LS Augmented Methods for Fluid and Porous Media Couplings

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Key Words & Related to Others

- IB (Immersed Boundary) → IIM (Immersed Interface) 1st to 2nd velocity & pressure, e.g., incompressible vesicle in incompressible fluid (2D, CAF). *Smoothing method → sharp interface method*
- Cartesian meshes & Fast Helmoltz/Poisson solvers
- Efficiency and accuracy; *augmented IIM*
- Applications: Fluid and Porous media; moving contact lines (E, Ren, Li)
- AMS classification: 65N, …
Outline

- **Introduction/Problem:**
  - Fluid and porous media (Stokes/Darcy or Navier-Stokes/Darcy)
  - Different governing equations on different regions
  - Interface conditions (including BJ or BJS)

- **Augmented IIM for Stokes-Darcy coupling**
  - Stokes/Darcy → *three Poisson Solvers*

- **Least squares augmented IIM for NS-D coupling**
  - Pressure incremental formulation
  - Least squares: more than augmented variables, why?

- **Moving contact line** sharp interface simulations (E/Ren’s model)

- **Numerical experiments & Conclusion**
  - Analytic, flow problems, corners, …
Fluid and Darcy Coupling

Fluid flow: Stokes eqns or \textbf{N-S eqns}

\[ \nabla p = \nabla \cdot \mu (\nabla u + \nabla u^T) + g \]
\[ \nabla \cdot u = 0 \]

\[ \rho \left( \frac{\partial u}{\partial t} + u \nabla u \right) + \nabla p = \mu \Delta u + g \]
\[ \nabla \cdot u = 0 \quad \text{Navier-Stokes/Darcy} \]

Porous media flow: Darcy’s law

\[ u = -\frac{K}{\mu} \nabla p \]
\[ \nabla \cdot u = 0 \quad \text{or} \quad \nabla \cdot u = \phi \]
Interface Conditions

\[ u_s \mathbf{g}_n = u_D \mathbf{g}_n \]  (normal velocity is continuous)

\[ [p] = p_s - p_D = 2\mu(n\mathbf{g}_s^T \mathbf{g}_n), \quad T_s = (\nabla u_s + \nabla u_s^T) / 2 \]

\[ n\mathbf{g}_s^T \mathbf{g}_\tau = -\frac{\sqrt{K}}{\mu \alpha} (u_s - u_D) \mathbf{g}_\tau \]  (BJ) or  \[ -\frac{\sqrt{K}}{\mu \alpha} u_s \mathbf{g}_\tau \]  (BJS)
Applications

- Flows across interfaces between soil and surface
- Oil reservoir
- Bio-medicine & cell deformation
- Blood motion in lungs, solid tumors and vessels
- Heat transfer in walls with fibrous insulation (firefighter)
Literature Review


- Numerical methods
  - FEM with domain decomposition, X. He, M. Gunzburg, …
    \[ \lambda_D K \nabla p_D g_n + g p_D = \eta_D \]
    Robin-Robin domain decomposition
  - Fictitious domain approach: P. Sun, … the interface conditions are built into the weak form & affects convergence order, larger system of saddle point system
  - Phase field method
Literature Review II

- **Cartesian mesh** method (BEM & Stokelet by R. Cortetzi for a circular interface)
- Cartesian method with local modified mesh (Z. Wang/Li, 2015), free-fem
- Almost no **non-trivial analytic solutions** with curved interfaces
- Most simulations are for straight interfaces.
The weak form

Assume $\phi_p$ and $u_f$ are 0 on the boundary $\partial \Omega$ and define the following functional spaces

\[
H_f = \{v_f \in (H^1(\Omega_f))^d \mid v_f = 0 \text{ on } \partial \Omega_f \setminus \Gamma\}, \quad (8)
\]

\[
Q = L^2(\Omega_f), \quad (9)
\]

\[
H_p = \{\psi_p \in H^1(\Omega_p) \mid \psi_p = 0 \text{ on } \partial \Omega_p \setminus \Gamma\}. \quad (10)
\]

