Random Mechanism Design on Multidimensional Domains

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Preliminaries: Preferences

- $I = \{1, \ldots, N\}, N \geq 2$: A finite set of voters;
- $A = \{a, b, c, \ldots\}, |A| \geq 3$: A finite set of alternatives;
- $P_i$: A preference, i.e., a linear order over $A$;
- $r_k(P_i)$: the $k$th ranked alternative in $P_i$;
- $\mathbb{D}$: The domain of preferences over $A$;
- $P \equiv (P_1, \ldots, P_N) \equiv (P_i, P_{-i}) \in \mathbb{D}^N$: A preference profile.

A domain $\mathbb{D}$ is **minimally rich** if for every $a \in A$, there exists $P_i \in \mathbb{D}$ with $r_1(P_i) = a$.

**Definition**

A Deterministic Social Choice Function (DSCF) is a map $f: \mathbb{D}^N \to A$. 
Preliminaries: Random Social Choice Functions

Definition

A Random Social Choice Function (RSCF) is a map \( \varphi : \mathbb{D}^N \rightarrow \Delta(A) \).

Definition

An RSCF \( \varphi : \mathbb{D}^N \rightarrow \Delta(A) \) is **unanimous** if for all \( a \in A \) and \( P \in \mathbb{D}^N \),

\[
[r_1(P_i) = a \text{ for all } i \in I] \Rightarrow [\varphi_a(P) = 1].
\]

Definition (Gibbard, 1977)

An RSCF \( \varphi : \mathbb{D}^N \rightarrow \Delta(A) \) is **strategy-proof** if for all \( i \in I; P_i, P'_i \in \mathbb{D} \) and \( P_{-i} \in \mathbb{D}^{N-1} \), lottery \( \varphi(P_i, P_{-i}) \) first-order stochastically dominates lottery \( \varphi(P'_i, P_{-i}) \) according to \( P_i \), i.e.,

\[
\sum_{k=1}^{t} \varphi_{r_k}(P_i, P_{-i}) \geq \sum_{k=1}^{t} \varphi_{r_k}(P'_i, P_{-i}), \quad t = 1, \ldots, |A|.
\]
Random dictatorships

**Definition**
An RSCF $\varphi^{\text{RD}} : \mathbb{D}^N \rightarrow \Delta(A)$ is a **random dictatorship** if there exists $\varepsilon_i \geq 0$ for each $i \in I$ with $\sum_{i \in I} \varepsilon_i = 1$ such that for all $P \in \mathbb{D}^N$ and $a \in A$, $\varphi_a^{\text{RD}}(P) = \sum_{i \in I : r_1(P_i) = a} \varepsilon_i$.

- A random dictatorship is unanimous and strategy-proof on any domains.
- A random dictatorship *never* admits compromise.
  For instance, let $r_1(P_1) = a$, $r_1(P_2) = b$ and $r_2(P_1) = r_2(P_2) = c$.
  However, $\varphi_{c}^{\text{RD}}(P_1, P_2) = 0$.
- Escape random dictatorships: Chatterji, Sen and Zeng (2014)
**Top-separability**

**Assumption**: Let \( A = \times_{s \in M} A_s \) where \( M \) is finite with \( |M| \geq 2 \), and \( A^s \) is finite with \( |A^s| \geq 2 \) for all \( s \in M \). We assume preferences satisfy **Top-separability**

\[
[r_1(P_i) = (a^s)_{s \in M}] \Rightarrow [(a^s, z^{-s}) P_i (b^s, z^{-s}) \text{ for all } s \in M, b^s \neq a^s \text{ and } z^{-s} \in A^{-s}].
\]

**Definition**

A domain is a **multidimensional domain** if all preferences are top-separable.

Every generalized dictatorship is strategy-proof if and only if all preferences are top-separable. Random generalized dictatorships however do not systematically admit compromise.
Two examples strengthening top-separability

Definition (Le Breton and Sen, 2009)

A preference $P_i$ is **separable** if there exists a (unique) marginal preference $[P_i]^s$ over $A^s$ for each $s \in M$ such that for all $a, b \in A$, we have

$$[a^s[P_i]^sb^s \text{ and } a^{-s} = b^{-s}] \Rightarrow [aP_ib].$$

Definition (Barberà, Gul and Stacchetti, 1993)

For each $s \in M$, let all elements of $A^s$ be located on a tree $G(A^s)$. A preference $P_i$ is **multidimensional single-peaked** on the product of trees $\times_{s \in M} G(A^s)$ if for all distinct $x, y \in A$, we have $[x \in \langle r_1(P_i), y \rangle] \Rightarrow [xP_iy]$.
The constrained compromise property

