/ w_0

$\boxed{\text{Note}}$ $w_0 \sim w_0'$ $\quad \underline{F_{w_0}(\beta) = \qquad\qquad}$ for $\beta \in \Phi_{\mathfrak{g}}^+$ $\rightsquigarrow$ ________ follows.

$\boxed{\text{Thm}}$ [Kleshchev-Ram, Hill-Melvin-Mondragon
Kato, McNamara, Brundan-Kleshchev-McNamara]

$\mathfrak{g}$: f-d simple Lie alg $\quad R^{\mathfrak{g}}$: q Hecke alg assoc to $\mathfrak{g}$

$\beta \in \Phi_{\mathfrak{g}}^+$ $\quad [w_0]$: comm class of w_0

(1) $\exists$ ____________________ module ________ over $R^{\mathfrak{g}}$ s.t. .

$\underline{\Phi(F_{[w_0]}^{up}(\beta))} \simeq$ ________ $\rightsquigarrow$ ________________ is defined!

(2) every simple module M appears as an image of ________

$S_{[w_0]}(\beta_1)^{\circ a_1} \circ \cdots \circ S_{[w_0]}(\beta_\ell)^{\circ a_\ell}$ (standard module)

composition of $\mathbf{r}_{a,b}$'s

$\xrightarrow{\mathbf{r}} S_{[w_0]}(\beta_\ell)^{\circ a_\ell} \circ \cdots \circ S_{[w_0]}(\beta_1)^{\circ a_1}$ (costandard module)

for a unique $\underline{a_1, \ldots, a_\ell} \in \mathbb{Z}_{\geq 0}$ and hence

$$M \simeq \mathrm{hd}\left(S_{[w_0]}(\beta_1)^{\circ a_1} \circ \cdots \circ S_{[w_0]}(\beta_\ell)^{\circ a_\ell} \right).$$

* Knowing $\mathfrak{d}(S_{[w_0]}(\beta), S_{[w_0]}(\gamma))$ is quite important!!
But computing $\mathfrak{d}(M,N)$ is quite hard in general!

$[w_0]$ was appeared in D-W combinatorics! For $Q = (\Delta, \sigma, \xi)$, let us

set $\underline{\qquad}$ $:= S_{[Q]}(\beta)$ $\underline{\qquad\qquad}$ $\in R^{\Delta}$-gmod.

s.t $\underline{\qquad} = (\beta, 0)$ for $\beta \in \Phi^+$ ($\rightsquigarrow$ $(i,p) \in (\Gamma^Q)_0$)

(Recall $[Q]$: the comm class of $w_0 \in W_\Delta$).

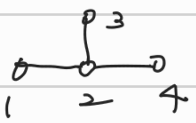
$\boxed{\text{Thm}}$ [Fujita, Fujita-O, Kashiwara-O]

- $Q = (\Delta, \sigma, \xi)$ Q-datum R^{Δ} : quiver Hecke alg asso w/ Δ
- $(i,p), (j,s) \in \Gamma^Q_0$ Let Λ_i : f'tal wt at $i \in \Delta_0$.
- τ_Q : the σ-Coxeter elt of Q.

Then we have

$$(\Lambda_i, \tau_Q^{(|p-s|+\xi_j-\xi_i)/2}(1-\tau_Q^{\#(j,Q)})\Lambda_j)$$

$$= \begin{cases} d(S_Q(i,p+2) \nabla S_Q(i,p), S_Q(j,s)) & \text{if } \begin{array}{l} \Delta \neq \mathrm{id} \\ i=n,\ \Delta = A_{2n-1},\ v \\ i \neq n, n+1\ \ \Delta = D_{n+1},\ v \end{array} \\ d(\mathrm{hd}(S_Q(2,p+4) \circ S_Q(2,p+2) \circ S_Q(p)), S_Q(j,s)) & \text{if } (\Delta_{D_4}, \tilde{v}) \\ & \quad i = 2. \\ d(S_Q(i,p), S_Q(j,s)) & \text{otherwise!!} \end{cases}$$



(the invariant can be read
from AR-quiver Γ^Q !!!).

Moreover, we know

$S_{\mathfrak{a}}(\alpha) \circ S_{\mathfrak{a}}(\beta)$ is simple or not!

