Basket Credit Derivatives Pricing in a Markov Chain Model with Interacting Intensities and Contagion Risk

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Basket Credit Derivatives

- Basket Credit Default Swap

Figure: The cash flow of $k$:th-to-default CDS
Basket Credit-Linked Notes

Figure: The cash flow of $k$th-to-default CLN
Zheng and Jiang (2009) proposed a factor contagion model for correlated defaults and the basket CDS rates can be computed analytically for homogeneous contagion portfolio.

Wu (2010) explored a reasonable coupon rate for basket credit linked notes (BCLN) with issuer default risk.

Herbertsson and Jang (2011) valued CDS spreads and k-th-to-default swap spreads in a tractable shot noise model.

Wang, Liang and Yang (2012) proposed a model for pricing a basket Loan-only CDS with the negative correlation between prepayment and default under the reduced form framework and Botton Up method.

Guo, Dong and Wang (2018) employed an intensity-based credit risk model with regime-switching to study the valuation of basket CDS in a homogeneous portfolio.
Our Results

- We analyze basket credit derivatives (BCDS and BCLN) with counterparty risk using a Markov chain with interacting default intensities and contagion risk.
  - Leung and Kwok (2009) presented a Markov chain model with stochastic interacting intensities for the pricing of single-name CDS.

- The default intensities of the protection seller and the references are affected by an external shock event and contagion risk.

- We derive recursive formulas for the joint default probabilities of reference assets. The pricing formulas of BCDS and BCLN are presented.

- We examine how the correlated default risks between the protection seller and the underlying entities may affect the premium rates.
Suppose that there’re three kinds of entities concerned in our model:

\[ \mathcal{L} = \left\{ R_1, R_2, \cdots, R_N, C, S \right\}. \]

where \( R_i, i \in I, I = \{1, 2, \cdots, N\} \) represent reference entities, \( C \) is the counterparty and \( S \) is an external shock event.

Each element in \( \mathcal{L} \) can only trigger the default of name \( R_i, C, S \) individually.

The basket credit derivatives default process

\[ H_t = (H_t^{R_1}, H_t^{R_2}, \cdots, H_t^{R_N}, H_t^C, H_t^S) \in \{0, 1\}^{N+2} \]

is a Markov chain with \( 2^{N+2} \) states, where \( H_t^C = 1_{\{\tau_C \leq t\}}, H_t^S = 1_{\{\tau_S \leq t\}}, H_t^{R_i} = 1_{\{\tau_{R_i} \leq t\}} (i = 1, 2, \ldots, N) \), \( \tau_C, \tau_{R_i} \) denote the random default times of the counterparty and reference entity, and \( \tau_S \) denotes the random time of arrival of external shock \( S \). \( \tau_S \) is independent of \( \tau_C \) and \( \tau_{R_i} \).
The macroeconomic variables are described by a stochastic process $\Psi = (\Psi_t)_{t \in [0,T]}$.

The information available to the investor in the market at time $t$ includes the history of macroeconomic variables and default status of the portfolio up to time $t$. The filtration $(\mathcal{F}_t)_{t \geq 0}$ is given by

$$\mathcal{F}_t = \mathcal{F}_t^\Psi \bigvee \mathcal{F}_t^{R_1} \bigvee \cdots \bigvee \mathcal{F}_t^{R_N} \bigvee \mathcal{F}_t^C \bigvee \mathcal{F}_t^S,$$

where $\mathcal{F}_t^\Psi = \sigma(\Psi_s : 0 \leq s \leq t)$, $\mathcal{F}_t^{R_i} = \sigma(H_{s_i}^{R_i} : 0 \leq s \leq t)$, $\mathcal{F}_t^C = \sigma(H_{s_i}^C : 0 \leq s \leq t)$, $\mathcal{F}_t^S = \sigma(H_{s_i}^S : 0 \leq s \leq t)$. 
The default intensities of reference entity $i$ and counterparty can be respectively described as $\lambda^{R_i}(\Psi_s, H_s)$, $\lambda^{C}(\Psi_s, H_s)$, which have the property that

$$H^{R_i}_t - \int_0^{t \wedge \tau_{R_i}} \lambda^{R_i}(\Psi_s, H_s) ds, \quad H^C_t - \int_0^{t \wedge \tau_{C}} \lambda^{C}(\Psi_s, H_s) ds$$

are $\mathcal{F}_t$-martingales.

The arrival of the shock event is modeled as a Cox process with stochastic intensity $\{\lambda^S_t : t \leq 0\}$.

Prior to the arrival of external shock $S$, the default intensities of $\lambda^R_t, \lambda^C_t$ are assumed respectively to be $a^R_t, a^C_t$. Upon the arrival of external shock $S$, the default intensity processes $\lambda^R_t, \lambda^C_t$ respectively jump to $\alpha_R a^R_t, \alpha_C a^C_t$.

