K-SEMISTABLE FANO MANIFOLDS WITH THE SMALLEST ALPHA INVARIANT

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Abstract. In this short note, we show that K-semistable Fano manifolds with the smallest alpha invariant are projective spaces. Singular cases are also investigated.

1. INTRODUCTION

Throughout the article, we work over the complex number field \( \mathbb{C} \). A \( \mathbb{Q} \)-Fano variety is a normal projective variety \( X \) with log terminal singularities such that the anti-canonical divisor \( -K_X \) is an ample \( \mathbb{Q} \)-Cartier divisor. It has been known that a Fano manifold (i.e., a smooth \( \mathbb{Q} \)-Fano variety) admits Kähler–Einstein metrics if and only if \( X \) is K-polystable by the works [DT92, Tia97, Don02, CT08, Sto09, Mab08, Mab09, Ber16] and [CDS15a, CDS15b, CDS15c, Tia15].

On the other hand, the existence of Kähler–Einstein metrics and K-stability are related to the alpha invariants \( \alpha(X) \) of \( X \) defined by Tian [Tia87] (see also [TY87, Zel98, Lu00, Dem08]). Tian [Tia87] proved that for a Fano manifold \( X \), if \( \alpha(X) > \dim X/(\dim X + 1) \), then \( X \) admits Kähler–Einstein metrics. Odaka and Sano [OS12, Theorem 1.4] (see also its generalizations [Der16, BHJ15, FO16, Fuj16c]) proved a variant of Tian’s theorem: if a \( \mathbb{Q} \)-Fano variety \( X \) satisfies that \( \alpha(X) > \dim X/(\dim X + 1) \) (resp. \( \geq \dim X/(\dim X + 1) \)), then \( X \) is K-stable (resp. K-semistable). We are interested in the relation of alpha invariants and K-semistability.

Recall that Fujita and Odaka proved that there exists a lower bound of alpha invariants for K-semistable \( \mathbb{Q} \)-Fano varieties.

**Theorem 1.1** ([FO16, Theorem 3.5]). Let \( X \) be a K-semistable \( \mathbb{Q} \)-Fano variety of dimension \( n \).

Then \( \alpha(X) \geq \frac{1}{n+1} \).

It is natural and interesting to ask when the equality holds. For example, it is well-known that \( \mathbb{P}^n \) is K-semistable with \( \alpha(\mathbb{P}^n) = \frac{1}{n+1} \). The main theorem of this paper is the following.

**Theorem 1.2.** Let \( X \) be a K-semistable Fano manifold of dimension \( n \).

Then \( \alpha(X) = \frac{1}{n+1} \) if and only if \( X \cong \mathbb{P}^n \).

This is an application of Birkar’s answer to Tian’s question [Bir16, Theorem 1.5], and Fujita–Li’s criterion for K-semistability [Li15, Fuj16b].
It is natural to ask whether the same statement holds true for K-semistable \( \mathbb{Q} \)-Fano varieties instead of manifolds. However, this is no longer true even in dimension 2. We are grateful to Kento Fujita for kindly providing the following example:

**Example 1.3.** Consider the cubic surface \( X = (x_0^3 = x_1x_2x_3) \subseteq \mathbb{P}^3 \), which is a toric log del Pezzo surface (i.e., a \( \mathbb{Q} \)-Fano variety of dimension 2) with 3 du Val singularities of type \( A_2 \). On one hand, it is well-known that \( X \) admits a Kähler–Einstein metric (cf. [DT92]), hence is K-semistable. On the other hand, \( \alpha(X) = \frac{1}{3} \) (cf. [PW10]).

In fact, by the classification of possible du Val singularities of K-semistable log del Pezzo surfaces (cf. [Liu16, Corollary 6]) and explicit computation of alpha invariants (cf. [Par03, PW10, CK14]), we have the following theorem.

**Theorem 1.4.** Let \( X \) be a K-semistable log del Pezzo surface with at worst du Val singularities. Then \( \alpha(X) = \frac{1}{3} \) if and only if \( X \cong \mathbb{P}^2 \) or \( X \subseteq \mathbb{P}^3 \) is a cubic surface with at least 2 singularities of type \( A_2 \).