Definition

An RSCF \( \varphi : \mathbb{D}^N \rightarrow \Delta(A) \) satisfies the constrained compromise property if there exists \( \hat{I} \subseteq I \) with \( |\hat{I}| = \frac{N}{2} \) if \( N \) is even and \( |\hat{I}| = \frac{N+1}{2} \) if \( N \) is odd, such that given \( P_i, P_j \in \mathbb{D} \), we have

\[
\begin{align*}
&\begin{cases}
  r_1(P_i) \equiv (x^s, a^{-s}) \neq (y^s, a^{-s}) \equiv r_1(P_j) \\
  r_2(P_i) = r_2(P_j) \equiv (z^s, a^{-s}) \text{ where } z^s \notin \{x^s, y^s\}
\end{cases} \\
\Rightarrow \left[ \varphi(z^s, a^{-s}) \left( \frac{P_i}{\hat{I}}, \frac{P_j}{I \setminus \hat{I}} \right) > 0 \right].
\end{align*}
\]

The constrained compromise property focuses on non-assemblable compromise alternatives, and hence weakens the compromise property of Chatterji, Sen and Zeng (2016).

Question

Suppose a multidimensional domain admits a unanimous, strategy-proof RSCF which also satisfies the constrained compromise property: What can we infer about the structure of such a domain?
Let $\Gamma(P_i, P'_i) = \{(a, b) \in A^2 | aP_i b \text{ and } bP'_i a\}$.


Preferences $P_i$ and $P'_i$ are adjacent, denoted $P_i \sim P'_i$, if we have $\Gamma(P_i, P'_i) = \{(a, b)\}$ for some $a, b \in A$. 

\[
\begin{array}{cccccccc}
 & & & & \ & \ & \ & \\
 P_i & & & & P'_i & & & \ \\
 a & \rightarrow & c & & c & & c & \rightarrow & b \\
 c & \rightarrow & a & & a & \rightarrow & b & \rightarrow & c \\
 d & \rightarrow & d & \rightarrow & b & \rightarrow & a & \rightarrow & d \\
 b & \rightarrow & b & \rightarrow & d & \rightarrow & a & \rightarrow & a \\
 P^1_i & & P^2_i & & P^3_i & & P^4_i & & P^5_i & & P^6_i \\
\end{array}
\]
After imposing top-separability, preferences $P_i^2$ and $P_i^5$ are excluded.

Besides adjacency (e.g., $P_i^3 \sim P_i^4$), multiple local switchings occurs simultaneously between $P_i^1$ and $P_i^3$: $\Gamma(P_i^1, P_i^3) = \{(a, c), (d, b)\}$.

Preferences $P_i$ and $P_i'$ are adjacent+, denoted $P_i \sim^+ P_i'$, if we have

(i) $P_i$ and $P_i'$ are separable preferences, and
(ii) $\Gamma(P_i, P_i') = \{(a^s, z^{-s}), (b^s, z^{-s})\}$ for some $s \in M, a^s, b^s \in A^s$. 

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A path \( \{P^1_i, \ldots, P^t_i\} \) is a sequence of preferences such that \( P^k_i \sim P^{k+1}_i \) or \( P^k_i \sim^+ P^{k+1}_i \) for all \( k = 1, \ldots, t - 1 \).

Grandmont (1978): The notion of betweenness is stronger than a path as it requires the inclusion of all preferences between two preferences. Monjardet (2009), Sato (2013) and Cho (2016): Only adjacency.

We introduce some parsimony in the lengths of these paths via the notion of a connected$^+$ domain.
Definition (The Interior\(^{+}\) property)

Given \(P_i, P'_i \in \mathbb{D}\) with \(r_1(P_i) = r_1(P'_i) \equiv a\), there exists a path \(\{P^k_i\}_{k=1}^q \subseteq \mathbb{D}\) connecting \(P_i\) and \(P'_i\) such that \(r_1(P^k_i) = a, k = 1, \ldots, q\).

Definition (The Exterior\(^{+}\) property)

Given \(P_i, P'_i \in \mathbb{D}\) with \(r_1(P_i) \neq r_1(P'_i)\), and \(a, b \in A\) with \(aPib\) and \(aP'_ib\), there exists a path \(\{P^k_i\}_{k=1}^q \subseteq \mathbb{D}\) connecting \(P_i\) and \(P'_i\) such that \(aP^k_ib, k = 1, \ldots, q\). In particular, when \(r_1(P_i) \equiv (a^s, z^{-s}) \neq (b^s, z^{-s}) \equiv r_1(P'_i)\), the path \(\{P^k_i\}_{k=1}^q\) satisfies the non-detour property, i.e., \(r_1(P^k_i) \in (A^s, z^{-s}), k = 1, \ldots, q\).

