Near-critical percolation with heavy-tailed impurities and forest fire processes

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Bernoulli percolation

Model for random media: Bernoulli percolation (Broadbent, Hammersley, 1957)

Site percolation on $\mathbb{Z}^2$

Site percolation on $\mathbb{T}$
Bernoulli percolation

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For some parameter $p \in [0, 1]$, vertices ("sites") independently

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Percolation: phase transition as $p$ varies

\[ p_{c}^{\text{site}}(\mathbb{Z}^d) / p_{c}^{\text{site}}(\mathbb{T}) \]
Bernoulli percolation

Percolation: phase transition as $p$ varies

$0 \quad p_c^{\text{site}}(\mathbb{Z}^d) \; / \; p_c^{\text{site}}(\mathbb{T}) \quad 1$

*subcritical regime*

- no $\infty$ cluster
- exponential decay for cluster size
- only tiny clusters

*trivial large scale behavior*
Bernoulli percolation

Percolation: phase transition as $p$ varies

$\frac{p_c^{\text{site}}(\mathbb{Z}^d)}{p_c^{\text{site}}(\mathbb{T})}$

$0$ $1$ $p$

subcritical regime
- no $\infty$ cluster
- exponential decay for cluster size
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supercritical regime
- unique $\infty$ cluster
- exponential decay for finite clusters
- only tiny finite clusters
- trivial large scale behavior
Forest fire processes

We consider processes on a 2D lattice ($\mathbb{Z}^2$ or $\mathbb{T}$), constructed from 2 Poisson point processes: on each vertex, **births** (rate 1) and **ignitions** (rate $\zeta > 0$, typically very small)

- Initially, all vertices vacant
- Each vertex vacant; occupied at birth times: pure birth process ($\leftrightarrow$ Bernoulli site percolation with parameter $p(t) = 1 - e^{-t}$)
- $N$-volume-frozen percolation: occupied clusters stop growing if their volume (= # vertices) gets $\geq N$, i.e. all vertices along the outer boundary then stay vacant forever
- Forest fire process: occupied clusters burn when hit by lightning, i.e. all vertices become vacant instantaneously without recovery: burnt vertices then stay vacant forever
- Forest fire process with recovery: burnt vertices can become occupied again, at later birth times
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Frozen percolation

$N = 200$-volume-frozen percolation on $\mathbb{T}$

Final configuration at time $t = \infty$  (Fig. Demeter Kiss)
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Forest fire process on $\mathbb{Z}^2$ (Drossel, Schwabl, 1992)

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We can consider forest fire processes with or without recovery.
Forest fire processes

Forest fire process *without recovery*, rate $\zeta = 0.01$
Relevant “macroscopic” behavior starts to occur around critical time $t_c$ (defined by $1 - e^{-t_c} = p_c$).

→ Instance of self-organized criticality (well-known phenomenon in statistical physics).
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Critical regime

Percolation: phase transition as $p$ varies

\[ p_{c}^{\text{site}}(T) \]

\begin{align*}
\text{subcritical regime} & \quad \text{supercritical regime} \\
\text{no } \infty \text{ cluster} & \quad \text{unique } \infty \text{ cluster} \\
\text{exponential decay} & \quad \text{exponential decay} \\
\text{for cluster size} & \quad \text{for finite clusters} \\
\text{only tiny clusters} & \quad \text{only tiny finite clusters} \\
\text{trivial large scale behavior} & \quad \text{trivial large scale behavior} \\
\text{critical regime} & \\
\text{non-trivial scaling limits} & \\
\text{conformal invariance} & \\
\text{connection with SLE}(6) & \\
\text{(Lawler, Schramm, Werner, Smirnov 1999-2001)} & \ldots
\end{align*}
Near-critical regime

Critical regime \((p = p_c)\)

e.g. \(\mathbb{P}_{p_c}(\text{graph}) = N^{-5/48+o(1)}\) \((N \to \infty)\)
**Near-critical regime**

- Critical regime ($p = p_c$)
- Near-critical regime ($p \approx p_c$)

**Scaling relations** (Kesten 1987)

- E.g. $\mathbb{P}_{p_c} \left( \begin{array}{c} N \\ 0 \end{array} \right) = N^{-5/48+o(1)} (N \to \infty)$
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scaling relations
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near-critical regime \((p \simeq p_c)\)

e.g. density \(\theta(p) = (p - p_c)^{5/36 + o(1)} \)

\((p \searrow p_c)\)

