Matter Waves in Disorder: from Localization to Thermalization and Condensation

Christian Miniatura
UMI 3654 MajuLab
CNRS-UNS-NUS-NTU International Joint Research Unit
Singapore
7.5.2. **Thermalization and condensation in disordered systems**

(...)

The thermalization of a nonlinear wave in a random potential $V(x)$ is expected to be strongly affected by the spatially localized nature of Anderson modes. The WT theory can shed new light into this vast problematic.

A natural important question to be addressed is to see whether a weak disorder can prevent the thermalization process to take place. If such thermalization process can occur, then one may expect a process of classical wave condensation on a localized Anderson mode.

(...)

This is what we have tried to address in this work.
Usual Optical Wave Turbulence (WT) theory

Randomized wavefront + wave mixing ⇒ Thermalization (possibly condensation)

Our idea

- Take an initial perfectly coherent field (plane wave)
- Let scattering by disorder randomize the wave.
- Look at thermalization and possibly condensation when GP-interactions are present.
Physical and Numerical Framework

At time $t = 0$, a global excitation of the system (a plane wave state $k_0$) is subjected to a random (2D here) potential

\[ i\hbar \frac{\partial}{\partial t} \Psi(r, t) = -\frac{\hbar^2}{2m} \nabla^2 \Psi(r, t) + V(r)\Psi(r, t) + g|\Psi(r, t)|^2\Psi(r, t) \]

\[ |\psi(t = 0)\rangle = |k_0\rangle \]

Disorder strength

\[ V(r')V(r' + r) = V_0^2 C(r/\zeta) \]

2-point correlator

Disorder Correlation length (speckle grain)

What is the time evolution of the disorder-averaged momentum distribution when $g=0$ and when $g\neq 0$?

Repulsive disorder (speckle) $V_0 \geq 0$ Repulsive interactions ($g \geq 0$)

Measurable by Time-of-Flight experiments

\[ n(k, t) = |\bar{\Psi}(k, t)|^2 \]

(Bar = average over disorder)
Non-interacting particles in disordered potentials
Diffusion and Localization

Schrödinger equation with Random Potential

\[ i\hbar \frac{\partial |\Psi\rangle}{\partial t} = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) \right]|\Psi\rangle \]
In a *disordered* landscape, particles with momentum $k_0$ do not move along straight lines, but experience a *series of scatterings off obstacles sitting at random positions*.

⇒ Net result: **RANDOM WALK**
⇒ **Large-Scale motion = DIFFUSION**

The disorder-averaged equilibrium momentum distribution is isotropic and ring-shaped with radius $|k_0|$. 
1 Early times

\[ \mathbf{k}_0 = (2, 0) \]

(in units of correlation length)

Depletion of the initial mode

\[ t \simeq \tau_s \text{ scattering time} \]

\[ \tilde{\rho}(\mathbf{k}, t) \simeq e^{-t/\tau_s} \rho(\mathbf{k}, t = 0) \]

\( t \simeq \text{a few transport times } \tau \)

Progressive isotropization

of the distribution

Coherent peak emerges around

\[ \mathbf{k} = -\mathbf{k}_0 \]
Intermediate times

Isotropic background: fast (diffusive) **randomization** of the direction of propagation

Peak:

**Enhancement** of the back scattering probability (coherent back scattering or CBS effect)
Numerical experiment

Long times: localization

$t \sim \tau_{H}$
The coherent forward scattering (CFS) peak twins the CBS peak

$t \sim \tau_{loc}$
Emergence of a second coherent peak at $k = k_0$

$\tau_{loc} = \frac{\xi_{loc}^2}{D}$
Localization time

$\tau_{H} = 2\pi \nu \xi_{loc}^2$
Heisenberg time

[Karpiuk, NC, Lee, Grémaud, Müller, Miniatura, PRL (2012)]
Wave picture: interference ⇒ weak and strong localization

\[ n(k)/\zeta^2 \]

**CBS**

\[ -k_0 \]

**Weak Localization**

(Loop interference)

**Momentum distribution**

- **Multiple scattering**: isotropic ring
- **Interference effects**:
  - CBS = peak on top at \(-k_0\) (signature of coherent transport)
  - CFS = peak on top at \(k_0\) (signature of Anderson localization)

**Anderson Localization**

(Ergodicity breaking)
GP-interacting particles in disordered potentials: Thermalization & Condensation

Gross-Pitaevskii (aka Nonlinear Schrödinger) equation

\[ i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V(\mathbf{r})\Psi + g|\Psi|^2\Psi \]

\[ \Psi(\mathbf{r}, t = 0) = \sqrt{n} \exp(ik_0 \cdot \mathbf{r}) \]

Initial state = plane wave at constant number density.

