Stein’s Method on Wiener Chaos, and Level Sets of Gaussian Fields

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REFERENCES


Charles Stein and Paul Malliavin

In 2009, together with I. Nourdin, we discovered a connection between Stein’s method for probabilistic approximations (Stein, 1972) ...

... and the Malliavin calculus of variations on a Gaussian space (Malliavin, 1978).
THE CONNECTION

★ In the framework of normal approximations, Stein’s method requires indeed one to **uniformly bound** quantities such as

\[ |\mathbb{E}[Fg(F)] - \mathbb{E}[g'(F)]|, \]

over some class of smooth mappings \( g \).

★ When \( F \) is a smooth functional of a Gaussian field, then one can **integrate by parts** and obtain that

\[ \mathbb{E}[Fg(F)] = \mathbb{E}[g'(F) \langle DF, -DL^{-1}F \rangle], \]

where \( D \) is the **Malliavin derivative** and \( L^{-1} \) is the inverse of the generator of the Ornstein-Uhlenbeck semigroup.

★ Such a connection has been extremely useful for exploring the asymptotic structure of **Wiener chaos**.
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Such a connection has been extremely useful for exploring the asymptotic structure of Wiener chaos.
**Vignette: Wiener Chaos**

★ Consider a generic separable Gaussian field $G = \{ G(u) : u \in U \}$.
★ For every $q = 0, 1, 2...$, set

$$P_q := \text{v.s.} \left\{ p(G(u_1), ..., G(u_r)) : d^o p \leq q \right\}.$$  

Then: $P_q \subset P_{q+1}$.
★ Define the family of orthogonal spaces $\{C_q : q \geq 0\}$ as $C_0 = \mathbb{R}$ and $C_q := P_q \cap P_{q-1}^\perp$; one has

$$L^2(\sigma(G)) = \bigoplus_{q=0}^{\infty} C_q.$$  

★ $C_q = \text{Ker} \ (L + q I) = q\text{th Wiener chaos of } G.$
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A RIGID ASYMPTOTIC STRUCTURE

For fixed $q \geq 2$, let $\{F_k : k \geq 1\} \subset C_q$ (with unit variance).

- **Nourdin and Poly (2013):** If $F_k \Rightarrow Z$, then $Z$ has necessarily a density (and the set of possible laws for $Z$ does not depend on $G$).

- **Nualart and Peccati (2005):** $F_k \Rightarrow Z \sim \mathcal{N}(0, 1)$ if and only if $\mathbb{E}F_k^4 \to 3 (= \mathbb{E}Z^4)$.

- **Peccati and Tudor (2005):** Componentwise convergence to Gaussian implies joint convergence.

- **Nourdin, Nualart and Peccati (2015):** given $\{H_k\} \subset C_p$, then $F_k, H_k$ are asymptotically independent if and only if $\text{Cov}(H_k^2, F_k^2) \to 0$.

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Berry’s Random Waves (Berry, 1977)

★ Fix $E > 0$. The **Berry random wave model** on $\mathbb{R}^2$, with parameter $E$, written

$$B_E = \{ B_E(x) : x \in \mathbb{R}^2 \},$$

is the unique (in law) centred, isotropic Gaussian field on $\mathbb{R}^2$ such that

$$\Delta B_E + E \cdot B_E = 0,$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$.

★ Equivalently,

$$\mathbb{E}[B_E(x)B_E(y)] = \int_{S^1} e^{i\sqrt{E}(x-y,z)} \, dz = J_0(\sqrt{E}\|x - y\|).$$

(this is an infinite-dimensional Gaussian object).

★ Think of $B_E$ as a “canonical” Gaussian Laplace eigenfunction on $\mathbb{R}^2$, emerging as a universal local scaling limit for arithmetic and monochromatic RWs, random spherical harmonics... .
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Focus on the **length** $L_E$ of the **nodal set**:

$$B_E^{-1}(\{0\}) \cap \mathcal{Q} := \{ x \in \mathcal{Q} : B_E(x) = 0 \},$$

where $\mathcal{Q}$ is some fixed domain, as $E \to \infty$.