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Near-critical regime

$$0 \rightarrow p \rightarrow 1$$

$$p = p_c$$

$$p \approx p_c$$

scales "below" $$L(p) = |p - p_c|^{-4/3 + o(1)} (p \rightarrow p_c)$$

(characteristic length)

critical regime $$(p = p_c)$$

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near-critical regime $$(p \approx p_c)$$

e.g. density $$\theta(p) = (p - p_c)^{5/36 + o(1)} (p \downarrow p_c)$$

$$\theta(p)$$

0 1

1 1

0 0

$$p_c$$

$$p_c$$

1
Forest fire processes

Sequence of \textbf{exceptional scales}: for all \( k \geq 1 \),

\[
m_k(\zeta) = \zeta^{-\delta_k + o(1)}, \quad \text{with} \quad \delta_k \nearrow \delta_\infty = \frac{48}{55}
\]

\( \rightarrow \) highlight \textbf{non-monotonicity}, not predicted in the literature)
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($\rightarrow$ highlight **non-monotonicity**, not predicted in the literature)

**Theorem (van den Berg, N., 2018)**

*For forest fire process without recovery, in box $B_m(\zeta)$: as $\zeta \to 0$,*

- $m_1(\zeta) \approx m_k(\zeta)$
- $\lim \inf \mathbb{P}_\zeta^{B_m(\zeta)}(0 \text{ burns before } t) > 0$ (for $t > t_c$)
- burning on $(t_c, \infty)$
- $m_k(\zeta) \ll m(\zeta) \ll m_{k+1}(\zeta)$
- $\mathbb{P}_\zeta^{B_m(\zeta)}(0 \text{ burns before } t) \to 0$ (for $\zeta \to 0$)
- burning only near $t_c$

clusters in final configuration:

- macroscopic (volume $\asymp \zeta^{-1}$)
- microscopic (volume $O(1)$)
- mesoscopic (volume $\zeta^{-\delta + o(1)}$), $0 < \delta < 1$
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- \(\rightarrow\) We have to understand the effect of these “impurities” on the connectedness of the lattice. (not clear that they do not perturb too much the “near-critical picture”!)

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“Impurities” created by fires before time $t_c - \varepsilon$  ($\varepsilon = 0.1$)
Heavy-tailed impurities

→ New model: percolation with “heavy-tailed” impurities.
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Percolation with heavy-tailed impurities: random environment
Heavy-tailed impurities

We manage to obtain the **full “phase diagram”** as $\alpha, \beta$ vary:
Heavy-tailed impurities

For forest fires, $\alpha = \frac{55}{48}$ and $\beta > \alpha$ (most interesting regime)

Note: impurities have density $m^{-(\beta - \alpha)}$, $\beta - \alpha$ arbitrarily small
Heavy-tailed impurities

**Question**: do the impurities have a significant effect on connectedness of the lattice?
Heavy-tailed impurities

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- **classical case**: single-site updates ("impurities"), need $\beta > \frac{1}{\nu} = \frac{3}{4}$ ("$\alpha = -\infty$")
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  - density of impurities has to stay $\lesssim m^{-3/4+o(1)}$

▶ effect on pivotal sites: quite subtle balance (impurities "help" vacant arm / "hinder" occupied arms)
  - relies on inequality between arm exponents $\alpha_4 \leq \alpha_2 + 1$
    - hence, specific to $T$ so far.
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- here, any $\beta > \alpha > \frac{3}{4}$ work, density $m^{-(\beta - \alpha)}$
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  \[ \rightarrow \text{density of impurities has to stay} \lesssim m^{-3/4 + o(1)} \]

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- effect on **pivotal sites**: quite subtle balance (impurities “help” vacant arm / “hinder” occupied arms)
  
  \[ \rightarrow \text{relies on inequality between arm exponents} \]
  
  \[ \alpha_4 \leq \alpha_2 + 1 \]

  (hence, specific to \( \mathbb{T} \) so far).
Forest fire processes

Forest fire process at time $t_c + \varepsilon$, in a box with side length

$$M \gg m = L(t_c - \varepsilon) \asymp L(t_c + \varepsilon)$$

(typically, $m = \hat{M}$)

![Diagram](image)

- "lower bound" by percolation with heavy-tailed impurities
- near-critical behavior
- configuration at this time
Forest fire processes

In the full plane: existence of exceptional scales indicate a convoluted structure
Forest fire processes

In the full plane: existence of exceptional scales indicate a convoluted structure

→ **deconcentration** phenomenon as $\zeta \to 0$ (work in progress)
Forest fire processes

In the full plane: existence of exceptional scales indicate a convoluted structure

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**Theorem (van den Berg, N., 2019+)**

*For forest fire process without recovery, in full plane $\mathbb{T}$: for all $t > 0$,*

$$
\mathbb{P}_\zeta^T(0 \text{ burns before } t) \xrightarrow{\zeta \to 0} 0
$$

+ qualitative description of what happens right after $t_c$ (“avalanche” of successive fires surrounding 0, more and more localized).
Exceptional scales