Global excitation of the system.

Weak interactions

\[ V_0 \gg gn \]
Numerical experiment

\[ k_0 = 1.5 \quad V_0 = 5 \quad g = 1 \]

CBS peaks disappears

Diffusive ring disappears

Novel peak appears around \(|k| = 0\)

- \(t = 16\)
- \(t = 53\)
- \(t = 476\)
Numerical experiment

\[ k_0 = 1.5 \quad V_0 = 5 \quad g = 5 \]

Thermalization process as time grows

bimodal distribution shows up at long times: waves «condense» around \(|k| = 0\)?
Theory: nonlinear dynamics in disorder

Simplifying the problem:
- forget interference, only diffusion
- assume $t \gg \tau$ (isotropization)

**Kinetic equation** (including disorder average) derived from GPE:

$$- \frac{1}{2} \nabla^2 \Psi + V(r) \Psi + g|\Psi|^2 \Psi = i \partial_t \Psi$$

$$\partial_t f(r, t, \epsilon) - [\nabla - 2 \nabla g\bar{n}(r, t) \partial_\epsilon] D_\epsilon \left[\nabla - 2 \nabla g\bar{n}(r, t) \partial_\epsilon\right] f(r, t, \epsilon) = I_{\text{coll}}(r, t, \epsilon)$$

[Schwiete & Finkelstein (2010, 2011, 2013)]
[Cherroret & Wellens (2011)]

\[ A_\epsilon(k) = \text{Spectral function} \]
\[ \nu(\epsilon) = \int \frac{dk}{(2\pi)^d} A_\epsilon(k) = \text{Density of states} \]

\[ \bar{\rho}(k, t) = \int \frac{d\epsilon}{2\pi} \frac{A_\epsilon(k)}{2\pi \nu(\epsilon)} \int dr f(r, t, \epsilon) \]

proba that an atom with energy $\epsilon$ has momentum $k$

occupation number at energy $\epsilon$
Theory: nonlinear dynamics in disorder

Still complicated → start from a plane wave
→ System becomes translation invariant $f(\mathbf{r}, t, \epsilon) = f(t, \epsilon)$

$$\partial_t f(\epsilon, t) = I_{\text{coll}}(t, \epsilon)$$

$$I_{\text{coll}}(t, \epsilon) = 4\pi g^2 \int \frac{d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4}{(2\pi)^6} \int \frac{d\epsilon_2 d\epsilon_3 d\epsilon_4}{(2\pi)^3} A_\epsilon(\mathbf{k}_1) A_{\epsilon_2}(\mathbf{k}_2) A_{\epsilon_3}(\mathbf{k}_3) A_{\epsilon_4}(\mathbf{k}_4)$$

$$2\pi \nu(\epsilon) \delta(\epsilon - \epsilon_2 - \epsilon_3 - \epsilon_4) \delta(\mathbf{k} + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) \left[ (f'_{\epsilon} + f'_{\epsilon_2}) f'_{\epsilon_3} f'_{\epsilon_4} - f'_{\epsilon} f'_{\epsilon_2} (f'_{\epsilon_3} + f'_{\epsilon_4}) \right]$$

with $f'_{\epsilon_i} \equiv \frac{f(t, \epsilon_i)}{2\pi \nu(\epsilon_i)}$

At $t \to \infty$, $I_{\text{coll}} = 0$: the system has thermalized. This occurs for

``temperaturé'' (analogy with thermodynamics)

$$f'(t, \epsilon) = \frac{T}{\epsilon - \mu}$$

Rayleigh-Jeans distribution

``chemical potential''

idem weak turbulence theory of nonlinear waves

[Dyachenko, Newell, Pushkarev, Zakharov, Physica D (1992)]
Thermalized distribution

$$\bar{\rho}_{eq}(\mathbf{k}) = V \int \frac{d\epsilon}{2\pi} A_\epsilon(\mathbf{k}) \frac{T}{\epsilon - \mu}$$

$T(> 0)$ and $\mu(< -V_0)$ are determined from particle and energy conservation

$$\begin{cases} 
N \equiv \int \frac{d\mathbf{k}}{(2\pi)^2} \bar{\rho}_{eq}(\mathbf{k}) \\
\epsilon_0 \equiv \frac{k_0^2}{2} \equiv \int \frac{d\mathbf{k}}{(2\pi)^2} \frac{k^2}{2} \bar{\rho}_{eq}(\mathbf{k}) \\
\mu < -V_0 \\
T > 0
\end{cases}$$

Calculations for a 2D speckle potential $P(V) = e^{-V/V_0}/V_0$

- Energy scale
- $\sigma$ correlation length $\overline{V(0)V(\mathbf{r})} = 2V_0^2 J_1(r/\sigma)$

$V(\mathbf{r})$

0

$-V_0$

edge of the spectrum
Results for the chemical potential

When $V_0 = 0$ there is a solution $\mu < 0$ for all energies

When $V_0 \neq 0$ there is a solution $\mu < -V_0$ only for $\epsilon_0 > \epsilon_c(V_0)$

suggests a condensation phenomenon (analogy with Bose condensation of quantum particles)
Results for the chemical potential

- In the absence of disorder, no wave condensation in 2D
  (well known in weak turbulence theory in 2D)

Remark: 3D case is different!