Contagion risk. If reference $i$ defaults, the intensities of other reference entities and counterparty respectively jump to $\alpha_{R_j} a^R_j, \alpha_{C} a^C_t$. 
Default Intensities

The default intensity can be expressed as:

\[
\lambda^{R_i}_t = a^{R_i}_t \left[ (\alpha_{R_i} - 1) \mathbb{I}_{\tau_s \leq t} + 1 \right] \prod_{j \in I \setminus \{i\}} \left[ (\alpha_{R_j}^{R_i} - 1) \mathbb{I}_{\tau_{R_j} \leq t} + 1 \right],
\]

where \(\alpha_{R_i}, \alpha_{R_j}^{R_i} \geq 0 \), \(i, j \in I, i \neq j\).

\[
\lambda^{C}_t = a^{C}_t \left[ (\alpha_C - 1) \mathbb{I}_{\tau_s \leq t} + 1 \right] \prod_{j \in I} \left[ (\alpha_{C}^{R_j} - 1) \mathbb{I}_{\tau_{R_i} \leq t} + 1 \right],
\]

where \(\alpha_{C}, \alpha_{C}^{R_i} \geq 0 \), \(i \in I\).

The effect of external shock \(S\): \(\alpha_{R_i}, \alpha_{C}\).

The contagion between reference entities and counterpart:

\(\alpha_{R_j}^{R_i}, \alpha_{C}^{R_j}\).
Suppose there’re only two reference entities, we have

\[
\begin{align*}
\lambda_t^{R_1} &= a_t^{R_1} \cdot \left[ (\alpha_{R_1} - 1) I_{\tau_s \leq t} + 1 \right] \cdot \left[ (\alpha_{R_2}^{R_2} - 1) I_{\tau_{R_2} \leq t} + 1 \right] \\
\lambda_t^{R_2} &= a_t^{R_2} \cdot \left[ (\alpha_{R_2} - 1) I_{\tau_s \leq t} + 1 \right] \cdot \left[ (\alpha_{R_1}^{R_1} - 1) I_{\tau_{R_1} \leq t} + 1 \right] \\
\lambda_t^{C} &= a_t^{C} \cdot \left[ (\alpha_{C} - 1) I_{\tau_s \leq t} + 1 \right] \prod_{i \in \{1, 2\}} \left[ (\alpha_{R_i}^{R_i} - 1) I_{\tau_{R_i} \leq t} + 1 \right]
\end{align*}
\]

The infinitesimal generator matrix function
\[
\Lambda_{[\psi^s]}(t) = (\Lambda_{ij}(t))_{2^4 \times 2^4}
\]
of the Markov chain
\[
H_t = (H_t^{R_1}, H_t^{R_2}, H_t^{C}, H_t^{S})
\]
is as follows.

This matrix satisfies the sum of elements in each row equals to zero.
**Infinitesimal Generator Matrix**

\[
\begin{array}{c|cccccccccccccc}
 & (0,0,0) & (1,0,0) & (0,1,0) & (0,0,1) & (1,1,0) & (1,0,1) & (0,1,1) & (0,0,1) & (1,1,0) & (1,0,1) & (0,1,1) & (0,0,1) & (1,1,0) & (1,0,1) & (0,1,1)\\
\hline
(0,0,0) & \lambda_1 & a_{t}^{R_1} & a_{t}^{R_2} & a_{t}^{C} & \lambda_{t}^{S} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
(1,0,0) & 0 & \lambda_{2,2} & 0 & 0 & 0 & 0 & 0 & a_{t}^{R_1} & a_{t}^{R_2} & a_{t}^{C} & \lambda_{t}^{S} & 0 & 0 & 0 & 0 & 0 & 0\\
(0,1,0) & 0 & 0 & \lambda_{3,3} & 0 & 0 & a_{t}^{R_1} & a_{t}^{R_2} & 0 & 0 & a_{t}^{C} & \lambda_{t}^{S} & 0 & 0 & 0 & 0 & 0 & 0\\
(0,0,1) & 0 & 0 & 0 & \lambda_{4,4} & 0 & 0 & a_{t}^{R_1} & a_{t}^{R_2} & 0 & \lambda_{t}^{S} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
(0,0,1) & 0 & 0 & 0 & 0 & \lambda_{5,5} & 0 & 0 & a_{t}^{R_1} & a_{t}^{R_2} & a_{t}^{C} & \lambda_{t}^{S} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
(1,1,0) & 0 & 0 & 0 & 0 & 0 & \lambda_{6,6} & 0 & 0 & 0 & 0 & \lambda_{t}^{S} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
(1,0,1) & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_{7,7} & 0 & 0 & 0 & 0 & \lambda_{t}^{S} & 0 & 0 & 0 & 0 & 0 & 0\\
(1,0,1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_{8,8} & 0 & 0 & 0 & 0 & \lambda_{t}^{S} & 0 & 0 & 0 & 0 & 0\\
(0,1,1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_{9,9} & 0 & 0 & 0 & \lambda_{t}^{S} & 0 & 0 & 0 & 0 & 0\\
(0,1,1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_{10,10} & 0 & 0 & 0 & \lambda_{t}^{S} & 0 & 0 & 0 & 0 & 0\\
(0,1,1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_{11,11} & 0 & 0 & 0 & \lambda_{t}^{S} & 0 & 0 & 0 & 0 & 0\\
(1,1,0) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_{12,12} & 0 & 0 & 0 & \lambda_{t}^{S} & 0 & 0 & 0 & 0 & 0\\
(1,1,0) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_{13,13} & 0 & 0 & 0 & \lambda_{t}^{S} & 0 & 0 & 0 & 0 & 0\\
(1,0,1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_{14,14} & 0 & 0 & 0 & \lambda_{t}^{S} & 0 & 0 & 0 & 0 & 0\\
(0,1,1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_{15,15} & 0 & 0 & 0 & \lambda_{t}^{S} & 0 & 0 & 0 & 0 & 0\\
(1,1,1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_{16,16} & 0 & 0 & 0 & \lambda_{t}^{S} & 0 & 0 & 0 & 0 & 0\\
\end{array}
\]
By the forward Kolmogorov equation, the conditional transition probability matrix