*Images: D. Belyaev*
A Cancellation Phenomenon

★ Berry (2002): an application of Kac-Rice formulae leads to

\[ \mathbb{E}[L_E] = \text{area } Q \times \sqrt{\frac{E}{8}}, \]

and a legitimate guess for the order of the variance is

\[ \text{Var}(L_E) \asymp \sqrt{E}. \]

★ However, Berry showed that

\[ \text{Var}(L_E) \sim \frac{\text{area } Q}{512\pi} \log E, \]

whereas the length variances of non-zero level sets display the “correct” order of \( \sqrt{E} \).

★ Such a variance reduction “... results from a cancellation whose meaning is still obscure... ” (Berry (2002), p. 3032).
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Spherical Case

- Berry’s constants were confirmed by I. Wigman (2010) in the related model of random spherical harmonics.

- Here, the Laplace eigenvalues are the integers \( n(n + 1) \), \( n \in \mathbb{N} \).

Picture: A. Barnett
Let $T = \mathbb{R}^2 / \mathbb{Z}^2 \simeq [0, 1)^2$ be the 2-dimensional flat torus.

We are again interested in real (random) eigenfunctions of $\Delta$, that is, solutions of the Helmholtz equation

$$\Delta f + Ef = 0,$$

for some adequate $E > 0$ (eigenvalue).

A $L^2$-complete orthonormal set of eigenfunctions of $\Delta$ is obtained as:

$$(x_1, x_2) \mapsto \exp \left\{ 2i\pi(\lambda_1 x_1 + \lambda_2 x_2) \right\},$$

with $(\lambda_1, \lambda_2) \in \mathbb{Z}^2$. Each one is associated with the eigenvalue $E = 4\pi^2(\lambda_1^2 + \lambda_2^2)$. 

ARITHMETIC RANDOM WAVES
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The eigenvalues of $\Delta$ are therefore given by the set

$$\{E_n := 4\pi^2 n : n \in S\},$$

where

$$S = \{n : n = a^2 + b^2; a, b \in \mathbb{Z}\}.$$

For $n \in S$, the dimension of the corresponding eigenspace is $\mathcal{N}_n = r_2(n) := \#\Lambda_n$, where $\Lambda_n := \{(\lambda_1, \lambda_2) : \lambda_1^2 + \lambda_2^2 = n\}$.

We know e.g. that $r_2(n) \ll n^\epsilon, \forall \epsilon > 0$, and “pathological” behaviours are possible.
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We define the arithmetic random wave of order $n \in S$ as:

$$f_n(x) = \frac{1}{\sqrt{N_n}} \sum_{\lambda \in \Lambda_n} a_{\lambda} e^{2i\pi \langle \lambda, x \rangle}, \quad x \in \mathbb{T},$$

where the $a_{\lambda}$ are i.i.d. complex standard Gaussian, except for the relation $a_{\lambda} = \overline{a_{-\lambda}}$.

We are interested in the behaviour, as $N_n \to \infty$, of the total nodal length

$$\mathcal{L}_n := \text{length } f_n^{-1}({\{0\}}).$$

Picture: J. Angst & G. Poly
NODAL LENGTHS AND SPECTRAL MEASURES

★ Crucial role played by the set of **spectral probability measures** on $S^1$

$$
\mu_n(dz) := \frac{1}{N_n} \sum_{\lambda \in \Lambda_n} \delta_{\lambda/\sqrt{n}}(dz), \quad n \in S
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(invariant with respect to $z \mapsto \bar{z}$ and $z \mapsto i \cdot z$.)

★ The set $\{\mu_n : n \in S\}$ is relatively compact and its adherent points are an **infinite strict subset** of the class of invariant probabilities on the circle (see Kurlberg and Wigman (2015)).