Also we can determine when

- $S_{\mathfrak{a}}(\alpha) \circ S_{\mathfrak{a}}(\beta)$ is of composition length 2 $\Longleftrightarrow \mathfrak{d}(\quad) = 1$

$$\Longleftrightarrow [S_{\mathfrak{a}}(\alpha) \circ S_{\mathfrak{a}}(\beta)] = [S_{\mathfrak{a}}(\alpha) \nabla S_{\mathfrak{a}}(\beta)] + [S_{\mathfrak{a}}(\alpha) \Delta S_{\mathfrak{a}}(\beta)]$$

$$\Longleftrightarrow S_{\mathfrak{a}}(\beta) = \frac{[S_{\mathfrak{a}}(\alpha) \nabla S_{\mathfrak{a}}(\beta)] + [S_{\mathfrak{a}}(\alpha) \Delta S_{\mathfrak{a}}(\beta)]}{[S_{\mathfrak{a}}(\alpha)]}$$

($\leftrightarrow$ exchange relation)

In this case (composition length 2)

$\rightsquigarrow \{S_{\mathfrak{a}}(\alpha), S_{\mathfrak{a}}(\alpha) \nabla S_{\mathfrak{a}}(\beta), S_{\mathfrak{a}}(\alpha) \Delta S_{\mathfrak{a}}(\beta)\}$

$\{S_{\mathfrak{a}}(\beta), S_{\mathfrak{a}}(\alpha) \nabla S_{\mathfrak{a}}(\beta), S_{\mathfrak{a}}(\alpha) \Delta S_{\mathfrak{a}}(\beta)\}$

are real simple families that are mutually commutative

* <u>Monoidal categorification.</u> (via symmetric quiver Hecke alg)

$\mathscr{C}$: full subcategory of R-gmod. (R-category!!)

$\mathscr{C} = \bigoplus_{\beta \in Q^-} \mathscr{C}_\beta$ where $\mathscr{C}_\beta = \mathscr{C} \cap R(-\beta)$-gmod.

<u>Def</u> A triple $\mathscr{S} = (\{M_j\}_{j \in J}, L, \tilde{B})$ quantum monoidal seed in $\mathscr{C}$ if the followings are satisfied

(1) $\tilde{B} := (b_{ij})_{i \in J, j \in J_{ex}}$ exchange matrix

(2) $\{M_j\}$ = <u>commuting family of real simples in $\mathscr{C}$</u>

monomials of $\{M_i\}$ are simple!

(3) $L = (\lambda_{ij})_{i,j \in J}$ s.t. $\underline{M_i \circ M_j \simeq q^{\lambda_{ij}} M_j \circ M_i}$

and $\underline{L \cdot \tilde{B} = 2\delta_{ij}}$ $(\Rightarrow \lambda_{ij} = -\Lambda(M_i, M_j))$

compatibility.

$\Rightarrow$ <u>These triple "categorifies" a quantum seed</u> ☆

Now we need "categorical notion" of mutation!

Def $\mathcal{S} = (\{M_j\}_{j \in J}, L, B)$: quantum monoidal seed in $\mathcal{C}$

(1) We say that $\mathcal{S}$ "admits" a mutation in direction $k \in J_{ex}$ if $\exists$ a self-dual simple module $M_k' \in \mathcal{C}$ s.t $\exists$ an exact seq in $\mathcal{C}$

(i) $0 \to q \bigodot_{b_{ik}>0} M_i^{\odot b_{ik}} \longrightarrow q^{\tilde{\Lambda}(M_k, M_k')} M_k \circ M_k' \to \bigodot_{b_{ik}<0} M_i^{\odot(-b_{ik})} \longrightarrow 0$

(ii) $\mu_k(\mathcal{S}) = (\{M_j\}_{j \neq k} \sqcup \{M_k'\}, \mu_k(L), \mu_k(\tilde{B}))$ is a quantum monoidal seed

(2) We say that $\mathcal{S} = (\{M_i\}_{i \in J}, L, \tilde{B})$ is admissible if $\exists$ a self-dual simple $M_k' \in \mathcal{C}$ satisfying (i) $\forall k \in J_{ex}$, and M_k' commutes with M_i $\forall i \neq k$

prop Let $\mathcal{S} = (\{M_i\}_{i \in J}, L, \tilde{B})$ is an admissible quantum monoidal seed in $\mathcal{C}$

$\Rightarrow$ (i) $d(M_k, M_k') = 1$ $\forall k \in J_{ex}$ and (ii) M_k' is real

Hence $\mu_k(\mathcal{S})$ is also an quantum monoidal seed.