\[ P(0, t | \psi^s) = \left( P_{ij}(0, t | \psi^s) \right)_{2(N+2) \times (N+2)} \]

is governed by

\[
\frac{dP(0, t | \psi^s)}{du} = P(0, t | \psi^s) \Lambda_{\psi^s}(t), \quad t \geq 0,
\]

that is

\[
\begin{bmatrix}
\frac{dP_{1,1}}{dt} & \frac{dP_{1,2}}{dt} & \cdots & \frac{dP_{1,2N+2}}{dt} \\
\frac{dP_{2,1}}{dt} & \frac{dP_{2,2}}{dt} & \cdots & \frac{dP_{2,2N+2}}{dt} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{dP_{2N+2,1}}{dt} & \frac{dP_{2N+2,2}}{dt} & \cdots & \frac{dP_{2N+2,2N+2}}{dt}
\end{bmatrix}
= \begin{bmatrix}
P_{1,1} & P_{1,2} & \cdots & P_{1,2N+2} \\
P_{2,1} & P_{2,2} & \cdots & P_{2,2N+2} \\
\vdots & \vdots & \ddots & \vdots \\
P_{2N+2,1} & P_{2N+2,2} & \cdots & P_{2N+2,2N+2}
\end{bmatrix}
\begin{bmatrix}
\Lambda_{1,1} & \Lambda_{1,2} & \cdots & \Lambda_{1,2N+2} \\
0 & \Lambda_{2,2} & \cdots & \Lambda_{2,2N+2} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \Lambda_{2N+2,2N+2}
\end{bmatrix}
\]

Since the matrix \( \Lambda_{\psi^s}(t) \) is upper triangular, the conditional transition probabilities \( P(0, t | \psi^s) \) can be solved successively in a sequential manner.
Notations for conditional transition intensity matrix

- $H_t$ is the state of the markov chain in time $t$.
- $H_{RM}$ represents the state that reference $i$ defaulted for all $i \in M \subset I$.
- $H_{RM \cup C}$ represents the state that reference $i$ and the counterparty $C$ defaulted for all $i \in M \subset I$.
- $H_{RM \cup S}$ represents the state that reference $i$ defaulted for all $i \in M \subset I$ and the external shock $S$ happened.
- $H_{RM \cup C \cup S}$ represents the state that reference $i$, counterparty $C$ defaulted for $i \in M \subset I$ and the external shock $S$ happened.
- $\Lambda_{H_t}$ represents the intensity that the state $H_t$ doesn’t change.
- $P_{H_t}(0, t)$ represents the probability that the state transfers from the initial state $H_{R_0}$ to $H_t$ during $(0, t]$. They are the first line of the transition probability matrix and can be used to deduce the joint default probability distribution during $(0, t]$. 
Firstly, we calculate the probability that none of the reference entities defaults during \((0, t]\) and set \(M = \emptyset\). The corresponding states are \(H^{R_\emptyset} = (0, \cdots, 0, 0)\) and \(H^S = (0, \cdots, 0, 1)\) where \(H^{R_i}_t = 0\) for all \(i \in I\). Some elements in the infinitesimal generator matrix are as follows:

\[
\begin{align*}
\Lambda_{H^{R_\emptyset}}(t) &= -\left[ \sum_{i \in I} a^{R_i}_t + a^C_t + \lambda^S_t \right], \\
\Lambda_{H^S}(t) &= -\left[ \sum_{i \in I} \alpha R_i a^{R_i}_t + \alpha C a^C_t \right].
\end{align*}
\]

\[
\begin{array}{cccccccccccc}
H^{R_\emptyset} & H^{R(1)} & \cdots & H^{R(N)} & \vdots & \vdots & H^S & \vdots & H^{R(i)US} & \vdots & H^{CUS} & \vdots & H^{R \cupCUS} \\
\hline
H^{R_\emptyset} & A_{H^{R_\emptyset}} & a^{R_1}_t & \cdots & a^{R_N}_t & a^C_t & \lambda^S_t & \vdots & 0 & \vdots & 0 & \vdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
H^S & 0 & 0 & \cdots & 0 & 0 & A_{H^S} & \alpha R_i a^{R_i}_t & \alpha C a^C_t & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
H^{R \cupCUS} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]
Probabilities of no-reference-default case

By the forward Kolmogorov equation, we have

\[
\begin{align*}
\frac{dP_{HR\phi}(0,u|\psi^s)}{du} &= P_{HR\phi}(0,u|\psi^s)\Lambda_{HR\phi}(u) \\
\frac{dP_{HS}(0,u|\psi^s)}{du} &= P_{HR\phi}(0,u|\psi^s)\lambda_u^S + P_{HS}(0,u|\psi^s)\Lambda_{HS}(u)
\end{align*}
\]

