Extended monstrous moonshine

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What is moonshine?
Strange connections between finite groups and modular forms

Symmetry
Finite groups

Moonshine

Complex analysis/
Number theory
Modular forms
What is moonshine?

Strange connections between finite groups and modular forms

The connections should be “very special”

Infinitely many cases $\Rightarrow$ not moonshine!
Classification of finite simple groups

Any finite simple group is one of the following

- A cyclic group of prime order
- An alternating group $A_n$ ($n \geq 5$)
- A group of Lie type (16 infinite families)
- One of 26 sporadic simple groups

Largest sporadic: Monster $\mathbb{M}$, about $8 \cdot 10^{53}$ elements (Griess 1982).
194 irreducible representations of dimension 1, 196883, 21296876, …
Hauptmodul (or principal modulus)

A Hauptmodul for a discrete subgroup $\Gamma \subset \text{SL}_2(\mathbb{R})$ is a holomorphic function $\mathcal{H} \to \mathbb{C}$ invariant under $\Gamma$, that generates the function field of $\Gamma \backslash \mathcal{H}$.

$J$-function as Hauptmodul

The quotient space $\text{SL}_2(\mathbb{Z}) \backslash \mathcal{H}$ has genus zero. Function field generated by $J$. Fourier expansion:

$q^{-1} + 196884q + 21493760q^2 + \cdots (q = e^{2\pi i z})$
Coefficients of $J$ and Irreducible Monster reps

\[
\begin{align*}
196884 & = 1 + 196883 \quad \text{(McKay, 1978)} \\
21493760 & = 1 + 196883 + 21296876 \quad \text{(Thompson, 1979)} \\
864299970 & = 2 \times 1 + 2 \times 196883 + 21296876 + 842609326 \\
& \vdots
\end{align*}
\]

How to continue this sequence?

McKay-Thompson conjecture: Natural graded rep

\[
\bigoplus_{n=0}^{\infty} V_n \text{ of } \mathbb{M} \text{ such that } \sum \dim V_n q^{n-1} = J.
\]
Idea: Physics forms a bridge

Conformal field theory

Monster (Vertex operator algebras) → J function

Solution: Frenkel, Lepowsky, Meurman 1988

Constructed a vertex operator algebra

\[ V^\natural = \bigoplus_{n \geq 0} V_n^\natural \] (the Moonshine Module), such that

\[ \sum_{n \geq 0} (\dim V_n^\natural) q^{n-1} = J \text{ and } \Aut V^\natural = \mathbb{M}. \]
Refined correspondence

Thompson’s suggestion: replace graded dimension with graded trace of non-identity elements.

Monstrous Moonshine Conjecture (Conway, Norton 1979)

There is a faithful graded representation $V = \bigoplus_{n \geq 0} V_n$ of the monster $\mathbb{M}$ such that for all $g \in \mathbb{M}$, the series $T_g(\tau) = \sum_{n \geq 0} \text{Tr}(g|V_n)q^{n-1}$ is the $q$-expansion of a congruence Hauptmodul (= “generates function field of genus 0 $\mathbb{H}$-quotient”).
First proof (Atkin, Fong, Smith 1980)

Theorem: A virtual representation of $\mathbb{M}$ exists yielding the desired functions. No construction.

Second proof (Borcherds 1992)

Theorem: The Conway-Norton conjecture holds for $\mathbb{V}^\dagger$. 
Outline of Borcherd's proof

**FLM construction:** $V^q$

Add torus and quantize

Lie algebra $m$

Isomorphism $m \cong L$

Twisted Denominator Identities

Automorph. $\infty$ prod.

gens. and rels.

Lie algebra $L$

Recursion relations

Hauptmoduln
Additional Moonshine Phenomena
Theorem (Ogg 1974)

The primes $p$ such that $X_0(p)^+ = X_0(p)/\langle w_p \rangle$ has genus zero are

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71$$

These are the primes $p$ such that all supersingular elliptic curves over $\overline{\mathbb{F}}_p$ have $j$ invariant in $\mathbb{F}_p$.

Ogg’s Jack Daniels problem

Explain why these are precisely the primes that divide the order of the Monster.
Borcherds’s half-solution

For each $p | \#\mathbb{M}$, there is a conjugacy class $pA$, such that $T_g(\tau)$ is a Hauptmodul of $X_0(p)^+$ for $g$ in $pA$.

The other half (still open)

Explain why $V^\frac{1}{2}$ has so many automorphisms.
Positivity phenomena

For $g$ in $pA$, the coefficients of $T_g(\tau)$ are non-negative integers.
E.g. for $2A$, $T_g(\tau) = q^{-1} + 4372q + 96256q^2 + \cdots$ is a Hauptmodul for $\Gamma_0(2)^+$. 