A connected\(^{+}\) domain: A multidimensional domains satisfying the Interior\(^{+}\) Property and the Exterior\(^{+}\) Property.
Connectedness\(^+\): Inclusions

- The top-separable domain
- The separable domain
- The multidimensional single-peaked domain
- The intersection of the separable domain and the multidimensional single-peaked domain
- The union of the separable domain and the multidimensional single-peaked domain(s)
Connectedness\(^+\): Exclusions

- The complete domain (Gibbard, 1973)
- The single-peaked domain (Moulin, 1980; Demange, 1982)
- The single-dipped domain (Barberà, Berga and Moreno, 2012)
- Single-crossing domains (Saporiti, 2009; Carroll, 2012)
- The lexicographically separable domain (Chatterji, Roy and Sen, 2012)
Characterization of multidimensional single-peakedness

Theorem
Let $\mathcal{D}$ be a minimally rich and connected domain. If it admits a unanimous and strategy-proof RSCF satisfying the constrained compromise property, it is multidimensional single-peaked. Conversely, a multidimensional single-peaked domain admits a unanimous and strategy-proof RSCF satisfying the constrained compromise property.

- Multidimensional domains were excluded by Chatterji, Sen and Zeng (2016). Furthermore, we endogenize the tops-only property here and work with a weaker notion of the compromise property.
Theorem

Let domain $\mathbb{D}$ be minimally rich and connected$^+$. If it admits a unanimous, anonymous and strategy-proof DSCF, it is multidimensional single-peaked.

- Generalize the results in Chatterji, Sanver and Sen (2013) and Chatterji and Massó (2018): No restriction on the number of voters, endogenize the tops-only property, and recover full single-peakedness.
Elaboration: Necessity (con.)

For instance,

Step 1 The constrained compromise property implies
\[ \varphi((a^1, 1), (b^1, 1)) = \alpha e_{(a^1, 1)} + (\beta - \alpha)e_{(c^1, 1)} + (1 - \beta)e_{(b^1, 1)}, \]
where \(0 \leq \alpha < \beta \leq 1\).

Since \((a^1, 1) \sim^+ (a^1, 0)\), from profile \(((a^1, 1), (b^1, 1))\) to \(((a^1, 0), (b^1, 1))\), strategy-proofness implies
\[ \varphi_{(c^1, 1)}((a^1, 0), (b^1, 1)) + \varphi_{(c^1, 0)}((a^1, 0), (b^1, 1)) = \beta - \alpha. \]

Step 2 Since \((a^1, 0) \sim^+ (b^1, 0)\), unanimity and strategy-proofness imply
\[ \varphi_{(a^1, 0)}((a^1, 0), (b^1, 0)) + \varphi_{(b^1, 0)}((a^1, 0), (b^1, 0)) = 1. \]

Since \((b^1, 0) \sim^+ (b^1, 1)\), from profile \(((a^1, 0), (b^1, 0))\) to \(((a^1, 0), (b^1, 1))\), strategy-proofness implies
\[ \varphi_{(c^1, 1)}((a^1, 0), (b^1, 1)) + \varphi_{(c^1, 0)}((a^1, 0), (b^1, 1)) = 0. \]
A contradiction to tops-onlyness.
5. Every separable preference is multidimensional single-peaked on $\times_{s \in M} G(A^s)$: A consequence of the constrained compromise property since each marginal preference is driven to be single-peaked.

6. Every preference is multidimensional single-peaked on $\times_{s \in M} G(A^s)$: A consequence of connectedness$^+$. 

Suppose that $P_i$ is not multidimensional single-peaked, e.g., $(x^s, z^{-s}) \in \langle r_1(P_i), (y^s, z^{-s}) \rangle$ but $(y^s, z^{-s}) P_i(x^s, z^{-s})$.

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<th>$P_i$</th>
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<th>$P^{k+1}_i$</th>
<th>$\ldots$</th>
<th>$P'_i$</th>
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1. Let $I = \{1, 2\}$. The multidimensional single-peaked domain $\mathbb{D}_{MSP}$.

2. A projection rule:
   Fix a threshold $z \in A$.
   Given $P_1, P_2 \in \mathbb{D}_{MSP}$, assume $r_1(P_1) = x$ and $r_1(P_2) = y$.
   Then, $f^z(P_i, P_j) = \pi(z, \langle x, y \rangle)$. 
3. **A mixed projection rule**: a mixture of all projection rules
   Let $\lambda^z > 0$ for all $z \in A$ and $\sum_{z \in A} \lambda^z = 1$.

