STANDARD MONOMIAL THEORY FOR HARMONICS IN CLASSICAL INVARIANT THEORY

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1. Introduction

The goal of this paper is to establish a standard monomial theory for the harmonic polynomials for a classical action in the sense of Weyl [Wy]. This provides a natural generalization of the theory of double tableaux developed in the 1980s for GL_n by Desarmenian-Kung-Rota [DKR] and DeConcini-Procesi [DP]. Moreover, it can be combined with our next paper [HKL2] or Lakshmibai et. al.'s [LRSS, LaSh] description of standard monomials for the invariants of a classical action to give a standard monomial theory for the full polynomial ring in the stable range [Ho2]. We will begin by providing a background for the current work.

Standard monomial theory (SMT), which originated with W.V.D. Hodge in the early 1940s [Hod], was generalized by Seshadri and co-workers (see [La1, Mu] and the references therein), and has come to be seen as a widely usable tool in representation theory ([BL, GL1, GL2, LaS, LaR, LLM1, LaSh, Stu, KM, Ki1, Ki2, Ki3, KL, KY, HL, HKL1]). Hodge's theory can be summarized as giving an explicit basis for a natural subalgebra of the flag algebra for GL_n.

We should explain this statement in a little more detail. Consider the standard maximal unipotent subgroup U_n of GL_n, consisting of the unipotent upper triangular matrices. The action by left translation of GL_n on the coset space GL_n/U_n defines a representation of GL_n on the algebra $R(GL_n/U_n)$ of regular functions on GL_n/U_n. Thanks to highest

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weight theory [GW], $R(GL_n/U_n)$ is known to contain exactly one copy of each irreducible (rational) representation of GL_n. We will call it the *flag algebra* of GL_n [1].

Among the irreducible representations of GL_n are ones which can be realized in the algebra $P(M_{nn})$ of polynomials on the $n \times n$ matrices—the so-called *polynomial representations*. The span of the polynomial representations inside $R(GL_n/U_n)$ is a subalgebra $R^+(GL_n/U_n)$—the *polynomial flag algebra* [2], Hodge's SMT describes a basis for $R^+(GL_n/U_n)$. This basis is in fact the union of standard bases for a finite number of polynomial subrings of $R^+(GL_n/U_n)$. Thus, at least implicitly, SMT also describes $R^+(GL_n/U_n)$ as a sum of polynomial subrings.

The abstract structure behind SMT was the subject of study in the latter half of the 20th century, culminating in the formulation of the notion of *algebra with straightening law* (ASL), [Eis1, DEP], and the identification by T. Hibi [Hi] of the key underlying structure, now often called a *Hibi ring*. A Hibi ring is constructed from a finite partially ordered set (aka *poset*). Precisely, it is the semigroup ring on the semigroup consisting of the non-negative, integer-valued, order-preserving functions on a poset X. We denote this semigroup by $(\mathbb{Z}^+)^{(X,\geq)}$, where $\geq$ indicates the partial ordering on X [3]. The corresponding semigroup ring is

$$R_H(X) = R_H(X, \geq) = \mathbf{C}((\mathbb{Z}^+)^{(X,\geq)}). \tag{1.1}$$

We call it the *Hibi ring* attached to the poset $(X, \geq)$. Hibi rings in some sense constitute the first level of complexity beyond polynomial rings (which can be thought of as the Hibi rings on clutters (posets with a trivial partial ordering), or alternatively, as the Hibi rings on totally ordered sets). A Hibi ring has a canonical set of generators, quadratic defining relations, and is spanned by a finite sum of polynomial rings [Ho3, Re].

Following [GL1, GL2] (see also [KM, Ki1]), we can now summarize Hodge's SMT as saying that the algebra $R^+(GL_n/U_n)$, the (polynomial subalgebra of) the flag algebra of GL_n, is a flat deformation ([BH, CHV]) of the Hibi ring attached to the Gelfand-Tsetlin (GT) poset Γ_{GL_n} [GT, GW, HL]. See Figure 1 and Figure 2 for illustrations of Γ_{GL_n}. More specifics of Hodge's SMT are given in Section 3.

It has turned out that Hibi rings are widely present in representation theory. The work of Gelfand and Tsetlin [GT] in the late 1940s offered an alternative description of $R^+(GL_n/U_n)$ in terms of successive restrictions of representations, from GL_n to GL_{n-1}, to GL_{n-2}, and so forth, down to GL_1. For GL_k embedded in GL_n as the upper right $k \times k$ block of matrices, consider the algebra

$$B(GL_n, GL_k) = R(GL_n/U_n)^{U_k} \tag{1.2}$$

of functions in $R(GL_n/U_n)$ that are invariant under translation on the left by U_k, the maximal unipotent subgroup of GL_k. Another appeal to the highest weight theory [GW] demonstrates that $B(GL_n, GL_k)$ determines the decomposition into irreducible GL_k representations of any irreducible representation of GL_n. It is called the (GL_n, GL_k)

[1] The space GL_n/U_n is a torus bundle over the variety of complete flags on GL_n.

[2] Any irreducible representation of GL_n can be expressed as a (perhaps negative) power of the one-dimensional representation det given by the determinant function times a polynomial representation. Thus $R(GL_n/U_n)$ is generated by $R^+(GL_n/U_n)$ and $\det^{-1}$.

[3] One can think of $(\mathbb{Z}^+)^{(X,\geq)}$ as a subset of the real vector space $\mathbf{R}^X$ of all real-valued functions on X. As such, it is the intersection of the cone $(\mathbf{R}^+)^{(X,\geq)}$of non-negative, real-valued, order preserving functions on X with the lattice $\mathbf{Z}^X$ integer-valued functions on X. For this reason, we call $(\mathbb{Z}^+)^{(X,\geq)}$ a *lattice cone*.

branching algebra [HTW1, HJLTW]. In the light of the ideas above, one can say that each branching algebra $B(GL_n, GL_k)$ has a standard monomial theory, and is a flat deformation of a Hibi ring attached to an appropriate poset (which in fact is a subset of the GT poset, cf. Figure 4).

Further, Hibi rings do not appear only in connection with GL_n; they help to describe the representation theory of all the classical groups. In the 1970s (cf. [Zh2, Kir]), Zhelobenko showed that there was a description of a basis for the flag algebra of the symplectic groups Sp_{2n} in terms of diagrams, and standard monomials for Sp_{2n} were also found [Be]. In [Ki1, Ki3], it was shown that the flag algebra for Sp_{2n}, as well as branching algebras $B(Sp_{2n}, Sp_{2k})$, were flat deformations of Hibi rings on the Zhelobenko-Gelfand-Tsetlin poset and subsets thereof. The situation for orthogonal groups is somewhat more complicated, but has been worked out recently in [Ki2, Ki3]. Here modest generalizations Hibi rings are needed.

Thus, Hibi rings help to understand key branching problems for all classical groups. In fact, it turns out that these branching algebras all bear a strong resemblance to the branching algebras $B(GL_n, GL_k)$ for GL_n[4], so it is not so surprising, given that Hibi rings apply to GL_n, they should also be of use for the other classical groups.

More surprising is that Hibi rings have also been found to describe certain tensor products for classical groups. For GL_n, this is predictable. Here the branching algebra $B(GL_n, GL_{n-1})$ is actually a polynomial ring on a canonical set of generators. Reciprocity laws ([Ho2, HTW1]) predict that $B(GL_n, GL_{n-1})$ (or more accurately, $B(GL_{n+1}, GL_n)$) can also be used to describe the tensor products of representations of GL_n with the "one-rowed" representations, that is, symmetric powers $S^k(\mathbb{C}^n)$ of the standard representation of GL_n on $\mathbb{C}^n$. The structure of such tensor products can also be encoded in an algebra, which we call in general a *tensor product algebra*[5], but because this special case is essentially a reformulation of the classical Pieri Rule, we call it a *Pieri algebra*.

One can obviously iterate the process that gives rise to the Pieri algebra, by tensoring with multiple copies of the $S^k(\mathbb{C}^n)$. We call the algebras that describe these multiple tensor products, *iterated Pieri algebras*. Again, because of reciprocity laws [Ho2], iterated Pieri algebras are closely related to the branching algebras $B(GL_n, GL_k)$, and can be described as flat deformations of appropriate Hibi rings, whose underlying posets are related to the Gelfand-Tsetlin posets.

One can also consider analogs of the Pieri Rule for the other classical groups. That is, one can consider the tensor products of a typical representation of, say, Sp_{2n} with a symmetric power $S^k(\mathbb{C}^{2n})$ [6]. Unlike the strong parallels between branching for GL_n and for the orthogonal and symplectic groups, the Pieri Rule for orthogonal and symplectic groups is considerably more complex than the one for GL_n. A description of it in terms of Young diagrams and tableaux is given in [Su1, Su2], but in contrast to the description for GL_n, which involves only adding boxes to a diagram, it involves both adding boxes and deleting boxes, making iteration not completely straightforward.

[4]Indeed, in [Ki3], it is shown that branching algebras $B(Sp_{2n}, Sp_{2k})$ deform to Hibi rings isomorphic to subalgebras of the corresponding deformations of $B(GL_{2n}, GL_{2k})$

[5]Tensor product algebras can also be described as branching algebras for appropriate pairs of groups [HKL1].

[6]For the orthogonal group O_m, the symmetric powers $S^k(\mathbb{C}^m)$ are not irreducible. The appropriate substitute is tensoring with $H^k(\mathbb{C}^m)$, the degree k spherical harmonics.

On the other hand, Kim and Lee in [KL] found that the Pieri Rules for orthogonal and symplectic groups could be described in terms of a Hibi ring (no longer a polynomial ring, as in the case of GL_n). Moreover, this allowed the construction to be iterated, so that one could describe also tensoring with multiple copies of $S^k(\mathbb{C}^{2n})$ (for Sp_{2n}, or of spherical harmonics for O_m). If the number of copies is not too large (the *stable range* [Ho2]), the rings involved in these descriptions are Hibi rings.

Furthermore, it was found that the "full" iterated Pieri Rule for GL_n, which is to say, a rule describing tensor products with several copies of either symmetric powers $S^k(\mathbb{C}^n)$ or with their duals $S^\ell((\mathbb{C}^n)^*)$, could also be described by the same family of Hibi rings. Indeed, in [HKL1], it is shown that there is a 4-parameter family of "double Pieri algebras" that describe the Pieri Rules for all the classical groups.