★ Quick demonstration (see Krishnapur, Kurlberg and Wigman (2013)): the adherent points of the set

$$
\hat{\mu}_n(4)^2 := \left( \int_{S^1} z^{-4} \mu_n(dz) \right)^2, \quad n \in S,
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are given by the whole interval $[0, 1]$. 
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Another Cancellation

★ Rudnick and Wigman (2008): For every \( n \in S \), \( \mathbb{E}[\mathcal{L}_n] = \frac{\sqrt{E_n}}{2\sqrt{2}} \).

Moreover, \( \text{Var}(\mathcal{L}_n) = O\left(\frac{E_n}{N_n^{1/2}}\right) \). Conjecture: \( \text{Var}(\mathcal{L}_n) = O\left(\frac{E_n}{N_n}\right) \).

★ Krishnapur, Kurlberg and Wigman (2013): if \( \{n_j\} \subset S \) is such that \( N_{n_j} \to \infty \), then

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\text{Var}(\mathcal{L}_{n_j}) = \frac{E_{n_j}}{N_{n_j}^2} \times c(n_j) + O(E_{n_j} R_5(n_j)),
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where

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c(n_j) = \frac{1 + \hat{\mu}_{n_j}(4)^2}{512}; \quad R_5(n_j) = \int_\mathbb{T} |r_{n_j}(x)|^5 dx = o\left(\frac{1}{N_{n_j}^2}\right).
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★ Two phenomena: (i) cancellation, and (ii) non-universality.
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★ Two phenomena: (i) **cancellation**, and (ii) **non-universality**.
INTERLUDE: CHLADNI PLATES (1787)
**Next Step: Second Order Results**

For $E > 0$ and $n \in S$, define the normalized quantities

$$\tilde{L}_E := \frac{L_E - \mathbb{E}(L_E)}{\text{Var}(L_E)^{1/2}} \quad \text{and} \quad \tilde{L}_n := \frac{L_n - \mathbb{E}(L_n)}{\text{Var}(L_n)^{1/2}}.$$

**Question**: Can we explain the above cancellation phenomena and, as $E, N_n \to \infty$, establish limit theorems of the type

$$\tilde{L}_E \xrightarrow{\text{LAW}} \gamma, \quad \text{and} \quad \tilde{L}_{n'} \xrightarrow{\text{LAW}} Z?$$

($\{n'_j\} \subset S$ is some subsequence)
For $E > 0$ and $n \in S$, define the normalized quantities

$$\tilde{L}_E := \frac{L_E - \mathbb{E}(L_E)}{\text{Var}(L_E)^{1/2}} \quad \text{and} \quad \tilde{L}_n := \frac{L_n - \mathbb{E}(L_n)}{\text{Var}(L_n)^{1/2}}.$$ 

**Question**: Can we explain the above cancellation phenomena and, as $E, N_n \to \infty$, establish limit theorems of the type

$$\tilde{L}_E \xrightarrow{\text{LAW}} \gamma, \quad \text{and} \quad \tilde{L}_{n_j} \xrightarrow{\text{LAW}} Z?$$

($\{n'_j\} \subset S$ is some subsequence)
**Step 1.** Let $V = f_n$ or $B_E$, and $L = L_E$ or $L_n$. Use the representation (based on the coarea formula)

$$L = \int \delta_0(V(x)) \|\nabla V(x)\| \, dx, \quad \text{in } L^2(\mathbb{P}),$$

to deduce the **Wiener chaos expansion** of $L$.

**Step 2.** Show that exactly one chaotic projection $L(4) := \text{proj}(L \mid C_4)$ dominates in the high-energy limit – thus accounting for the cancellation phenomenon.

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**A Common Strategy**
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A COMMON STRATEGY

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**Fluctuations for Berry’s Model**

Theorem (Nourdin, Peccati & Rossi, 2019)

1. **(Cancellation)** For every fixed $E > 0$,

$$\text{proj}(L_E \mid C_{2q+1}) = 0, \quad q \geq 0,$$

and $\text{proj}(\tilde{L}_E \mid C_2)$ reduces to a “negligible boundary term”, as $E \to \infty$.