   For all $P_1, P_2 \in \mathbb{D}_{MSP}$, $\varphi(P_1, P_2) = \sum_{z \in A} \lambda^z f^z(P_1, P_2)$.

4. A mixed projection rule is unanimous and strategy-proof, and satisfies the constrained compromise property. Moreover, a mixed projection rule also satisfies the compromise property of Chatterji, Sen and Zeng (2016).
Consider the top-separable domain $\mathbb{D}_{TS}$.

A two-voter \textit{point voting scheme} $\varphi : \mathbb{D}_{TS}^2 \rightarrow \Delta(A)$ introduced by Barberà (1979) is strategy-proof and satisfies the constrained compromise property:

- Fix $(\alpha_1, \alpha_2, \ldots, \alpha_{|A|}) \in \mathbb{R}_{+}^{|A|}$ such that $\alpha_1 > 0$, $\alpha_2 > 0$ and $\sum_{k=1}^{|A|} \alpha_k = \frac{1}{2}$.
- Given $P_i, P_j \in \mathbb{D}_{TS}$, if $a = r_s(P_i)$ and $a = r_t(P_j)$, then $\varphi(P_i, P_j) = \alpha_s + \alpha_t$. 
Consider the top-separable domain $\mathbb{D}_{TS}$.

A two-voter DSCF $f: \mathbb{D}_{TS}^2 \to A$

$$f(P_i, P_j) = \begin{cases} 
    a & \text{if } r_1(P_i) \neq r_1(P_j) \text{ and } r_2(P_i) = r_2(P_j) \equiv a; \\
    r_1(P_i) & \text{otherwise.}
\end{cases}$$

is unanimous and satisfies the constrained compromise property.
Consider the top-separable domain $\mathbb{D}_{TS}$.

A generalized random dictatorship is unanimous and strategy-proof.
Indispensability of top-separability

- Let $A = \times_{s \in M} A^s$ where $|A^s| = 2$ for all $s \in M$.
- The complete domain satisfies the Interior$^+$ and Exterior$^+$ properties.
- A random dictatorship is unanimous and strategy-proof, and satisfies the constrained compromise property *vacuously*. 
Let \( A = A^1 \times A^2, A^1 = \{0, 1, 2\} \) and \( A^2 = \{0, 1\} \). Specify domain \( \mathbb{D}_{\text{MSP}} \) on \( G(A^1) \times G(A^2) \) below. Remove all preferences with peak \((2, 0)\) or \((2, 1)\), i.e., let \( \hat{\mathbb{D}} = \{ P_i \in \mathbb{D}_{\text{MSP}} | r_1(P_i) \neq (2, 0) \) and \( r_1(P_i) \neq (2, 1) \} \). Add a new preference \( \bar{P}_i \).

![Diagram](image)

Domain \( \mathbb{D} = \hat{\mathbb{D}} \cup \{ \bar{P}_i \} \) is connected\(^+\) but never multidimensional single-peaked.

A two-voter mixed projection rule associating positive weights to all projectors other than \((2, 0)\) and \((2, 1)\) is unanimous and strategy-proof and satisfies the constrained compromise property \textit{vacuously}. 

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Indispensability of paths in connectedness

Let \( A = A^1 \times A^2 \times A^3 \), \( A^1 = \{0, 1, 2\} \) and \( A^2 = A^3 = \{0, 1\} \). Specify domain \( D_{MSP} \) on \( G(A^1) \times G(A^2) \times G(A^3) \) below. Moreover, add a new preference \( \bar{P}_i \).

Domain \( D = D_{MSP} \cup \{\bar{P}_i\} \) is minimally rich and top-separable, but never multidimensional single-peaked.

Domain \( D \) satisfies the Interior\(^+\) property but violates the Exterior\(^+\) property since there exists no path connecting \( \bar{P}_i \) and a preference with peak \((2, 1, 1)\) along which \((2, 1, 1)\) always ranks above \((1, 1, 1)\).

A two-voter mixed projection rule associating positive weight to every projector other than \((1, 1, 1)\) and \((2, 1, 1)\) is unanimous and strategy-proof, and satisfies the constrained compromise property.
Why do we adopt randomization?

Theorem

Let domain $\mathcal{D}$ be minimally rich and connected$^+$. If it admits a unanimous, anonymous and strategy-proof DSCF, it is multidimensional single-peaked.

Elicit a product of tree $\times s \in MG(A_s)$. Four cases:

- $f(a_1,0) = (a_1,0)$.
- $f(a_1,0) = (b_1,1)$.
- $f(a_1,0) = (b_1,0)$.
- $f(a_1,0) \in \{(a_1,0), (b_1,1), (b_1,0)\}$.