The following bilinear forms are defined as

\[
a_f(u_f, v_f) = 2\nu\left(\frac{1}{2}(\nabla u_f + \nabla^T u_f), \frac{1}{2}(\nabla v_f + \nabla^T v_f)\right) \quad \text{on } \Omega_f, \quad (11)
\]

\[
a_p(\phi_p, \psi_p) = (K\nabla \phi_p, \nabla \psi_p) \quad \text{on } \Omega_p, \quad (12)
\]

\[
b_f(v_f, p_f) = -(\nabla \cdot v_f, p_f) \quad \text{on } \Omega_f. \quad (13)
\]
Idea for Stokes-Darcy coupling

**Step 1:** Get Poisson equation for the pressure.

\[ \nabla p = \nabla \cdot \mu (\nabla u + \nabla u^T) + g; \quad u = -\frac{K}{\mu} \nabla p \]

\[ \nabla \cdot u = 0 \]

\[ \Delta p = \nabla \cdot \mathbf{g} \quad \text{in Stokes} \]

\[ \Delta p = 0 \quad \text{in Darcy} \]

\[ [p] = q_1, \quad [p_n] = q_2, \quad \text{(new unknown)} \]

\[ \Delta p = f(x) + \int_{\Gamma} q_2(s) \delta(x - X(s)) \, ds + \int_{\Gamma} q_1(s) \delta'(x - X(s)) \, ds \]
Idea for Stokes-Darcy coupling, II

- Step 2: Solve for the velocity

\[ \nabla p = \nabla \cdot \mu (\nabla u + \nabla u^T) + g \quad \Rightarrow \quad u = -\frac{K}{\mu} \nabla p \]
\[ \nabla \cdot u = 0 \]

\[ \Delta u = \begin{cases} 
\frac{(p_x - g_1)}{\mu} , & \\
-K \Delta p_x / \mu , & 
\end{cases} \]

\[ \Delta v = \begin{cases} 
\frac{(p_y - g_2)}{\mu} , & \\
-K \Delta p_y / \mu , & 
\end{cases} \]

\[ [u_n] = q_3 , \quad [v_n] = q_4 , \quad \text{(new unknown)} \]

\[ [u_{gn}] = 0 , \quad [u_{gr}] = q_5 \quad \Rightarrow \quad \text{get} \quad [u] \quad \& \quad [v] \]
Idea for Stokes-Darcy coupling III

- Introduce 5 (or 6) interface variables (from the primary variables), we get **three Poisson equations** for the pressure and velocity (**decoupled** the PDEs)

- Interface (augmented) variables

\[
[p] = q_1, \quad [p_n] = q_2, \\
[u_n] = q_3, \quad [v_n] = q_4, \\
[u \tau] = q_5
\]

\[\Omega_p \quad \Gamma \quad \Omega_f\]
Other Interface Conditions

- We set up 5 augmented variables, $q_1$-$q_5$, we need 5 augmented equations to close the system.

$$\begin{align*}
[p] &= 2\mu (ng_s gn) \\
u_s gn &= u_D gn \\
n g_s g &= -\frac{\sqrt{K}}{\mu \alpha} (u_s - u_D) gn \quad \text{or} \quad -\frac{\sqrt{K}}{\mu \alpha} u_s gn \\
\frac{\partial p_s}{\partial n} &= \mu \Delta u_s gn + g gn \\
\frac{\partial p_D}{\partial n} &= -\frac{\mu}{K} u_D gn
\end{align*}$$
### Outer Boundary Conditions

#### Darcy-Stokes (or Navier-Stokes)

- \( u_D g_n \) is given, \( \frac{\partial p_D}{\partial n} = -\frac{u_D g_n}{K} \) on \( \partial \Omega \)
- \( u_D = -K \nabla p_D \), Dirichlet BC for \((u,v)\) on \( \partial \Omega \)

