Wavelet Frames and Differential Operators: Bridging Discrete and Continuum for Image Restoration

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Outlines

I. Brief review of image restoration models

II. Bridging wavelet frame transforms and differential operators under variational and PDE framework

III. Applications of B-spline wavelet frames in medical imaging
Image Restoration Model

- Image Restoration Problems

\[ f = Au + \eta \]
Image Restoration Model

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- Denoising, when \( A \) is identity operator
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- Deblurring, when \( A \) is some blurring operator
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- Deblurring, when \( A \) is some blurring operator
- Inpainting, when \( A \) is some restriction operator
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- Denoising, when \( A \) is identity operator
- Deblurring, when \( A \) is some blurring operator
- Inpainting, when \( A \) is some restriction operator
- CT/MR Imaging, when \( A \) is partial Radon/Fourier transform
Image Restoration Model

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- Challenges: large-scale & ill-posed
Image Restoration Models: A Quick Review

- Image restoration: \[ f = Au + \eta \]
Image Restoration Models: A Quick Review

- Image restoration: \( f = Au + \eta \)
- Variational and Optimization Models
- PDEs and Iterative Algorithms
Image Restoration Models: A Quick Review

- Image restoration: $f = Au + \eta$
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  \[
  \min_u \lambda R(u) + \|Au - f\|^2
  \]
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  - Total variation (TV) and generalizations:
    \[
    R(u) = \|\nabla u\|_1 \quad \text{or} \quad \|Du\|_1
    \]
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  - Wavelet frame based: \( R(u) = \|Wu\|_1 \) or \( \|Wu\|_0 \)

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  - Others: total generalized variation, low rank, NLM, BM3D, dictionary learning, etc.
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- PDEs and Iterative Algorithms
  - Perona-Malik equation, shock-filtering (Rudin & Osher), etc
  \[
  u_t = \sum_{\ell=1}^{L} \frac{\partial}{\partial x} \alpha_\ell \Phi_\ell(Du, u) - A^*(Au - f), \quad \text{with } D = \left( \frac{\partial \beta_1}{\partial x \beta_1}, \ldots, \frac{\partial \beta_L}{\partial x \beta_L} \right)
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    \]
  - Iterative shrinkage algorithm
    \[
    u^k = \left( W W^T \right) S_{\alpha^{k-1}}(W u^{k-1}) - A^T (Au^{k-1} - f), \quad k = 1, 2, \ldots
    \]
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    u^k = \widetilde{W}^\top S_{\alpha_{k-1}}(Wu^{k-1}) - A^\top(Au^{k-1} - f), \quad k = 1, 2, \ldots
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- What do they have in common?
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  u_t = \sum_{\ell=1}^{L} \frac{\partial}{\partial x} \alpha_\ell \Phi_\ell(Du, u) - \alpha^* (Au - f), \quad \text{with } D = \left( \frac{\partial \beta_1}{\partial x}, \ldots, \frac{\partial \beta_L}{\partial x} \right)
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  u^k = \widetilde{W}^T S_{\alpha^{k-1}}(W u^{k-1}) - A^T (Au^{k-1} - f), \quad k = 1, 2, \ldots
  \]
- What do they have in common?
  **Shrinkage** in **sparse** domain under **transformation!**
Bridging discrete and continuum

WAVELET FRAME TRANSFORMS AND DIFFERENTIAL OPERATORS

MRA-Based Tight Wavelet Frames

Refinable and wavelet functions

\[ \phi = 2^d \sum a_0[k] \phi(2 \cdot -k) \quad \psi_\ell = 2^d \sum a_\ell[k] \phi(2 \cdot -k), \quad \ell = 1, 2, \ldots, q. \]
MRA-Based Tight Wavelet Frames

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- **Unitary extension principle (UEP)**

$$\sum_{\ell=0}^{q} |\hat{a}_\ell(\xi)|^2 = 1 \quad \text{and} \quad \sum_{\ell=0}^{q} \hat{a}_\ell(\xi) \hat{a}_\ell(\xi + \nu) = 0,$$

$$\nu \in \{0, \pi\}^d \setminus \{0\} \quad \text{and} \quad \xi \in [-\pi, \pi]^d$$
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\[ \nu \in \{0, \pi \}_r \setminus \{0\} \text{ and } \xi \in [-\pi, \pi]^d \]