2. **(4$^{th}$ chaos dominates)** Let $E \to \infty$. Then,

$$\tilde{L}_E = \text{proj}(\tilde{L}_E \mid C_4) + o_p(1).$$

3. **(CLT)** As $E \to \infty$,

$$\tilde{L}_E \Rightarrow Z \sim N(0,1).$$
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Theorem

Define, for $B = B_1$:

$$L_r := \text{length}(B^{-1}(\{0\}) \cap \text{Ball}(0, r)).$$

Then,

1. $\mathbb{E}[L_r] = \frac{\pi r^2}{2\sqrt{2}}$;
2. as $r \to \infty$, $\text{Var}(L_r) \sim \frac{r^2 \log r}{256}$;
3. as $r \to \infty$, 
   $$\frac{L_r - \mathbb{E}[L_r]}{\text{Var}(L_r)^{1/2}} \Rightarrow Z \sim N(0, 1).$$
Theorem (Marinucci, P., Rossi & Wigman, 2016)

1. **(Exact Cancellation)** For every fixed $n \in S$,

$$\text{proj}(\mathscr{L}_n | C_2) = \text{proj}(\mathscr{L}_n | C_{2q+1}) = 0, \quad q \geq 0.$$ 

2. **(4th chaos dominates)** Let $\{n_j\} \subset S$ be such that $N_{n_j} \to \infty$. Then,

$$\tilde{\mathscr{L}}_{n_j} = \text{proj}(\tilde{L}_{n_j} | C_4) + o_\mathbb{P}(1).$$

3. **(Non-Universal/Non-Gaussian)** If $|\hat{\mu}_{n_j}(4)| \to \eta \in [0, 1]$, where $\hat{\mu}_n(4) = \int z^4 \mu_n(dz)$, then

$$\tilde{\mathscr{L}}_{n_j} \Rightarrow M(\eta) := \frac{1}{2 \sqrt{1 + \eta^2}} \left( 2 - (1 - \eta)Z_1^2 - (1 + \eta)Z_2^2 \right),$$

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PHASE SINGULARITIES

Theorem (Dalmao, Nourdin, P. & Rossi, 2016)

For \( \hat{T} \) an independent copy, consider

\[
I_n := \#[T_n^{-1}(\{0\}) \cap \hat{T}_n^{-1}(\{0\})].
\]

1. As \( N_n \to \infty \),

\[
\text{Var}(I_n) \sim \frac{E_n^2}{N_n^2} \frac{3\hat{\mu}_n(4)^2 + 5}{128\pi^2}
\]

2. If \( |\hat{\mu}_n(4)| \to \eta \in [0, 1] \), then

\[
\tilde{I}_{n_j} \Rightarrow J(\eta) := \frac{1}{2\sqrt{10 + 6\eta^2}} \left( \frac{1 + \eta}{2} A + \frac{1 - \eta}{2} B - 2(C - 2) \right)
\]

with \( A, B, C \) independent s.t. \( A \overset{\text{law}}{=} B \overset{\text{law}}{=} 2X_1^2 + 2X_2^2 - 4X_3^2 \) and \( C \overset{\text{law}}{=} X_1^2 + X_2^2 \), where \( (X_1, X_2, X_3) \) is standard Gaussian.
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EXPlicit Bounds by Stein’s Method

★ (Arithmetic Case) One has that, for \( \eta = \mu_n(4) \)

\[
\text{Wass}_1(\tilde{L}_n, M(\eta)) \leq \frac{C}{|\Lambda_n|^{1/4}}.
\]

★ (Planar case) For every \( E > 0 \),

\[
\text{Wass}_1(\tilde{L}_E, N) \leq \frac{C}{(\log E)^{1/4}}.
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★ Technically challenging point: \( \delta_0 \) is a generalized function.
\textbf{Explicit Bounds by Stein’s Method}

\begin{itemize}
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Elements of Proof (BRW)

★ In view of Green’s identity, one has that

\[
\text{proj}(L_E | C_2) = \frac{1}{2\sqrt{E}} \int_{\partial Q} B_E(x) \langle \nabla B_E(x), n(x) \rangle \, dx,
\]

where \( n(x) \) is the outward unit normal at \( x \) (variance bounded).

