Stein’s Method and Random Matrices

Elizabeth Meckes

Case Western Reserve University

Symposium in Memory of Charles Stein
National University of Singapore
July, 2019
Stanford faculty members, from left, Pamela M. Lee, who wrote a petition opposing the appointment of Donald H. Rumsfeld as a distinguished visiting fellow at the Hoover Institution; Tom Wasow; Eric Roberts; Charles Stein; Charlotte Fonrobert; and Philip Zimbardo. Jim Wilson/The New York Times
Random orthogonal matrices

$O(n)$ denotes the orthogonal group:

$U \in M_n(\mathbb{R})$ 

$UU^T = I_n$

A random orthogonal matrix is a random element $U$ of $O(n)$ distributed according to Haar measure (uniform probability measure).

With high probability, the eigenvalues of a random orthogonal matrix are very evenly spaced around the circle.
Random orthogonal matrices

$O(n)$ denotes the orthogonal group:

$$U \in M_n(\mathbb{R}) \quad UU^T = I_n$$

A random orthogonal matrix is a random element $U$ of $O(n)$ distributed according to Haar measure (uniform probability measure).
Random orthogonal matrices

\( O(n) \) denotes the orthogonal group:

\[
U \in M_n(\mathbb{R}) \quad UU^T = I_n
\]

A random orthogonal matrix is a random element \( U \) of \( O(n) \) distributed according to Haar measure (uniform probability measure).

With high probability, the eigenvalues of a random orthogonal matrix are very evenly spaced around the circle.
In their proof that the eigenvalues of $U$ are close to uniform on the circle, Diaconis and Shahshahani showed that each mixed moment of the random vector

$$(\text{Tr}(U), \text{Tr}(U^2), \ldots, \text{Tr}(U^k))$$

is, for $n$ large enough, precisely that of a Gaussian random vector.
In their proof that the eigenvalues of $U$ are close to uniform on the circle, Diaconis and Shahshahani showed that each mixed moment of the random vector

$$(\text{Tr}(U), \text{Tr}(U^2), \ldots, \text{Tr}(U^k))$$

is, for $n$ large enough, precisely that of a Gaussian random vector.

In later work, Kurt Johansson and Charles Stein showed independently that, for $k$ fixed,

$$\text{Tr}(U^k) \xrightarrow{\text{very fast}} \mathcal{N}(\eta_k, k),$$

where for $\ell \in \mathbb{N}$, $\eta_{2\ell} = 1$ and $\eta_{2\ell+1} = 0$. 
Theorem (Stein ’95)

Let $U$ be a random orthogonal matrix and let $k \in \mathbb{N}$, and let $Z \sim \mathcal{N}(0, 1)$. For any $r$, there is a constant $c_{k,r}$ such that for any measurable $A \subseteq \mathbb{R}$,

$$\left| \mathbb{P} \left[ \frac{1}{\sqrt{k}} \left( \text{Tr}(U^k) - \eta_k \right) \in A \right] - \mathbb{P}[Z \in A] \right| \leq \frac{c_{k,r}}{n^r}.$$
Theorem (Stein ’95)

Let $U$ be a random orthogonal matrix and let $k \in \mathbb{N}$, and let $Z \sim \mathcal{N}(0,1)$. For any $r$, there is a constant $c_{k,r}$ such that for any measurable $A \subseteq \mathbb{R}$,

$$\left| \mathbb{P} \left[ \frac{1}{\sqrt{k}} \left( \text{Tr}(U^k) - \eta_k \right) \in A \right] - \mathbb{P}[Z \in A] \right| \leq \frac{c_{k,r}}{n^r}.$$ 

Theorem (Johansson ’97)

Let $U$ be a random orthogonal matrix and let $k \in \mathbb{N}$. There are constants $c, d$ such that

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P} \left[ \frac{1}{\sqrt{k}} \left( \text{Tr}(U^k) - \eta_k \right) \leq x \right] - \Phi(x) \right| \leq ce^{-dn}.$$
THE ACCURACY OF THE NORMAL APPROXIMATION TO THE
DISTRIBUTION OF THE TRACES OF POWERS OF RANDOM
ORTHOGONAL MATRICES

BY

CHARLES STEIN

TECHNICAL REPORT NO. 470

MARCH 1995
Stein’s method of exchangeable pairs

Let \((W, W')\) be an exchangeable pair of random variables, with \(\mathbb{E} W = 0\) and \(\mathbb{E} W^2 = 1\). Let \(\Delta = W' - W\), and suppose that there is a \(\lambda > 0\) such that

\[
\mathbb{E} [\Delta | W] = -\lambda W + R.
\]

Then if \(g \in C^1(\mathbb{R})\),

\[
|\mathbb{E} g(W) - \mathbb{E} g(Z)| \leq \frac{1}{\lambda} \|g - \mathbb{E} g(Z)\|_\infty \sqrt{\text{Var}(\mathbb{E}[\Delta^2 | W])} + \frac{\|g'\|_\infty}{2\lambda} \mathbb{E} |\Delta|^3
\]

\[
+ \frac{2\sqrt{2\pi}}{\lambda} \|g\|_\infty \mathbb{E} |R|.
\]
The exchangeable pair on $\mathfrak{O}(n)$

Let $U$ be a random orthogonal matrix.
The exchangeable pair on $\mathbb{O}(n)$

Let $U$ be a random orthogonal matrix.