- Discrete 2D transformation: \( W_u = \{ W_{l,i}u : 0 \leq l \leq L - 1, 0 \leq i_1, i_2 \leq r \} \)

\[ W_{l,i}u := a_{l,i}[\cdot] \ast u, \]
MRA-Based Tight Wavelet Frames

Reinfaible and wavelet functions

$$\phi = 2^d \sum a_0[k] \phi(2 \cdot -k) \quad \psi_{\ell} = 2^d \sum a_{\ell}[k] \phi(2 \cdot -k), \quad \ell = 1, 2, \ldots, q.$$ 

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  $$\sum_{\ell=0}^{q} |\hat{a}_{\ell}(\xi)|^2 = 1 \quad \text{and} \quad \sum_{\ell=0}^{q} \hat{a}_{\ell}(\xi) \overline{\hat{a}_{\ell}(\xi + \nu)} = 0, \quad \nu \in \{0, \pi\}^d \setminus \{0\} \quad \text{and} \quad \xi \in [-\pi, \pi]^d$$

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  $$W_{l,i} u := a_{l,i}[\cdot] \ast u,$$

  $$a_i[k] := a_{i_1}[k_1] a_{i_2}[k_2], \quad 0 \leq i_1, i_2 \leq r; \quad (k_1, k_2) \in \mathbb{Z}^2.$$ 

  $$a_{l,i} = \tilde{a}_{l,i} \ast \tilde{a}_{l-1,0} \ast \ldots \ast \tilde{a}_{0,0} \quad \text{with} \quad \tilde{a}_{l,i}[k] = \begin{cases} a_i[2^{-l} k], & k \in 2^l \mathbb{Z}^2; \\ 0, & k \notin 2^l \mathbb{Z}^2. \end{cases}$$
MRA-Based Tight Wavelet Frames

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\[ \phi = 2^d \sum a_0[k] \phi(2 \cdot -k) \quad \psi_\ell = 2^d \sum a_\ell[k] \phi(2 \cdot -k), \quad \ell = 1, 2, \ldots, q. \]

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\[ a_i[k] := a_{i_1}[k_1] a_{i_2}[k_2], \quad 0 \leq i_1, i_2 \leq r; \quad (k_1, k_2) \in \mathbb{Z}^2. \]

\[ a_{l,i} = \tilde{a}_{l,i} \oplus \tilde{a}_{l-1,0} \oplus \ldots \oplus \tilde{a}_{0,0} \quad \text{with} \quad \tilde{a}_{l,i}[k] = \begin{cases} a_i[2^{-l} k], & k \in 2^l \mathbb{Z}^2; \\ 0, & k \notin 2^l \mathbb{Z}^2. \end{cases} \]

- Perfect reconstruction: \( W^T W = I \)

- Example: B-spline tight wavelet frames and many others
Connections: Motivation

- Difference operators in wavelet frame transform:

Haar Filters

\[ h_{0,1} = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \quad h_{1,0} = \frac{1}{4} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad h_{1,1} = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \]

Transform

\[ W u = \{ h_{0,1}[-\cdot] \ast u; \quad h_{1,0}[-\cdot] \ast u; \quad h_{1,1}[-\cdot] \ast u \} \]
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  Approximation
  \[ h_{0,1}[-\cdot] \ast u \approx \frac{1}{2} \delta u_x, \quad h_{1,0}[-\cdot] \ast u \approx \frac{1}{2} \delta u_y, \quad h_{1,1}[-\cdot] \ast u \approx \frac{1}{4} \delta^2 u_{xy} \]
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  Wavelet Transform: \[ W u = \{ h_{0,1}[-\cdot] \ast u; h_{1,0}[-\cdot] \ast u; h_{1,1}[-\cdot] \ast u \} \]

  Approximation: \[ h_{0,1}[-\cdot] \ast u \approx \frac{1}{2} \delta u_x, \quad h_{1,0}[-\cdot] \ast u \approx \frac{1}{2} \delta u_y, \quad h_{1,1}[-\cdot] \ast u \approx \frac{1}{4} \delta^2 u_{xy} \]