#### Stokes (or Navier-Stokes)-Darcy

- \( u_S \) is given, \( \frac{\partial p_S}{\partial n} = \mu \Delta u_S g_n + F g_n \) on \( \partial \Omega \)
- Set \( \frac{\partial p_S}{\partial n} = q_6 \) another augmented variable
Discretization & Schur Complement

- Discretization of three Poisson Eqns with jump conditions (linear problem)
  \[ AU + BQ = F \]

- Discretization of the physical interface conditions
  \[ CU + DQ = F_2 \]

- \( Q \) is along the boundary \( O(5N) \)

- Schur Complement (direct or GMRES)
  \[ (D - C A^{-1} B)Q = F_2 - C A^{-1} F, \quad SQ=F_3 \]
Matrix-vector multiplication

- It is easy to get the matrix multiplication for a given augmented vector $\mathbf{Q}$ to get $S\mathbf{Q}$
  - Solve three Poisson equations $A\mathbf{U} + B\mathbf{Q} = \mathbf{F}$
  - Find the residual of interface conditions
    $$R(\mathbf{Q})=C\mathbf{U} + D\mathbf{Q} - F_2 \text{ via some interpolation}$$

- LU (SVD) or GMRES
  - Fixed interface, time independent, form the matrix $S$ and use LU (SVD)
  - Moving interface $\text{GMRES + preconditioning}$
It is challenging to construct non-trivial analytic solution for curved interface ($r=1$). In Darcy region, $u=0$, $v=0$, $p=1$, with slip jump. In fluid:

\[
\begin{align*}
    u(x, y) &= y(x^2 + y^2 - 1) - 2y, \\
    v(x, y) &= -x(x^2 + y^2 - 1) + 2x, \\
    p(x, y) &= x^2 + y^2, \\
    F_1(x, y) &= -8y + 2x, \\
    F_2(x, y) &= 8x + 2y,
\end{align*}
\]
Validation for Stokes/Darcy II

Average convergence rate: 1.8334 & 2.1388

\[ \mathbf{n} = [x, y]^T, \quad \mathbf{\tau} = [-y, x]^T, \quad p_s = 1, \quad p_D = 1, \quad u_s = -2y, \quad v_s = 2x, \]
\[ u_D = 0, \quad v_D = 0, \quad \frac{\partial u_s}{\partial n} = 0, \quad \frac{\partial v_s}{\partial n} = 0, \quad \frac{\partial u_s}{\partial \mathbf{\tau}} = -2x, \quad \frac{\partial v_s}{\partial \mathbf{\tau}} = -2y, \]
\[ D_s \mathbf{n} \cdot \mathbf{n} = 0 \quad D_s \mathbf{\tau} \cdot \mathbf{n} = -2. \]

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Flow Test I

(a) $\mu=1, K=0.02, u=1, \beta=5.5$

(b) $\mu=1, K=0.02, u=1, \beta=5.5$

(c) $\mu=1, K=0.02, u=1, \beta=5.5$

(d) $\mu=1, K=0.02, u=1$
Flow Test (fluid inside)
Flow tests: Stokes-Darcy

Figure 2. (a)-(b): Velocity plot of a porous media and a fluid (outside the interface $x^2 + y^2 = 0.5^2$) interaction with different parameters. (a): $K_1 = 1$, $\mu = 1$, $K_2 = 1$; (b): $K_1 = 0.2$, $\mu = 0.2$, $K_2 = 1$. (c): The mesh plot of the pressure corresponding to (b). (d): The mesh plot of the $v$ component of the velocity corresponding to (b).
Equivalence of two systems

- **Original system** (Stokes/Darcy) —> via augmented (interface) variables —> **New system** (*three Poisson eqns*):
  
  \[
  \begin{align*}
  [p] &= q_1, \quad [p_n] = q_2, \\
  [u_n] &= q_3, \quad [v_n] = q_4, \\
  [u \times \tau] &= q_5
  \end{align*}
  \]

- The soln to the *original* is also a soln to the *new*. 
Equivalence of two systems

Is the solution to the new system also the soln to the original system?