Certain of these double Pieri algebras are closely related to the "double tableaux" studied by Desarmenian, Kung, and Rota [DKR], and then used by DeConcini and Procesi [DP] to establish a characteristic free version of classical invariant theory for GL_n. Indeed, what the theory of double tableaux accomplishes is essentially to show that the ring $P(M_{nm})$ of polynomials on the $n \times m$ matrices can be deformed to a Hibi ring in a manner consistent with the decomposition of $P(M_{nm})$ into irreducible representations under the joint action of $GL_n \times GL_m$ via multiplication on the left and on the right. The Hibi ring involved is one of the double Pieri algebras.

The joint action of $GL_n \times GL_n$ on $P(M_{nm})$ is the most visible example of a dual pair in the sense of [Ho1]. As is explained in [Ho1] (see also [GW]), for any other classical action of a group G on a vector space V, there is a dual Lie algebra g', that generates the full algebra $PD(V)^G$ of polynomial coefficient differential operators on V commuting with the action of G. The Lie algebra g' has a grading

$$g' \simeq g'^{(2,0)} \oplus g'^{(1,1)} \oplus g'^{(0,2)}, \tag{1.3}$$

such that $g'^{(2,0)}$ consists of multiplication operators by quadratic polynomials, $g'^{(0,2)}$ consists of second order constant coefficient differential operators, and $g'^{(1,1)}$ consists of (slightly modified) vector fields with first order coefficients, and is isomorphic to gl_m for suitable m.

The Lie algebra $g'^{(2,0)}$ is commutative as a Lie algebra, and generates the associative algebra $P(V)^G = J(V,G)$ of all polynomials on V that are invariant under the action of G. Since the Lie algebra $g'^{(0,2)}$ consists of constant coefficient differential operators, its elements reduce degree of all polynomials, and so must annihilate many polynomials. A polynomial that is annihilated by <u>all</u> elements of $g'^{(0,2)}$ is called *harmonic* for G. The space of harmonic polynomials for G is denoted $H(V,G)$.

The following facts are known about this situation [Ho1, GW].

(1) The harmonics have a decomposition

$$H(V,G) \simeq \sum_D \sigma_G^D \otimes \rho_m^D; \tag{1.4}$$

where:
 (a) D is a Young diagram satisfying appropriate constraints, depending on G and V;
 (b) ρ_m^D is the representation of $gl_m \simeq g'^{(1,1)}$ corresponding to D; and
 (c) σ_G^D is the representation of G labeled by D.

(2) The "multiplication map"

$$\mu : H(V,G) \otimes J(V,G) \to P(V) \tag{1.5}$$

defined by $\mu(h \otimes g) = hf$, for h in $H(V,G)$ and f in $J(V,G)$, is surjective. That is, every polynomial is a sum of products of a harmonic and an invariant.

These facts together can be thought of as a generalization of the theory of spherical harmonics (which is the case of $G = O_n$ acting on $\mathbb{C}^n = V$).

When $G = GL_n$ and $V = M_{nm}$, then $g' = g^{(1,1)} \simeq gl_m$, and $g'^{(2,0)} = \{0\} = g'^{(0,2)}$. Thus in this case, the harmonics are the full polynomial ring, and the harmonic decomposition above simply amounts to (GL_n, GL_m) duality:

$$P(M_{nm}) \simeq \sum_D \rho_n^D \otimes \rho_m^D \tag{1.6}$$

with the diagram D ranging over all diagrams with at most $\min(n,m)$ rows. Thus, the theory of double tableaux mentioned above amounts to a standard monomial theory for the harmonics for the dual pair (GL_n, GL_m). The main goal of this paper is to exhibit an analogous standard monomial theory for the harmonics $H(V,G)$ for any classical action.

Here is the plan for the rest of the paper. In Section 2, we will discuss some general issues about Hibi rings. In Section 3, we will recall SMT for the classical groups. Then in Sections 4-7, we will combine the results of Sections 2 and 3 to describe SMT for the harmonics.

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2. Some general constructions with Hibi rings

Let X be a finite set, and let

$$X = X_1 \cup X_2, \tag{2.1}$$

be a decomposition of X into two disjoint subsets, X_1 and X_2. Let $f : X \to \mathbb{Z}^+$ be a non-negative, integer-valued function, and let $f_j = f_{|X_j}$ be the restriction of f to X_j. Then f_j is a function on X_j, but we can extend it to a function $\tilde{f}_j$ on X, by letting $\tilde{f}_j$ be zero on $X - X_j (= X_{3-j})$. With this notation, it is clear that $f = \tilde{f}_1 + \tilde{f}_2$, and that this decomposition is the unique way to express f as a sum of functions, one of which vanishes on $X - X_1 = X_2$, and the other of which vanishes on $X - X_2 = X_1$. The mappings $f \to (f_1, f_2)$, and $(f_1, f_2) \to \tilde{f}_1 + \tilde{f}_2$ define mutually inverse isomorphisms of semigroups

$$(\mathbb{Z}^+)^X \simeq (\mathbb{Z}^+)^{X_1} \oplus (\mathbb{Z}^+)^{X_2}. \tag{2.2}$$

Now suppose that X is a poset, with order relation $\geq$. We regard the subsets X_1 and X_2 as posets with order relation $\geq_j$ inherited from X. That is, if x and x' both belong to X_j, then $x \geq_j x'$ if and only if $x \geq x'$. Then if f belongs to $(\mathbb{Z}^+)^{(X,\geq)}$, that is, if f is order-preserving, then clearly f_j will belong to $(\mathbb{Z}^+)^{(X_j,\geq_j)}$. Thus, we see that the forward direction of the isomorphism (2.2) will map $(\mathbb{Z}^+)^{(X,\geq)}$ into $(\mathbb{Z}^+)^{(X_1,\geq_1)} \oplus (\mathbb{Z}^+)^{(X_2,\geq_2)}$.

What happens if we want to go in the other direction? Let f_j belong to $(\mathbb{Z}^+)^{(X_j,\geq_j)}$, and consider $\tilde{f}_j$. This belongs to $(\mathbb{Z}^+)^X$, but will it be order-preserving? Suppose that $\tilde{f}_2$ is order-preserving as a function on X. Suppose there are a pair of points x_j, with x_j in X_j, and that $x_1 \geq x_2$. Then, $0 = \tilde{f}_2(x_1) \geq \tilde{f}_2(x_2) = f_2(x_2) \geq 0$, since $\tilde{f}_2$ is order preserving. We conclude that $f_2(x_2) = 0$. But this will not necessarily be true for all f_2 in $(\mathbb{Z}^+)^{(X_2,\geq_2)}$. For example, the constant function 1 on X_2 is always in $(\mathbb{Z}^+)^{(X_2,\geq_2)}$.

Thus, if any element of X_1 dominates any element of X_2 with respect to $\geq$, there will be some functions f_2 in $(\mathbb{Z}^+)^{(X_2,\geq_2)}$ whose extensions by zero to X are not order-preserving. Similar remarks apply with X_1 and X_2 reversed.

On the other hand, suppose it is never the case that for a pair (x_1, x_2) with x_j in X_j, that $x_1 \geq x_2$, or that $x_2 \geq x_1$. We will describe this situation by saying that X_1 and X_2 are *totally incomparable.* In [Sta1, §3.2], X is called the *direct sum* of X_1 and X_2, with notation $X = X_1 + X_2$. When $X = X_1 + X_2$, a function f on X such that f_j is order-preserving (with respect to $\geq_j$), will in fact be order-preserving on X, because for any pair (x, x') such that $x \geq x'$, we must have either x and x' both lie in X_1, or both lie in X_2, and so we can conclude that $f(x) \geq f(x')$ by the parallel fact for f_1 or f_2.

Thus, this discussion has demonstrated the following fact.

Proposition 2.1. *If the poset $X = X_1 + X_2$ is the disjoint union/direct sum of two totally incomparable sets X_j, as in* (2.1)*, then the isomorphism* (2.2) *restricts to an isomorphism*

$$(\mathbb{Z}^+)^{(X,\geq)} \simeq (\mathbb{Z}^+)^{(X_1,\geq_1)} \oplus (\mathbb{Z}^+)^{(X_2,\geq_2)}, \tag{2.3}$$

where $\geq_j$ is the restriction to X_j of the order relation on X. Furthermore,

$$R_H(X) \simeq R_H(X_1) \otimes R_H(X_2), \tag{2.4}$$

where $R_H(X)$ is the Hibi ring attached to X, as defined in formula (1.1)*.*

Proof. Statement (2.3) follows from the discussion preceding the statement of the proposition. Relation (2.4) follows from relation (2.3) by general facts about semigroup rings [BH]. □

We have discussed above the direct sum, or unordered disjoint union, of two posets. In [Sta1, §3.2], besides the direct sum, a second way of combining two posets into a larger one is mentioned. This way is to proclaim every element of one set larger than every element of the other set. This is called the *ordinal sum.* Let us write

$$X = X_1 \oplus > X_2 \tag{2.5}$$

for the ordinal sum of X_1 and X_2, with X_1 being the larger set - every element of X_1 dominates every element of X_2. (We have added the inequality sign to Stanley's notation to emphasize which set is larger.) Then the following proposition is clear.

Proposition 2.2. *Let $X = X_1 \oplus > X_2$. Given f in $(\mathbb{Z}^+)^{X,\geq}$, let $f_i = f_{X_i}$ for $i = 1, 2$. Then*

i) *f_i belongs to $(\mathbb{Z}^+)^{(X_i,\geq)}$, and*
ii) $\max f_2 \leq \min f_1$.

Conversely, given f_i satisfying i) and ii), there is a unique f in $(\mathbb{Z}^+)^{(X,\geq)}$ whose restrictions to X_i are the f_i.