Extra phenomena (Conway-Norton, Queen)

The coefficients of $T_g(\tau)$ appear to be dimensions of representations of centralizers.
E.g., For $g$ in $2A$, $C_M(g) \cong 2.B$, with irreducible representations: $1, 4371, 96255, 96256, \ldots$
Two explanations conjectured!

1. Generalized Moonshine (Norton 1987): Graded representations $V(g)$ of $C_M(g)$ in characteristic zero, traces of centralizing elements are Hauptmoduln.

**Complementary history**

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**Key advance:**

Theory of cyclic orbifolds of vertex operator algebras.
What is a vertex operator algebra?

1. $V = \bigoplus_{n \in \mathbb{Z}} V_n$, a graded vector space
2. $1 \in V_0$, an “identity element”
3. $\omega \in V_2$ a “Virasoro element”
4. $Y : V \otimes V \to V((z))$, “multiplication”

satisfying

1. $Y(1, z)x = x, \ Y(x, z)1 \in x + zV[[z]]$
2. coefficients of $Y(\omega, z)$ give Virasoro action
3. “commutativity and associativity”.
4. $\forall n, \dim V_n < \infty$. For $n \ll 0$, $V_n = 0$. 
Lattice vertex operator algebras

$L$ an even positive definite lattice.

$V_L = \mathbb{C}[L] \otimes \text{Sym}_\mathbb{C} x (L \otimes \mathbb{C})[x].$

Graded dimension is $\frac{\theta_L(\tau)}{\eta^{\text{rank } L}}$.

E.g., For Leech lattice, get $J + 24$. 
V-Modules

For $V$ - vertex operator algebra, a $V$-module is a vector space $M$ with an action map $Y^M : V \otimes M \rightarrow M((z))$ satisfying some compatibility. $V$ is **holomorphic** if all $V$-modules are direct sums of $V$.

**Theorem (Dong 1994)**

All $V_L$-modules are direct sums of $V_{L+\alpha}$ for $\alpha \in L^\vee/L$. In particular, $V_L$ holomorphic $\Leftrightarrow L$ unimodular.
Theorem (van Ekeren, Möller, Scheithauer 2015)

$V$ holomorphic VOA, $g$ “anomaly-free” automorphism of order $n$. Then there exist:
- a “generalized VOA” $g V = \bigoplus_{i,j \in \mathbb{Z}/n\mathbb{Z}} g V^{i,j}$.
- a pair of commuting automorphisms $(g, g^*)$ giving decomposition into $V^{i,j}$.
- An isomorphism $V = \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} g V^{i,0}$
- a holomorphic VOA $V/g = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} g V^{0,j}$

Cyclic orbifold duality: $(V, g) \leftrightarrow (V/g, g^*)$
Special case: First construction of $V^\Lambda$ (1988)

$V_\Lambda$ - Leech lattice VOA

$\sigma$ lifted from the $-1$-automorphism of $\Lambda$.

Then $V^\Lambda = (V_\Lambda)/\sigma$.

Now we have 51 constructions

We can take any $\sigma$ that is fixed-point free, with “no massless states”. (51 algebraic conjugacy classes).
Generalized Monstrous Moonshine
Generalized Moonshine Conjecture (Norton 1987):

- \( g \in M \Rightarrow V(g) \) graded proj. rep. of \( C_M(g) \)
- \( (g, h), gh = hg \Rightarrow Z(g, h; \tau) \) holomorphic on \( \mathcal{H} \)

1. \( q \)-expansion of \( Z(g, h; \tau) \) is graded trace of (a lift of) \( h \) on \( V(g) \).
2. \( Z \) is invariant under simultaneous conjugation of the pair \( (g, h) \) up to scalars.
3. \( Z(g, h; \tau) \) constant or a Hauptmodul.
4. \( Z(g, h; \frac{a\tau + b}{c\tau + d}) \) proportional to \( Z(g^ah^c, g^bh^d; \tau) \).
5. \( Z(g, h; \tau) = J(\tau) \) if and only if \( g = h = 1 \).
Brute force solution (like Atkin-Fong-Smith)?

This is a finite problem:

- Finitely many conjugacy classes of commuting pairs, and possible levels are bounded.
- Central extensions of centralizers “can be computed”.

Not finite enough for 2018

- We still haven’t classified the commuting pairs.
- We still don’t know character tables of all centralizers, let alone central extensions.
Physics Language (Dixon, Ginsparg, Harvey 1988)

\( V(g) \) - twisted sectors of a conformal field theory with \( \mathbb{M} \)-symmetry.

\( Z(g, h; \tau) \) - genus 1 partition functions (with \((g, h)\)-twisted boundary conditions).

All except Hauptmodul claim (3) “follow” from conformal field theory considerations.