- Thus,

\[
\frac{2}{\delta} W u \approx \nabla u
\]

\[ |\nabla u| \approx \left( \frac{1}{4} \left[ (D_x^+ u_{i,j})^2 + (D_x^+ u_{i,j+1})^2 + (D_y^+ u_{i,j})^2 + (D_y^+ u_{i+1,j})^2 \right] \right)^{\frac{1}{2}}
\]

\[ + \left[ \frac{(D_x^+ u_{i,j} + D_y^- u_{i,j+1})^2}{4} + \frac{(D_x^+ u_{i,j} + D_y^+ u_{i+1,j})^2}{4} \right] \right)^{\frac{1}{2}} \]
Connections: Motivation

- Works for every tensor product B-spline wavelet frame transforms in a weaker setting [Cai, Dong, Osher and Shen, 2012]
- More general cases by [Dong, Xie and Shen, 2017; Choi, Dong and Zhang, 2017]

**Proposition.** Let a tensor product framelet function $\psi_\alpha \in L_2(\mathbb{R}^2)$ have vanishing moments of order $\alpha$ with $|\alpha| \leq s$, and let $\text{supp}(\psi_\alpha) = [a_1, a_2] \times [b_1, b_2]$. For $n \in \mathbb{N}$ and $k \in \mathbb{Z}^2$ with $\text{supp}(\psi_{\alpha,n-1,k}) \subseteq \bar{\Omega}$, we have

$$
\langle u, \psi_{\alpha,n-1,k} \rangle = (-1)^{|\alpha|} 2^{|\alpha|(1-n)} \langle \partial^\alpha u, \varphi_{\alpha,n-1,k} \rangle
$$

for every $u \in W^s_1(\Omega)$ with $\int \varphi_\alpha \neq 0$ and $\text{supp}(\varphi_\alpha) = \text{supp}(\psi_\alpha)$. 
Connections: Analysis Based Model and Variational Model

- [Cai, Dong, Osher and Shen, 2012]:

\[
\| \lambda \cdot Wu \|_1 + \frac{1}{2} \| Au - f \|_2^2 \quad \quad \lambda \| D(u) \|_1 + \frac{1}{2} \| Au - f \|_{L^2(\Omega)}^2
\]
Connections: Analysis Based Model and Variational Model

- [Cai, Dong, Osher and Shen, 2012]:

\[ \lambda W u_1 + \frac{1}{2} \| Au - f \|_2^2 \quad \text{Converges} \quad \lambda \| D(u) \|_1 + \frac{1}{2} \| Au - f \|_{L^2(\Omega)}^2 \]

For any differential operator when proper parameter is chosen.
Connections: Analysis Based Model and Variational Model

- [Cai, Dong, Osher and Shen, 2012]:

\[
\lambda \| W u \|_1 + \frac{1}{2} \| A u - f \|_2^2 \quad \text{Converges} \quad \lambda \| D(u) \|_1 + \frac{1}{2} \| A u - f \|_{L^2(\Omega)}^2
\]

For any differential operator when proper parameter is chosen.

**Theorem.** Let the objective functionals of the analysis based model and the variational model be \( E_n(u) \) and \( E(u) \) respectively, then:

1. \( E_n(u) \to E(u) \) for each \( u \in W^s_1(\Omega) \);
2. \( E_n(u_n) \to E(u) \) for every sequence \( u_n \to u \). Consequently, \( E_n \) \( \Gamma \)-converges to \( E \);
3. If \( u_n^* \) is an \( \epsilon \)-optimal solution to \( E_n \), i.e. \( E_n(u_n^*) \leq \inf_u E_n(u) + \epsilon \), then

\[
\limsup_{n} E_n(u_n^*) \leq \inf_u E(u) + \epsilon.
\]
Connections: Analysis Based Model and Variational Model

- [Cai, Dong, Osher and Shen, 2012]:

\[
\lambda \| Du \|_1 + \frac{1}{2} \| Au - f \|_2^2 \quad \text{Converges} \quad \lambda \| Du \|_1 + \frac{1}{2} \| Au - f \|_{L^2(\Omega)}^2
\]

For any differential operator when proper parameter is chosen.

- The connections give us
Connections: Analysis Based Model and Variational Model

- [Cai, Dong, Osher and Shen, 2012]:

\[
\lambda \| W u \|_1 + \frac{1}{2} \| Au - f \|_2^2 \quad \text{Converges} \quad \lambda \| D(u) \|_1 + \frac{1}{2} \| Au - f \|_{L^2(\Omega)}^2
\]

For any differential operator when proper parameter is chosen.