The solutions of the above equations are

\[
\begin{align*}
P_{HR\phi}(0,t|\psi^s) &= e^{\int_0^t \Lambda_{HR\phi}(u)du} \\
P_{HS}(0,t|\psi^s) &= e^{\int_0^t \Lambda_{HS}(u)du} \int_0^t \lambda_u^S e^{-\int_0^u \Lambda_{HS}(r)du} P_{HR\phi}(0,u|\psi^s)du
\end{align*}
\]
The states are $\mathbb{H}^{R\{i\}} = (0, \cdots, 0, 1, 0, \cdots, 0, 0)$ where $H_t^{R_i} = 1$, $\mathbb{H}^{R\{i\} \cup S} = (0, \cdots, 0, 1, 0, \cdots, 0, 1)$ where $H_t^{R_i} = 1, H_t^S = 1$.

\[
\begin{pmatrix}
... & H^{R\{i\}}_t & ... & H^{R\{i\}}_{j \in I \setminus \{i\}} & ... & H^{R\{i\} \cup C}_t & ... & H^{R\{i\} \cup S}_t & ... & H^{R\{i\} \cup C \cup S}_t & ... & H^{R\{i\} \cup C \cup S}_t \\
H^0 & ... & \alpha^{R_i}_t & ... & 0 & ... & 0 & ... & 0 & ... & 0 & ... & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
H^S & ... & 0 & ... & 0 & ... & 0 & ... & 0 & ... & 0 & ... & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
H^{R\{i\}} & ... & \Lambda_{H^{R\{i\}}_t} & ... & \alpha^{R_i} R_j & ... & \alpha^{R_i}_C & ... & \lambda^S & ... & 0 & ... & 0 & ... & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
H^{R\{i\} \cup S} & ... & 0 & ... & 0 & ... & 0 & ... & \Lambda_{H^{R\{i\} \cup S}_t} & ... & \alpha^{R_j} R_i & ... & \alpha^{R_j} a^C & ... & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
H^{R\{i\} \cup C \cup S} & ... & 0 & ... & 0 & ... & 0 & ... & 0 & ... & 0 & ... & 0 & ... & 0 & ...
\end{pmatrix}
\]

where
\[
\Lambda_{H^{R\{i\}}_t} (t) = -\left[ \sum_{j \in I \setminus \{i\}} \alpha^{R_j} R_i a^R_t + \alpha^{R_j} a^C + \lambda^S_t \right],
\]
\[
\Lambda_{H^{R\{i\} \cup S}_t} (t) = -\left[ \sum_{j \in I \setminus \{i\}} \alpha^{R_j} R_i a^R_t + \alpha a^C \alpha^{R_i}_C a^C_t \right].
\]
By the forward Kolmogorov equation, we have

\[
\begin{align*}
\frac{dP_{\mathcal{HR}\{i\}}(0,u|\psi_s)}{du} &= P_{\mathcal{HR}\phi}(0,u|\psi_s)a^R_i + P_{\mathcal{HR}\{i\}}(0,u|\psi_s)\Lambda_{\mathcal{HR}\{i\}}(u) \\
\frac{dP_{\mathcal{HR}\{i\} \cup S}(0,u|\psi_s)}{du} &= P_{\mathcal{HR}\{i\}}(0,u|\psi_s)\lambda_u^S + P_{\mathcal{HS}}(0,u|\psi_s)\alpha_{R_i}a^R_i \\
&\quad + P_{\mathcal{HR}\{i\} \cup S}(0,u|\psi_s)\Lambda_{\mathcal{HR}\{i\} \cup S}(u)
\end{align*}
\]

The solutions of the above equations are

\[
\begin{align*}
P_{\mathcal{HR}\{i\}}(0,t|\psi_s) &= e^{\int_0^t \Lambda_{\mathcal{HR}\{i\}}(u)du} \int_0^t a^R_i e^{\int_0^u \Lambda_{\mathcal{HR}\{i\}}(r)dr} P_{\mathcal{HR}\phi}(0,u|\psi_s)du \\
P_{\mathcal{HR}\{i\} \cup S}(0,t|\psi_s) &= e^{\int_0^t \Lambda_{\mathcal{HR}\{i\} \cup S}(u)du} \left[ \int_0^t \lambda_u^S e^{\int_0^u \Lambda_{\mathcal{HR}\{i\} \cup S}(r)dr} P_{\mathcal{HR}\{i\}}(0,u|\psi_s)du \\
&\quad + \int_0^t \alpha_{R_i}a^R_i e^{\int_0^u \Lambda_{\mathcal{HR}\{i\} \cup S}(r)dr} P_{\mathcal{HS}}(0,u|\psi_s)du \right]
\end{align*}
\]
The corresponding states are for $|M| = k$:

- $\mathbb{H}_R^{M} = (\cdots, 1, \cdots, 0, 0)$ where $H_t^R_i = 1$ for $i \in M$.
- $\mathbb{H}_R^{M \cup S} = (\cdots, 1, \cdots, 0, 1)$ where $H_t^R_i = 1$ for $i \in M$, $H_t^S = 1$.
Probabilities of $k$-reference-default case