Algebraic Interpretation

\( V(g) = \) irreducible \( g \)-twisted \( V^\mathbb{M} \)-module \( V^\mathbb{M}(g) \)

\( Z(g, h; \tau) = \text{Tr}(\tilde{h} q^L(0)^{-1} | V(g)) \) for a lift \( \tilde{h} \).
**Geometric interpretation of $Z$**

Physicists draw boundary conditions as colorings.

Commuting pair $(g, h)$ gives $\text{hom } \pi_1(E, e) \to \mathcal{M}$. $SL_2(\mathbb{Z})$ action changes generating pair. Ignoring scalar ambiguities, claims (2) and (4) say that $Z$ is a function on the moduli space of elliptic curves with principal $\mathcal{M}$-bundles.
First Breakthrough (Dong, Li, Mason 1997)

- Existence and uniqueness (up to isom.) of $V^g$.  
- Convergence of power series defining $Z$.  
- Settles claims (1), (2), (5).  
- Reduces $SL_2(\mathbb{Z})$ claim (4) to “$g$-rationality”.

Theorem (C, Miyamoto 2016)

Category of $g$-twisted $V^g$-modules is semisimple. This resolves the $SL_2(\mathbb{Z})$-compatibility claim (4).
On to claim (3)

We now need to show that all $Z(g, h; \tau)$ are Hauptmoduln or constant.

Second Breakthrough (Höhn 2003)

Generalized Moonshine for 2A (Baby monster case).
- Gives outline for proving Hauptmodul claim (3).
Borcherds-Höhn program for Hauptmoduln

Ab. intertw. alg. $\frac{g}{N} V^{\frac{q}{N}}$

Add torus and quantize

Lie algebra $m_g$

Automorph. $\infty$ prod.

gens. and rels.

Lie algebra $L_g$

Isom. $m_g \cong L_g$

Twisted Denominator Identities

Recursion relations

Hauptmoduln

Extended monstrous moonshine
Borcherds products of the form:

\[ T_g(\sigma) - T_g(-1/\tau) = p^{-1} \prod_{m>0, n \in \frac{1}{N}\mathbb{Z}} \left(1 - p^m q^n\right)^{c_{m,n}^g(mn)} \]

- Exponent \( c_{m,n}^g(mn) \) is \( q^{mn} \)-coefficient of a v.v. modular function formed from \( \{T_g^i(\tau)\}_{i=0}^{N-1} \).
- \( L_g \) is a \( \mathbb{Z} \oplus \frac{1}{N}\mathbb{Z} \)-graded BKM Lie algebra.
- Simple roots of multiplicity \( c_{1,n}^g(n) \) in degree \((1, n)\).
Add a torus and quantize

- Take a graded tensor product with a lattice
- Get conformal VA, $c = 26$, graded by 2d lattice, has invariant form.
- Apply a bosonic string quantization functor.
- For Fricke $g$ (i.e., $T_g(\tau) = T_g(-1/N\tau)$), get a
- graded by $II_{1,1}(-1/N) \cong \mathbb{Z} \oplus \frac{1}{N}\mathbb{Z}$. 
Comparison

Borcherds-Kac-Moody Lie algebras:
- $m_g$ has canonical projective action of $C_M(g)$.
- $L_g$ has “nice shape”: known simple roots, good homology.

Isomorphism from matching root multiplicities:
$$\dim(L_g)_{m,n} = (m_g)_{m,n} = c_{g_{m,n}}(mn).$$

Transport de structure $\Rightarrow$ $L_g$ gets $\widehat{C_M(g)}$ action.
Virtual $\tilde{C}_M(g)$-module isom $H_*(E_g, \mathbb{C}) \cong \wedge^* E_g$

implies equivariant Hecke operators $n \hat{T}_n$ given by

$$n \hat{T}_n Z(g, h, \tau) = \sum_{ad=n, 0 \leq b < d} Z(g^d, g^{-b} h^a, \frac{a \tau + b}{d})$$

act by monic polynomials on $Z(g, h, \tau)$.

- Hauptmodul condition follows (C 2008).
- Constants come from $(g, h)$ such that all $g^a h^c$ are non-Fricke when $(a, c) = 1$, using claim (4).