- The momentum equations is satisfied
- The Darcy’s law in porous media region is also enforced.
- For Stokes and Darcy’s coupling, it works fine
Why not work well for NSE/Darcy?

(c) Comparison of the condition number.

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Why & where Least Squares

- The method (5 aug. variables) works fine for Stokes-Darcy’s coupling, but barely works for Navier-Stokes & Darcy’s coupling.

- Why? Is the velocity divergence free in flow region? We have $\Delta(\text{div}(u))=0 \Rightarrow \text{div}(u)=0$? Yes, if it true along $\Gamma$ & $\partial \Omega$.

- Our solution: enforce the divergence condition on $\Gamma$ & $\partial \Omega$: 5 aug variables, six eqn. Least squares!

- Soln exists & it is unique if the original problem is well-posed!
Algorithm for NSE/Darcy

- Time marching + pressure incremental

\[ \Delta p_{k+1}^+ = \begin{cases} \nabla \cdot F_{k+1/2}^{k+1/2} - \rho \nabla \cdot (u \cdot \nabla u)^{k+1/2} , & x \in \Omega_f, \\ 0 & x \in \Omega_D, \end{cases} \]

\[ \begin{bmatrix} p_{k+1} \\ q_{1+1}^k \end{bmatrix} = q_{1+1}^k, \quad \begin{bmatrix} \frac{\partial p_{k+1}^+}{\partial n} \end{bmatrix} = q_{2+1}^k, \quad \text{on } \Gamma. \]
Algorithm for NSE/Darcy II

$$
\Delta u^* - \frac{2\rho}{\mu \Delta t} u^* = \begin{cases} 
\frac{2}{\mu} \left( p^{k+1}_x - \frac{\rho}{\Delta t} u^k \right) - \Delta u^k + \frac{2}{\mu} \left( \rho (u \cdot \nabla u)^{k+1/2} - F_1^{k+1/2} \right), & \mathbf{x} \in \Omega_f, \\
-K_1 \Delta p^{k+1}_x + \frac{2K_1 \rho}{\mu \Delta t} p^{k+1}_x, & \mathbf{x} \in \Omega_D, 
\end{cases}
$$

$$
[u^*] = -q_5^{k+1} \sin \theta, \quad \left[ \frac{\partial u^*}{\partial n} \right] = q_3^{k+1}, \quad \text{on } \Gamma;
$$

$$
\Delta v^* - \frac{2\rho}{\mu \Delta t} v^* = \begin{cases} 
\frac{2}{\mu} \left( p^{k+1}_y - \frac{\rho}{\Delta t} v^k \right) - \Delta v^k + \frac{2}{\mu} \left( \rho (u \cdot \nabla v)^{k+1/2} - F_2^{k+1/2} \right), & \mathbf{x} \in \Omega_f, \\
-K_1 \Delta p^{k+1}_y + \frac{2K_1 \rho}{\mu \Delta t} p^{k+1}_y, & \mathbf{x} \in \Omega_D, 
\end{cases}
$$

$$
[v^*] = -q_5^{k+1} \cos \theta, \quad \left[ \frac{\partial v^*}{\partial n} \right] = q_4^{k+1}, \quad \text{on } \Gamma;
$$
Important Details

- How to make the Helmholtz eqn have the same magnitude in both fluid & porous media region? Add an artificial term in the Darcy’s domain $u^*/\Delta t$

- Reinforce the Darcy’s law after solving the Helmholtz eqns.

- With and without projection step ($O(h^2)$).