Let

$$A_\epsilon = \begin{bmatrix} \sqrt{1-\epsilon^2} & \epsilon \\ -\epsilon & \sqrt{1-\epsilon^2} \end{bmatrix} \oplus I_{n-2}. $$
The exchangeable pair on $\mathbb{O}(n)$

Let $U$ be a random orthogonal matrix.

Let
\[
A_\epsilon = \begin{bmatrix}
\sqrt{1 - \epsilon^2} & \epsilon \\
-\epsilon & \sqrt{1 - \epsilon^2}
\end{bmatrix} \oplus I_{n-2}.
\]

Let $V$ be a random orthogonal matrix (independent of $U$), and define $U_\epsilon$ by
\[
U_\epsilon = \begin{bmatrix} VA_\epsilon V^T \end{bmatrix} U.
\]
The exchangeable pair on $\mathbb{O}(n)$

Let $U$ be a random orthogonal matrix.

Let
$$A_\epsilon = \begin{bmatrix} \sqrt{1-\epsilon^2} & \epsilon \\ -\epsilon & \sqrt{1-\epsilon^2} \end{bmatrix} \oplus I_{n-2}.$$ 

Let $V$ be a random orthogonal matrix (independent of $U$), and define $U_\epsilon$ by

$$U_\epsilon = [VA_\epsilon V^T] U.$$

$(U, U_\epsilon)$ is a parametrized family of Stein pairs.
In proving Stein’s CLT for $W_k := \text{Tr}(U^k)$, take

$$U_\epsilon = [VA_\epsilon V^T]U \quad \text{and} \quad W_{k,\epsilon} = \text{Tr}(U^k_\epsilon).$$
In proving Stein’s CLT for $W_k := \text{Tr}(U^k)$, take

$$U_\epsilon = [VA_\epsilon V^T]U \quad \quad W_{k,\epsilon} = \text{Tr}(U_{\epsilon}^k).$$

Expanding $VA_\epsilon V^T$ in $\epsilon$, one can show that $U_\epsilon - U = O(\epsilon)$ and

$$\mathbb{E} [U_\epsilon - U|U] = \mathbb{E} [VA_\epsilon V^T - I_n] U = -\frac{\epsilon^2}{n} U + O(\epsilon^4).$$
In proving Stein’s CLT for $W_k := \text{Tr}(U^k)$, take

$$U_\epsilon = \begin{bmatrix} VA_\epsilon & V^T \end{bmatrix} U \quad \quad W_{k,\epsilon} = \text{Tr}(U^k_\epsilon).$$

Expanding $VA_\epsilon V^T$ in $\epsilon$, one can show that $U_\epsilon - U = O(\epsilon)$ and

$$\mathbb{E} \left[ U_\epsilon - U \mid U \right] = \mathbb{E} \left[ VA_\epsilon V^T - I_n \right] U = -\frac{\epsilon^2}{n} U + O(\epsilon^4).$$

It is then immediate that $\mathbb{E} |\Delta|^3 = O(\epsilon^3)$, and some computation shows that

$$\frac{n}{\epsilon^2 k} \mathbb{E} \left[ W_{k,\epsilon} - W_k \mid U \right]$$

$$= -W_k + \frac{1}{n-1} \left[ \begin{array}{c} \text{complicated but} \\ \text{bounded expression} \end{array} \right] + O(\epsilon^2).$$
The multivariate setting

Let \((X, X')\) be an exchangeable pair of random vectors. Let 
\(\Delta = X' - X\), and suppose that there is an invertible \(\Lambda\), a symmetric, nonnegative definite \(\Sigma\), and a \(\sigma(X)\)-measureable random matrix \(E\) such that

\[
\mathbb{E} [\Delta | X] = -\Lambda X \quad \text{and} \quad \mathbb{E} [\Delta \Delta^T | X] = 2\Lambda \Sigma + E.
\]

