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Abstracts

Computation, Analysis and Applications of PDEs with Nonlocal and Singular Operators

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1 Andrea Bonito

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Numerical approximations of elliptic fractional operators: Part I and II

Abstract

We review numerical algorithms for the approximation of fractional elliptic operators with a particular attention on their analysis and implementations. Our main emphasis is on methods using the Dunford-Taylor representations of fractional diffusion problems, but other methods are addressed as well.

In the case of spectral fractional powers of an elliptic operator (Part I), the Dunford-Taylor representation consists of an improper integral. The latter is approximated using an exponentially convergent sinc quadrature method. The integrand at each quadrature point is approximated using a standard finite element method. The method is easily parallelizable and consists of a straightforward modification of standard finite element methods for reaction-diffusion problems.

For the integral fractional laplacian (Part II), the Dunford-Taylor integral representation is instrumental to derive novel variational formulations. As in the spectral case, sinc quadrature formulas coupled with finite element discretizations on parameter dependent truncated domains are put in place. This yields a non-conforming method where the action of the stiffness matrix on a vector is approximated. The efficiency of the method is illustrated in three dimensions.

2 Andrea Bonito

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Elliptic fractional operators in applications

Abstract

In the third part of this tutorial, we present two applications involving fractional operators: The surface-quasi geostrophic (SQG) flow model and electro-convection of thin liquid crystals. If time permits, we will also discuss the numerical simulation of perpetual American options and Gaussian fields.

The SQG flows are modeled by a nonlinear partial differential equation coupling transport and fractional diffusion phenomena. The time discretization consists of an explicit strong-stability-preserving three-stage Runge-Kutta method while a flux-corrected-transport (FCT) method coupled with Dunford-Taylor representations of fractional operators for the space discretization. In the inviscid case, we show that the resulting scheme satisfies a discrete maximum principle property under a standard CFL condition and observe, in practice, its second-order accuracy in space. The algorithm successfully approximates several benchmarks with sharp transitions and fine structures typical of SQG flows. In addition, theoretical Kolmogorov energy decay rates are observed on a freely decaying atmospheric turbulence simulation.

In the electroconvection model, a thin liquid crystal is located in between two concentric circular electrodes which are either assumed to be of infinite height or slim. Each configuration results in a different nonlocal electro-magnetic model (spectral or integral fractional laplacians) defined on a two-dimensional bounded domain. The numerical method consists in approximating the surface charge density, the liquid velocity and pressure, and the electric potential in the two-dimensional liquid region. Finite elements for the space discretization coupled with standard time stepping methods are put forward. Unlike for the infinite electrodes configuration, the numerical simulations indicate that slim electrodes are favorable for electroconvection to occur and are able to sustain the phenomena over long period of time.

3 Li-Lian Wang

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A tutorial introduction to spectral methods for some singular and nonlocal problems

Abstract

It is known that the accuracy and performance of a usual spectral method can be deleteriously degraded when the solution exhibits local singular behaviors with very limited regularity. In practice, the singularity may occur in many problems under various scenarios such as PDEs in nonsmooth computational domains with sharp corners or with degenerate or discontinuous coefficients, non-matching boundary conditions, singular kernels or potentials, non-differentiable nonlinear terms, and among others.

We shall give a tutorial introduction to spectral methods for various singular and nonlocal problems with a focus on the fundamental spectral approximation theory and efficient computation of the related stiffness and mass matrices. The overall plan is as follows. We start with collecting some singular problems leading to some typical types of singularities. Then we focus on the Jacobi spectral approximation of such singular functions with the regularity characterized by certain fractional Sobolev spaces. Finally, we introduce efficient spectral algorithms for some fractional PDEs.