- Given $Q^{k+1}$, solve to get:
  $$AU^{k+1} + B Q^{k+1} = F^k$$
Algorithm Revisit II

- Discretization of three Poisson Eqns (ignore $k$)
  
  \[ AU + BQ = F \]

- Discretization of the physical interface conditions
  
  \[ CU + DQ = F_2 \]

- $Q$ is along the boundary $O(5N)$

- Schur Complement (direct or GMRES)
  
  \[ (D - CA^{-1}B)Q = F_2 - CA^{-1}F, \quad SQ = F_3 \]

- For fixed $\Gamma$, $h$, $S$ is a constant matrix.
Matrix-vector form at one step

\[
\begin{bmatrix}
A & B \\
C & D \\
\end{bmatrix}
\begin{bmatrix}
\tilde{u}^{k+1} \\
Q^{k+1} \\
\end{bmatrix}
= 
\begin{bmatrix}
\tilde{F}_1^{k+1} \\
\tilde{F}_2^{k+1} \\
\end{bmatrix}.
\]

Therefore, the Schur complement for \(Q^{k+1}\) is

\[
(D - CA^{-1}B)Q^{k+1} = \tilde{F}^{k+1}_2 - CA^{-1}\tilde{F}^{k+1}_1 = \bar{F}^{k+1}, \quad \text{or} \quad SQ^{k+1} = \bar{F}^{k+1}.
\]
Validation of NSE/Darcy

\[ u_f = g(t) \left( y(x^2 + y^2 - 1) + 2y \right), \]
\[ v_f = g(t) \left( -x(x^2 + y^2 - 1) - 2x \right), \]
\[ p_f = g(t) \left( x^2 + y^2 \right), \]

\[ n = [x, y]^T, \quad \tau = [-y, x]^T, \quad p_f = g(t), \quad p_D = g(t), \quad u_f = 2y \, g(t), \quad v_f = -2x \, g(t), \]
\[ u_D = 0, \quad v_D = 0, \quad \frac{\partial u_f}{\partial n} = 4y \, g(t), \quad \frac{\partial v_f}{\partial n} = -4x \, g(t), \quad \frac{\partial u_f}{\partial \tau} = 2x \, g(t), \quad \frac{\partial v_f}{\partial \tau} = 2y \, g(t), \]
\[ n \cdot D_f \cdot n = 0, \quad \tau \cdot D_f \cdot n = 0. \]
Grid refinement analysis

- Tangent slip, pressure is constant in Darcy. Average convergence: 2.0221 & 3.2110

(a) Comparison of the pressure error and accuracy order.

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(b) Comparison of velocity error and accuracy order.

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Another example

A continuous tangential velocity but discontinuous pressure along the interface. More importantly, the velocity and pressure are non-trivial in both regions and the normal derivatives of the velocity components are also discontinuous across the interface.

\[ u_f = g(t) \left( y(x^2 + y^2 - 1) + 2x \right), \]
\[ v_f = g(t) \left( -x(x^2 + y^2 - 1) - 2y \right), \]
\[ p_f = 3g(t) \left( x^2 - y^2 \right), \]
# Grid Refinement Analysis

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<td>2.2540</td>
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<td>2.0642</td>
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Table 2: The residual of the six interface conditions (15)-(20) of the computed solution. The last row is the average convergence order of the six interface equations.
Orientation Effect (flow inside)
Orientation Effect (flow outside)
Corner Effect (flow outside)
Transient Behaviors
More on flow test with corners

Figure 9: Contour plots of the pressure and the magnitude of the velocity at $t = 2$ with the set-up in Figure 8.
FSI (Fluid and Porous Media)

- Model: incompressible Navier-Stokes or Stokes equations coupled with Darcy’s law. Multi-connected domain
Signed distance function

- Re-initialization of a 2D level set function in a tube: *blue line* is the interface. Treat boundary is challenging!
Multi-particles & multi-scales

- Many inclusions with different permeability. Flow from right
Moving Contact Line (E, Ren, Li)