Loosely speaking, all these four cases are covered simultaneously in the random setting.
Why do we adopt randomization?

**Theorem**

Let domain $\mathbb{D}$ be minimally rich and connected$^\dagger$. If it admits a unanimous, anonymous and strategy-proof DSCF, it is multidimensional single-peaked.

- Generalize the results in Chatterji, Sanver and Sen (2013) and Chatterji and Massó (2018): No restriction on the number of voters, endogenize the tops-only property, and recover the full single-peakedness.

- Elicit a product of tree $\times_{s \in M} G(A^s)$.

- Four cases:

  - $f((a^1, 0), (b^1, 1)) = (a^1, 0)$.
  - $f((a^1, 0), (b^1, 1)) = (b^1, 1)$.
  - $f((a^1, 0), (b^1, 1)) = (b^1, 0)$.
  - $f((a^1, 0), (b^1, 1)) \notin \{(a^1, 0), (b^1, 1), (b^1, 0)\}$.

- Loosely speaking, all these four cases are covered simultaneously in the random setting.
Summary

- Generalized random dictatorships and top-separability.
- Connectedness\(^+\) and a characterization of multidimensional single-peakedness in both random and deterministic settings.
- The characterization of multidimensional single-peakedness remains robust to the voting under constraints.
Assumption: Let \( A = \times_{s \in M} A^s \) where \( M \) is finite with \( |M| \geq 2 \), and \( A^s \) is finite with \( |A^s| \geq 2 \) for all \( s \in M \).

- For each \( s \in M \), a voter \( i^s \in I \) is fixed. A voter sequence: \( i \equiv (i^s)_{s \in M} \).
- A generalized dictatorship: For instance, fix voter sequence \((1, 2)\). Let \( r_1(P_1) = (a^1, a^2) \) and \( r_1(P_2) = (b^1, b^2) \), we have \( f^i(P_1, P_2) = (a^1, b^2) \).

Definition

An RSCF \( \varphi^{GRD} : \mathbb{D}^N \rightarrow \Delta(A) \) is a generalized random dictatorship if there exists \( \gamma(i) \geq 0 \) for each \( i \in I^N \) with \( \sum_{i \in I^N} \gamma(i) = 1 \) such that for all \( P \in \mathbb{D}^N \),

\[
\varphi^{GRD}(P) = \sum_{i \in I^N} \gamma(i) f^i(P)
\]
A GRD does not admit non-assemblable compromise

For instance, assume $\gamma(i) > 0$ for all $i \in I^N$.

Let $r_1(P_1) = (x^1, x^2)$, $r_1(P_2) = (y^1, y^2)$ and $r_2(P_1) = r_2(P_2) = (x^1, y^2)$. Thus, the compromise alternative $(x^1, y^2)$ can be assembled via voter sequence $(1, 2)$ at $(P_1, P_2)$, i.e., $f^{(1,2)}(P_1, P_2) = (x^1, y^2)$. Hence, $\varphi_{GRD}^{(x^1, y^2)}(P_1, P_2) > 0$.

Let $r_1(P_1) = (x^1, a^2)$, $r_1(P_2) = (y^1, a^2)$ and $r_2(P_1) = r_2(P_2) = (z^1, a^2)$. Thus, the compromise alternative $(z^1, a^2)$ is unable to be assembled via any voter sequence at $(P_1, P_2)$, i.e., $f^i(P_1, P_2) \neq (z^1, a^2)$ for all $i \in I^2$. Hence, $\varphi_{GRD}^{(z^1, a^2)}(P_1, P_2) = 0$. 
Two preferences disagreeing on peaks are never adjacent: A consequence of top-separability.

Every unanimous and strategy-proof RSCF satisfies the tops-only property: given \( P, P' \) in \( \mathbb{D}^N \),

\[
\big[r_1(P_i) = r_1(P_i'); \text{ for all } i \big] \Rightarrow \varphi(P) = \varphi(P'): \nonumber
\]

A consequence of connectedness\(^+\). Degenerate \( P_i \sim^+ P_i' \) with \( r_1(P_i) \equiv a \neq b \equiv r_1(P_i') \) to \( a \sim^+ b \).

If \( \lvert A^s \rvert = 2 \) for all \( s \) in \( M \), top-separability implies multidimensional single-peakedness immediately.

If \( \lvert A^s \rvert > 2 \) for some \( s \) in \( M \), we elicit a product of tree

\( \times_{s \in M} G(A^s) \): A consequence of the constrained compromise property.