This resolves the final claim (3).
Extended monstrous moonshine

Modular Moonshine

Generalized Monstrous Moonshine

Additional Monstrous Phenomena

Modular Moonshine

Unification?
Ryba’s conjecture

For each $g$ in conjugacy class $pA \ (p \nmid \# \mathbb{M})$, there is a vertex algebra $V^p = \bigoplus_{n \geq 0} V_n^p$ over $\mathbb{F}_p$ with an action of $C_{\mathbb{M}}(g)$, such that for all $p$-regular elements $h$, the Brauer character

$$\sum_{n \geq 0} \hat{\text{Tr}}(h | V_n^p) q^{n-1}$$

is the $q$-expansion of the Hauptmodul $T_{gh}(\tau)$. 
Borcherds-Ryba interpretation 1996

\[ V^p = H^0(g, V^\mathbb{Z}_{\mathbb{Z}}), \text{ where } V^\mathbb{Z}_{\mathbb{Z}} \text{ is a self-dual integral form of } V^\mathbb{Z} \text{ (i.e., a VOA over } \mathbb{Z} \text{ with } \mathbb{M} \text{-symmetry).} \]

Theorem (Borcherds-Ryba 1996, Borcherds 1998)

If \( V^\mathbb{Z}_{\mathbb{Z}} \) exists, then \( V^p := H^0(g, V^\mathbb{Z}_{\mathbb{Z}}) \) works.

Theorem (C 2017)

\( V^\mathbb{Z}_{\mathbb{Z}} \) exists.
How to construct a self-dual integral form?

Existing constructions (e.g., by cyclic orbifold) have denominators.
For example, order $n$ orbifold construction requires $\frac{1}{n}$ and $e^{\pi i/n}$.

Solution

Do orbifold constructions of many different orders, and glue using faithfully flat descent.
Recall

Conjugacy classes \( pA \) yield reps. of \( C_\mathbb{M}(g) \):
\( V(g) \) (char. 0) and \( V^g \) (char. \( p \))

Same \( p \)-regular characters!

In fact, for any \( g \in \mathbb{M} \), we get reps of \( C_\mathbb{M}(g) \):
\( V(g) \) (char. 0) and \( \hat{H}^*(g, V_\mathbb{Z}^\perp) \) (char. \( |g| \)).

Question

Is there an integral structure that produces both?
Main obstruction

Sometimes $\hat{H}^1(g, V_{\mathbb{Z}}^l) \neq 0$.

Conjecture (Borcherds-Ryba 1996)

$\hat{H}^1(g, V_{\mathbb{Z}}^l) = 0$ if and only if $T_g(\tau) = \sum \text{Tr}(g|V^l_n)q^{n-1}$ has a pole at 0.

Definition

We say $g$ is Fricke if $T_g(\tau)$ has a pole at 0. Equivalently, $T_g(\tau)$ is invariant under the Fricke involution $w_N : \tau \mapsto -\frac{1}{N\tau}$. $g$ is non-Fricke if $T_g(\tau)$ is regular at 0.
Fricke versus non-Fricke

- $\mathcal{M}$ has 141 Fricke classes, and 53 non-Fricke classes.
- $T_g(\tau)$ non-negative coeffs. $\iff g$ Fricke.
- $Z(g, h, \tau)$ has a pole at $\infty$ if and only if $g$ is Fricke.
- $V^{\frac{1}{2}}/g \cong \begin{cases} V^{\frac{1}{2}} & g \text{ is Fricke} \\ V_\Lambda & g \text{ non-Fricke} \end{cases}$
Conjecture (Borcherds 1998)

There is a rule that assigns to any $g \in \mathbb{M}$ of order $n$, a $\frac{1}{n}\mathbb{Z}$-graded $\mathbb{Z}[e^{2\pi i/n}]$-module $V_g$ with an action of a central extension $\mathbb{Z}/n\mathbb{Z}.C_{\mathbb{M}}(g)$, such that

1. $V_g \otimes_{\mathbb{Z}[e^{2\pi i/n}]} \mathbb{C} \cong V(g)$ as $\mathbb{Z}/n\mathbb{Z}.C_{\mathbb{M}}(g)$-reps.

2. $g \not\sim \text{Fricke} \Rightarrow V_g \otimes_{\mathbb{Z}[e^{2\pi i/n}]} \mathbb{Z}/n\mathbb{Z} \cong \hat{H}^0(g, V_{\mathbb{Z}}^h)$ as $\mathbb{Z}/n\mathbb{Z}.C_{\mathbb{M}}(g)$-reps.

3. $\hat{H}^*(h, V_g) = V_{gh} \otimes \mathbb{Z}/|h|\mathbb{Z}$ when $g, h$ commute and have coprime order.

4. (additional compatibilities)
Current progress

Twisted modules $V(g)$ can be defined over $\mathbb{Z}[\frac{1}{n}, e^{2\pi i/n}]$, but removing $\frac{1}{n}$ is tricky.
Looks like $\hat{H}^1(g, V_{\mathbb{Z}})$ = 0 for Fricke $g$ (in progress)

What we really need

- Canonical lifts of $\hat{H}^*(g, V_{\mathbb{Z}})$ to characteristic zero.
- Meaningful interpretation of these objects.