- Free boundary problems (drop spreading)

- Navier Stokes equation for incompressible fluid(s)
- Two phase or one phase
- Navier BC along the contact line

\[ \beta_i u_i = -\mu_i \frac{\partial u_i}{\partial y} \]
Governing Eqns and BC

- Navier-Stokes equation for the fluid
  \[
  \rho \left( \frac{\partial u}{\partial t} + u \nabla u \right) + \nabla p = \nabla \cdot \mu (\nabla u + \nabla u^T) + G
  \]
  \[\n  \nabla \cdot u = 0
  \]

- Free boundary condition
  \[
  -p + n^T \cdot \mu (\nabla u + \nabla u^T) \cdot n = -p_{\text{air}} + \sigma \kappa + f_n, \quad x \in \partial \Omega
  \]
  \[
  \tau^T \cdot \mu (\nabla u + \nabla u^T) \cdot n = f_\tau, \quad x \in \partial \Omega
  \]

- Navier (slip) BC:
  \[\beta_i u_i = -\mu_i \frac{\partial u_i}{\partial y}\]

- Tip velocity (W. Ren & W. E):
  \[\alpha u = \gamma (\cos \theta - \cos \theta^*)\]

- What is the consistent boundary condition at the triple junctions?
The Augmented Algorithm

\[ \rho \frac{u^* - u^k}{\Delta t} = \begin{cases} 
-\nabla p^{k-\frac{1}{2}} - \rho (u \cdot \nabla u)^{k+\frac{1}{2}} + \frac{\mu}{2} \left( \Delta u^* + \Delta u^k \right) + G^{k+\frac{1}{2}}, & x \in (\Omega \cap Z^R) \\
-\nabla p^k - \rho (u \cdot \nabla u)^k + \mu \Delta u^* + G^k, & x \in (\Omega \cap Z^I) \\
-\rho (u \cdot \nabla u)^{k+\frac{1}{2}} + \frac{\mu}{2} \left( \Delta u^* + \Delta u^k \right), & x \in (\Omega^c \cap Z^R) \\
-\rho (u \cdot \nabla u)^k + \mu \Delta u^*, & x \in (\Omega^c \cap Z^I) 
\end{cases} \]

\[ \frac{\partial u^*}{\partial n} = 0 \quad \text{at} \ x = a, \ x = b, \ \text{and} \ y = d, \quad v = 0 \ \text{and} \ \beta u = -\mu \frac{\partial u}{\partial y} \ \text{at} \ y = c. \quad (2.10) \]

\[ [u^*]_{\partial \Omega} = 0, \quad \begin{bmatrix} \frac{\partial u^*}{\partial n} \end{bmatrix}_{\partial \Omega} = q^{k+1}, \quad \begin{bmatrix} A & B \\ S & E \end{bmatrix} \begin{bmatrix} U^{k+1} \\ G^{k+1} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \]

\[ \begin{cases} 
\Delta \phi^{k+1} = \frac{\nabla \cdot u^*}{\Delta t}, & x \in R, \\
\frac{\partial \phi^{k+1}}{\partial n} \Big|_{\partial R} = 0, & \phi^{k+1} \Big|_{\partial \Omega} = 0, \quad \frac{\partial \phi^{k+1}}{\partial n} \Big|_{\partial \Omega} = 0,
\end{cases} \]

\[ u^{k+1} = u^* - \Delta t \nabla \phi^{k+1}, \quad x \in R, \]

\[ \nabla p^{k+\frac{1}{2}} = \nabla p^{k-\frac{1}{2}} + \nabla \phi^{k+1}, \quad x \in \Omega, \]

\[ p + n^T \cdot \frac{\mu}{2} \nabla u + \nabla u^T \cdot n = p_{air} - \sigma_k, \quad x \in \partial \Omega, \]

\[ \tau^T \cdot \frac{\mu}{2} \nabla u + \nabla u^T \cdot n = 0, \quad x \in \partial \Omega, \]
Drop spreading and contracting