Then if \(g \in C^3(\mathbb{R})\),

\[
\left| \mathbb{E} g(X) - \mathbb{E} g(\Sigma^{1/2} Z) \right| \\
\leq \|\Lambda^{-1}\|_{op} \left[ \frac{1}{4} M_2(g) \mathbb{E} \|E\|_{\text{H.S.}} + \frac{1}{9} M_3(g) \mathbb{E} |\Delta|^3 \right],
\]

where

\[
M_2(g) = \sup_x \|\text{Hess } g(x)\|_{\text{H.S.}} \quad M_3(g) = \sup_x \|D^3 g(x)\|_{op}.
\]
The multivariate setting

**Theorem (Döbler–Stolz, 2011)**

Let $U \in \mathbb{O}(n)$ be a random orthogonal matrix,

$$
W = (\text{Tr}(U), \text{Tr}(U^2) - 1, \ldots, \text{Tr}(U^k) - \mathbb{1}_{2|k}),
$$

with $Z_1, \ldots, Z_k$ i.i.d. standard Gaussian random variables.
The multivariate setting

Theorem (Döbler–Stolz, 2011)

Let \( U \in O(n) \) be a random orthogonal matrix, \( W = (\text{Tr}(U), \text{Tr}(U^2) - 1, \ldots, \text{Tr}(U^k) - 1_{2|k}) \),

and

\( Z = (Z_1, \sqrt{2}Z_2, \ldots, \sqrt{k}Z_k) \),

with \( Z_1, \ldots, Z_k \) i.i.d. standard Gaussian random variables.
The multivariate setting

Theorem (Döbler–Stolz, 2011)

Let $U \in \mathbb{O}(n)$ be a random orthogonal matrix,

$$W = (\text{Tr}(U), \text{Tr}(U^2) - 1, \ldots, \text{Tr}(U^k) - 1_{2|k}),$$

and

$$Z = (Z_1, \sqrt{2}Z_2, \ldots, \sqrt{k}Z_k),$$

with $Z_1, \ldots, Z_k$ i.i.d. standard Gaussian random variables.

Then

$$W_1(W, Z) = O\left(\frac{k^{7/2}}{n}\right).$$
The entries of a random orthogonal matrix

Heuristic: Random orthogonal matrices are kind of like matrices with i.i.d. Gaussian entries.

Theorem (Jiang–Ma, Stewart)
Let $U$ be a random orthogonal matrix in $O(n)$ and let $U_{p,q}$ be the upper-left $p \times q$ block of $U$.
Let $Z_{p,q}$ denote a $p \times q$ matrix of i.i.d centered Gaussian random variables, of variance $1/n$.
Then $d_{TV}(U_{p,q}, Z_{p,q}) \xrightarrow{n \to \infty} 0$ as long as $pq = o(n)$. 
The entries of a random orthogonal matrix

Heuristic: Random orthogonal matrices are kind of like matrices with i.i.d. Gaussian entries.
The entries of a random orthogonal matrix

**Heuristic:** Random orthogonal matrices are kind of like matrices with i.i.d. Gaussian entries.

**Theorem (Jiang–Ma, Stewart)**

Let $U$ be a random orthogonal matrix in $O(n)$ and let $U_{p,q}$ be the upper-left $p \times q$ block of $U$.

Let $Z_{p,q}$ denote a $p \times q$ matrix of i.i.d centered Gaussian random variables, of variance $\frac{1}{n}$.

Then $d_{\text{TV}}(U_{p,q}, Z_{p,q}) \xrightarrow{n \to \infty} 0$ as long as $pq = o(n)$. 

The entries of a random orthogonal matrix

Heuristic: Random orthogonal matrices are kind of like matrices with i.i.d. Gaussian entries.

Theorem (Jiang–Ma, Stewart)

Let $U$ be a random orthogonal matrix in $O(n)$ and let $U_{p,q}$ be the upper-left $p \times q$ block of $U$. Let $Z_{p,q}$ denote a $p \times q$ matrix of i.i.d centered Gaussian random variables, of variance $\frac{1}{n}$.
The entries of a random orthogonal matrix

Heuristic: Random orthogonal matrices are kind of like matrices with i.i.d. Gaussian entries.

Theorem (Jiang–Ma, Stewart)

Let $U$ be a random orthogonal matrix in $O(n)$ and let $U_{p,q}$ be the upper-left $p \times q$ block of $U$. Let $Z_{p,q}$ denote a $p \times q$ matrix of i.i.d centered Gaussian random variables, of variance $\frac{1}{n}$. Then

$$d_{TV}(U_{p,q}, Z_{p,q}) \xrightarrow{n \to \infty} 0$$

as long as $pq = o(n)$. 
The entries of a random orthogonal matrix

**Theorem (Chatterjee–M. ’08)**

Let $U \in \mathbb{O}(n)$ be a random orthogonal matrix, let $A_1, \ldots, A_k \in M_n(\mathbb{R})$ be orthonormal (w.r.t. $\langle A, B \rangle = \text{Tr}(AB^T)$), and let

$$X = (\text{Tr}(A_1 U), \ldots, \text{Tr}(A_k U)).$$
The entries of a random orthogonal matrix