Moreover, by classification of \( \mathbb{Q} \)-Fano 3-fold with \( \mathbb{Q} \)-factorial terminal singularities and \( \rho(X) = 1 \) with large Fano index due to Prokhorov [Pro10, Pro13], we prove the following:

**Theorem 1.5.** Let \( X \) be a K-semistable \( \mathbb{Q} \)-Fano 3-fold with \( \mathbb{Q} \)-factorial terminal singularities and \( \rho(X) = 1 \). Assume that \( h^0(-K_X) \geq 22 \). Then \( \alpha(X) = \frac{1}{4} \) if and only if \( X \cong \mathbb{P}^3 \).

Finally, we propose the following much stronger conjecture. For some evidence in dimension 3, we refer to [CS08] and [Fuj16a].

**Conjecture 1.6.** Let \( X \) be a K-semistable Fano manifold. Then \( \alpha(X) < \frac{1}{n} \) if and only if \( X \cong \mathbb{P}^n \).

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2. Preliminaries

We adopt the standard notation and definitions in [KM98] and will freely use them.

**Definition 2.1.** Let \( X \) be a \( \mathbb{Q} \)-Fano variety. The *alpha invariant* \( \alpha(X) \) of \( X \) is defined by the supremum of positive rational numbers \( \alpha \) such that the pair \((X, \alpha D)\) is log canonical for any effective \( \mathbb{Q} \)-divisor \( D \) with \( D \sim \mathbb{Q} - K_X \).

In other words,

\[
\alpha(X) := \inf \{ \text{lct}(X; D) | 0 \leq D \sim \mathbb{Q} - K_X \}.
\]

Tian [Tia90] asked whether whether the infimum is a minimum, which is answered by Birkar affirmatively.
**Theorem 2.2** ([Bir16, Theorem 1.5]). Let $X$ be a $\mathbb{Q}$-Fano variety. Assume that $\alpha(X) \leq 1$. Then there exists an effective $\mathbb{Q}$-divisor $D$ such that $D \sim_{\mathbb{Q}} -K_X$ and $\text{lct}(X; D) = \alpha(X)$.

**Definition 2.3** ([Fuj16b]). Let $X$ be a $\mathbb{Q}$-Fano variety of dimension $n$. Take any projective birational morphism $\sigma : Y \to X$ with $Y$ normal and any prime divisor $F$ on $Y$, that is, $F$ is a prime divisor over $X$.

1. Define the log discrepancy of $F$ as $A(F) := \text{mult}_F(K_Y - \sigma^*K_X) + 1$;
2. Define $\text{vol}_X(-K_X - xF) := \text{vol}_Y(-\sigma^*K_X - xF)$;
3. Define $\beta(F) := A(F) \cdot (-K_X)^n - \int_0^{\infty} \text{vol}_X(-K_X - xF) \, dx$.

Note that the definitions do not depend on the choice of birational model $Y$.

Instead of recalling the original definition, we use the following criterion to define K-semistability.

**Definition-Proposition 2.4** ([Fuj16b, Corollary 1.5], [Li15, Theorem 3.7]). Let $X$ be a $\mathbb{Q}$-Fano variety. $X$ is $K$-semistable if $\beta(F) \geq 0$ for any divisor $F$ over $X$.

Note that K-semistability is known to be equivalent to Ding-semistability by [BBJ15].

### 3. Proof of main theorem

**Proposition 3.1.** Let $X$ be a $K$-semistable $\mathbb{Q}$-Fano variety of dimension $n$. Assume that $\alpha(X) = \frac{1}{n+1}$, then there exists a prime divisor $E$ on $X$ such that $-K_X \sim_{\mathbb{Q}} (n+1)E$ and $(X, E)$ is plt.