- Calculate

\[
\Lambda_{H_{RM}}(t) = -\left[ \sum_{j \in I \setminus M} \left( \prod_{i \in M} \alpha_{R_i}^{R_j} \right) a_t^{R_j} + \left( \prod_{i \in M} \alpha_{C_i}^{R_i} \right) a_t^{C} + \lambda_t^S \right]
\]

\[
\Lambda_{H_{RM} \cup S}(t) = -\left[ \sum_{j \in I \setminus M} \alpha_{R_j} \left( \prod_{i \in M} \alpha_{R_i}^{R_j} \right) a_t^{R_j} + \alpha_C \left( \prod_{i \in M} \alpha_{C_i}^{R_i} \right) a_t^{C} \right]
\]

- By the forward Kolmogorov equation, we have

\[
\begin{aligned}
\frac{dP_{H_{RM}}(0, u|\psi^s)}{du} &= \sum_{i \in M} \left[ P_{H_{RM}\{i\}}(0, u|\psi^s) \left( \prod_{j \in M \setminus \{i\}} \alpha_{R_j}^{R_i} \right) a_u^{R_i} \right] \\
&\quad + P_{H_{RM}}(0, u) \Lambda_{H_{RM}}(u|\psi^s) \\
\frac{dP_{H_{RM} \cup S}(0, u|\psi^s)}{du} &= \sum_{i \in M} \left[ P_{H_{RM}\{i\} \cup S}(0, u|\psi^s) \left( \prod_{j \in M \setminus \{i\}} \alpha_{R_j}^{R_i} \right) \alpha_{R_i} a_u^{R_i} \right] \\
&\quad + P_{H_{RM}}(0, u|\psi^s) \lambda_u^S + P_{H_{RM} \cup S}(0, u|\psi^s) \Lambda_{H_{RM} \cup S}(u)
\end{aligned}
\]
The solutions of the above equations are

\[
\begin{align*}
P_{\mathbb{H} \setminus RM} (0, t | \psi^s) &= e^\int_0^t \Lambda_{\mathbb{H} \setminus RM} (u) du \sum_{i \in M} \left[ \int_0^t \left( \prod_{j \in M \setminus \{i\}} \alpha_{R_j} R_i \right) a_{R_i} e^{-\int_0^u \Lambda_{\mathbb{H} \setminus RM} (r) dr} \right]
\times P_{\mathbb{H} \setminus RM \setminus \{i\}} (0, u | \psi^s) du
\end{align*}
\]

\[
\begin{align*}
P_{\mathbb{H} \setminus RM \cup S} (0, t | \psi^s) &= e^\int_0^t \Lambda_{\mathbb{H} \setminus RM \cup S} (u) du \left\{ \int_0^t \lambda_u S e^{-\int_0^u \Lambda_{\mathbb{H} \setminus RM \cup S} (r) dr} P_{\mathbb{H} \setminus RM} (0, u | \psi^s) du \right. \\
&+ \sum_{i \in M} \left[ \alpha_{R_i} \left( \prod_{j \in M \setminus \{i\}} \alpha_{R_j} \right) \right] \int_0^t a_{R_i} e^{-\int_0^u \Lambda_{\mathbb{H} \setminus RM \cup S} (r) dr} \\
&\left. \times P_{\mathbb{H} \setminus RM \setminus \{i\} \cup S} (0, u | \psi^s) du \right\}
\end{align*}
\]
Construction of the arrival of the external shock

- Taking expectation for probabilities containing $\lambda_t^S$, we can get the unconditional transition probabilities.

- To compute $E_\psi [e^{-\int_0^t \lambda_u^S du}]$ and $E_\psi [\lambda_t^S e^{-\int_0^t \lambda_u^S du}]$, we assume that the process $\lambda_t^S$ satisfies a CIR Process with jump (see Duffie and Gârleanu (2001)). That is

$$d\lambda_t^S = k(\theta - \lambda_t^S)dt + \sigma \sqrt{\lambda_t^S}dZ_t + \Delta J_t,$$

where $\Delta J_t$ is the jump of the pure jump process $J_t$ which is independent of Brownian motion $Z_t$, the jump size is an independent conditional distribution with mean $\mu$ and the jump times are independent Poisson process with jump arrival rate $l$. 

It can be shown that

\[
E_{\psi S}[e^{-\int_0^t \lambda_u^S du}] = e^{\alpha(t) + \beta(t) \lambda_0^S},
\]

\[
E_{\psi S}[\lambda_t^S e^{-\int_0^t \lambda_u^S du}] = -[\bar{\alpha}(t) + \bar{\beta}(t) \lambda_0^S] e^{\alpha(t) + \beta(t) \lambda_0^S},
\]

where

\[
\alpha(t) = \frac{k \theta (c_1 + d_1)}{b_1 c_1 d_1} \ln \frac{c_1 + d_1 e^{b_1 t}}{c_1 + d_1} + \frac{k \theta}{c_1} t + \frac{l(a_2 c_2 - d_2)}{b_2 c_2 d_2} \ln \frac{c_2 + d_2 e^{b_2 t}}{c_2 + d_2} + \left( \frac{l}{c_2} - l \right) t,
\]

\[
\beta(t) = \frac{1 - e^{b_1 t}}{c_1 + d_1 e^{b_1 t}},
\]

\[
\bar{\alpha}(t) = \frac{k \theta (c_1 + d_1)}{c_1} \frac{e^{b_1 t}}{c_1 + d_1 e^{b_1 t}} + \frac{k \theta}{c_1} t + \frac{l(a_2 c_2 - d_2)}{c_2} \frac{e^{b_2 t}}{c_2 + d_2 e^{b_2 t}} + \left( \frac{l}{c_2} - l \right),
\]

\[
\bar{\beta}(t) = \frac{-(k^2 + 2\sigma^2) e^{b_1 t}}{(c_1 + d_1 e^{b_1 t})^2}.
\]
The premium for $k$th-to-default CDS