We want to extend the above discussion to cover a situation where the union (2.1) may not be disjoint. For this, it is convenient to think of independent posets Z_1 and Z_2, such that there are order preserving injections

$$\alpha_j : Z_j \to X_1 \subset X, \tag{2.6}$$

such that $X = X_1 \cup X_2$, but the union may not be disjoint. In this situation, if we set

$$Y = X_1 \cap X_2, \tag{2.7}$$

and

$$Y_1 = \alpha_1^{-1}(X_1 \cap X_2), \quad Y_2 = \alpha_2^{-1}(X_1 \cap X_2), \tag{2.8}$$

then we have poset isomorphisms

$$\begin{array}{ccc} & Y & \\ {}^{\alpha_1}\nearrow & & \nwarrow^{\alpha_2} \\ Y_1 & \underset{\alpha}{\longleftrightarrow} & Y_2 \end{array} \tag{2.9}$$

Let $\succeq_j$ be the order relation on Z_j. Then α_j is an order isomorphism between Z_j equipped with $\succeq_j$ and X_j equipped with $\geq_j$. Thus, if (x, x') is a pair of points in X_j, we will have $x \geq_j x'$ if and only if $\alpha_j^{-1}(x) \succeq_j \alpha_j^{-1}(x')$.

What can we say if x belongs to X_1 and x' belongs to X_2? We can give a sufficient condition that $x \geq x'$, as follows. Suppose there is y in Y, such that $x \geq_1 y$, and $y \geq_2 x'$. Then by transitivity of $\geq$, we must have $x \geq x'$. This can be translated back to the Z_j as follows: if $z_1 = \alpha_1^{-1}(x)$, and $z_2 = \alpha_2^{-1}(x')$, and $y_1 = \alpha_1^{-1}(y)$, then we can express the relation above, by saying that $z_1 \succeq_1 y_1$, and $\alpha(y_1)) \succeq_2 z_2$.

Definition 2.3. *Given*

1. *posets Z_1, Z_2 with order relations $\succeq_j$,*
2. *subsets $Y_j \subset Z_j$, and*
3. *an order isomorphism $\alpha : Y_1 \to Y_2$,*

define

$$X = Z_1 \cup Z_2/\alpha, \tag{2.10}$$

to be the disjoint union $Z_1 \cup Z_2$, subject to the equivalence relation $\alpha(y) = y$ for y in Y_1.

Define the relation $\geq$ on X by the conditions:

1. *If x, x' are in Z_j, then $x \geq x' \Leftrightarrow x \succeq_j x'$;*
2. *if x is in Z_1 and x' is in Z_2, then $x \geq x' \Leftrightarrow$ there is y in Y_1 such that $x \succeq_1 y$ and $\alpha(y) \succeq_2 x'$; and*
3. *the symmetric condition if x is in Z_2 and x' is in Z_1.*

Proposition 2.4. *With notations as in Definition 2.3, the relation $\geq$ is a partial order relation on X.*

Proof. We must show that the relation $\geq$ as defined above is transitive. Consider three points x, x', x'' in X, such that $x \geq x'$ and $x' \geq x''$. We need to show that also $x \geq x''$. Suppose that x is in Z_1. Then there are four possibilities for x' and x'':

a) $x' \in Z_1$, and $x'' \in Z_1$;
b) $x' \in Z_1$, and $x'' \in Z_2$;
c) $x' \in Z_2$, and $x'' \in Z_1$;

d) $x' \in Z_2$, and $x'' \in Z_2$.

For a), we conclude that $x \succeq_1 x'$ and $x' \succeq_1 x''$, whence $x \succeq_1 x''$ since $\succeq_1$ is an order relation. But since x and x'' are both in Z_1, this implies $x \geq x''$, as desired.

For b), we have that $x' \geq x''$ implies that there is y in Y so that $x' \succeq_1 y$ and $y \succeq_2 x''$. Then since $x \geq x'$ and both x and x' are in Z_1, we know that $x \succeq_1 x'$. Hence also $x \succeq_1 y$ since x and y are both in Z_1 and $\succeq_1$ is an order relation. Hence, $x \geq x''$ by definition of $\geq$.

For c), we know that there is y in Y such that $x \succeq_1 y$ and $y \succeq_2 x'$. Also, there is $\tilde{y}$ in Y such that $x' \succeq_2 \tilde{y}$ and $\tilde{y} \succeq_1 x''$. Since $y \succeq_2 x' \succeq_2 \tilde{y}$, we know that $y \succeq_2 \tilde{y}$, by transitivity of $\succeq_2$. Hence also $y \succeq_1 \tilde{y}$, since the relations $\succeq_1$ and $\succeq_2$ agree on Y. Therefore, we have $x \succeq_1 y \succeq_1 \tilde{y} \succeq_1 x''$, whence $x \succeq_1 x''$, by transitivity of $\succeq_1$.

For d), the argument is quite similar to that for b).

If x belongs to Z_2, then there are again 4 cases, and the arguments are essentially the same as those above. This concludes the proof of the proposition. □

Terminology and Definition 2.5. *We call $\geq$ the minimal extension of the partial orders $\succeq_j$ on the Z_j amalgamated along the Y_j. We write*

$$(X, \geq) = \mu((Z_1, \succeq_1), (Z_2, \succeq_2), (Y_1, Y_2, \alpha)).$$

Given the set-up as above, with two posets Z_1, Z_2, and an order isomorphism of subsets Y_j of each, we can construct the minimal extension X of the Z_j subject to identifying the Y_j. Associated with these posets, we have the Hibi semigroups $(\mathbb{Z}^+)^{(Z_j \succeq_j)}$ and $(\mathbb{Z}^+)^{(X,\geq)}$. Since X is described in terms of the Z_j, it should be possible to describe $(\mathbb{Z}^+)^{(X,\geq)}$ in terms of the $(\mathbb{Z}^+)^{(Z_j,\succeq_j)}$. The following result does this.

Proposition 2.6. *If $(X \geq) = \mu((Z_1, \succeq_1), (Z_2, \succeq_2), (Y_1, Y_2, \alpha))$ is the minimal extension of the Z_j amalgamated along the Y_j, then $(\mathbb{Z}^+)^{(X,\geq)}$ is isomorphic to the subsemigroup of $(\mathbb{Z}^+)^{(Z_1,\succeq_1)} \oplus (\mathbb{Z}^+)^{(Z_2,\succeq_2)}$ consisting of pairs of functions (f_1, f_2), with f_j in $(\mathbb{Z}^+)^{(Z_j \succeq_j)}$, and such that*

$$(f_1)_{|Y_1} = f_2 \circ \alpha. \tag{2.11}$$

That is, for y in Y_1, $f_1(y) = f_2(\alpha(y))$. The isomorphism is given by the product of restriction mappings

$$(R_1 \times R_2)(\mathbb{Z}^+)^{(X,\geq)} \to (\mathbb{Z}^+)^{(Z_1,\succeq_1)} \oplus (\mathbb{Z}^+)^{(Z_2,\succeq_2)}$$

where

$$R_j(f)(z) = f(z), \tag{2.12}$$

for z in Z_j.

Proof. If f is in $(\mathbb{Z}^+)^{(X,\geq)}$, it is clear that $f_j = f_{|Z_j}$ will be in $(\mathbb{Z}^+)^{(Z_j,\succeq_j)}$. It is also clear that $f_1(y) = f_2(\alpha(y))$ for y in Y_1. Thus, $R_1 \times R_2(f)$ will satisfy the stipulated conditions. The main thing to verify is that a pair of functions (f_1, f_2) in $(\mathbb{Z}^+)^{(Z_1,\succeq_1)} \oplus (\mathbb{Z}^+)^{(Z_2,\succeq_2)}$ that satisfies the compatibility condition (2.11) is of the form $(R_1 \times R_2)(f)$ for some f in $(\mathbb{Z}^+)^{(X,\geq)}$.

It is clear also that the compatibility condition guarantees that $(f_1, f_2) = (R_1 \times R_2)(f)$ for a (uniquely determined) f in $(\mathbb{Z}^+)^X$. The burden is to show that this f is order preserving on X for the minimal extension ordering $\geq$. So consider a pair of points x and x' in X, with $x \geq x'$. We want to know that $f(x) \geq f(x')$. If x and x' both belong to Z_1 or both to Z_2, then this is clear, since $f_{Z_j} = f_j$, which is assumed to be order-preserving. Suppose that x is in Z_1, and x' is in Z_2. Then the dominance condition $x \geq x'$ means that there is a y_1 in Y_1 such that $x \succeq_1 y_1$, and $\alpha(y_1) \succeq_2 x'$. But then $f(x) = f_1(x) \geq f_1(y_1)$ since f_1 is order-preserving, and $f_1(y_1) = f_2(\alpha(y_1)) \geq f_2(x') = f(x')$, by the compatibility condition, and because f_2 is order-preserving. Thus, we do have $f(x) \geq f(x')$. The reasoning in the case when x is in Z_2 and x' is in Z_1 is symmetrical to this. We conclude that f is indeed order-preserving, so the proposition is established. □

A third topic we will need to know about is, what happens if we want to restrict our attention to functions in $(\mathbb{Z}^+)^{(X,\geq)}$ that take the same value at two or more different points? If $f(x_1) = f(x_2)$, and $x_1 \leq y \leq x_2$, then we must have $f(x_1) \leq f(y) \leq f(x_2) = f(x_1)$; hence $f(y) = f(x_1) = f(x_2)$. We call the set

$$I(x_1, x_2) = \{y \in X : x_1 \leq y \leq x_2\} \tag{2.13}$$

the *order interval* (or when we are feeling lazy, just the *interval*) between x_1 and x_2. The observation above shows that if f in $(\mathbb{Z}^+)^{(X,\geq)}$ takes equal values at x_1 and x_2, then it takes this same value at all points of the order interval $I(x_1, x_2)$.

Given points $x_1 \leq x_2$ in X, we define a new poset $X_{x_1 \simeq x_2}$ as follows: As set,

$$X_{x_1 \simeq x_2} = (X - I(x_1, x_2)) \cup \{I(x_1, x_2)\}. \tag{2.14}$$

That is, we get $X_{x_1 \simeq x_2}$ by deleting $I(x_1, x_2)$ from X, then replacing it with a single point. We could also think of $X_{x_1 \simeq x_2}$ as resulting from X by collapsing $I(x_1, x_2)$ to a point.

We define a relation $\leq'$ on $X_{x_1 \simeq x_2}$ by the following rules. We abbreviate $I(x_1, x_2) = I$.

(1) If x and y are in $X - I(x_1, x_2)$, then $x \leq' y$ iff:

$$\text{a) } x \leq y; \quad \text{or} \quad \text{b) } \ x \leq x_2 \text{ and } \ x_1 \leq y;$$

(2) $x \leq' I$ iff $x \leq x_2$, and $I \leq' x$ iff $x_1 \leq x$;

(3) $I \leq' I$.