**Proof.** Let $X$ be a K-semistable $\mathbb{Q}$-Fano variety of dimension $n$ with $\alpha(X) = \frac{1}{n+1}$. By Theorem 2.2, there is a divisor $D \sim_{\mathbb{Q}} -K_X$ such that $\text{lct}(X; D) = \frac{1}{n+1}$. Take $F$ to be a non-klt place of $(X, \frac{1}{n+1}D)$, then there is a resolution $\sigma : Y \to X$ such that $F$ is a divisor on $Y$.

Denote $\mu$ to be the multiplicity of $F$ in $\sigma^*D$. Note that $\mu > 0$ since $X$ is klt. By assumption,

$$\text{mult}_F \left( K_Y - \sigma^* \left( K_X + \frac{1}{n+1}D \right) \right) = -1,$$

which means that

$$A(F) = \frac{\mu}{n+1}.$$  

By Definition-Proposition 2.4, $\beta(F) \geq 0$, which means that

$$\frac{1}{n+1}(-K_X)^n = \frac{A(F)}{\mu} (-K_X)^n$$

$$\geq \frac{1}{\mu} \int_0^{\infty} \text{vol}_X(-K_X - xF) \, dx$$

$$= \int_0^{\infty} \text{vol}_X(-K_X - x\mu F) \, dx.$$
\[
\geq \int_0^\infty \text{vol}_X(-K_X - xD) \, dx
= \int_0^1 (1 - x)^n(-K_X)^n \, dx
= \frac{1}{n+1}(-K_X)^n.
\]

The second equality holds since \( \sigma^*D \geq \mu F \). Hence all inequalities become equalities. In particular,

\[
\text{vol}_X(-K_X - x\mu F) = \text{vol}_X(-K_X - xD) = (1 - x)^n(-K_X)^n
\]

for almost all \( x \). By differentiability of volume functions ([BFJ09, Corollary C]),

\[
\mu \cdot \text{vol}_{Y|F}(-\sigma^*K_X)
= -\frac{1}{n} \left. \frac{d}{dx} \text{vol}_Y(-\sigma^*K_X - x\mu F) \right|_{x=0}
= -\frac{1}{n} \left. \frac{d}{dx} (1 - x)^n(-K_X)^n \right|_{x=0}
= (-K_X)^n.
\]

Here \( \text{vol}_{Y|F} \) is the restricted volume, we refer to [ELMNP09] for definition and properties. Since \( \text{vol}_{Y|F}(-\sigma^*K_X) > 0 \), \( F \not\subseteq B_{\infty}(-\sigma^*K_X) \) by [ELMNP09, Theorem C]. Hence by [ELMNP09, Corollary 2.17],

\[
\text{vol}_{Y|F}(-\sigma^*K_X) = (-\sigma^*K_X)^{n-1} \cdot F = (-K_X)^{n-1} \cdot \sigma_*F.
\]

In other words, we have

\[
(-K_X)^{n-1}(D - \mu \sigma_*F) = (-K_X)^n - \mu \cdot \text{vol}_{Y|F}(-\sigma^*K_X) = 0.
\]

This implies that \( D = \mu \sigma_*F \) since \( D \geq \mu \sigma_*F \) and \(-K_X\) is ample. In particular, \( F \) is not \( \sigma \)-exceptional and \( \sigma_*F \) is a prime divisor on \( X \). Denote \( E := \sigma_*F \). Moreover, since \( F \) is a non-klt place of \( (X, \frac{1}{n+1}D) \), \( \text{mult}_E \frac{1}{n+1}D = 1 \), that is, \( \mu = n + 1 \). In particular, \(-K_X \sim_{\mathbb{Q}} D = (n+1)E \). Finally, this argument shows that \( F \) is the only non-klt place of \( (X, E) \), which means that \( (X, E) \) is plt. \( \square \)

**Corollary 3.2.** Let \((X, E)\) as in Proposition 3.1. Then \( X \simeq \mathbb{P}^n \) if one of the following condition holds:

1. \( X \) is factorial;
2. \((E)^n \geq 1\);
3. \( E \) is Cartier in codimension two and \( E \simeq \mathbb{P}^{n-1} \).