(a), $\theta^* = \pi/4$, $\theta^0 = \pi/2$; (b), $\theta^* = 3\pi/4$, $\theta^0 = \pi/2$
Another Example

For this one, the Crank-Nicholson approach fails.
Effect of Gravity
Drop spreading with perturbation

Cui

\[ \theta_0 \approx \frac{\pi}{2} \]
How to solve Poisson Eqn. (regular)

- Regular domain (rectangular, circles, spheres, ..), no interface/singularity

\[ \Delta u = f(x) \]

BC (e.g. Dirichlet, Neuman, Mixed)

- The FD scheme at \((x_i, y_j)\)

\[ \frac{u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j}}{h^2} = L_h u_{i,j} = f_{ij} \]

- \(AU=F; \quad A\): Discrete Laplacian. Can be solved by a fast Poisson solver (e.g. FFT, \(O(N^2)\log(N)\)), e.g., Fish-pack, or structured multigrid
How to Solve Poisson Eqn. with Jumps

- **Interface problems**, simplified *Peskin’s IB* model

\[ \Delta u = f(x) + \int_{\Gamma} c(s)\delta(x - X(s))\,ds + \int_{\Gamma} w(s)\delta'(x - X(s))\,ds \]

BC (e.g., Dirichlet, Neuman, Mixed)

- **Equivalent Problem**

\[ \Delta u = f(x), \quad x \in \Omega \setminus \Gamma, \quad [u]_{\Gamma} = w(s), \quad [\nabla u \cdot n]_{\Gamma} = \left[ \frac{\partial u}{\partial n} \right]_{\Gamma} = C(s) \]

BC (e.g., Dirichlet, Neuman, Mixed)

- **FD scheme** \((x_j, y_j)\), regular/irregular

\[ \frac{u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j}}{h^2} = L_h u_{i,j} = f_{ij} \]

\[ \frac{u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j}}{h^2} = L_h u_{i,j} = f_{ij} + C_{ij} \]
Solve Poisson Eqn. with Jumps using IB method

Discrete delta function approach

\[ \frac{u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j}}{h^2} = L_h u_{i,j} = f_{ij} + C_{ij} \]

\[ C_{ij} = f_{ij} + \sum_k c_k \delta_h(x_i - X_k)\delta_h(y_j - Y_k)\Delta s + \sum_k w_k \delta'_h(x_i - X_k)\delta'_h(y_j - Y_k)\Delta s \]
Analysis of IB Method

- It is inconsistent! The local truncation error is $O(1/h)$!

$$\Delta u = f(x) + \int_\Gamma c(s) \delta(x - X(s)) \, ds + g$$

BC (e.g., Dirichlet, Neuman, Mixed)

$$\frac{u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j}}{h^2} = L_h u_{i,j} = f_{ij} + C_{ij}$$

$$C_{ij} = \sum_k C_k \delta_h(x_i - X_k) \delta_h(y_j - Y_k) \Delta s_k$$

- But it is first order convergent in the infinity norm!
- It has better convergence order in $L^2$ (average) norm
- Rigorous proof by Z. Li, MathCom, 2015.
- Often local adaptive mesh is used
More Accurate Method: IIM

- IIM: Immersed Interface Method (LeVeque/Li)
  - Replace the Dirac delta function with jump conditions
  - For Poisson equations (coef=1), the FD scheme (LHS) is the same
  - Better correction terms only the 5-point FD stencil has points from both sides (smaller support than IB method)
More Accurate Method: IIM

- Use Taylor expansion from each side of the interface to minimize the local truncation error
- Second order accurate solution and gradient at all grid points (strictly proved, T. Beale, Li)
- One fast Poisson solver for one variable
An example

- A Poisson equation with general BC on a star region

Table 2: (a), A grid refinement analysis with $\beta^- = 1000$, $\beta^+ = 1$. The average convergence orders are 3.3288, 4.2931, and 4.2812, respectively for the three quantities.

