Mean-Risk Portfolio Selection
with Law-Invariant Coherent Risk Measure

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Mean-risk preference

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- We study the explicit way, which is more intuitive.
Mean-risk portfolio selection in continuous-time market

- We consider our mean-risk problem in a continuous-time, *arbitrage-free*, and *complete* financial market, with interest rate $r \equiv 0$.
  - A standard Black-Scholes market is an often-used example.
  - Here we do not need many details of the market other than the unique *pricing kernel*, denoted as $\xi$, which satisfies $\mathbb{E}[\xi] = 1$.
  - For any terminal wealth $X_T$, we have $X_0 = \mathbb{E}[\xi X_T]$.
- Assumption: $\xi > 0$ admits *no atom*, i.e., $P(\xi = x) = 0$ for $\forall x \in \mathbb{R}$. 

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- Assumption: $\xi > 0$ admits no atom, i.e., $P(\xi = x) = 0$ for $\forall x \in \mathbb{R}$.

- For an investor with initial wealth $x > 0$, the mean-risk portfolio selection in the time period $[0, T]$ can be formulated as

$$
\begin{align*}
\text{Min} & \quad \rho(X_T) \\
\text{s.t.} & \quad (X, \pi) \text{ is a wealth-portfolio process with } X_0 = x, \\
& \quad \mathbb{E}X_T \geq l.
\end{align*}
$$

where $\rho$ is the investor’s sense on the risk, $l > x$ is a target level.
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- *Comonotonic additive risk measure*: $\rho(X) + \rho(Y) = \rho(X + Y)$ for any comonotonic $X$ and $Y$. 
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• **Comonotonic additive** risk measure: $\rho(X) + \rho(Y) = \rho(X + Y)$ for any comonotonic $X$ and $Y$.

• **Law-invariant** risk measure: risk is fully described by distribution.
Examples of law-invariant risk measures

- Variance $\text{Var}(X) = \mathbb{E}(X - \mathbb{E}X)^2$.
  - $\text{Var}(X)$ is NOT a risk measure!
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- **Conditional Value-at-risk**: $\text{CV@R}_\alpha(X) = \frac{1}{\alpha} \int_0^\alpha \text{V@R}_\beta(X) d\beta$.
  - $\text{CV@R}_\alpha$ is a risk measure. It is law-invariant, coherent, and comonotonic additive.
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- Any convex combination of $CV@R$ is law-invariant, coherent, and comonotonic.
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- We aim at the mean-risk portfolio selection with law-invariant coherent risk measure.
Representation of law-invariant coherent risk measures

**Theorem 1**: Denote $\mathcal{P}([0, 1])$ as the set of probability measure on $[0, 1]$. With some regularity condition, $\rho$ is a law-invariant convex risk measure if and only if

$$\rho(X) = \sup_{\mu \in \mathcal{A}} \left\{ \int_{[0,1]} CV@R_z(X) \mu(dz) \right\}$$

for some closed set $\mathcal{A} \subset \mathcal{P}([0, 1])$. 
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$\rho$ is furthermore comonotonic if and only if

$$\rho(X) = \int_{[0,1]} CV@R_z(X) \mu(dz)$$

for some $\mu \in \mathcal{P}([0,1])$. 
Weighted V@R

- Law-invariant coherent risk measures are generated by $CV@R$. 
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• Recall that $CV@R$ is an average of $V@R$, we have

$$\int_0^1 CV@R_z(X) \mu(dz) = \int_0^1 V@R_z(X) m(dz)$$

for $m(dz) = \int_z^1 \frac{1}{\beta} \mu(d\beta)$ is a probability measure on $[0, 1]$. 
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  for $m(dz) = \int_z^1 \frac{1}{\beta} \mu(d\beta)$ is a probability measure on $[0, 1]$.
- For $m \in \mathcal{P}([0, 1])$, define the weighted $V@R$ by
  \[ WV@R_m(X) = \int_0^1 V@R_z(X)m(dz), \]
  then a law-invariant coherent risk measure can be written as
  \[ \sup_{\mu \in \mathcal{B}} WV@R_{\varphi(\mu)}(X), \]
  where $\varphi(\mu)$ is defined as the probability measure
  \[ \varphi(\mu)(dz) = \int_z^1 \frac{1}{\beta} \mu(d\beta). \]
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- For $m \in \mathcal{P}([0, 1])$, define the weighted V@R by
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  \varphi(\mu)(dz) = \int_z^1 \frac{1}{\beta} \mu(d\beta).
  \]
- $WV@R$ is the building block for law-invariant coherent risk measure.
Martingale approach

- By the **completeness** of the market, any random payoff $X$ at time $T$ can be replicated by some portfolio $\pi$. starting from initial wealth $\mathbb{E}[\xi X]$. 
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For the portfolio selection problem (1), we can firstly solve the optimal terminal wealth $X^*$ by

$$\begin{align*}
\text{Min} & \quad \rho(X) \\
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• Since replication in a complete market is theoretically easy by, e.g., martingale representation, we focus the first step for optimal terminal wealth $X^*$. 
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• If $X$ is optimal (2) with distribution function $F$, then for any other random variable $Y \sim F$, we should have $\mathbb{E}[\xi X] \leq \mathbb{E}[\xi Y]$.