We claim that $\leq'$ is an order relation; that is, it is transitive. To check this, take 3 points x, y and z in X, and suppose that $x \leq' y \leq' z$. We want to show that $x \leq' z$. We must distinguish several cases.

First suppose that all of x, y and z are in $X - I$. Then we may have:

a) $x \leq y$ and $y \leq z$. Then $x \leq z$, so that $x \leq' z$ also.

b) $x \leq y$ and $y \leq x_2$ and $x_1 \leq z$. Then $x \leq x_2$, and $x_1 \leq z$, so again $x \leq' z$.

c) $x \leq x_2$ and $x_1 \leq y$ and $y \leq z$. This case is similar to b).

d) $x \leq x_2$ and $x_1 \leq y$ and $y \leq x_2$ and $x_1 \leq z$. This implies that $x_1 \leq y \leq x_2$, so y belongs to I, so this case does not in fact occur.

Next, suppose that $x = I$. Then $x \leq' y$ means that $x_1 \leq y$, and we have two possibilities:

e) $x_1 \leq y$ and $y \leq z$. This implies $x_1 \leq z$, which means that $I \leq z$, which means that $x \leq' z$.

f) $x_1 \leq y$ and $y \leq x_2$ and $x_1 \leq z$. This again entails that $x_1 \leq y \leq x_2$, so this case does not occur.

Taking $z = I$ gives cases parallel to e) and f). We do not write them. Finally, consider the possibility that $y = I$. Then $x \leq' y$ means that $x \leq x_2$, and $y \leq' z$ means that $x_1 \leq z$, so that the conditions for $x \leq' z$ are fulfilled. The above discussion covers all cases, so we conclude that $\leq'$ is indeed a partial ordering.

Now we will show that $X_{x_1 \simeq x_2}$ is the natural home for elements of $(\mathbb{Z}^+)^{(X,\geq)}$ that agree on x_1 and x_2. We can define a natural mapping

$$Q : X \to X_{x_1 \simeq x_2} \tag{2.15}$$

by the rule

$$Q(x) = x \quad \text{for} \quad x \notin I(x_1, x_2) \quad \text{and} \quad Q(x) = \{I(x_1, x_2)\} \quad \text{for } x \in I(x_1, x_2).$$

Proposition 2.7. (1) *The map Q of formula* (2.15) *is order preserving.*

(2) *The map*

$$Q^* : (\mathbb{Z}^+)^{(X_{x_1 \simeq x_2}, \geq')} \to (\mathbb{Z}^+)^{(X,\geq)}, \qquad Q^*(f) = f \circ Q$$

defines an isomorphism between $(\mathbb{Z}^+)^{(X_{x_1 \simeq x_2}, \geq')}$ and the subsemigroup of elements of $(\mathbb{Z}^+)^{(X,\geq)}$ that take equal values on x_1 and x_2.

Remark 2.8. *Consider the integer lattice $\mathbb{Z}^2$ in the plane with the usual partial order, and let $\vec{z_1} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\vec{z_2} = \vec{z_1} + \begin{bmatrix} a \\ b \end{bmatrix}$ with a and b non-negative, so that $\vec{z_1} \leq \vec{z_2}$. Define a mapping*

$$\phi : \mathbb{Z}^2 \to \mathbb{Z}^2 \tag{2.16}$$

by the following recipe.

$$\phi(\begin{bmatrix} x \\ y \end{bmatrix}) = \vec{z_2} \quad \textit{if} \quad \vec{z_1} \leq \begin{bmatrix} x \\ y \end{bmatrix} \leq \vec{z_2};$$

$$\phi(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} x \\ y \end{bmatrix} \quad \textit{if} \quad \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} x_1 - 1 \\ y_1 + b \end{bmatrix} \quad \textit{or} \quad \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} x_1 + a \\ y_1 - 1 \end{bmatrix};$$

$$\phi(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} x + a \\ y + b \end{bmatrix} \quad \textit{if} \quad \begin{bmatrix} x \\ y \end{bmatrix} \geq \begin{bmatrix} x_1 + a + 1 \\ y_1 \end{bmatrix} \quad \textit{or} \quad \begin{bmatrix} x \\ y \end{bmatrix} \geq \begin{bmatrix} x_1 \\ y_1 + b + 1 \end{bmatrix};$$

$$\phi(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} x + a \\ y \end{bmatrix} \quad \textit{if} \quad x > x_1 + a \quad \textit{and} \quad y < y_1;$$

$$\phi(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} x \\ y + b \end{bmatrix} \quad \textit{if} \quad x < x_1 \quad \textit{and} \quad y > y_1 + b.$$

The mapping ϕ collapses the interval $I(\vec{z_1}, \vec{z_2})$ to the point $\vec{z_2}$, and is an order isomorphism of $\mathbb{Z}^2 - I(\vec{z_1}, \vec{z_2})$ into $\mathbb{Z}^2$, in such a way that the set $\phi(\mathbb{Z}^2 - I(\vec{z_1}, \vec{z_2})) \cup \{\vec{z_2}\}$ is isomorphic to $\mathbb{Z}^2_{\vec{z_1} \simeq \vec{z_2}}$. Thus, the operation of forming $X_{\vec{z_1}, \vec{z_2}}$ will take a subposet of $\mathbb{Z}^2$ to another subposet of $\mathbb{Z}^2$.

3. Standard monomial theory for classical flag algebras

Since the space $H(G, V)$ of G-harmonic polynomials on V is multiplicity-free as a representation of $G \times g'^{(1,1)}$, as can be seen from formula (1.4), its structure should be related to that of the flag algebra for $G \times g'^{(1,1)}$. Therefore, to prepare for study of $H(G, V)$, we will recall SMT for the flag algebras of the classical groups.

For a connected reductive group G and its maximal unipotent subgroup U, we consider the ring $\mathbf{F}(G)$ of regular functions on G invariant under the left action of U. This space contains every irreducible regular representation of G with multiplicity one ([Ho2, GW]). The structure of $\mathbf{F}(GL_n)$, or more accurately of its polynomial subalgebra $\mathbf{F}^+(GL_n)$, is the subject of Hodge's original work. This has been put in the form we want by a series

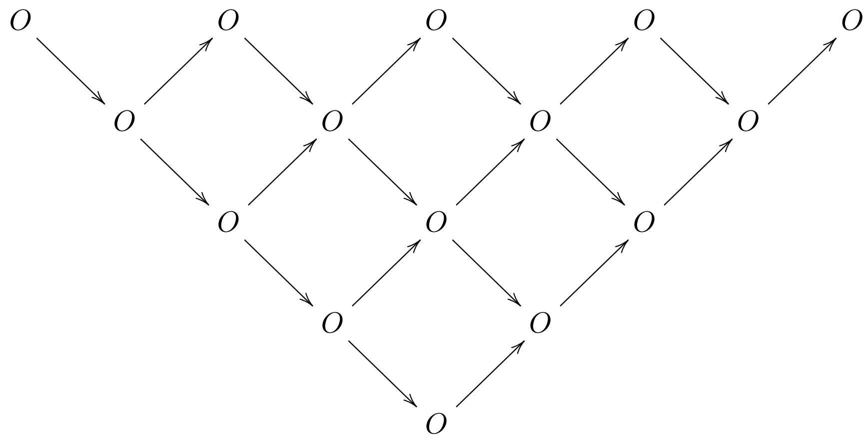

FIGURE 1. The GT poset Γ_{GL_5}.

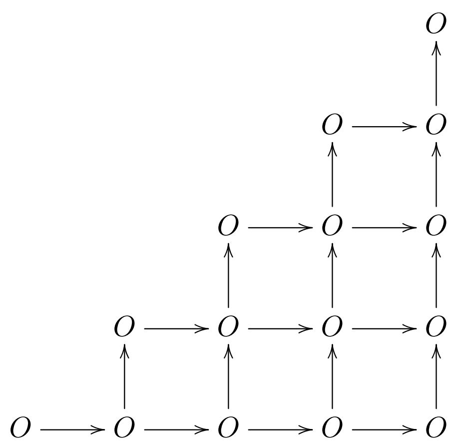

FIGURE 2. The GT poset Γ_{GL_5} in $\mathbb{Z}^2$.

of papers, of which we mention especially [GL1, GL2] and [KM], with the final form that we state here coming from [Ki1].

We need to define the Gelfand-Tsetlin(GT) poset Γ_{GL_n}. This is a triangular array of points that has been traditionally illustrated in Figure 1. Arrows indicating the elements that are "next smallest" from a given element are inserted to describe the order relation in Γ_{GL_5}:

In [HKL1] (see also [Ho5]), we found it convenient to describe Γ_{GL_n} slightly differently, so that it can be regarded as a subset of $\mathbb{Z}^2$, with its standard partial order:

$$\begin{bmatrix} a \\ b \end{bmatrix} \leq \begin{bmatrix} a' \\ b' \end{bmatrix} \quad \Leftrightarrow \quad a \leq a' \quad \text{and} \quad b \leq b', \tag{3.1}$$

Here the upper left corner in Figure 1 has become the upper corner in Figure 2. The upper right corner in Figure 1 has become the lower left corner in Figure 2. The left, downward sloping side in Figure 1 is the right vertical side in Figure 2, and the right upward sloping side in Figure 1 is the bottom side of Figure 2. The top row of Figure 1 has become the top diagonal in Figure 2.

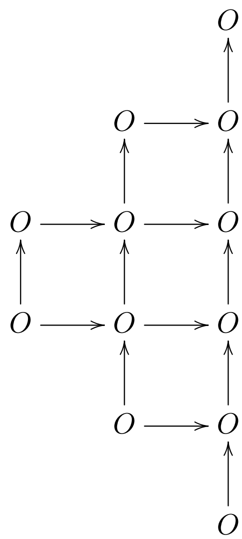

FIGURE 3. The GT poset Γ_{Sp_6} in $\mathbb{Z}^2$.

It doesn't matter where in the plane this triangle is located - it can be translated without affecting its poset properties. However, in [HKL1], the position was normalized so that the upper right corner of Figure 2 was the point $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$. In more detail, given two integers $c < d$, we define the "subdiagonal triangle" $T(c, d)$ by:

$$T(c,d) = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{Z}^2 : c \leq a \leq d; c \leq b \leq a \right\}. \tag{3.2}$$

Then we can declare that the "standard model" for Γ_{GL_n} is $T(-n, -1)$.