[Connaughton, Josserand, Picozzi, Pomeau, Rica, PRL (2005)]

- In the presence of disorder, «condensation»: For $\epsilon_0 < \epsilon_c(V_0)$
[equivalently for $T < T_c(V_0)$] one (maybe several) low-energy states
are macroscopically occupied and relation $N = V \int \frac{dk}{2\pi} \int \frac{d\epsilon}{2\pi} A_\epsilon(k) \frac{T}{\epsilon - \mu}$
no longer holds
Difference between disorder and absence of disorder?

\[ N = V \int \frac{dk}{2\pi} \int \frac{d\epsilon}{2\pi} A_\epsilon(k) \frac{T}{\epsilon - \mu} = VT \int d\epsilon \frac{\nu(\epsilon)}{\epsilon - \mu} \]

**Absence** of disorder \( \nu(\epsilon \to 0^+) = \text{const} \) (2D)

→ for \( \mu \to 0^- \) integral **diverges** and \( T \) (and \( \epsilon_0 \)) must go to zero

**Presence** of disorder \( \nu(\epsilon \to -V_0^+) \propto \exp[-\alpha \frac{\epsilon \sigma}{\epsilon + V_0}] \) («Lifshitz» tail)

→ for \( \mu \to -V_0^- \) integral **converges** and \( T \) (and \( \epsilon_0 \)) is finite

• Solving for the 1-body density matrix

\[ \rho^{(1)}(k, k', t) = \overline{\Psi^*(k', t)} \Psi(k, t) - \overline{\Psi^*(k', t)} \overline{\Psi(k, t)} = \sum_i \frac{N_i}{N} \phi^*(k', t) \phi(k, t) \]

\[ n = 2/\zeta^2 \quad V_0 = 0.75\epsilon_\zeta \]

\[ E_0 = 0.12\epsilon_\zeta \quad L = 64\zeta \]

\[ N_{\text{BEC}}/N \]

**Caveat:** Condensation in the Lifshitz tail address exponentially rare (extreme) events…
Take-Home Messages & Perspectives

- GP-interactions induce \textbf{dephasing} and lead the system into a \textbf{thermal Rayleigh-Jeans energy distribution}.

- Disorder seems to enforce a \textbf{condensation} in 2D at E=0 for suitable parameter values contrary to the homogeneous case.

- Full momentum distribution is \textbf{bimodal} and \textbf{Penrose-Onsager criterion is fulfilled}.

\textbf{To be addressed:}
- Detailed analysis of the dynamics of thermalization and condensation (kinetic theory), fate of localization.
- Analysis of the condensate state (1-body density matrix, scaling, understand E=0 state in Lifshitz tail)
- Effect of residual interactions in the equilibrated system
- Universality class.
My Precious Collaborators

B. Grémaud  S. Ghosh  K. L. Lee  C. Müller  T. Karpiuk  N. Cherroret  D. Delande

Group papers on these topics:

N. Cherroret et al., PRA 85, 011604 (2012)
K. L. Lee et al., PRA 90, 043605 (2014)
S. Ghosh et al., PRL 115, 200602 (2015)

T. Karpiuk et al., PRL 109, 190601 (2012)
S. Ghosh et al., PRA 90, 063602 (2014)
N. Cherroret et al., PRA 92, 063614 (2015)
When \( g \neq 0 \), atoms are scattered both from disorder fluctuations and from the nonlinear potential \( g|\Psi(r, t)|^2 \).

Collisions redistribute energy over a time scale \( \tau_{\text{coll}} \sim 1/g^2 \) and Thermalization time

Fully-developed diffusion is established after \( \tau_B \sim 1/V_0^2 \) and Transport Boltzmann time

**Collisional and isotropization time scales**

**Nonlinearity challenges the Anderson transition**

Cherroret et al., PRL 112, 170603 (2014)

The \( g|\Psi|^2 \) term induces a dephasing that leads to a crossover from localization to subdiffusion.