**Proof.** (1) If \( X \) is factorial, then \( E \) is a Cartier divisor. In particular, \((E)^n \geq 1\). Hence this is a special case of (2).

(2) If \((E)^n \geq 1\), then

\[
(-K_X)^n = (n+1)^n(E)^n \geq (n+1)^n.
\]

By [Liu16, Theorem 1.1] or [LZ16, Theorem 9], \( X \simeq \mathbb{P}^n \).
A cubic surface with at worst singularities of type by Theorems 4.1 and 4.2. To see the “if” part, one just notice that any (cf. [OSS16, Theorem 4.3]). □

by Theorem 1.2. If $X$ is K-semistable log del Pezzo surface with at worst du Val singularities. Proof of Theorem 1.2. It follows directly from Proposition 3.1 and Corollary 3.2(1) (or [KO73]).

Recall the following theorem on classification of possible du Val singularities of a K-semistable log del Pezzo surface.

**Theorem 4.1** ([Liu16, Theorem 23, Proof of Corollary 6]). Let $X$ be a K-semistable log del Pezzo surface with at worst du Val singularities.

1. If $(-K_X)^2 = 1$, then $X$ has at worst singularities of type $A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8,$ or $D_4$;
2. If $(-K_X)^2 = 2$, then $X$ has at worst singularities of type $A_1, A_2$, or $A_3$;
3. If $(-K_X)^2 = 3$, then $X$ has at worst singularities of type $A_1$ or $A_2$;
4. If $(-K_X)^2 = 4$, then $X$ has at worst singularities of type $A_1$;
5. If $(-K_X)^2 \geq 5$, then $X$ is smooth.

We remark that in [Liu16, Corollary 6], log del Pezzo surfaces are assumed to be admitting Kähler–Einstein metrics, but the proof works well for K-semistable log del Pezzo surfaces. The only part that the existence of Kähler–Einstein metrics is needed is to exclude the case that $(-K_X)^2 = 1$ and $X$ has singularities of type $A_8$.

Recall the following theorem on explicit computation of alpha invariants.

**Theorem 4.2** ([Par03], [PW10, Theorems 1.4, 1.5, and 1.6], [CK14, Theorem 1.26, Example 1.27]). Let $X$ be a log del Pezzo surface with at worst du Val singularities. Assume that $X$ is singular, then $\alpha(X) = \frac{1}{3}$ if and only if one of the following holds:

1. $(-K_X)^2 = 6$ and $\text{Sing}(X) = \{A_1\}$;
2. $(-K_X)^2 = 5$ and $\text{Sing}(X) = \{A_2\}$ or $\{2A_1\}$;
3. $(-K_X)^2 = 4$ and $\text{Sing}(X) = \{A_3\}$ or $\text{Sing}(X) \supseteq \{A_1 + A_2\}$;
4. $(-K_X)^2 = 3$ and $\text{Sing}(X) \supseteq \{A_4\}, \{2A_2\}$, or $\text{Sing}(X) = \{D_4\}$;
5. $(-K_X)^2 = 2$ and $\text{Sing}(X) \supseteq \{D_5\}, \{A_5\'}, \{A_7\}$;
6. $(-K_X)^2 = 1$ and $\text{Sing}(X) \supseteq \{D_9\}$ or $\{E_6\}$.

**Proof of Theorem 1.4.** Let $X$ be a K-semistable log del Pezzo surface with at worst du Val singularities and $\alpha(X) = \frac{1}{3}$. If $X$ is smooth, then $X \simeq \mathbb{P}^2$ by Theorem 1.2. If $X$ is singular, then $(-K_X)^2 = 3$ and $\text{Sing}(X) \supseteq \{2A_2\}$ by Theorems 4.1 and 4.2. To see the “if” part, one just notice that any cubic surface with at worst singularities of type $A_1$ or $A_2$ is K-semistable (cf. [OSS16, Theorem 4.3]). □

4. Singular surfaces

Recall the following theorem on classification of possible du Val singularities of a K-semistable log del Pezzo surface.