With these preliminaries, we can state SMT for $\mathbf{F}^+(GL_n)$ as follows [GL1, GL2, KM, Ki1].

Theorem 3.1. *The polynomial flag algebra $\mathbf{F}^+(GL_n)$ has a flat deformation to the Hibi ring $H(\Gamma_{GL_n})$.*

SMT for the symplectic groups Sp_{2n} can be formulated in a parallel fashion [Ki1]. In place of Γ_{GL_n}, we need the Zhelobenko poset $\Gamma_{Sp_{2m}}$. Roughly speaking, this is the big left hand side of $\Gamma_{GL_{2n}}$ in Figure 1, or the part weakly above the central antidiagonal in the Figure 2. See Figure 3 for an illustration of Γ_{Sp_6} in the presentation parallel to Figure 2.

Theorem 3.2. *The polynomial flag algebra $\mathbf{F}(Sp_{2n})$ has a flat deformation to the Hibi ring $H(\Gamma_{Sp_{2n}})$.*

The description of the flag algebra of O_n is somewhat more complicated than that for GL_n or Sp_{2n}, and its most natural deformation is not quite a Hibi ring. (However, it contains many natural subalgebras which are Hibi rings.) The exact description can be found in, for example, [DT, GL2, LT1, LT2]. We quote the result from [Ki2] here.

For $n = 2m$ or $n = 2m + 1$, and $k \leq m$, starting from the joint action of $O_n \times GL_k$ on $P(M_{nk})$, we can construct the isotropic flag algebra $\mathbf{F}^{iso}(O_n)$, in the sense of Lancaster and Towber [LT1, LT2] for the special orthogonal group. The algebra $\mathbf{F}^{iso}(O_n)$ contains all the polynomial representations of O_n labeled by Young diagrams of depth at most k. When $n = 2m$ and $k = m$, the multiplicative structure of $\mathbf{F}^{iso}(O_n)$ is different from the actual flag algebra of SO_n, in that the product of two irreducible SO_n representations may not be the Cartan product of the two representations.

Theorem 3.3. *For $\frac{n}{2} \geq k$, the isotropic flag algebra $\mathbf{F}^{iso}(O_n)$ has a flat deformation to the quotient of the Hibi ring $H(\Gamma_{G_n,k})$ by an ideal J_{nk} generated by quadratic monomials.*

See (4.5) for the notation $\Gamma_{GL_n,k}$. Recall that the ideal $I(M_{nk}, SO_n)$ in $P(M_{nk})$ generated by SO_n invariants without constant terms is prime when $n > 2k$. When $n = 2k$, the ideal $I(M_{nk}, SO_n)$ is the intersection of two prime ideals, and therefore the SO_n nullcone in M_{nk} contains two irreducible components (see, e.g., [DT, Ho2]). By considering separately the ring of regular functions supported on each of the irreducible components, we can obtain individual irreducible representations of SO_n. We will consider the standard monomial theory of these rings in a separate article.

4. Standard monomial theory for harmonics: Standard action of GL_n

In this and following sections, we will combine the results of Section 2 and Section 3 to describe a standard monomial theory for the space of G-harmonic polynomials for a classical action of G.

Consider first the case of GL_n acting by multiplication on the left on the space M_{nm} of $n \times m$ matrices, as described in, for example, [HKL1]. Here there are no invariants, and $g' = g'^{(1,1)} = g\ell_m$, and may be taken to be the Lie algebra of Aronhold polarization operators [Wy]. This is the Lie algebra of GL_m acting by multiplication on the right. The harmonics comprise the full space of polynomials, and are decomposed according to (GL_n, GL_m)-duality [Ho1, GW]:

$$P(M_{nm}) \simeq \sum_D \rho_n^D \otimes \rho_m^D \tag{4.1}$$

where ρ_n^D is the irreducible representation of GL_n labeled by the Young diagram D, which is arbitrary subject to the restriction that it have at most $\min(n, m)$ rows.

The subalgebra $P(M_{nm})^{U_m}$, of polynomials on M_{nm} that are invariant under the maximal unipotent subgroup U_m of GL_m, provides a model for the polynomial flag algebra $\mathbf{F}^+(GL_n)$ (or a sub algebra thereof if $m < n$). In this situation, SMT says that $P(M_{nm})^{U_m}$ is generated by the polynomials

$$\delta_T = \det \begin{bmatrix} x_{r_1,1} & x_{r_1,2} & \cdots & x_{r_1,k} \\ x_{r_2,1} & x_{r_2,2} & \cdots & x_{r_2,k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{r_k,1} & x_{r_k,2} & \cdots & x_{r_k,k} \end{bmatrix} \tag{4.2}$$

(with $k \leq \min(n, m)$), labeled by any strictly increasing sequence

$$T = \{r_1 < r_2 < r_3 < \cdots < r_k\} \tag{4.3}$$

of whole numbers between 1 and n. Since the δ_T generate $P(M_{nm})^{U_m}$ as algebra, the monomials in the δ_T form a spanning set for $P(M_{nm})^{U_m}$ as a complex vector space.

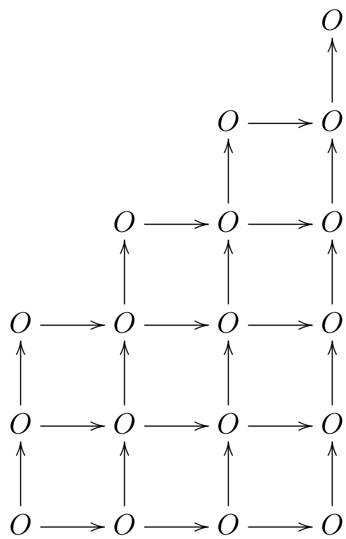

FIGURE 4. The poset $\Gamma_{GL_6,4}$.

SMT further specifies how to refine this spanning set to a basis. One can put a partial ordering on the sequences T as follows: If $T = \{r_j\}$ as above has k elements, and $T' = \{r'_j\}$ has k' elements, then we say that $T \le T'$ provided that

$$i)\ k \ge k' \quad \text{and} \quad ii)\ r_j \le r'_j \quad \text{for} \quad 1 \le j \le k'. \tag{4.4}$$

With this language in place, SMT says that:

(1) a set of generators $\{\delta_{T_i}\}$ such that the sequences T_i can be totally ordered with respect to the partial order (4.4) generates a polynomial subring of $P(M_{nm})^{U_m}$; and
(2) $P(M_{nm})^{U_m}$ is spanned by these polynomial subrings.

One approach (cf. [Ki1, HKL1]) to proving this proceeds by defining a term order on $P(M_{nm})$ and determining the semigroup generated by the highest terms of all the δ_T. It can be mapped isomorphically to the Hibi lattice cone $(\mathbb{Z}^+)^{(\Gamma_{GL_n},\ge)}$ when $n \le m$.

When $n > m$, then as noted, $P(M_{nm})^{U_m}$ does not give the full polynomial flag algebra $\mathbf{F}^+(GL_n)$, but instead a "band limited" subalgebra $\mathbf{F}^+(GL_n)_m$, spanned by the representations ρ_n^D for diagrams D having no more than m rows. The general scheme shows that this also is a Hibi ring, on a subposet $\Gamma_{GL_n,m}$ of Γ_{GL_n}, obtained by deleting the lower left triangle (isomorphic to $\Gamma_{GL_{n-m}}$), so that the all the horizontal rows ,and all the diagonals of the subset $\Gamma_{GL_n,m}$ have length at most m. For example $\Gamma_{GL_6,4}$ is shown in Figure 4.

If Γ_{GL_n} is realized by its "standard model" $T(-n,-1)$ (cf. formula (3.2) - do not confuse this use of T with that of formula (4.3)!) , then we can also describe

$$\Gamma_{GL_n,m} \simeq T(-n,-1) - T(-n,-m-1). \tag{4.5}$$

Now turn attention to the full polynomial ring, as described by equation (4.1). The first point we should note is that, while the sub algebra

$$P(M_{nm})^{U_m} \simeq \left(\sum_D \rho_n^D \otimes \rho_m^D\right)^{U_m} = \sum_D \rho_n^D \otimes (\rho_m^D)^{U_m}$$

is graded by the Young diagram D, in the sense that

$$(\rho_n^D \otimes (\rho_m^D)^{U_m}) \cdot (\rho_n^E \otimes (\rho_m^E)^{U_m}) \subset \rho_n^F \otimes (\rho_m^F)^{U_m},$$

where $F = D + E$ in the usual sense of adding diagrams (i.e., by adding row lengths), the analogous statement for the full ring polynomials $P(M_{nm})$ is false. That is, the analogous inclusion for the summands $\rho_n^D \otimes \rho_m^D$ fails:

$$(\rho_n^D \otimes \rho_m^D) \cdot (\rho_n^E \otimes \rho_m^E) \not\subseteq \rho_n^F \otimes \rho_m^F.$$

For example the space $M_{nm} \simeq \rho_n^{(1)} \otimes \rho_m^{(1)}$, where (1) indicates the Young diagram with exactly one box, consists of all the coördinate functions x_{ij}, and generates the whole polynomial ring.

To make $P(M_{nm})$ look more like a flag algebra, we filter $P(M_{nm})$ using the direct sum decomposition (4.1). Let $\preceq$ be the dominance order on Young diagrams [Ma, Sta2]. This coincides with the standard order on highest weights for a general Lie algebra [GW]. For a fixed diagram E, set

$$P(M_{nm})_E = \sum_{D \preceq E} \rho_n^D \otimes \rho_m^D. \tag{4.6}$$

Then well-known facts about tensor products (e.g.,[Po]) tell us that this is an algebra filtration:

$$(P(M_{nm})_E) \cdot (P(M_{nm})_F) \subseteq P(M_{nm})_{(E+F)}, \tag{4.7}$$

for Young diagrams E and F, and their sum $E + F$.