Consider \( N \)-volume-frozen percolation, in a box with side length \( C \sqrt{N} \) \((C > 1)\).
Exceptional scales

Consider \( N \)-volume-frozen percolation, in a box with side length \( C \sqrt{N} \) (\( C > 1 \)). For \( t \) just above \( t_c \) \((1 - e^{-t_c} = p_c)\)

\[
C \sqrt{N} \simeq L(t) \quad \text{volume} \simeq \theta(t) \cdot (C \sqrt{N})^2
\]

(Borgs, Chayes, Kesten, Spencer, 2001)
Exceptional scales

Consider $N$-volume-frozen percolation, in a box with side length $C \sqrt{N}$ ($C > 1$). For $t$ just above $t_c$ ($1 - e^{-t_c} = p_c$) the volume freezes at a time very close to $\bar{t} = \bar{t}(C) := \theta^{-1}(\frac{1}{C^2})$.
Exceptional scales

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- freezes at a time very close to $\bar{t} = \bar{t}(C) := \theta^{-1}\left(\frac{1}{C^2}\right)$
- leaves holes with volume $\lesssim L(\bar{t})^2 \ll N$
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- freezes at a time very close to $\bar{t} = \bar{t}(C) := \theta^{-1}(1/C^2)$
- leaves holes with volume $\lesssim L(\bar{t})^2 \ll N$
- nothing else freezes: only 1 giant cluster freezes, “spanning” the box
Exceptional scales

In a box with side length $m = L(t)$ ($t = t(N) \searrow t_c$): for $t'$ just above $t$,

\[ L(t) \approx L(t') \]

\[ \text{volume} \approx \theta(t') \cdot (L(t))^2 \]

\[ \implies \text{freezes at a time very close to } \hat{t} \text{ s.t. } L(t)^2 \theta(\hat{t}) = N, \]
Exceptional scales

In a box with side length \( m = L(t) \) \((t = t(N) \searrow t_c)\): for \( t' \) just above \( t \),

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L(t) \approx L(t') \\
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\]

▶ freezes at a time very close to \( \hat{t} \) s.t. \( L(t)^2 \theta(\hat{t}) = N \),
Exceptional scales

In a box with side length \( m = L(t) \) \((t = t(N) \searrow t_c)\): for \( t' \) just above \( t \),

\[
\text{volume} \simeq \theta(t') \cdot (L(t))^2
\]

- freezes at a time very close to \( \hat{t} \) s.t. \( L(t) \theta(\hat{t}) = N \),
- leaves a hole around 0 with diameter \( \asymp L(\hat{t}) \),
Exceptional scales

In a box with side length \( m = L(t) \) \( (t = t(N) \searrow t_c) \): for \( t' \) just above \( t \),

\[
\text{volume } \simeq \theta(t') \cdot (L(t))^2
\]

- freezes at a time very close to \( \hat{t} \) s.t. \( L(t)^2 \theta(\hat{t}) = N \),
- leaves a hole around 0 with diameter \( \asymp L(\hat{t}) \),
- \( \rightarrow \) next scale \( \hat{m} = L(\hat{t}) \).
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**Exceptional scales**: define $m_1(N) = \sqrt{N}$, then $m_2(N)$ s.t. $\hat{m}_2 = m_1$, then $m_3(N)$ s.t. $\hat{m}_3 = m_2$, and so on.
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From $m_{k+1}^2 \pi_1(m_k) \asymp N$, we obtain

$$m_k(N) = N^{\delta_k + o(1)}, \quad \text{with } \delta_k \nearrow \delta_\infty = \frac{48}{91}$$
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→ condition $m^2 \pi_1(m) \ll N$, i.e.

$$m \ll m_\infty(N) = N^{\delta_\infty + o(1)}$$
Exceptional scales

For forest fire processes: we can again start with $m_1(\zeta) = \zeta^{-1/2}$, and try to follow the same reasonings.
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For $m = L(t)$, $\hat{t} > t$ such that

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In order to make this reasoning rigorous, we use the model with impurities.
Forest fire processes

Conclusion:

- By studying **percolation with heavy tailed impurities**, we show that early fires do not perturb too much connectedness of the forest.

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- We also obtain a similar *deconcentration* phenomenon around $t_c$, and a rather complete understanding of the final configuration (*work in progress*).

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► For forest fires with recovery, the same behavior should hold, up to a time $t_c + \delta$ where $\delta > 0$ universal (using also properties of “self-destructive percolation”\(^4\)) → precise description beyond $t_c$.

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▶ This should improve our understanding of the long-term ($t \to \infty$) behavior, but limited progress so far.

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Thank you!