- The connections give us
  - Geometric interpretations of the wavelet frame transform (WFT)
Connections: Analysis Based Model and Variational Model

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\[
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  - Geometric interpretations of the wavelet frame transform (WFT)
  - WFT provides flexible and good discretization for differential operators
Connections: Analysis Based Model and Variational Model

- [Cai, Dong, Osher and Shen, 2012]:
  \[ \lambda \| W u \|_1 + \frac{1}{2} \| Au - f \|_2^2 \rightarrow \lambda \| D(u) \|_1 + \frac{1}{2} \| Au - f \|_{L_2(\Omega)}^2 \]

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![Graphs showing standard discretization and piecewise linear WFT](image)
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- The connections give us
  - Geometric interpretations of the wavelet frame transform (WFT)
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  - Different discretizations affect reconstruction results (or from modified equation perspective)
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  - Different discretizations affect reconstruction results (or from modified equation perspective)
  - Good regularization should contain differential operators with varied orders (e.g., total generalized variation [Bredies, Kunisch, and Pock, 2010])
Connections: Analysis Based Model and Variational Model

- [Cai, Dong, Osher and Shen, 2012]:

\[
\gamma W u_1 + \frac{1}{2} \| A u - f \|_2^2 \quad \text{Converges} \quad \gamma \| D(u)_1 + \frac{1}{2} \| A u - f \|_{L_2(\Omega)}^2
\]

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- The connections give us:
  - Geometric interpretations of the wavelet frame transform (WFT)
  - WFT provides flexible and good discretization for differential operators
  - Different discretizations affect reconstruction results (or from modified equation perspective)
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- Leads to new applications of wavelet frames:
  - Image segmentation: [Dong, Chien and Shen, 2010]
  - Surface reconstruction from point clouds: [Dong and Shen, 2011]
Relations: Wavelet Shrinkage and Nonlinear PDEs

- Earlier work
  - 2nd-order diffusion and Haar wavelet: [Mrazek, Weickert and Steidl, 2003&2005]
  - High-order diffusion and tight wavelet frames in 1D: [Jiang, 2011]
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- Questions yet to be answered
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  - How general the connections can be?
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  - Can we theoretically justify such connection, at least for some PDEs?
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- Questions yet to be answered
  - How general the connections can be?
  - Can we theoretically justify such connection, at least for some PDEs?
  - Can we see something new and useful from such connection?
Relations: Wavelet Shrinkage and Nonlinear PDEs

[Dong, Jiang and Shen, 2017]

\[ u^k = \tilde{W}^\top S_{\alpha^{k-1}}(W u^{k-1}), \quad k = 1, 2, \ldots \]

\[ u_t = \sum_{\ell=1}^{L} \frac{\partial \alpha_\ell}{\partial x^\alpha_\ell} \Phi_\ell(Du, u), \quad \text{with } Du = \left( \frac{\partial \beta_1}{\partial x^{\beta_1}}, \ldots, \frac{\partial \beta_L}{\partial x^{\beta_L}} \right) \]
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Relations: Wavelet Shrinkage and Nonlinear PDEs

[Dong, Jiang and Shen, 2017]

\[ u^k = \hat{W}^\top S_{\alpha_{k-1}}(W u^{k-1}), \quad k = 1, 2, \ldots \]

\[ u_t = \sum_{\ell=1}^{L} \frac{\partial \alpha_{\ell}}{\partial x_{\alpha_{\ell}}} \Phi_{\ell}(D u, u), \quad \text{with} \ D u = \left( \frac{\partial \beta_1}{\partial x_{\beta_1}}, \ldots, \frac{\partial \beta_L}{\partial x_{\beta_L}} \right) \]

Theoretical justification available for quasilinear parabolic equations.
Relations: Wavelet Shrinkage and Nonlinear PDEs

- [Dong, Jiang and Shen, 2017]

\[ u^k = \overrightarrow{W} S_{\alpha^{k-1}}(W u^{k-1}), \quad k = 1, 2, \cdots \]

\[ u_t = \sum_{\ell=1}^{L} \frac{\partial \alpha_{\ell}}{\partial x_{\alpha_{\ell}}} \Phi_{\ell}(Du, u), \quad \text{with} \quad Du = \left( \frac{\partial \beta_1}{\partial x_{\beta_1}}, \cdots, \frac{\partial \beta_L}{\partial x_{\beta_L}} \right) \]

- Theoretical justification available for quasilinear parabolic equations.
- Lead to new PDE models such as:
Relations: Wavelet Shrinkage and Nonlinear PDEs