**Theorem 4.1** ([Liu16, Theorem 23, Proof of Corollary 6]). Let $X$ be a K-semistable log del Pezzo surface with at worst du Val singularities.

1. If $(-K_X)^2 = 1$, then $X$ has at worst singularities of type $A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8,$ or $D_4$;
2. If $(-K_X)^2 = 2$, then $X$ has at worst singularities of type $A_1, A_2$, or $A_3$;
3. If $(-K_X)^2 = 3$, then $X$ has at worst singularities of type $A_1$ or $A_2$;
4. If $(-K_X)^2 = 4$, then $X$ has at worst singularities of type $A_1$;
5. If $(-K_X)^2 \geq 5$, then $X$ is smooth.

We remark that in [Liu16, Corollary 6], log del Pezzo surfaces are assumed to be admitting Kähler–Einstein metrics, but the proof works well for K-semistable log del Pezzo surfaces. The only part that the existence of Kähler–Einstein metrics is needed is to exclude the case that $(-K_X)^2 = 1$ and $X$ has singularities of type $A_8$.

Recall the following theorem on explicit computation of alpha invariants.

**Theorem 4.2** ([Par03], [PW10, Theorems 1.4, 1.5, and 1.6], [CK14, Theorem 1.26, Example 1.27]). Let $X$ be a log del Pezzo surface with at worst du Val singularities. Assume that $X$ is singular, then $\alpha(X) = \frac{1}{3}$ if and only if one of the following holds:

1. $(-K_X)^2 = 6$ and $\text{Sing}(X) = \{A_1\}$;
2. $(-K_X)^2 = 5$ and $\text{Sing}(X) = \{A_2\}$ or $\{2A_1\}$;
3. $(-K_X)^2 = 4$ and $\text{Sing}(X) = \{A_3\}$ or $\text{Sing}(X) \supseteq \{A_1 + A_2\}$;
4. $(-K_X)^2 = 3$ and $\text{Sing}(X) \supseteq \{A_4\}, \{2A_2\}$, or $\text{Sing}(X) = \{D_4\}$;
5. $(-K_X)^2 = 2$ and $\text{Sing}(X) \supseteq \{D_5\}, \{A_5\'}, \{A_7\}$;
6. $(-K_X)^2 = 1$ and $\text{Sing}(X) \supseteq \{D_9\}$ or $\{E_6\}$.

**Proof of Theorem 1.4.** Let $X$ be a K-semistable log del Pezzo surface with at worst du Val singularities and $\alpha(X) = \frac{1}{3}$. If $X$ is smooth, then $X \simeq \mathbb{P}^2$ by Theorem 1.2. If $X$ is singular, then $(-K_X)^2 = 3$ and $\text{Sing}(X) \supseteq \{2A_2\}$ by Theorems 4.1 and 4.2. To see the “if” part, one just notice that any cubic surface with at worst singularities of type $A_1$ or $A_2$ is K-semistable (cf. [OSS16, Theorem 4.3]). □
5. Singular threefolds

In this section, we prove Theorem 1.5. Recall the following theorem on the upper bound of volumes.

**Theorem 5.1** (cf. [Liu16, Theorem 25]). Let $X$ be a $K$-semistable $\mathbb{Q}$-Fano 3-fold with at worst terminal singularities. Let $p \in X$ be an isolated singularity with local index $r$. Then

$$(-K_X)^3 \leq \frac{(r+2)(4+4r)^3}{(3r)^3}. $$

**Proof.** Denote by $m_p$ the maximal ideal at $p$. We may take a log resolution of $(X, m_p)$, namely $\pi : Y \to X$ such that $\pi$ is an isomorphism away from $p$ and $\pi^{-1}m_p \cdot O_Y$ is an invertible ideal sheaf on $Y$. Let $E_i$ be exceptional divisors of $\pi$. We define the numbers $a_i$ and $b_i$ by