We can now form the graded algebra associated to this filtration [Po]. Set

$$P(M_{nm})^{(D)} = P(M_{nm})_D / (\sum_{E \prec D} P(M_{nm})_E), \tag{4.8}$$

and

$$GrP(M_{nm}) = \sum_D P(M_{nm})^{(D)}. \tag{4.9}$$

It is obvious from the definitions that $\rho_n^D \otimes \rho_m^D \subset P(M_{nm})_D$. It is not hard to check that the projection of $P(M_{nm})_D$ to $P(M_{nm})^{(D)}$ defines an isomorphism:

$$\rho_n^D \otimes \rho_m^D \simeq P(M_{nm})^{(D)}, \tag{4.10}$$

so that $GrP(M_{nm})$ is isomorphic to $P(M_{nm})$ as a vector space and as a $GL_n \times GL_m$ module. The difference is in the multiplication: the lower order terms in a product have been suppressed, so that $GrP(M_{nm})$ is a graded algebra with respect to the index D. Specifically, we have

$$P(M_{nm})^{(E)} \otimes P(M_{nm})^{(F)} \subset P(M_{nm})^{(E+F)}, \tag{4.11}$$

The algebra $GrP(M_{nm})$ is in fact a flat deformation of $P(M_{nm})$ [Po].

The algebra $P(M_{nm})^{U_n\times U_m}$ of $GL_n \times GL_m$ highest weight vectors in $P(M_{nm})$ is of course graded, by the semigroup of $A_n \times A_m$ weights attached to the irreducible representations appearing in $P(M_{nm})$, which in this situation is isomorphic to the semigroup of diagrams, with at most $\min(n,m)$ rows, under diagram addition, and is a free semigroup on the diagrams with a single column, of length up to $\min(n,m)$. It is easy to argue that $P(M_{nm})^{U_n\times U_m}$ projects isomorphically to $GrP(M_{nm})$, onto the sub algebra $GrP(M_{nm})^{U_n\times U_m}$. Both these algebras are in fact polynomial rings, and isomorphic to the semigroup ring on the semigroup of Young diagrams.

On the other hand, the polynomial flag algebra for $GL_n \times GL_m$, which is just the tensor product $\mathbf{F}^+(GL_n)\otimes\mathbf{F}^+(GL_m)$, is also a graded algebra, graded by the product of the semigroups of Young diagrams with n rows and diagrams with m rows. It is a sum of $GL_n\times GL_m$ representations of the form $\rho_n^E\otimes\rho_m^F$. We want to consider the subalgebra $\Delta\left(\mathbf{F}^+(GL_n)\otimes\mathbf{F}^+(GL_m)\right)$ of $\mathbf{F}^+(GL_n)\otimes\mathbf{F}^+(GL_m)$ consisting of summands $\rho_n^D\otimes\rho_m^D$, that is, such that both factors have highest weights labeled by the same diagram.

Proposition 4.1. *$GrP(M_{nm})$ is isomorphic to $\Delta\left(\mathbf{F}^+(GL_n)\otimes\mathbf{F}^+(GL_m)\right)$ as algebra with $GL_n\times GL_m$ action.*

Proof. Indeed, suppose that we have an algebra A that

a) allows $GL_n\times GL_m$ action by algebra automorphisms, and
b) is graded, with the irreducible $GL_n\times GL_m$ modules as homogeneous components, and
c) is multiplicity free as $GL_n\times GL_m$ module, and
d) such that the highest weights of the $GL_n\times GL_m$ modules contained in A in form a given free semigroup S, and
e) is a domain (i.e., has no zero divisors).

Then we claim that the algebra structure on A is uniquely determined up to isomorphism by the semigroup S. Thus, A is isomorphic to the subalgebra of $\mathbf{F}^+(GL_n)\otimes\mathbf{F}^+(GL_m)$ with the same highest weights as A.

To see this, consider the set $\{s_i\}$ of generators for the free semigroup S. For each s_i, let f_i be a $GL_n\times GL_n$ highest weight vector in A, with weight equal to s_1. Then $A^{U_n\times U_m}$ is the polynomial ring generated by the f_i.

We claim further that the algebra structure on A is completely determined by the algebra structure on $A^{U_n\times U_m}$. Indeed, consider the multiplication on A as a mapping

$$M : A\otimes A\to A.$$

Since A is graded by the highest weights of its irreducible $GL_n\times GL_m$ components, the multiplication mapping must take

$$m : (\rho_n^D\otimes\rho_m^E)\otimes(\rho_n^{D'}\otimes\rho_m^{E'})\to\rho_n^{(D+D')}\otimes\rho_m^{(E+E')}.$$

The fact that $GL_n\times GL_m$ acts on A by automorphisms means that m is a $GL_n\times GL_m$ module map. It is well known that the representation $\rho_n^{(D+D')}\otimes\rho_m^{(E+E')}$ occurs in the tensor product $(\rho_n^D\otimes\rho_m^E)\otimes(\rho_n^{D'}\otimes\rho_m^{E'})$ with multiplicity one - it is the "Cartan component" of $(\rho_n^D\otimes\rho_m^E)\otimes(\rho_n^{D'}\otimes\rho_m^{E'})$. Therefore, by Schur's Lemma, up to scalar multiples, there is only one $GL_n\times GL_m$ module map from $(\rho_n^D\otimes\rho_m^E)\otimes(\rho_n^{D'}\otimes\rho_m^{E'})$ to $\rho_n^{(D+D')}\otimes\rho_m^{(E+E')}$. And the multiple given by m will be determined by the product of the highest weight vectors of $\rho_n^D\otimes\rho_m^E)$ and of $\rho_n^{D'}\otimes\rho_m^{E'}$, which product will necessarily be a highest weight vector for $\rho_n^{(D+D')} \otimes \rho_m^{(E+E')}$. So, as claimed, the algebra structure is already determined on $A^{U_n \times U_m}$. □

Remark 4.2. *A result analogous to Proposition 4.1 clearly holds for any reductive group G. That is, if A is a G-algebra that satisfies the analogs of conditions a) to e) of the proposition, then A is isomorphic to the appropriate subalgebra of the flag algebra $\mathbf{F}(G)$.*

From Proposition 4.1, we may conclude that the associated graded algebra $GrP(M_{nm})$ may be considered as a subalgebra of the tensor product $\mathbf{F}^+(GL_n) \otimes \mathbf{F}^+(GL_m)$ (which we may call the polynomial flag algebra for $GL_n \times GL_m$). Since $\mathbf{F}^+(GL_n)$ has a flat deformation to the Hibi ring $R_H(\Gamma_{GL_n})$, general properties of flat deformation imply that $\mathbf{F}^+(GL_n) \otimes \mathbf{F}^+(GL_m)$ has a flat deformation to $R_H(\Gamma_{GL_n}) \otimes R_H(\Gamma_{GL_m})$, which is in turn isomorphic to $R_H(\Gamma_{GL_n} \cup \Gamma_{GL_m})$, where the union is the totally incomparable union in the sense of Proposition 2.3. In this deformation, the subalgebra $GrP(M_{nm})$ will get deformed to some subalgebra of $R_H(\Gamma_{GL_n} \cup \Gamma_{GL_m})$. We want to identify this subalgebra.

Since $H(\Gamma_{GL_n} \cup \Gamma_{GL_m})$ is multiplicity-free as a $GL_n \times GL_m$ module,with each irreducible submodule comprising one homogeneous component, the subalgebra isomorphic to $GrP(M_{nm})$ is determined by the homogeneous components that it contains. From the formula (4.1) for (GL_n, GL_m) duality, we know that these are exactly the homogeneous components $\rho_n^D \otimes \rho_m^E$ in which $E = D$.

It turns out that it is easy to interpret the condition $E = D$ in terms of the posets Γ_{GL_n} and Γ_{GL_m}. It is well-known that, in the diagram (2), the entries of the highest diagonal record the row lengths of D. More precisely, the r-th entry from the top corner [7] records the length of the r-th row of D. Thus, two GT diagrams corresponding to the same Young diagram agree on their top diagonals. Now we can recognize that we are in the context of Proposition 2.6. We state the result formally.

Proposition 4.3. *The algebra $GrP(M_{nm})$ also deforms to a Hibi ring, corresponding to the poset Γ_{GL_n,Gl_m} that is the minimal extension of Γ_{GL_n} and Γ_{GL_m} with their top diagonals identified.*

Remark 4.4.

a) *In* [HKL1] *and* [Ho5] *this is described in a concrete although somewhat* <u>*ad hoc*</u> *way by flipping Γ_{GL_m} over its top diagonal, then gluing the two GT posets together. Thus turns out just to be an $n \times m$ rectangle. For example, the poset Γ_{GL_6,GL_4} is Here, the diagonal along which the two GT posets are glued is indicated by Θs.*

b) *As noted, in the literature of the 1980s (*[DKR, DP]*), this was described, not in terms of Hibi rings, but in terms of "double tableaux". The double tableaux version of SMT for the full polynomial ring $P(M_{nm})$ was used to establish that some standard results remain valid in characteristic p. This more or less amounts to the fact that the standard monomials form an integral basis for the polynomial ring with integer coefficients. This in turn amounts to the fact that the coefficient of the leading monomial in any standard monomial for $P(M_{nm})^{U_m}$ is 1, which follows from the standard formula for determinant.*

c) *The treatment of SMT in* [Ho5] *uses strongly the leading monomials of elements of $P(M_{nm})^{U_m}$. One can analyze $\mathbf{F}^+(GL_n) \otimes \mathbf{F}^+(GL_m)$ in a similar fashion if*

[7] In the standard model $T(-n,-1)$ for Γ_{GL_n}, this would be the entry at $\begin{bmatrix} -r \\ -r \end{bmatrix}$.

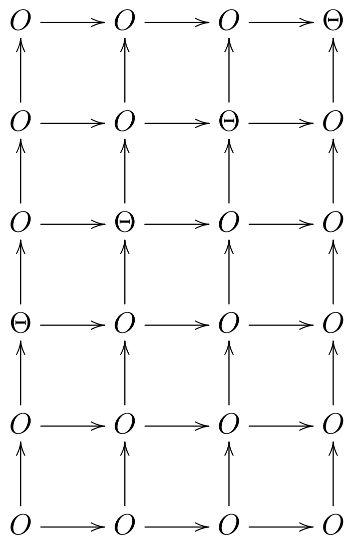

Figure 5. The poset Γ_{GL_6,GL_4}.