[Dong, Jiang and Shen, 2017]

\[ u^k = \tilde{W}^\top S_{\alpha_k^{-1}} (W u^{k-1}), \quad k = 1, 2, \ldots \]

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- Theoretical justification available for quasilinear parabolic equations.
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\[ u_{tt} + Cu_t = \sum_{\ell=1}^{L} (-1)^{1 + |\beta_{\ell}|} \frac{\partial \beta_{\ell}}{\partial x_{\beta_{\ell}}} \left[ g_{\ell} \left( u, \frac{\partial \beta_1 u}{\partial x_{\beta_1}}, \ldots, \frac{\partial \beta_L u}{\partial x_{\beta_L}} \right) \frac{\partial \beta_{\ell}}{\partial x_{\beta_{\ell}}} u \right] - \kappa A^\top (Au - f) \]
Relations: Wavelet Shrinkage and Nonlinear PDEs

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\[ u^k = \hat{W}^\top S_{\alpha^{k-1}} (W u^{k-1}), \quad k = 1, 2, \ldots \]

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Lead to new wavelet frame shrinkage algorithms:
Relations: Wavelet Shrinkage and Nonlinear PDEs

- **[Dong, Jiang and Shen, 2017]**

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- Lead to new wavelet frame shrinkage algorithms:

\[ u^k = (I - \mu A^\top A) W^\top S_{\alpha^{k-1}}(W u^{k-1}) + \mu A^\top f \]

where

\[ S_{\alpha^{k-1}}(W u^{k-1}) = \left\{ S_{\alpha_{l},\ell,n}(W_l u^{k-1}) : 0 \leq l \leq \text{Lev} - 1, 1 \leq \ell \leq L \right\} \]

\[ S_{\alpha_{l},\ell,n}(d_1,n,d_2,n) = d_{\ell,n} \left( 1 - \frac{4\tau}{h^2} g \left( \frac{4(d_1,n)^2 + 4(d_2,n)^2}{h^2} \right) \right) \]
Relations: Wavelet Shrinkage and Nonlinear PDEs

- Advantage of the new models for image deblurring:
Relations: Wavelet Shrinkage and Nonlinear PDEs

- Advantage of the new models for image deblurring:
  - WFT v.s. standard finite differencing

<table>
<thead>
<tr>
<th>Image Name</th>
<th>PM-SD</th>
<th>PM-Haar</th>
<th>PM-Linear</th>
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<tbody>
<tr>
<td>Barbara</td>
<td>24.8097</td>
<td>24.9080</td>
<td>24.9625</td>
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<tr>
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<td>23.5915</td>
<td>23.6089</td>
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Advantage of the new models for image deblurring:

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- New models are generally better
Relations: Wavelet Shrinkage and Nonlinear PDEs

- Advantage of the new models for image deblurring:
  - WFT v.s. standard finite differencing
  - New models are generally better

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- Acceleration really works

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<th>AMT-Haar</th>
<th>AMT-Linear</th>
<th>AST-Haar</th>
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<th>A-IRED-FD</th>
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Further Developments – Piecewise Smooth Model

- Piecewise-smooth image restoration model: analysis and applications. [Cai, Dong and Shen, 2016]
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\[
\inf_{u, \Gamma} \| [\lambda \cdot W u]_{\Gamma^c} \|^2_2 + \| [\gamma \cdot W u]_{\Gamma} \|^1_1 + \frac{1}{2} \| A u - f \|^2_2
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\]

\[
\inf_{u \in H^{1,s}(\Omega_{j,j}), \{\Gamma_{j}, \{\Gamma_{j,j}\}}} \left\| \nu \cdot D u \right\|_{2}^2 + \sum_{j=1}^{m} \left[ \mu_1 \int_{\Gamma_j} \left| \xi_j^+(u) - \xi_j^-(u) \right| ds \right. \\
+ \left. \mu_2 \sum_{j=1}^{\tilde{m}_j} \int_{\Gamma_{j,\bar{j}}} \left( \sum_{i=1}^{n} \left| \xi_{j,\bar{j}}^+(D_i u) - \xi_{j,\bar{j}}^-(D_i u) \right|^2 \right)^{\frac{1}{2}} ds \right] + \frac{1}{2} \left\| A u - f \right\|_{L^2(\Omega)}^2
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\]