★ The term \( \text{proj}(\tilde{L}_E | C_4) \) is a l.c. of \( 4^{\text{th}} \) order terms, among which

\[
V_E := \sqrt{E} \int_Q H_4(B_E(x)) \, dx,
\]

for which one has that

\[
\text{Var}(V_E) = \frac{24}{E} \int_{(\sqrt{E}Q)^2} J_0(||x - y||)^4 \, dx \, dy \sim \frac{18}{\pi^2} \log E,
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using e.g. \( J_0(r) \sim \sqrt{\frac{2}{\pi r}} \cos(r - \pi/4), r \to \infty. \)
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⋆ Write $\mathcal{L}_n(u) = \text{length } f_n^{-1}(u)$. One has that

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$$= c \frac{e^{-u^2/2}u^2}{\mathcal{N}_n} \sum_{\lambda \in \Lambda^n} (|a_\lambda|^2 - 1)$$

(this is the dominating term for $u \neq 0$; it verifies a CLT).

⋆ Prove that proj$(\mathcal{L}_n \mid C_4)$ has the form

$$\sqrt{\frac{E_n}{\mathcal{N}_n^2}} \times Q_n,$$

where $Q_n$ is a quadratic form, involving sums of the type

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FURTHER RESULTS

★ Benatar and Maffucci (2017) and Cammarota (2017): fluctuations on nodal volumes for ARW on $\mathbb{R}^3 / \mathbb{Z}^3$.

★ The nodal length of random spherical harmonics verifies a Gaussian CLT (Marinucci, Rossi, Wigman (2017)).

★ Analogous non-central results hold for nodal lengths on shrinking balls (Benatar, Marinucci and Wigman, 2017).
Further Results

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Suppose \( \{ K_\lambda : \lambda > 0 \} \) is a collection of covariance kernels on \( \mathbb{R}^2 \) such that, for \( \lambda \to \infty \), some \( r_\lambda \to \infty \) and every \( \alpha, \beta \),

\[
\sup_{|x|,|y| \leq r_\lambda} \left| \partial^\alpha \partial^\beta (K_\lambda(x, y) - J_0(\|x - y\|)) \right| := \eta(\lambda) = o(1)
\]

Let \( Y_\lambda \sim K_\lambda \) and \( B \sim J_0 \).

Typical example: \( Y_\lambda = \frac{1}{\sqrt{2\pi}} \times \) Canzani-Hanin’s pullback random wave (dim. 2) at a point of isotropic scaling (needs \( r_\lambda = o(\lambda) \)).
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**Beyond Explicit Models (W.I.P.)**

- Write $L(\lambda, r) := \text{length}\{\lambda^{-1}(\{0\}) \cap \text{Ball}(0, r)\}$, and $L_r := \text{length}(B_1 \cap \text{Ball}(0, r))$.

- Then, one can couple $\lambda$ and $B$ on the same probability space, in such a way that, if $r_{\lambda} \eta(\lambda)^{\beta} \to 0$ (say, $\beta \simeq 1/30$),

\[
\left| \frac{L(\lambda, r) - \mathbb{E}L(\lambda, r)}{\text{Var}(L_{r_{\lambda}})^{1/2}} - \frac{L_r - \mathbb{E}L_r}{\text{Var}(L_r)^{1/2}} \right| \to 0,
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in $L^2$.

- For instance, if $\eta(\lambda) = O(1/ \log \lambda)$ (expected for pullback waves coming from manifolds with no conjugate points), then the statement is true for $r_{\lambda} = (\log \lambda)^{\beta}$, $\beta \simeq 1/30$. 
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THANK YOU FOR YOUR ATTENTION!