Continuous Time Random Matching

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Random Matching Markets

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Continuous Time Random Matching
Many researchers have worked with continuous-time models that assume a large number of agents who meet their partners randomly according to a Poisson process with a given arrival rate.

The intuition is that when a large number of agents conduct searches without explicit coordination, random searches by different agents can be considered to be independent.

By the law of large numbers, there should be an almost-sure constant cross-sectional distribution of types, which will simplify the analysis dramatically.
However, the matching processes cannot be mathematically independent, as long as there are only finitely many agents in the economy.

This naturally leads to the consideration of infinitely many agents in random matching models.
Why Continuum?

- If the agent space is countable, the measure can not be countably additive.

- Failure of convergence theorems.

- Non-existence of equilibrium of simple models in games and economies: Khan, Qiao, Rath and Sun (working paper)
Reliance on Continuous-Time Random Matching


Let \((I, \mathcal{I}, \lambda)\) be an atomless probability space of agents.

\[ S = \{1, \ldots, K\} \] a set of finite types.

\[ \alpha : I \rightarrow S \] a type function with type distribution \(p\) on \(S\).

\((\Omega, \mathcal{F}, P)\), another probability space modeling randomness in matching.
A mapping $\pi$ from $I$ to $I$ is called a (deterministic) matching if for any $i \in I$, $\pi(\pi(i)) = i$. If $\pi(i) = i$, then $i$ is not matched.

A mapping $\pi$ from $I \times \Omega$ to $I$ is called a random matching if for any $\omega \in \Omega$, $\pi_\omega$ is a (deterministic) matching.

For $k, l \in S$, let $q_{kl} \in \mathbb{R}$ be the matching probability for a type-$k$ agent to meet a type-$l$ agent. Assume that $p_k q_{kl} = p_l q_{lk}$ for any $k, l \in S$ and $\sum_{l=1}^{K} q_{kl} \leq 1$ for each $k \in S$.

Measurability problem of a continuum of independent random variables.
Let \((I \times \Omega, \mathcal{W}, Q)\) be a probability space extending the usual product \((I \times \Omega, \mathcal{I} \otimes \mathcal{A}, \lambda \times P)\). The extension is said to be a **Fubini extension** of the usual product probability space if it retains the Fubini property, i.e., for any real-valued \(\mathcal{W}\)-integrable function \(f\) on \(T \times \Omega\),

\[
\int_{I \times \Omega} f \, dQ = \int_I \left( \int_{\Omega} f_t \, dP \right) \, d\lambda = \int_{\Omega} \left( \int_I f_\omega \, d\lambda \right) \, dP.
\]

The extension is denoted by \((I \times \Omega, \mathcal{I} \otimes \mathcal{A}, \lambda \otimes P)\).

**Exact Law of Large Numbers** [Sun 1998, 2006]: Let \(f\) be a process defined on a Fubini extension. If \(f\) is essentially pairwise independent, then for \(P\)-almost all \(\omega \in \Omega\),

\[
P f_\omega^{-1} = (\lambda \otimes P) f^{-1}.
\]
Let $\pi$ be a mapping from a Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to $I$. $\pi$ is said to be a independent random matching if:

- For any $\omega \in \Omega$, $\pi_\omega$ is a (deterministic) matching.
- Let
  $$g(i, \omega) = \begin{cases} 
  \alpha(\pi(i, \omega)) & \pi(i, \omega) \neq i \\
  J & \pi(i, \omega) = i,
  \end{cases}$$

  and $g$ is $\mathcal{I} \boxtimes \mathcal{F}$ measurable.
- For any type-$k$ agent $i \in I$, $P(g_i = l) = q_{kl}$.
- The process $g$ is essentially pairwise independent in the sense that for $\lambda$-almost all $i, j \in I$, $g_i$ and $g_j$ are independent.
Proposition (Duffie, Qiao and Sun, 2018)

(a) Let $\pi$ be an independent random matching with parameters $(p, q)$. Then, for $P$-almost every $\omega \in \Omega$, we have $\lambda(\{ i : \alpha(i) = k, g_\omega(i) = l \}) = p_k q_{kl}$ for any $k, l \in S$.

(b) For any given $(p, q)$, there exists an independent random matching.
In our most basic model:

- $S = \{1, \ldots, K\}$ is a set of finite types.
- $\alpha^0 : (I, I, \lambda) \rightarrow S$ is the initial a type function.
- $p^0$ is the distribution of $\alpha^0$. 
Continuous-Time Random Matching

- $\alpha(i, t)$ is the random type of agent $i$ at time $t$.
- $\varphi(i, t)$ is the last partner of agent $i$ up to time $t$.
- $h(i, t)$ is the type of the last partner of agent $i$ up to time $t$. 
Continuous-Time Random Matching

- $d^n_i$ is the $n$-th matching time for agent $i$.

- For any agent $i$ and any matching time $d^n_i$ for agent $i$, if $\varphi(i, d^n_i) = j$, then $\varphi(j, d^n_i) = i$ $P$-almost surely, or equivalently
  \[ \varphi(\varphi(i, d^n_i), d^n_i) = i \]

$P$-almost surely.
Continuous-Time Random Matching

- Mutation intensity \( \eta_{kl} \).