<table>
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<tr>
<th>$N$</th>
<th>$E(u)$</th>
<th>$r$</th>
<th>$E(u^-)$</th>
<th>$r$</th>
<th>$E(u^+)$</th>
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(b) The number of iterations without and with the two preconditioners:

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<th>320</th>
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<td>37</td>
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</tr>
</tbody>
</table>
Conclusions

New augmented methods for fluid flow and porous media (*Stokes-Darcy or NES-Darcy*)

- Different governing equations are transformed to the same type equations via *augmented interface variables*
- **Three Poisson equations** with jump in the soln and normal derivative
- Use *least squares* to get equivalent systems
- Can utilize the FFT based *fast Poisson solver*
Conclusions

- Second order accurate in both pressure and velocity
- Equivalence has been proved under stronger regularity assumptions
- What are the best augmented variables that have the same No. of unknowns and equations?
Why augmented approach?

- Can make the solver faster, e.g., fast IIM for elliptic interface problems with piecewise constant coefficient, IIM for irregular domains.

- Can decouple problems, e.g., the Stokes or NSE equations with discontinuous viscosity, the augmented approach enable us to decouple the jump conditions in the pressure and the velocity.

- For some problems, it is the only way to get accurate discretization.

- No need to have the Green functions, independent of BC, source terms, domains etc.

- Can couple problems: deal with Stokes-Darcy coupling.
Key Words & Related to others

- IB (Immersed Boundary) → IIM (Immersed Interface) 1\textsuperscript{st} to 2\textsuperscript{nd} velocity & pressure
- Cartesian meshes & Fast Helmoltz/Poisson solvers
- Efficiency and accuracy; augmented IIM
- Applications: Fluid and Porous media; moving contact lines (E, Ren, Li)
- AMS classification: 65N, …
Thank you!
How to Solve Poisson Eqn. with jumps

- **Interface problems**, simplified *Peskin’s IB* model
  \[ \Delta u = f(x) + \int_{\Gamma} c(s) \delta(x - X(s)) ds + g \]
  BC (e.g., Dirichlet, Neuman, Mixed)

- **Equivalent Problem**
  \[ \Delta u = f(x), \quad x \in \Omega \setminus \Gamma, \quad [u]_{\Gamma} = 0, \quad [\nabla u \cdot n]_{\Gamma} = \left[ \frac{\partial u}{\partial n} \right]_{\Gamma} = C(s) \]
  BC (e.g., Dirichlet, Neuman, Mixed)

- **FD scheme** \((x_i, y_j)\), regular/irregular
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  \frac{u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j}}{h^2} = L_h u_{i,j} = f_{ij}
  \]

- **IB**, first order, IIM second order (C’s depend on curvature!)

- **AU=F+BC**; \(A\): Discrete Laplacian. Can be solved by fast Poisson solver

- IIM is *second order* both in solution & gradient (T. Beale …)
Algorithm Revisit I

Given $Q$, solve $p$, $u$, $v$: $AU+BQ=F$

$$\nabla p = \nabla \cdot \mu (\nabla u + \nabla u^T) + g$$
$$\nabla \cdot u = 0$$

$$u = -\frac{K}{\mu} \nabla p$$
$$\nabla \cdot u = 0$$

$$\Delta u = \begin{cases} (p_x - g_1) / \mu \\ -K\Delta p_x / \mu \end{cases}$$
$$\Delta v = \begin{cases} (p_y - g_2) / \mu \\ -K\Delta p_y / \mu \end{cases}$$

$$[u_n] = q_3, \quad [v_n] = q_4, \quad \text{(new unknown)}$$

$$[u \mid n] = 0, \quad [u \mid \tau] = q_5 \Rightarrow \text{get } [u] \& [v]$$