• Hence $X$ also solves $\min_{Y \sim F} \mathbb{E}[\xi Y]$. 
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**Proposition 1**: If $X$ is optimal for (2) with distribution function $F$, then $X = G(1 - F_\xi(\xi))$, where $G = F^{-1}$ is quantile function of $F$, $F_\xi$ is the distribution function of $\xi$. 
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- Redefine $\rho(G) := \rho(X)$ with $G = F_X^{-1}$, and denote $Z = F_\xi(\xi)$.
- Then (2) can be reformulated into

$$\begin{align*}
\operatorname{Min} & \quad \rho(G) \\
\text{s.t.} & \quad E[\xi G(1 - Z)] = x_0, \\
& \quad E[G(1 - Z)] \geq l.
\end{align*}$$

(3)
When $\rho$ is a weighted V@R

- Consider (3) when $\rho(X) = WV@R_\mu(X)$, which means

$$\rho(G) = - \int_0^1 G(z) \mu(dz).$$
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$$\gamma^* := \sup_{0 < c < 1} \frac{\mu((c, 1])}{\int_c^1 F_{\xi}^{-1}(1 - z)dz}. $$
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**Theorem 2:** Denote $V$ as the optimal value for problem (3), and suppose $\text{essinf } \xi = 0$. Then

(i) If $\gamma^* > 1$, then $V = -\infty$; If $\gamma^* \leq 1$, then $V = -x$.

(ii) If $\gamma^* < 1$, there exists a sequence of $X_n = a_n + b_n 1_{\xi \leq c_n}$ asymptotically optimal, where $c_n \downarrow 0$, $b_n \uparrow +\infty$ and $a_n \to x$.

(iii) If $\gamma^* = 1$ and achieved by $c^*$, then $X^* = a + b 1_{\xi \leq F_\xi^{-1}(1-c^*)}$ for some $a \in \mathbb{R}$, $b > 0$. 
Weighted V@R with no-bankruptcy

- In (3), The optimal value does neither depend on \( m \) nor \( l \).
- The no bankruptcy constraint \( X \geq 0 \) makes the trade-off better.
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- Denote \( \hat{\nu} \) as the optimal value for the problem with no-bankruptcy constraint.
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- Denote $\hat{V}$ as the optimal value for the problem with no-bankruptcy constraint.

**Theorem 3:** Suppose $\text{essinf} \xi = 0$. Then

(i') If $\gamma^* \leq 1$, then $\hat{V} = -x$. If $\gamma^* \in (1, +\infty)$, then $\hat{V} = -\gamma^* x$.

(ii') There exists an optimal solution $X^*$ iff $V > -\infty$ and $\gamma^*$ is obtained by some $c^* \in (0, 1)$, in which case $X^* = b1_{\xi \leq F_\xi^{-1}(1-c^*)}$ for some $b$. 
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(ii') There exists an optimal solution $X^*$ iff $V > -\infty$ and $\gamma^*$ is obtained by some $c^* \in (0, 1)$, in which case $X^* = b1_{\xi \leq F^{-1}_\xi (1-c^*)}$ for some $b$.

• The optimal value $\hat{V}$ does not depend on $l$, but does depend on $\mu$.
• $\hat{V}$ may not be asymptotically approached by $X_n = a_n + b_n 1_{\xi \leq c_n}$. 

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When $\rho$ is coherent and law-invariant

- If $\rho$ is coherent and law-invariant, then

$$\rho(G) = \sup_{\mu \in A} \int_{0}^{1} G(z) \varphi(\mu) (dz)$$

for some closed $A \subset \mathcal{P}([0, 1])$. 
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- The optimal terminal wealth problem turns into
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  \\
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s.t.

$$\mathbb{E}[\xi G(1 - Z)] = x_0,$$

$$\mathbb{E}[G(1 - Z)] \geq l. \quad (4)$$

- If we can swap min and sup, then the minimization over $G$ is the same as that for WV@R.
When $\rho$ is coherent and law-invariant

**Theorem 4**: We can exchange the order of $\min$ and $\sup$ in problem (4) with or without the extra no-bankruptcy constraint, i.e., $G(0) = 0$. 
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**Theorem 4:** We can exchange the order of $\min$ and $\sup$ in problem (4) with or without the extra no-bankruptcy constraint, i.e., $G(0) = 0$.

- For problem (4) w/o no-bankruptcy constraint, we have another critical quantity

\[
\gamma_A := \inf_{\mu \in A} \sup_{0 < c < 1} \frac{\varphi(\mu)((c,1])}{\int_c^1 F_{\xi}^{-1}(1-z)dz}.
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**Theorem 5:** Suppose $\text{essinf} \xi = 0$. Denote $V_c$ and $\hat{V}_c$ as the optimal value for problem (4) without and with the no-bankruptcy constraint.

- For problem (4) without no-bankruptcy constraint, $V_c > -\infty$ iff $\gamma_A \leq 1$. When $V_c > -\infty$, we have $V_c = -x$.

- For problem (4) with no-bankruptcy constraint, $\hat{V}_c > -\infty$ iff $\gamma_A < +\infty$. When $V_c > -\infty$, we have $V_c = -x \max(\gamma_A, 1)$. 
Questions and Comments