**Theorem (Chatterjee–M. ’08)**

Let \( U \in O(n) \) be a random orthogonal matrix, let \( A_1, \ldots, A_k \in M_n(\mathbb{R}) \) be orthonormal (w.r.t. \( \langle A, B \rangle = \text{Tr}(AB^T) \)), and let

\[
X = (\text{Tr}(A_1 U), \ldots, \text{Tr}(A_k U)).
\]

Let \( Z = (Z_1, \ldots, Z_k) \) a vector of i.i.d. standard Gaussians. Then

\[
W_1(X, Z) \leq \frac{\sqrt{2k}}{n - 1}.
\]
Classical random matrix ensemble: GOE

Random matrices

\[ X = \begin{bmatrix} X_{ij} \end{bmatrix}_{n \times n} \]

where \{X_{ii}\} \cup \{X_{ij}\}_{i < j}

are independent, \( X_{ii} \sim \mathcal{N}(0, 1) \) if \( i < j \), \( X_{ij} \sim \mathcal{N}(0, 1^2) \), and \( X_{ij} = X_{ji} \).

The random matrix \( X \) can thus be written as

\[ X = \sum_{1 \leq i \leq j \leq n} Z_{ij} B_{ij} \]

for \( B_{ii} = \begin{pmatrix} i & 1 \end{pmatrix} \) \( B_{ij} = \begin{pmatrix} i & j \end{pmatrix} \sqrt{2} \) \( j & 1 \sqrt{2} \) and the \( \{Z_{ij}\}_{1 \leq i \leq j \leq n} \)
i.i.d. standard Gaussian.
Classical random matrix ensemble: GOE

Random matrices $X = [X_{ij}]_{i,j=1}^n$, where $\{X_{ii}\} \cup \{X_{ij}\}_{i<j}$ are independent, $X_{ii} \sim \mathcal{N}(0, 1)$, if $i < j$, $X_{ij} \sim \mathcal{N}(0, \frac{1}{2})$, and $X_{ij} = X_{ji}$.
Classical random matrix ensemble: GOE

Random matrices $X = [X_{ij}]_{i,j=1}^n$, where $\{X_{ii}\} \cup \{X_{ij}\}_{i<j}$ are independent, $X_{ii} \sim \mathcal{N}(0, 1)$, if $i < j$, $X_{ij} \sim \mathcal{N}(0, \frac{1}{2})$, and $X_{ij} = X_{ji}$.

The random matrix $X$ can thus be written as

$$X = \sum_{1 \leq i \leq j \leq n} Z_{ij} B_{ij},$$

for

$$B_{ii} = i \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad B_{ij} = i \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad i, \quad j$$

and the $\{Z_{ij}\}_{1 \leq i \leq j \leq n}$ i.i.d. standard Gaussian.
Norm-dependent random matrices

Theorem (E.M.–M. Meckes (to be posted someday soon...))

Let $X$ be a random $n \times n$ Hermitian matrix distributed uniformly in

$$\{X \in M_n(\mathbb{R}) : X = X^T, \|X\|_{H.S.} = \sqrt{n}\}.$$  

For $k \in \mathbb{N}$, let $W_k := \text{Tr}(X^k)$. Fix $d \in \mathbb{N}$, and let

$$W := (W_1, \ldots, W_d) = (\text{Tr}(X), \ldots, \text{Tr}(X^d)).$$

Then $W - \mathbb{E} W$ converges in distribution to $\Sigma^{1/2} Z$, where $Z$ is a standard Gaussian random vector in $\mathbb{R}^d$ and $\Sigma$ is a nonnegative definite symmetric matrix of rank $d - 1$. 
The exchangeable pair: rotations in matrix space

The random matrix \( X \) can be written as in the GOE case:

\[
X = \sum_{1 \leq i \leq j \leq n} X_{ij} B_{ij}
\]

with \( \tilde{X} = \{ X_{ij} \} \) chosen uniformly from the sphere of radius \( \sqrt{n} \) in \( \mathbb{R}^n \) \((n+1)/2\).

If \( V \in O(n(n+1)/2) \) is a random unitary matrix and \( A \in \text{as before} \), then \((\tilde{X}, VA \epsilon V^T \tilde{X})\) is an exchangeable pair of uniformly random points on the sphere, and so if \( X \in \mathcal{X} = \sum_{i \leq j} \left[ VA \epsilon V^T \tilde{X} \right]_{ij} B_{ij} \), \((X, X)\) is a family of Stein pairs.
The exchangeable pair: rotations in matrix space

The random matrix $X$ can be written as in the GOE case:

$$X = \sum_{1 \leq i \leq j \leq n} X_{ij}B_{ij}$$

with $\tilde{X} = \{X_{ij}\}$ chosen uniformly from the sphere of radius $\sqrt{n}$ in $\mathbb{R}^{n(n+1)/2}$. 
The exchangeable pair: rotations in matrix space

The random matrix $X$ can be written as in the GOE case:

$$X = \sum_{1 \leq i \leq j \leq n} X_{ij} B_{ij}$$

with $\tilde{X} = \{X_{ij}\}$ chosen uniformly from the sphere of radius $\sqrt{n}$ in $\mathbb{R}^{n(n+1)/2}$.