Mumford-Shah Functional

\[
\nu \int_{\Omega \setminus \Gamma} | \nabla u |^2 + \mu | \Gamma | + \frac{1}{2} \| u - f \|_{L^2(\Omega)}^2
\]
Further Developments – Piecewise Smooth Model

- Analysis based model
  \[ \text{PSNR}=31.72 \]

- Piecewise smooth model
  \[ \text{PSNR}=34.27 \]
Further Developments – General Frame Based Model

- A general wavelet frame based image restoration model. [Dong, Shen and Xie, 2017]
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\[
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- Special case I: balanced model

\[
\inf_{v} \left\{ a\|(Id - WW^T)v\|^{2}_{\ell_2(O)} + b\|v\|_{\ell_1(O)} + \frac{1}{2}\|AW^Tv - f\|^{2}_{\ell_2(O)} \right\}
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- Special case II: wavelet packet model

$$\inf_{u_1,u_2} \left\{ a\|Wu_1\|_{l_1(o)} + b\|W^2u_2\|_{l_1(o)} + \frac{1}{2}\|A(u_1 + u_2) - f\|_{l_2(o)}^2 \right\}$$
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\]

- **Gamma-limit** of the energy functional:

\[
\nu_1 \| D' u - v \|_{L^p(\Omega;\ell_2)} + \nu_2 \| D'' v \|_{L^q(\Omega;\ell_2)} + \frac{1}{2} \| A u - f \|_{L^2(\Omega)}^2
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- TGV(2) is a special case
Summary

Continuum
Variational Model
PDEs

Discrete
Optimization
Iterative Algorithms
Summary

Continuum

Variational Model

PDEs

Discrete

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Summary


Significance of the series of studies:

- Merging of the two separately developed fields for the first time
- PDE/Variational models also have the property of sparse approximation
- Illustration of “redundancy is good” from PDE/variational viewpoint
- Granting geometric interpretations to wavelet/wavelet frame models
- Giving birth to new and effective models
APPLICATION IN MEDICAL IMAGING

Take full advantage of sparse approximation of B-spline tight wavelet frames
Workflow of Medical Image Analysis

Medical Problem:
- Imaging
- Diagnosis
- Treatment

Modeling:
- Large scale ill-posed inverse problems
- Segmentation
- Registration
- Knowledge Extraction
- Optimization
CT image reconstruction: safer imaging

- B-spline wavelet frame based algorithm with GPU speed-up
  [Jia, Dong, Lou, and Jiang, PMB, 2011]

Image size: 512 x 512 x 70; Time: 2 min and 20 sec
CT image reconstruction: safer imaging

- B-spline wavelet frame based algorithm with GPU speed-up
  【Jia, Dong, Lou, and Jiang, PMB, 2011】

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- B-spline wavelet frame based spatial-Radon reconstruction model 【Dong, Li and Shen, JSC, 2013】
Imaging

- CT image reconstruction: safer imaging
  - Data-driven spatial-Radon reconstruction model 【Zhan and Dong, SIIMS, 2016】

![Non data-driven vs. Data-driven CT images](image-url)
Imaging

- **CT image reconstruction: safer imaging**
  - Data-driven spatial-Radon reconstruction model  
    【Zhan and Dong, SIIMS, 2016】

- **B-spline wavelet frame based metal artifact reduction**
  【Zhang, Dong, Liu, SIIMS, 2017】
Segmentation

- Segmentation for accurate quantification and diagnosis of abnormalities

- B-spline wavelet frame based variational model for medical image segmentation 【Dong, Chien and Shen, CMS, 2010】
Segmentation

- Segmentation for accurate quantification and diagnosis of abnormalities
  - B-spline wavelet frame based variational model for ultrasound video segmentation with shape priors [Liu, Zhang, Dong, Shen and Gu, SIIMS, 2016]

---

**Chan-Vese**

**Our model**

**Radiologist**
Segmentation

- Segmentation for accurate quantification and diagnosis of abnormalities
  - B-spline wavelet frame based variational model for ultrasound video segmentation with shape priors 【Liu, Zhang, Dong, Shen and Gu, SIIMS, 2016】
Conclusion

- Established generic connection between wavelet frame transform and differential operators
- Obtained unified view of the models and algorithms in image restoration
- Give rise to new insights and new models
- Successful applications in medical imaging
- Extension to challenging tasks in data science – combining applied math with deep learning
Happy Birthday Professor Carl de Boor!