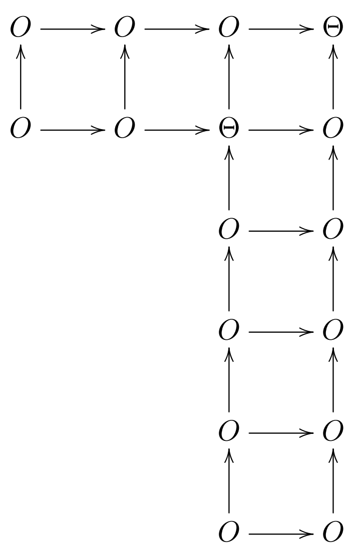

Figure 6. The poset $\Gamma_{GL_6,GL_4,2}$.

one realizes it as $P(M_{np})^{U_p} \otimes P(M_{mq})^{U_q}$ for some convenient p and q. But we do not know how to use term orders and leading monomials to establish SMT for $P(M_{nm})$ directly. Rather, the indirect approach using the constructions of Section 2 seems more effective.

d) *If instead of the full polynomial ring, we wanted to limit the diagrams D in the sum (4.1) to have depth at most $\ell \leq m$, then the poset Γ_{GL_n,GL_m} should be replaced by the smaller set $\Gamma_{GL_n,GL_m,\ell}$, in which there are only ℓ columns in the GL_n part of the set, and only ℓ rows in the GL_m part. This is the result of gluing together the two sets $\Gamma_{GL_n,\ell}$ and (the reflection of) $\Gamma_{GL_m,\ell}$ (cf. Figure 4) along their upper diagonals. Thus, $\Gamma_{GL_6,GL_4,2}$ looks like Figure 6.*

5. Standard monomial theory for harmonics: Sp_{2n} case

Turn now to the case of the symplectic group. Let Sp_{2n} denote the group of $2n \times 2n$ matrices that preserve the pairing

$$\langle \vec{v}, \vec{v'} \rangle = \sum_{j=1}^{n} (x_j y'_{n+1-j} - y_j x'_{n+1-j}), \tag{5.1}$$

where

$$\vec{v} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \qquad \text{and} \qquad \vec{v'} = \begin{bmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \\ y'_1 \\ y'_2 \\ \vdots \\ y'_n \end{bmatrix} \tag{5.2}$$

are two vectors in $\mathbb{C}^{2n} = Y$. With this choice of coordinates, the intersection

$$Sp_{2n} \cap B_{2n}, \tag{5.3}$$

of Sp_{2n} with the standard upper triangular Borel subgroup B_{2n} of GL_{2n} is a Borel subgroup of Sp_{2n}.

The irreducible representations of Sp_{2n} can be labeled by Young diagrams, just as for GL_n. The diagrams needed to label representations of Sp_{2n} are those with depth at most n. The irreducible representation σ_{2n}^D of Sp_{2n} is the representation generated by the action of Sp_{2n} on the highest weight vector in ρ_{2n}^D.

We should also note that, parallel to the case of GL_n, the Zhelobenko poset $\Gamma_{Sp_{2n}}$ of Figure 3 can be modified either

i) to describe bases for representations labeled by Young diagrams with at most m rows; or
ii) to describe the branching algebras $B(Sp_{2n}, Sp_{2m})$ for any $m \leq n$.

If we want to restrict the depth of the labeling diagrams to be at most m, then we should delete the part of the diagram to the left of a vertical line, so that all the rows that are left are of length at most m. For example, if we only consider representations of Sp_6 defined by diagrams of depth at most 2, we get the poset in Figure 7.

On the other hand, if we want to describe branching from Sp_{2n} to Sp_{2m}, we should eliminate the lowermost $2m-1$ diagonals. So for example, to describe branching from Sp_6 to Sp_4, we use the poset in Figure 8.

We want to use these considerations, plus the information from Section 2 and Section 3 to describe the harmonic polynomials for Sp_{2n} acting on $V = P(M_{2n,m})$. According to [Ho1], and as reviewed in the introduction (cf. see (1.4)), the harmonics $H(V, Sp_{2n})$ have a decomposition

$$H(V, Sp_{2n}) \simeq \sum_{D} \sigma_{2n}^D \otimes \rho_m^D, \tag{5.4}$$

as $Sp_{2n} \times GL_m$ module. Here D runs through Young diagrams with up to $\min(n, m)$ rows.

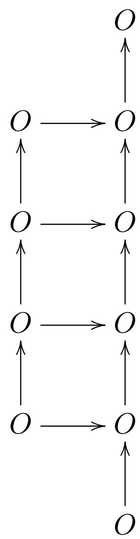

FIGURE 7. The poset for σ_6^D with $\ell(D) \leq 2$.

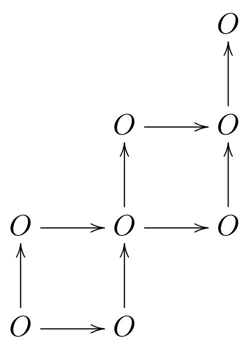

FIGURE 8. The poset describing branching from Sp_6 to Sp_4.

We want to apply in this situation reasoning similar to what we did above for GL_n. However, we must deal with the issue that the harmonics are not actually a subalgebra of $P(V)$. In order to have an algebra to work with, we use the fact (1.5). It implies that, if we take $J^2(V,G) = g'^{(2,0)}$, the space of degree 2 homogeneous, G-invariant polynomials, and form the ideal $IJ(V,G)$ generated by $J^2(V,G)$, then $H(V,G)$ projects isomorphically to the quotient ring $P(V)/IJ(V,G)$. Thus the standard monomials described here will actually form a basis for this quotient ring, which we will denote by $\overline{H}(V,G)$.

With this understanding, and using Remark 4.2, we can see that the module structure (5.4) implies that this ring is isomorphic to a subring of the tensor product $\mathbf{F}(Sp_{2n}) \otimes \mathbf{F}^+(GL_m)$ of flag algebras (which of course is a subalgebra of the flag algebra for the product $Sp_{2n} \times GL_m$). We know from [Ki1] that each of the factor algebras is a Hibi ring, on posets as described above. Also, we can go through the process of filtering with respect to highest weights and passing to the associated graded algebra $Gr\overline{H}(V,G)$.

In $H(V, Sp_{2n})$ we have the joint $Sp_{2n} \times GL_m$ highest weight vectors, which are in fact the same as the joint $GL_{2n} \times GL_m$ highest weight vectors. This is a polynomial ring, generated by the elements δ_T of formula (4.2), with $T = T_k = \{1, 2, 3, ..., k\}$ for

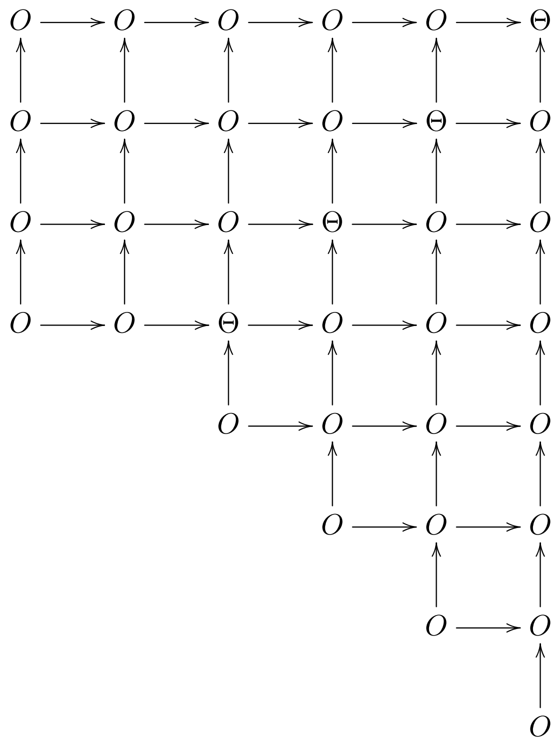

FIGURE 9. The poset $\Gamma_{GL_6,4}$.

$1 \leq k \leq \min(n, m)$. It then follows that $Gr\overline{H}(V, G)$ is isomorphic to the subalgebra of $\mathbf{F}(Sp_{2n}) \otimes \mathbf{F}^+(GL_m)$ spanned by the $Sp_{2n} \times GL_m$ modules appearing in formula (5.4).

In turn, this subalgebra can be described in terms of the minimal extension of the GT poset for GL_m (or the appropriate subposet when $n < m$) amalgamated with the Zhelobenko poset along their top diagonals. As with the case of SMT for $GL_n \times GL_n$ acting on $P(M_{nm})$, we can describe this by flipping one of the posets across its top diagonal and gluing the two together inside $\mathbb{Z}^2$. If we flip the GT poset $\Gamma_{GL_m,n}$ across the diagonal, we get diagrams that look like the one in Figure 9. See also [Ki4].

Again, the diagonal along which the two posets are glued together is indicated by replacing O with Θ. If we reduce m, we chop columns off the left side of the diagram. More precisely, if we replace m with $m' < m$, then we chop off the leftmost $m - m'$ columns.

6. STANDARD MONOMIAL THEORY FOR HARMONICS: O_n CASE

Let O_n denote the group of $n \times n$ matrices preserving the standard inner product on $\mathbb{C}^n$. Then, as in the case of the symplectic group, we can set $V = M_{nk}$, and can consider the harmonic polynomials $H(V, O_n)$ for this action. We will always take $k \leq \frac{n}{2}$. Again from [Ho1], we know that we have a decomposition

$$H(V, O_n) \simeq \sum_D \tau_n^D \otimes \rho_k^D \tag{6.1}$$

as $O_n \times GL_k$ module. The sum here ranges over all Young diagrams with up to k rows.

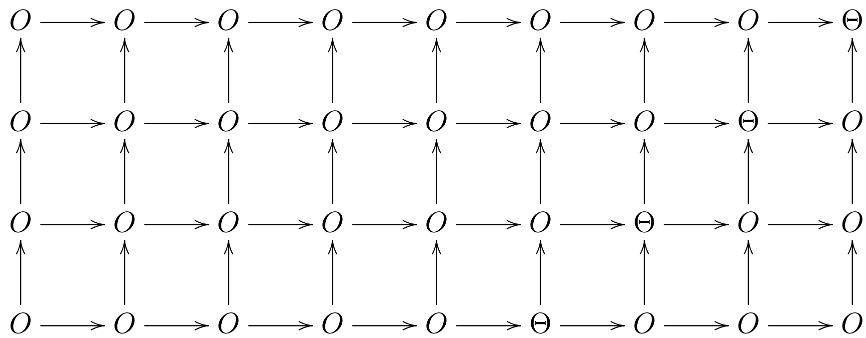

FIGURE 10. The poset $\Gamma_{GL_9,4}$.