If $V \in O \left( \frac{n(n+1)}{2} \right)$ is a random unitary matrix and $A_\epsilon$ is as before, then $(\tilde{X}, VA_\epsilon V^T \tilde{X})$ is an exchangeable pair of uniformly random random points on the sphere, and so if

$$X_\epsilon = \sum_{i \leq j} [VA_\epsilon V^T \tilde{X}]_{ij} B_{ij},$$

$(X, X_\epsilon)$ is a family of Stein pairs.
Random matrices with prescribed eigenvalues

In classical random matrix theory, we specify the distribution of the entries and try to understand the distribution of the eigenvalues:

- Wigner $\rightarrow$ semi-circle law
- Ginibre $\rightarrow$ circular law (uniform on disc)
- Wishart $\rightarrow$ Marchenko–Pastur law
- Haar unitary/orthogonal/etc. $\rightarrow$ uniform on circle
Random matrices with prescribed eigenvalues

In classical random matrix theory, we specify the distribution of the entries and try to understand the distribution of the eigenvalues:

- **Wigner**: $\leadsto$ semi-circle law
- **Ginibre**: $\leadsto$ circular law (uniform on disc)
- **Wishart**: $\leadsto$ Marchenko–Pastur law
- **Haar unitary/orthogonal/etc.**: $\leadsto$ uniform on circle
Random matrices with prescribed eigenvalues

In classical random matrix theory, we specify the distribution of the entries and try to understand the distribution of the eigenvalues:

- Wigner $\leadsto$ semi-circle law
Random matrices with prescribed eigenvalues

In classical random matrix theory, we specify the distribution of the entries and try to understand the distribution of the eigenvalues:

- **Wigner** ⇝ semi-circle law
- **Ginibre** ⇝ circular law (uniform on disc)
Random matrices with prescribed eigenvalues

In classical random matrix theory, we specify the distribution of the entries and try to understand the distribution of the eigenvalues:

- Wigner $\rightsquigarrow$ semi-circle law
- Ginibre $\rightsquigarrow$ circular law (uniform on disc)
- Wishart $\rightsquigarrow$ Marchenko–Pastur law
Random matrices with prescribed eigenvalues

In classical random matrix theory, we specify the distribution of the entries and try to understand the distribution of the eigenvalues:

- Wigner $\rightsquigarrow$ semi-circle law
- Ginibre $\rightsquigarrow$ circular law (uniform on disc)
- Wishart $\rightsquigarrow$ Marchenko–Pastur law
- Haar unitary/orthogonal/etc. $\rightsquigarrow$ uniform on circle
Turning the question around

If you specify the eigenvalues of a random matrix, what are the entries like? Let $\lambda_1 \leq \cdots \leq \lambda_n$ be real numbers, and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Let $U \in U(n)$ be distributed according to Haar measure, and consider the random matrix $M = U \Lambda U^*$. 
If you specify the eigenvalues of a random matrix, what are the entries like?
Turning the question around

If you specify the eigenvalues of a random matrix, what are the entries like?

Let $\lambda_1 \leq \cdots \leq \lambda_n$ be real numbers, and

$$\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n).$$
Turning the question around

If you specify the eigenvalues of a random matrix, what are the entries like?

Let \( \lambda_1 \leq \cdots \leq \lambda_n \) be real numbers, and

\[
\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n).
\]

Let \( U \in \mathbb{U}(n) \) be distributed according to Haar measure, and consider the random matrix

\[
M = U \Lambda U^*.
\]
Theorem 1 (E. M.–M. Meckes)

Let $\Lambda$ be diagonal with $\text{Tr}(\Lambda) = 0$, $U \sim \text{Haar}(U(n))$, and

$$M = U\Lambda U^*.$$
Theorem 1 (E. M.–M. Meckes)