Figure: The cash flow of $k$th-to-default CDS
The premium for $k$th-to-default CDS

Suppose the nominal value of CDS is one, $\tau_k$ is the time of $k$ reference entities default, $r$ is the risk-free interest rate and $\rho$ is the recovery rate.

The swap premium payments are assumed to be made continuously at a constant CDS premium rate $c$ in our model.

From the pricing formula of CDS

$$E\left[\int_0^T ce^{-rs} I_{\tau_k > s} ds\right] = E\left[(1 - \rho)e^{-r\tau_k} I_{\tau_k \leq T}\right],$$

we can calculate the premium rate

$$c = \frac{E\left[(1 - \rho)e^{-r\tau_k} I_{\tau_k \leq T}\right]}{E\left[\int_0^T e^{-rs} I_{\tau_k > s} ds\right]}.$$
Case of 1th-to-default CDS

For the 1th-to-default CDS, there are two possible scenarios during \((0, t]\): non-occurrence of the shock event \(S\) or occurrence of \(S\). Given that there is no default of reference entities during \((0, t]\) and one reference entity defaults during the next infinitesimal time interval \((t, t + dt]\), the probability of such occurrence is given by

\[
Q_1(t)dt = \left[ P_{HR_0}(0, t) \left( \sum_{i \in I} a_{R_i}^t \right) + P_{HS}(0, t) \left( \sum_{i \in I} \alpha_{R_i} a_{R_i}^t \right) \right] dt.
\]

Over the whole period \([0, T]\), the expected present value of compensation payment by the protection seller is given by

\[
\int_0^T (1 - \rho) e^{-rt} Q_1(t) dt.
\]
The premium for 1th-to-default CDS

- The swap premium payment of 1th-to-default CDS continues when the Markov chain state is either $R_\emptyset$ or $S$. The swap premium payment paid by the protection buyer within $(t, t + dt]$ is given as

$$ce^{-rt}D_1(t))dt,$$

where

$$D_1(t) = P_{R_\emptyset}(0, t) + P_S(0, t).$$

- Over the whole period $[0, T]$, the expected present value of the premium payment is as follows

$$c \int_0^T e^{-rt}D_1(t)dt.$$
The premium for 1th-to-default CDS

By equating the expected present value of two counterparties, the 1th-to-default CDS premium rate is obtained as

\[ c = (1 - \rho) \frac{\int_0^T e^{-rt} Q_1(t) dt}{\int_0^T e^{-rt} D_1(t) dt}. \]
The premium for 2th-to-default CDS

- For the 2th-to-default CDS, there are two possible scenarios during \((0, t]\): non-occurrence of the shock event \(S\) or occurrence of \(S\) while one reference entity has defaulted. Given that one reference entity has defaulted during \((0, t]\) and another reference entity defaults during the next infinitesimal time interval \((t, t + dt]\), the probability of such occurrence is given by

\[
Q_2(t)dt = \sum_{i \in I} \left[ P_{HR_i}(0, t) \left( \sum_{j \in I \setminus \{i\}} \alpha_{R_i}^{R_j} a_t^{R_j} \right) + P_{HR_i \cup S}(0, t) \left( \sum_{j \in I \setminus \{i\}} \alpha_{R_j}^{R_i} a_t^{R_j} \right) \right] dt.
\]

- Over the whole period \([0, T]\), the expected present value of compensation payment by the protection seller is given by

\[
\int_0^T (1 - \rho)e^{-rt}Q_2(t)dt.
\]
The premium for 2th-to-default CDS

The swap premium payment of 2th-to-default CDS continues when the Markov chain state is $\mathbb{H}^R_\emptyset$, $\mathbb{H}^S$, $\mathbb{H}^R_\{i\}$ or $\mathbb{H}^R_\{i\} \cup S$. The swap premium payment paid by the protection buyer within $(t, t + dt]$ is given as

$$ce^{-rt} D_2(t) dt,$$

where

$$D_2(t) = P_{\mathbb{H}^R_\emptyset}(0, t) + P_{\mathbb{H}^R_\emptyset \cup S}(0, t) + \sum_{i \in I} [P_{\mathbb{H}^R_\{i\}}(0, t) + P_{\mathbb{H}^R_\{i\} \cup S}(0, t)].$$

Over the whole period $[0, T]$, the expected present value of the premium payment is as follows

$$c \int_0^T e^{-rt} D_2(t) dt.$$
By equating the expected present value of two counterparties, the 2th-to-default CDS premium rate is obtained as

\[ c = (1 - \rho) \frac{\int_0^T e^{-rt} Q_2(t) dt}{\int_0^T e^{-rt} D_2(t) dt} \]
The premium for \( k \)th-to-default CDS