Def (Monoidal categorification)

We say that $\mathcal{C}$ is a monoidal categorification of a quantum cluster alg $\mathcal{A}_{q^{1/2}}$ if

(i) [Categorification]

$$\mathbb{Z}[q^{\pm 1/2}] \otimes_{\mathbb{Z}[q]} K(\mathcal{C}) \simeq \mathcal{A}_{q^{1/2}}$$

(ii) $\exists$ a quantum monoidal seed $\mathcal{S} = (\{M_i\}_{i \in J}, L, B)$ in $\mathcal{C}$ s.t

$[\mathcal{S}] = (\{q^{*}[M_i]\}, L, B)$ is a quantum seed of $\mathcal{A}_{q^{1/2}}$

(iii)* $\mathcal{S}$ admits successive mutation in all directions

(Criterion theorem)

Thm [K^{3}O]

If ① $\exists$ $\mathcal{S} = (\{M_i\}_{i \in J}, L, \tilde{B})$ admissible seed in $\mathcal{C}$ and

② $\mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1/2}]} K(\mathcal{C}) \simeq \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1/2}]} \mathcal{A}_{q^{1/2}}([\mathcal{S}])$

$\Rightarrow$ $\mathcal{S}$ admits "successive" mutation in all directions!

Hence $\mathcal{C}$ is a monoidal categorification of $\mathcal{A}_{q^{1/2}}([\mathcal{S}])$.

Cor (Consequence)

(*) Any cluster monomial in $\mathbb{Z}[q^{\pm 1/2}] \otimes K(\mathcal{C})$ is the iso class of a real simple module in $\mathcal{C}$ (up to $q^{1/2}$)

$\Rightarrow$ Dream comes true (with result of VV, R).

Thm [Geiss-Leclerc-Schröer + K³O]

R : symm quiver Hecke alg of type ADE. Δ

Let $Q = (\Delta, \sigma, \xi)$ of ADE Δ. $\exists$ an admissible quantum monoidal seed

$$\mathscr{S}_Q = (\{M_Q(i,p)\}_{(i,p)\in \Gamma_0^Q}, L_Q, \widetilde{B}_Q) \text{ in } R\text{-gmod}.$$

where $M_Q(i,p) = \mathrm{hd}(S_Q(i,\xi_i) \circ S_Q(i, \xi_i - 2\#(i,\sigma)) \circ \cdots \circ S(i,p))$

Thus R-gmod is a monoidal categorification of $A_q(\mathfrak{n}) \simeq \mathcal{A}_{q^{1/2}}[\mathscr{S}_Q]$

Hence any cluster monomial in $\mathbb{Z}[q^{\pm 1/2}] \otimes_{\mathbb{Z}[q^{\pm 1}]} K(R\text{-gmod})$ corresponds to an element in $\mathbb{B}$.

$$\begin{tikzcd}
(i \setminus p & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 \\
1 & & & {[5]^\dagger} & & & & {[1,4]} \arrow[llll, dotted] \arrow[drr] & & & & \\
2 & & & & & {[1,5]^\dagger} \arrow[urr] \arrow[dr] & & & & {[2,4]} \arrow[llll, dotted] \arrow[dr] & & \\
3 & & {[3]^\dagger} & & {[1,2]} \arrow[ll, dotted] \arrow[ur] & & {[3,5]} \arrow[ll, dotted] \arrow[dr] & & {[2]} \arrow[ll, dotted] \arrow[ur] & & {[3,4]} \arrow[ll, dotted] \arrow[dr] & \\
4 & & & {[1,3]^\dagger} \arrow[ur] \arrow[drr] & & & & {[2,5]} \arrow[llll, dotted] \arrow[ur] \arrow[drr] & & & & {[4]} \arrow[llll, dotted] \\
5 & {[1]^\dagger} & & & & {[2,3]} \arrow[llll, dotted] \arrow[urr] & & & & {[4,5]} \arrow[llll, dotted] \arrow[urr] & &
\end{tikzcd}$$