Let $\Lambda$ be diagonal with $\text{Tr}(\Lambda) = 0$, $U \sim \text{Haar}(\mathbb{U}(n))$, and

$$M = U \Lambda U^*.$$

Let $B_1, \ldots, B_d \in H_n(\mathbb{C})$ with $\text{Tr}(B_i B_j) = \delta_{ij}$ and $\text{Tr}(B_j) = 0$, and let

$$X = (\text{Tr}(MB_1), \ldots, \text{Tr}(MB_d)).$$
Theorem 1 (E. M.–M. Meckes)
Let $\Lambda$ be diagonal with $\text{Tr}(\Lambda) = 0$, $U \sim \text{Haar}(\mathbb{U}(n))$, and

$$M = U \Lambda U^*.$$ 

Let $B_1, \ldots, B_d \in H_n(\mathbb{C})$ with $\text{Tr}(B_i B_j) = \delta_{ij}$ and $\text{Tr}(B_j) = 0$, and let

$$X = (\text{Tr}(MB_1), \ldots, \text{Tr}(MB_d)).$$

For $g = (g_1, \ldots, g_d)$ a standard Gaussian random vector,

$$W_1 \left( \frac{n}{\|\Lambda\|_{HS}} X, g \right) \leq Cd\sqrt{n} \left( \frac{\|\Lambda\|_{op}^2}{\|\Lambda\|_{HS}^2} \right) = \frac{Cd\sqrt{n}}{sr(\Lambda)}.$$
Corollary

Let $V$ be a $d$-dimensional subspace of the space of $n \times n$ traceless Hermitian matrices, and let $\pi_V$ denote orthogonal projection onto $V$. If $M = U\Lambda U^*$ and $G$ is a GUE random matrix,

$$W_1 \left( \pi_V \left( \frac{n}{\|\Lambda\|_{HS}} M \right), \pi_V(G) \right) \leq \frac{Cd\sqrt{n}}{sr(\Lambda)}.$$

E.g., if half the $\lambda_i$ are $\sqrt{n}$ and half are $-\sqrt{n}$, then $sr(\Lambda) = n$. If $d \ll \sqrt{n}$.
Corollary

Let $V$ be a $d$-dimensional subspace of the space of $n \times n$ traceless Hermitian matrices, and let $\pi_V$ denote orthogonal projection onto $V$. If $M = U \Lambda U^*$ and $G$ is a GUE random matrix,

$$W_1 \left( \pi_V \left( \frac{n}{\|\Lambda\|_{HS}} M \right), \pi_V(G) \right) \leq \frac{Cd \sqrt{n}}{sr(\Lambda)}.$$  

E.g., if half the $\lambda_i$ are $\sqrt{n}$ and half are $-\sqrt{n}$, then $sr(\Lambda) = n$:

$$W_1 (\pi_V(M), \pi_V(G)) \to 0$$

if $d \ll \sqrt{n}$. 

Theorem 2 (E. M.–M. Meckes)

Let $\Lambda = \text{diag}(1, 0, \ldots, 0)$, $U \sim \text{Haar}(\mathbb{U}(n))$, and

$$M = U\Lambda U^*.$$
Theorem 2 (E. M.–M. Meckes)

Let \( \Lambda = \text{diag}(1, 0, \ldots, 0) \), \( U \sim \text{Haar}(\mathbb{U}(n)) \), and

\[
M = U \Lambda U^*.
\]

Let \( B_1, \ldots, B_d \in H_n(\mathbb{C}) \) with \( \text{Tr}(B_i B_j) = \delta_{ij} \) and \( \text{Tr}(B_j) = 0 \).
Theorem 2 (E. M.–M. Meckes)

Let $\Lambda = \text{diag}(1, 0, \ldots, 0)$, $U \sim \text{Haar}(\mathbb{U}(n))$, and

$$M = U\Lambda U^*.$$ 

Let $B_1, \ldots, B_d \in H_n(\mathbb{C})$ with $\text{Tr}(B_i B_j) = \delta_{ij}$ and $\text{Tr}(B_j) = 0$. Let

$$X = (\text{Tr}(MB_1), \ldots, \text{Tr}(MB_d))$$ 

and $g = (g_1, \ldots, g_d)$ a standard Gaussian.
Theorem 2 (E. M.–M. Meckes)

Let $\Lambda = \text{diag}(1, 0, \ldots, 0)$, $U \sim \text{Haar}(U(n))$, and

$$M = U\Lambda U^*.$$ 

Let $B_1, \ldots, B_d \in H_n(\mathbb{C})$ with $\text{Tr}(B_iB_j) = \delta_{ij}$ and $\text{Tr}(B_j) = 0$. Let

$$X = (\text{Tr}(MB_1), \ldots, \text{Tr}(MB_d))$$

and $g = (g_1, \ldots, g_d)$ a standard Gaussian. Then

$$W_1(nX, g) \leq C \sum_{j=1}^d \|B_j\|_4^2 \leq (C\sqrt{n}) \sum_{j=1}^d \frac{1}{\text{sr}(B_j)}.$$
Application 1: A probabilist’s Schur–Horn theorem
Application 1: A probabilist’s Schur–Horn theorem

The Schur–Horn Theorem

If $A$ is a real symmetric or complex Hermitian matrix with eigenvalues

$$\lambda_1 \leq \cdots \leq \lambda_n$$

and diagonal entries

$$d_1 \leq \cdots \leq d_n,$$

then $(d_j)_{1 \leq j \leq n} \prec (\lambda_j)_{1 \leq j \leq n}$; i.e., $(d_j)_{1 \leq j \leq n}$ is a convex combination of permutations of $(\lambda_j)_{1 \leq j \leq n}$.