- For the \( k \)th-to-default CDS, there are two possible scenarios during \((0, t]\): non-occurrence of the shock event \( S \) or occurrence of \( S \) while \( k - 1 \) reference entities have defaulted. Given that \( k - 1 \) reference entity have defaulted during \((0, t]\) and another reference entity defaults during the next infinitesimal time interval \((t, t + dt]\), the probability of such occurrence is given by

\[
Q_k(t)dt = \sum_{M \subset I, |M| = k-1} \left\{ P_{RM}(0, t) \left[ \sum_{j \in I \setminus M} \left( \prod_{i \in M} \alpha_{R_i} R_j \right) a_t^R_j \right] \right.
\]

\[
+ P_{RM \cup S}(0, t) \left[ \sum_{j \in I \setminus M} \alpha_{R_j} \left( \prod_{i \in M} \alpha_{R_i} R_j \right) a_t^R_j \right] \right\} dt
\]

- Over the whole period \([0, T]\), the expected present value of the premium payment is as follows

\[
\int_0^T (1 - \rho)e^{-rt}Q_k(t)dt.
\]
The premium for $k$th-to-default CDS

- The swap premium payment of $k$th-to-default CDS continues when the Markov chain state is either $HR_M$ or $HR_{M\cup S}$ where $M \subset I$ and $0 \leq |M| \leq k - 1$. The swap premium payment paid by the protection buyer within $(t, t + dt]$ is given as

$$ce^{-rt}D_k(t)dt,$$

where

$$D_k(t) = \sum_{M \subset I, 0 \leq |M| \leq k-1} \left[ P_{HR_M}(0, t) + P_{HR_{M\cup S}}(0, t) \right].$$

- Over the whole period $[0, T]$, the expected present value of the premium payment is as follows

$$c \int_0^T e^{-rt}D_k(t)dt.$$
By equating the expected present value of two counterparties, the \( k \)-th-to-default CDS premium rate is obtained as

\[
c = (1 - \rho) \frac{\int_0^T e^{-rt} Q_k(t) dt}{\int_0^T e^{-rt} D_k(t) dt}.
\]
The premium for $k$th-to-default CLN

**Figure:** $k$th-to-default CLN cash flow
Suppose the face value of CLN is one, $c$ is the coupon rate, $\tau_k$ is the default time of $k$ reference entities, $r$ is the risk-free interest rate and $\rho$ is the recovery rate. The counterparty is the CLN issuer.

The expected cash flow of CLN is as follows

$$1 = E\left[ \int_0^T c e^{-rs} I_{\tau_k > s} ds + \rho e^{-r\tau_k} I_{\tau_k \leq T} + e^{-rT} I_{\tau_k > T} \right].$$

We can get the premium rate as follows

$$c = \frac{1 - E[\rho e^{-r\tau_k} I_{\tau_k \leq T}] - E[e^{-rT} I_{\tau_k > T}]}{E[\int_0^T e^{-rs} I_{\tau_k > s} ds]}.$$
The premium for $k$th-to-default CLN

Similarly to CDS case, the default probabilities can be calculated as follows.

\[
E[\rho e^{-rt} I_{\tau_k \leq T}] = \int_0^T \rho e^{-rt} Q_k(t) dt,
\]
\[
E[\int_0^T e^{-rs} I_{\tau_k > s} ds] = \int_0^T e^{-rt} D_k(t) dt,
\]
\[
E[e^{-rT} I_{\tau_k > T}] = e^{-rT} D_k(T),
\]

where

\[
Q_k(t) dt = \sum_{M \subset I, |M| = k-1} \left\{ P_{HR_M}(0, t) \left[ \sum_{j \in I \setminus M} \left( \prod_{i \in M} \alpha_{R_i} a_t^{R_j} \right) a_t^{R_j} \right] \\
+ P_{HR_M \cup S}(0, t) \left[ \sum_{j \in I \setminus M} \alpha_{R_j} \left( \prod_{i \in M} \alpha_{R_i} a_t^{R_j} \right) a_t^{R_j} \right] \right\} dt,
\]
\[
D_k(t) = \sum_{M \subset I, 0 \leq |M| \leq k-1} \left[ P_{HR_M}(0, t) + P_{HR_M \cup S}(0, t) \right].
\]
The premium for $k$th-to-default CLN

The $k$th-to-default CLN premium rate is given by

$$c = \frac{1 - \rho \int_0^T e^{-rt} Q_k(t) dt - e^{-rT} D_k(T)}{\int_0^T e^{-rt} D_k(t) dt}.$$
Numerical results about $k$th-to-default CDS

We take $N = 10$ and $I = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. The parameter values used in our calculation are as follows:

\[ \rho = 0.4, \ r = 0.04, \ \alpha_{R_i} = 1.15, \ \alpha_C = 1.15, \ \lambda_0^S = 0.05, \ \sigma = 0.2, \ k = 0.3, \ \theta = 0.02, \ l = 0.3, \]

\[ \mu = 0.15, \ \alpha_{R_i}^{R_j} = 0.003, \ \alpha_{R_i}^{R_C} = 0.003, \ a_{R_i}(t) = a_C(t) = 0.2. \]

The table below lists the $k$th-to-default CDS premium rate for $k = 1, \cdots, 10$.