$$K_Y = \pi^*K_X + \sum a_i E_i$$

and

$$\pi^{-1}m_p \cdot O_Y = O_Y(-\sum b_i E_i).$$

It is clear that $lct(X; m_p) = \min_i 1 + \frac{a_i}{b_i}$. Since $\pi$ is an isomorphism away from $p$, we have $b_i \geq 1$ for any $i$. Since $X$ is terminal at $p$, by [Kaw93], there exists an index $i_0$ such that $a_{i_0} = \frac{1}{r}$. Hence

$$\text{lct}(X; m_p) \leq \frac{1 + a_{i_0}}{b_{i_0}} \leq 1 + \frac{1}{r}. $$

On the other hand, by [Kak00] (see also [TW04, Proposition 3.10]), $\text{mult}_p X \leq r + 2$. Hence by [Liu16, Theorem 16],

$$(-K_X)^3 \leq \left(1 + \frac{1}{3}\right)^3 \text{lct}(X; m_p)^3 \text{mult}_p X \leq \frac{(r+2)(4+4r)^3}{(3r)^3}. $$

$\square$

Now let $X$ be a $K$-semistable $\mathbb{Q}$-Fano 3-fold with $\mathbb{Q}$-factorial terminal singularities and $\rho(X) = 1$ with $\alpha(X) = \frac{1}{4}$. By Proposition 3.1, there exists a prime divisor $E$ on $X$ such that $-K_X \sim_{\mathbb{Q}} 4E$.

Recall that we may define ([Pro10])

$$qW(X) := \max\{q \mid -K_X \sim qA, A \text{ is a Weil divisor}\},$$

$$q\mathbb{Q}(X) := \max\{q \mid -K_X \sim_{\mathbb{Q}} qA, A \text{ is a Weil divisor}\}. $$

It is known by [Suz04, Pro10] that

$$qW(X), q\mathbb{Q}(X) \in \{1, \ldots, 11, 13, 17, 19\}. $$

Moreover, by [Pro10, Lemma 3.2], in our case, $4q\mathbb{Q}(X)$. Hence there are 2 cases: (i) $q\mathbb{Q}(X) = 8$; (ii) $q\mathbb{Q}(X) = 4$.

Now assume that $h^0(-K_X) \geq 22$. Define the genus $g(X) := h^0(-K_X) - 2 \geq 20$.

If $q\mathbb{Q}(X) = 8$, since $g(X) > 10$, then by [Pro13, Theorem 1.2(ii)], either $X \simeq X_6 \subset \mathbb{P}(1,2,3,3,5)$ or $X \simeq X_{10} \subset \mathbb{P}(1,2,3,5,7)$. But in either case,
$-K_X \sim 8A$ where $A$ is an effective divisor, which implies that $\alpha(X) \leq \frac{1}{8}$ since $(X, A)$ is not klt, a contradiction.

Now assume that $q\mathcal{Q}(X) = 4$, by [Pro13, Lemma 8.3], $\text{Cl}(X)$ is torsion-free and $qW(X) = q\mathcal{Q}(X)$, hence there is a Weil divisor $A$ such that $-K_X \sim 4A$. If $g(X) \geq 22$, then by [Pro13, Theorem 1.2(vi)], $X \cong \mathbb{P}^3$ or $X_4 \subset \mathbb{P}(1, 1, 1, 2, 3)$. The latter is absurd, since it has a singularity of index 3, and $(-K_{X_4})^3 = 128/3$, which contradicts to Theorem 5.1. If $20 \leq g(X) \leq 21$, then we have the following possibilities due to computer computation (see [GRD], or [BS07, Pro10, Pro13]):

<table>
<thead>
<tr>
<th>$g(X)$</th>
<th>$\mathbf{B}$</th>
<th>$A^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>21</td>
<td>${3}$</td>
<td>$2/3$</td>
</tr>
<tr>
<td>20</td>
<td>${5, 7}$</td>
<td>$22/35$</td>
</tr>
</tbody>
</table>

Here $\mathbf{B}$ is the set local indices of singular points. It is easy to see that both cases contradict to Theorem 5.1.

In summary, Theorem 1.5 is proved.

**References**


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