Conversely, if $(d_j)_{1 \leq j \leq n} \prec (\lambda_j)_{1 \leq j \leq n}$, then there is a real symmetric matrix with diagonal entries $(d_j)_{1 \leq j \leq n}$ and eigenvalues $(\lambda_j)_{1 \leq j \leq n}$. 
Application 1: A probabilist’s Schur–Horn theorem
Application 1: A probabilist’s Schur–Horn theorem

For each \( n \in \mathbb{N} \), let

\[ \Lambda_n = \text{diag}(\lambda_1^{(n)}, \ldots, \lambda_n^{(n)}) , \]

be a fixed diagonal matrix, and let \( \mu_n \) be the spectral measure of \( n^{-1/2} \Lambda_n \).
Application 1: A probabilist’s Schur–Horn theorem

For each $n \in \mathbb{N}$, let

$$\Lambda_n = \text{diag}(\lambda_1^{(n)}, \ldots, \lambda_n^{(n)})$$

be a fixed diagonal matrix, and let $\mu_n$ be the spectral measure of $n^{-1/2}\Lambda_n$.

Suppose that there is a probability measure $\mu$ with mean $m$ and variance $\sigma^2 > 0$, such that $\mathcal{W}_2(\mu_n, \mu) \to 0$. 
Application 1: A probabilist’s Schur–Horn theorem

For each $n \in \mathbb{N}$, let

$$\Lambda_n = \text{diag}(\lambda_1^{(n)}, \ldots, \lambda_n^{(n)}),$$

be a fixed diagonal matrix, and let $\mu_n$ be the spectral measure of $n^{-1/2}\Lambda_n$.

Suppose that there is a probability measure $\mu$ with mean $m$ and variance $\sigma^2 > 0$, such that $W_2(\mu_n, \mu) \to 0$.

Let $A_n = U_n\Lambda_n U_n^*$ with $U_n$ Haar-distributed in $\mathbb{U}(n)$, and let $\nu_n$ be the empirical measure of the diagonal entries of $A_n$:

$$\nu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{a_i^{(n)}}.$$
Application 1: A probabilist’s Schur–Horn theorem

Theorem

If \( \| \Lambda_n - \frac{\text{Tr}(\Lambda_n)}{n} I_n \|_{op} = o(n) \), then

\[ \nu_n \to \mathcal{N}(m, \sigma^2) \text{ weakly in probability.} \]

If \( \| \Lambda_n - \frac{\text{Tr}(\Lambda_n)}{n} I_n \|_{op} = o\left( \frac{n}{\sqrt{\log n}} \right) \), then

\[ \nu_n \to \mathcal{N}(m, \sigma^2) \text{ weakly almost surely.} \]
Application 1: A probabilist’s Schur–Horn theorem

Theorem

If \( \| \Lambda_n - \frac{\text{Tr}(\Lambda_n)}{n} I_n \|_{op} = o(n) \), then

\[ \nu_n \to \mathcal{N}(m, \sigma^2) \text{ weakly in probability.} \]

If \( \| \Lambda_n - \frac{\text{Tr}(\Lambda_n)}{n} I_n \|_{op} = o\left( \frac{n}{\sqrt{\log n}} \right) \), then

\[ \nu_n \to \mathcal{N}(m, \sigma^2) \text{ weakly almost surely.} \]

If for some constants \( k, K > 0 \) and for all \( n \),
\( k \leq \| \Lambda_n - \frac{\text{Tr}(\Lambda_n)}{n} I_n \|_{op} \leq K \), then there are constants \( \kappa_1, \kappa_2, \kappa_3 > 0 \) depending only on \( k, K \) such that

\[ \kappa_1 \sqrt{\log n} \leq \mathbb{E} \left( \max_{1 \leq i \leq n} a_{ii}^{(n)} - \frac{1}{n} \text{Tr} \Lambda_n \right) \leq \kappa_2 \sqrt{\log n}. \]
Application 2: Random quantum states

A density matrix $\rho$ is an $n \times n$ Hermitian matrix with nonnegative eigenvalues and $\text{Tr} \rho = 1$. $\rho$ represents the state of a quantum mechanical system. If $\rho = \psi \psi^*$ for a unit vector $\psi$, $\rho$ is called a pure state. Any density matrix $\rho$ necessarily has the form $\rho = \text{Tr}_2 (\psi \psi^*)$, where $\psi$ is a unit vector in $\mathbb{C}^n \otimes \mathbb{C}^s \approx \mathbb{C}^{ns}$ and the partial trace $\text{Tr}_2$ is defined by $\text{Tr}_2 (A \otimes B) = \text{Tr}(B) A$. 
Application 2: Random quantum states