<table>
<thead>
<tr>
<th>$k$th</th>
<th>premium rate(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1th</td>
<td>60.3724</td>
</tr>
<tr>
<td>2th</td>
<td>28.7716</td>
</tr>
<tr>
<td>3th</td>
<td>17.2977</td>
</tr>
<tr>
<td>4th</td>
<td>13.5195</td>
</tr>
<tr>
<td>5th</td>
<td>8.3919</td>
</tr>
<tr>
<td>6th</td>
<td>4.4503</td>
</tr>
<tr>
<td>7th</td>
<td>1.8347</td>
</tr>
<tr>
<td>8th</td>
<td>0.4272</td>
</tr>
<tr>
<td>9th</td>
<td>0.0565</td>
</tr>
<tr>
<td>10th</td>
<td>0.0000009</td>
</tr>
</tbody>
</table>

Table: $k$th-to-default CDS premium rate

The premium rates decrease with the increase of $k$. 
The premium rates increase with the increase of $\alpha_{R_i}$. 
Numerical results about $k$th-to-default CDS

The premium rates decrease with the increase of $\alpha_C$. 
Numerical results about \(k\)th-to-default CDS

- The premium rates increase with the increasement of \(\frac{R_j}{R_i}\).
Numerical results about $k$th-to-default CDS

- The premium rates decrease with the increase of $\alpha_C^{R_i}$. 

---

### Numerical Results

<table>
<thead>
<tr>
<th>CDS Premium Rate (%)</th>
<th>$\alpha_C^{R_i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>61.5</td>
<td>0</td>
</tr>
<tr>
<td>61.0</td>
<td>1</td>
</tr>
<tr>
<td>60.5</td>
<td>2</td>
</tr>
<tr>
<td>60.0</td>
<td>3</td>
</tr>
</tbody>
</table>

**1st-to-default CDS**

<table>
<thead>
<tr>
<th>CDS Premium Rate (%)</th>
<th>$\alpha_C^{R_i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>27.6</td>
<td>0</td>
</tr>
<tr>
<td>27.5</td>
<td>1</td>
</tr>
<tr>
<td>27.4</td>
<td>2</td>
</tr>
<tr>
<td>27.3</td>
<td>3</td>
</tr>
</tbody>
</table>

**2nd-to-default CDS**

<table>
<thead>
<tr>
<th>CDS Premium Rate (%)</th>
<th>$\alpha_C^{R_i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.9</td>
<td>0</td>
</tr>
<tr>
<td>9.8</td>
<td>1</td>
</tr>
<tr>
<td>9.7</td>
<td>2</td>
</tr>
<tr>
<td>9.6</td>
<td>3</td>
</tr>
</tbody>
</table>

**3rd-to-default CDS**

<table>
<thead>
<tr>
<th>CDS Premium Rate (%)</th>
<th>$\alpha_C^{R_i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.4</td>
<td>0</td>
</tr>
<tr>
<td>6.3</td>
<td>1</td>
</tr>
<tr>
<td>6.2</td>
<td>2</td>
</tr>
<tr>
<td>6.1</td>
<td>3</td>
</tr>
</tbody>
</table>

**4th-to-default CDS**
Numerical results about $k$th-to default CLN

We take $N = 10$ and $I = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. The table below lists the $k$th-to-default CLN premium rate for $k = 1, \cdots, 10$.

<table>
<thead>
<tr>
<th>$k$th</th>
<th>premium rate(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1th</td>
<td>60.9980</td>
</tr>
<tr>
<td>2th</td>
<td>33.4078</td>
</tr>
<tr>
<td>3th</td>
<td>18.3288</td>
</tr>
<tr>
<td>4th</td>
<td>13.3264</td>
</tr>
<tr>
<td>5th</td>
<td>9.6668</td>
</tr>
<tr>
<td>6th</td>
<td>7.4503</td>
</tr>
<tr>
<td>7th</td>
<td>5.8347</td>
</tr>
<tr>
<td>8th</td>
<td>4.9583</td>
</tr>
<tr>
<td>9th</td>
<td>4.0034</td>
</tr>
<tr>
<td>10th</td>
<td>4.00000853</td>
</tr>
</tbody>
</table>

Table: $k$th-to-default CLN premium rate

The premium rates of BCLN have similar trends as the premium rates of BCDS.
Numerical results about $k$th-to-default CLN

- The premium rates increase with the increase of $\alpha_{R_i}$. 
Numerical results about $k$th-to-default CLN

- The premium rates increase with the increasement of $\alpha_C$. 
Numerical results about $k$th-to-default CLN

- The premium rates increase with the increase of $\alpha^{R_i}_{R_j}$. 
Numerical results about $k$-th-to-default CLN

- The premium rates increase with the increase of $\alpha^R_i$. 

![Graphs showing the increase in CLN premium rates with $\alpha^R_i$.]
Conclusion

- We use Markov chain model with stochastic intensities and contagion risk to price basket credit derivatives (BCDS and BCLN).

- The default correlation between the reference entities and counterparty is modeled by an external shock and contagion.

- Our analysis indicates that the impact of correlated risks between the counterparty and reference entities on the fair premium rates can be quite substantial under a high arrival rate of external shock and contagion risk.

- The contagion risk doesn't affect the 1th-to-default premium.
Thank you!