- Matching intensity \( \theta_{kl} : \Delta(S) \to \mathbb{R}_+ \), continuous, satisfying the balance identity
  \[ p_k \theta_{kl}(p) = p_l \theta_{lk}(p). \]

- Match-induced type changing probability distribution \( \gamma_{kl} \in \Delta(S) \).
Random Matching with Enduring Partnership

\[ \approx \delta \eta_{kl} \]

\[ \approx \delta \theta(p^t)_{kl} \]

mutation

match, type change
Fixing any parameters \((p^0, \eta, \theta, \gamma)\), there exists a continuous-time dynamical system with random mutation, matching and type changing such that

- The expected cross-sectional type distribution \(\bar{p}^t\) satisfies

\[
\frac{d\bar{p}^t}{dt} = \bar{p}^t R^t, \quad \bar{p}^0 = p^0,
\]

where

\[
R^t_{kl} = \eta_{kl} + \sum_{l=1}^{K} \theta_{kr}(\bar{p}^t) \gamma_{kr}(l)
\]

for any \(k \neq l\), and \(R^t_{kk} = -\sum_{l \in S, l \neq k} R^t_{kl}\).
Continuous-Time Random Matching: Theorem

- With probability one, the realized cross-sectional type $p_t^t$ is equal to the expected cross-sectional type distribution $\bar{p}^t$.

- $\{\alpha_i\}_{i \in I}$ forms a continuum of independent continuous-time Markov chains with transition intensity matrix $R^t$ at time $t$, where

$$R^t_{kl} = \eta_{kl} + \sum_{l=1}^K \theta_{kr}(\bar{p}^t)\gamma_{kr}(l)$$

for any $k \neq r$, and $R^t_{kk} = -\sum_{l \in S, l \neq k} R^t_{kl}$. 

Continuous-Time Random Matching: Theorem

- \{ (\alpha_i, h_i) \}_{i \in I} forms a continuum of independent continuous-time Markov chains.

- For any \((\eta, \theta, \gamma)\), there exists \(p^0\) such that with probability one, the realized cross-sectional type distribution \(p^t_\omega = p^0\) for all \(t \geq 0\).
Continuous-Time Random Matching: Theorem

Let $N_{ikl}(t)$ be the number of matches by agent $i$ up to time $t$, when of type $k$, to an agent of type $l$.

Then the cumulative total quantity $\Theta_{kl}(\omega, t)$ of matches can be defined as

$$\int_I N_{ikl}(\omega, t) \, d\lambda(i).$$

For $P$-almost all $\omega \in \Omega$, for any types $k$ and $l$, the cumulative total quantity $\Theta_{kl}(\omega, t)$ equals to its expectation $\mathbb{E}(\Theta_{kl}(t))$ and grows at the rate $\dot{\Theta}_{kl}(\omega, t) = \bar{p}_k \theta_{kl}(\bar{p}^t)$.
Random Matching with Enduring Partnership

\[ \xi_{kl} \]

$\beta_{kl}$

$\text{match}$

$\text{breakup}$

$t$ $s$ $T$
Transition Intensity Matrix for Extended Types

\{ (\alpha_i, g_i) \}_{i \in I} \) forms a continuum of independent continuous-time Markov chains with transition intensity matrix \( Q^t \) at time \( t \), where

\[
Q^t_{(k_1l_1)(k_2l_2)} = \eta_{k_1k_2} \delta_{l_1}(l_2) + \eta_{l_1l_2} \delta_{k_1}(k_2),
\]

\[
Q^t_{(k_1l_1)(k_2J)} = \beta_{k_1l_1} \gamma_{k_1l_1}(k_2),
\]

\[
Q^t_{(k_1J)(k_2l_2)} = \sum_{l_1=1}^{K} \theta_{k_1l_1}(\tilde{p}(t)) \xi_{k_1l_1} \sigma_{k_1l_1}(k_2, l_2),
\]

\[
Q^t_{(k_1J)(k_2J)} = \eta_{k_1k_2} + \sum_{l_1=1}^{K} \theta_{k_1l_1}(\tilde{p}(t))(1 - \xi_{k_1l_1}) \gamma_{k_1l_1}(k_2),
\]

\[
Q^t_{(kl)(kl)} = - \sum_{(k',l') \neq (k,l)} Q^t_{(kl')(k'l')},
\]

where \( \tilde{p}(t) = \mathbb{E} (\hat{p}(t)) \)
Compact Type Space

- $S$ is a compact metric space.

\[
Q^{t}_{(kl)}(\hat{A}) = \eta_k(\hat{A}_l) + \eta_l(\hat{A}^T_k) + \varrho^{S}_{kl}(\hat{A}_J)
\]

\[
Q^{t}_{(k, J)}(\hat{B}) = \eta_k(\hat{B}_J) + \int_{\nu \in S} \xi_{kl'} \sigma_{kl'}(\hat{B} \cap (S \times S))d\theta(k, \hat{p}(t))
\]

\[
+ \int_{\nu \in S} (1 - \xi_{kl'})\varsigma^{S}_{kl'}(\hat{B}_J)d\theta(k, \hat{p}(t))
\]

- \[
\frac{d\hat{p}^t}{dt} = \int_{\hat{S}} Q^{t}_{k, l}d\hat{p}^t, \quad \hat{p}^0 = \hat{p}^0.
\]
\section*{\(\sigma\)-Compact Type Space}

- \(S\) is a \(\sigma\)-compact metric space.
- **Boundedness:** \(\eta(\cdot)(S), \theta(\cdot, \cdot)(S), \vartheta(\cdot, \cdot)\) are bounded.
- **Compact Tightness:** For any \(t \in \mathbb{R}_+\), for any \(\epsilon > 0\) and any compact \(K \in S\), there exists a compact \(K'\) in \(S\), such that

\[
\eta_k(K') > 1 - \epsilon, \\
\sigma^S_{kl}(K') > 1 - \epsilon, \\
\varsigma_{kl}(K') > 1 - \epsilon
\]

for any \(k \in K, l \in S\).
Thanks!