A density matrix $\rho$ is an $n \times n$ Hermitian matrix with nonnegative eigenvalues and $\text{Tr} \rho = 1$. $\rho$ represents the state of a quantum mechanical system. If $\rho = \psi \psi^\ast$ for a unit vector $\psi$, $\rho$ is called a pure state. Any density matrix $\rho$ necessarily has the form $\rho = \text{Tr}_2 (\psi \psi^\ast)$, where $\psi$ is a unit vector in $\mathbb{C}^n \otimes \mathbb{C}^s \approx \mathbb{C}^{ns}$ and the partial trace $\text{Tr}_2$ is defined by $\text{Tr}_2 (A \otimes B) = \text{Tr} (B) A$. 
Application 2: Random quantum states

A density matrix $\rho$ is an $n \times n$ Hermitian matrix with nonnegative eigenvalues and $\text{Tr} \, \rho = 1$.

$\rho$ represents the state of a quantum mechanical system.

If $\rho = \psi \psi^*$ for a unit vector $\psi$, $\rho$ is called a pure state.
Application 2: Random quantum states

A density matrix $\rho$ is an $n \times n$ Hermitian matrix with nonnegative eigenvalues and $\text{Tr} \, \rho = 1$.

$\rho$ represents the state of a quantum mechanical system.

If $\rho = \psi \psi^*$ for a unit vector $\psi$, $\rho$ is called a pure state.

Any density matrix $\rho$ necessarily has the form

$$\rho = \text{Tr}_2(\psi \psi^*),$$

where $\psi$ is a unit vector in $\mathbb{C}^n \otimes \mathbb{C}^s \simeq \mathbb{C}^{ns}$ and the partial trace $\text{Tr}_2$ is defined by $\text{Tr}_2(A \otimes B) = \text{Tr}(B)A$. 
Application 2: Random quantum states

An observable is an $n \times n$ Hermitian matrix $B$. If the system is in mixed state $\rho$, the expectation value of the observable $B$ is $\langle B \rangle = \text{Tr}(\rho B)$.

If we know the eigenvalues $0 \leq \lambda_1 \leq \cdots \leq \lambda_n$ of a quantum state but nothing else, a natural model to consider is $\rho = U \Lambda U^*$, with $U$ Haar distributed.

Our main theorem gives that if $B_1, \ldots, B_d$ are observables, then under some conditions, $(\langle B_1 \rangle, \ldots, \langle B_d \rangle)$ are approximately jointly Gaussian.
Application 2: Random quantum states

An observable is an $n \times n$ Hermitian matrix $B$. 
Application 2: Random quantum states

An observable is an $n \times n$ Hermitian matrix $B$.

If the system is in mixed state $\rho$, the expectation value of the observable $B$ is

$$\langle B \rangle = \text{Tr}(\rho B).$$
Application 2: Random quantum states

An observable is an $n \times n$ Hermitian matrix $B$.

If the system is in mixed state $\rho$, the expectation value of the observable $B$ is

$$\langle B \rangle = \text{Tr}(\rho B).$$

If we know the eigenvalues $0 \leq \lambda_1 \leq \cdots \leq \lambda_n$ of a quantum state but nothing else, a natural model to consider is

$$\rho = U \Lambda U^*,$$

with $U$ Haar distributed.
Application 2: Random quantum states

An observable is an $n \times n$ Hermitian matrix $B$.

If the system is in mixed state $\rho$, the expectation value of the observable $B$ is
\[ \langle B \rangle = \text{Tr}(\rho B). \]

If we know the eigenvalues $0 \leq \lambda_1 \leq \cdots \leq \lambda_n$ of a quantum state but nothing else, a natural model to consider is
\[ \rho = U \Lambda U^*, \]
with $U$ Haar distributed.

Our main theorem gives that if $B_1, \ldots, B_d$ are observables, then under some conditions, $(\langle B_1 \rangle, \ldots, \langle B_d \rangle)$ are approximately jointly Gaussian.
One last extension of Stein’s idea

Theorem (M., ’06)

Let $M$ be a compact Riemannian manifold, $f$ an eigenfunction of the Laplace-Beltrami operator on $M$ with eigenvalue $-\lambda$. Let $X$ be a uniform random point of $M$ and suppose that $f$ is normalized such that $\mathbb{E}f^2(X) = 1$. Then

\[ d_{TV}(f(X), Z) \leq \frac{1}{\lambda} \sqrt{\text{Var}(\|\nabla f(X)\|^2)}. \]
Thank you!