Dephasing time \( \tau_\phi \sim 1/g \)
• Time scale hierarchy in the **weak** nonlinear regime

\[ \tau_\phi \ll \tau_{coll} \quad \frac{1}{g} \ll \frac{1}{g^2} \quad \text{Discard interference effects (CBS, CFS) in momentum space.} \]

\[ \tau_B \ll \tau_{coll} \quad \frac{1}{V_0^2} \ll \frac{1}{g^2} \quad \text{Disorder isotropizes atomic momenta before nonlinearity plays a role} \]

• **Momentum Distribution**

\[ n(k, t \gg \tau_B) = \frac{1}{N} \int dE \left( A(E, k) \frac{f(E, t)}{n} \right) \]

Energy distribution

Normalization:

\[ \int dE \nu(E)f(E, t) = n = \frac{N}{L^d} \]

Interaction-Free **Spectral Function**

\[ A(E, k) = -2 \text{Im} G^R(E) \]

\[ \nu(E) = \int \frac{dk}{(2\pi)^d} A(E, k) \quad \text{Density of States (DoS)} \]
What happens when the Critical Line $\mu = 0$ is crossed?

Physically, the critical line signals a **saturation of the population in the excited states**, which can no longer accommodate particles when the initial energy $E_0$ is decreased below $E_c$ and thus when $T$ becomes smaller than $T_c$.

**Cure:** allow for a **condensation** in the lowest-energy state.

$T \leq T_c$: the gas consists of a thermal Rayleigh-Jeans part and a condensate.

Solving the self-consistent equations with $\mu = 0$ gives $T$ and the condensate fraction.

\[
\frac{n(k)}{N} = \frac{n_{BEC}(k)}{N} + \frac{k_BT}{n} \int_0^T \frac{dE}{E} A(k, E)
\]

\[
\frac{N_{BEC}}{N} = 1 - \frac{k_BT}{n} \int_0^T \frac{\nu(E)}{E} dE
\]

This ratio drives the convergence properties of the integral at $E=0$. 

**Historical Einstein’s BEC argument**
Disorder strongly affects DOS at low energies compared to the uniform case (V=0).

\[
\frac{N_{BEC}}{N} = 1 - \frac{k_B T}{n} \int_0^T \frac{\nu(E)}{E} dE
\]

Convergent integral at E=0 due to the Lifshitz tail. This explains 2D condensation here contrary to the usual 2D WT theory in uniform media.

Condensation in the Lifshitz Tail

\[\nu(E) \sim e^{-\epsilon_\zeta/E}\]

\[\nu_{uni}^{(2D)}(E) = \frac{m}{2\pi h^2} = \text{cte}\]

\[V = 0.75\epsilon_\zeta\]

\[k_0 = 1.7/\zeta\]
A Bimodal Momentum Distribution

- Momentum Density at different times:
  - $t=1$
  - $t=16$
  - $t=132$

- Momentum Density at $t=600$
  - Pedestal

- Graphs showing evolution of density over momentum with time milestones.
**Theory: nonlinear dynamics in disorder**

**Kinetic equation** (including disorder average) derived from GPE:

\[-\frac{1}{2} \nabla^2 \Psi + V(\mathbf{r}) \Psi + g |\Psi|^2 \Psi = i \partial_t \Psi\]

\[\partial_t f(\mathbf{r}, t, \epsilon) - [\nabla - 2\nabla g\bar{n}(\mathbf{r}, t) \partial_\epsilon] D_\epsilon [\nabla - 2\nabla g\bar{n}(\mathbf{r}, t) \partial_\epsilon] f(\mathbf{r}, t, \epsilon) = I_{\text{coll}}(\mathbf{r}, t, \epsilon)\]

[Schwiete & Finkelstein (2010, 2011, 2013)]
[Cherroret & Wellens (2011)]

<table>
<thead>
<tr>
<th>GPE</th>
<th>KE</th>
</tr>
</thead>
<tbody>
<tr>
<td>wave function $\Psi(\mathbf{r}, t)$</td>
<td>phase-space function $f(\mathbf{r}, t, \epsilon)$</td>
</tr>
<tr>
<td>random potential $V(\mathbf{r})$</td>
<td>diffusion constant $D_\epsilon$</td>
</tr>
<tr>
<td>nonlinear (SC) pot. $g</td>
<td>\Psi(\mathbf{r}, t)</td>
</tr>
<tr>
<td>averaged part</td>
<td>averaged part</td>
</tr>
<tr>
<td>fluctuating part</td>
<td>fluctuating part</td>
</tr>
<tr>
<td>collision integral $I_{\text{coll}}(\mathbf{r}, t, \epsilon)$</td>
<td>collision integral $I_{\text{coll}}(\mathbf{r}, t, \epsilon)$</td>
</tr>
</tbody>
</table>