As in the case of Sp_{2m}, we can consider the quotient ring $\overline{H}(V, O_n)$ associated with the space of harmonics, which is isomorphic to the subring of the tensor product $\mathbf{F}_k^{iso}(O_n) \otimes \mathbf{F}^+(GL_k)$. Here, as in Theorem 3.3, $\mathbf{F}_k^{iso}(O_n)$ is the isotropic flag algebra for O_n, as defined by Lancaster and Towber [LT1, LT2]. The arguments used for the symplectic case (but using the elements of the quotient $H(\Gamma_{GL_n,k})/J_{nk}$ for O_n, as described in Theorem 3.3, instead of the Hibi ring $H(\Gamma_{Sp_{2n}})$ for the Zhelobenko poset), we know that the associated graded algebra $Gr\overline{H}(V, O_n)$ can be described by the minimal extension Γ_{GL_n,GL_k} of two GT posets $\Gamma_{GL_n,k}$ and Γ_{GL_k} amalgamated along their top diagonals, as in (4.11). For instance, if $n = 9$ and $k = 4$, we get the rectangle in Figure 10.

Note that this poset is just the flip over the diagonal of the poset Γ_{GL_n,GL_k} for $GL_n \times GL_k$ acting on $P(M_{nk})$, in the case when $n > k$ (cf. Figure 5). The flipped poset (Figure 10), however, is associated with the degeneration of the isotropic flag algebra $\mathbf{F}_k^{iso}(O_n)$, and therefore with O_n modules. Therefore, we conclude that the associated graded algebra $Gr\overline{H}(V, O_n)$ can be described by the quotient of the Hibi ring corresponding to Γ_{GL_n,GL_k} by the ideal extended from J_{nk}.

7. Standard monomial theory for harmonics: Mixed actions of GL_n

We now consider GL_n acting on

$$V = V_{n,p,q} = ((\mathbb{C}^n)^* \otimes (\mathbb{C}^p)^*) \oplus (\mathbb{C}^n \otimes (\mathbb{C}^q)^*), \tag{7.1}$$

with both p and q positive. Here the harmonics look like

$$H(V_{n,p,q}, GL_n) \simeq \sum_{D,E} \rho_n^{D,E} \otimes \rho_p^D \otimes \rho_q^E. \tag{7.2}$$

The notation $\rho_n^{D,E}$ is explained in, e.g., [HKL1, Ho5]. The diagram D is describing the positive part of the highest weight, and the diagram E is describing the negative part. The diagrams D and E are subject to the restrictions

$$k = depth\ D \leq p, \qquad \ell = depth\ E \leq q, \qquad k + \ell \leq n. \tag{7.3}$$

7.1. **Stable range ($p + q < n$).** We consider first the stable range; precisely, the case when $p + q < n$. We can form the quotient algebra $\overline{H}(V, GL_n)$, just as when G is the symplectic group. Also, as in the discussion of GL_n acting on $P(M_{nm})$, we can filter according to the highest weights and then form the associated graded algebra, thereby converting the problem into describing the subalgebra of $\mathbf{F}(GL_n) \otimes \mathbf{F}^+(GL_p) \otimes \mathbf{F}^+(GL_q)$ defined by the sum (7.2).

There are two new issues here, beyond the case of $P(M_{nm})$. One is that there are three factors rather than just two. The other is that the representations of GL_n are not polynomial representations; the GL_n highest weights have some negative components. This means that the associated GT patterns will have negative entries, so that they do not immediately fit into the Hibi cone paradigm.

We deal with the second issue first. Thus we want to consider the subalgebra of $\mathbf{F}(GL_n)$ defined by the sum

$$\sum_{D,E} \rho_n^{D,E}$$

with depth $D \leq p$ and depth $E \leq q$. These will correspond to GT patterns for which the entries on the top diagonal satisfy

$$f(\begin{bmatrix} -j \\ -j \end{bmatrix}) = d_j \qquad \text{for} \quad 1 \leq j \leq p,$$
$$f(\begin{bmatrix} -n-1+j \\ -n-1+j \end{bmatrix}) = -e_j \qquad \text{for} \quad 1 \leq j \leq q, \tag{7.4}$$
$$f(\begin{bmatrix} -j \\ -j \end{bmatrix}) = 0 \qquad \text{for} \quad p < j \leq n-q.$$

All GT patterns having top diagonal as specified by (7.4) will have all entries equal to zero in the whole subdiagonal triangle $T(-n+q, -p-1)$ (cf. (3.2)). Thus, they are effectively functions on the set

$$\Gamma'_{n,p,q} = T(-n, -1) - T(-n+q, -p-1). \tag{7.5}$$

See Figure 11 for a picture of $\Gamma'_{9,3,3}$.

However, not all order preserving functions on $\Gamma'_{n,p,q}$ will come from GT patterns, because the fact that the entries below the deleted triangle $T(-n+q, -p-1)$ must be non-positive is not enforced by the order relation on $\Gamma'_{n,k,\ell}$, even if all the diagonal elements weakly below $\begin{bmatrix} -n-1+q \\ -n-1+q \end{bmatrix}$ are non-positive. Similarly, the entries to the right of $T(-n+q, -p-1)$ are not forced to be non-negative. To make sure that all entries in $\Gamma'_{n,k,\ell}$ have the correct signs, we can add the element $\begin{bmatrix} -p-1 \\ -n+q \end{bmatrix}$ (which is the lower right hand corner of the triangular set $T(-n+q, -p-1)$) and assign it the value 0. Thus, we set

$$\Gamma''_{n,p,q} = (T(-n, -1) - T(-n+q, -p-1)) \cup \left\{ \begin{bmatrix} -p-1 \\ -n+q \end{bmatrix} \right\}. \tag{7.6}$$

Then the GT patterns that we need for our representations are the $\mathbb{Z}$-valued, order preserving functions on that take the value 0 at $\begin{bmatrix} -p-1 \\ -n+q \end{bmatrix}$. This is the content in this situation of the construction of Proposition 2.7. In the notation of (2.14) and Proposition 2.7, $X = T(-n, -1)$, $x_1 = \begin{bmatrix} -n+q \\ -n+q \end{bmatrix}$, $x_2 = \begin{bmatrix} -p-1 \\ -p-1 \end{bmatrix}$, and $I(x_1, x_2) = T(-n+q, -p-1)$. We also have $X_{x_1 \simeq x_2} = \Gamma''_{n,p,q}$ where we have identified the element $I(x_1, x_2)$ of $X_{x_1 \simeq x_2}$ with the point $\begin{bmatrix} -p-1 \\ -n+q \end{bmatrix}$. For a picture of $\Gamma''_{9,3,3}$, see Figure 12.

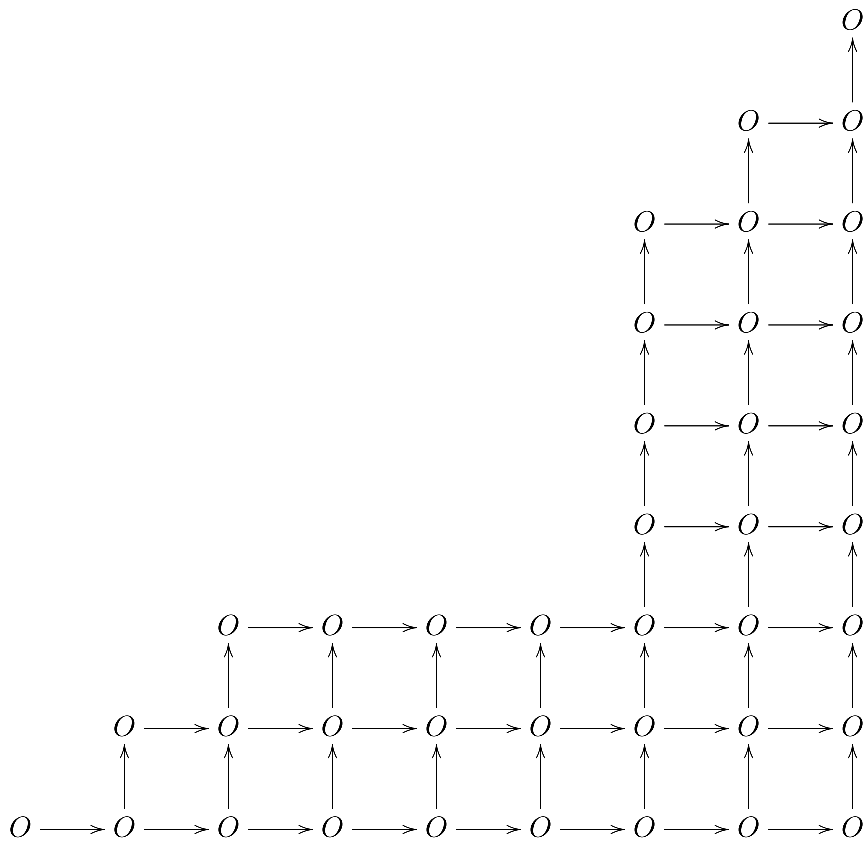

FIGURE 11. The poset $\Gamma'_{9,3,3}$.

By adding the extra point to $\Gamma'_{n,p,q}$, and requiring it to have value zero, we ensure that the order-preserving functions on $\Gamma''_{n,p,q}$ will be GT patterns of representations of the form $\rho_n^{D,E}$, with D and E as specified. However, they still take negative values, and we now have the added issue of a point where all functions must vanish. It turns out that we can deal with both these issues by an easy construction. The key observation is that $\Gamma''_{n,p,q}$ has a minimum element, namely $\begin{bmatrix} -n \\ -n \end{bmatrix}$.

Lemma 7.1. a) *Let X be a poset with minimum element x_o. That is, $x_o \leq x$ for all x in X. Let $\mathbb{Z}^{X,\geq}$ be the semigroup of all integer-valued, order-preserving functions on X. Then the mapping*

$$\nu : f \to \nu(f) \tag{7.7}$$

where $\nu(f)$ is the function on $X - \{x_o\}$ given by

$$\nu(f)(x) = f(x) - f(x_o) \tag{7.8}$$

is a surjective mapping from the lattice cone $\mathbb{Z}^{X,\geq}$ to the Hibi cone $(\mathbb{Z}^+)^{(X-\{x_o\},\geq)}$. The kernel of ν consists of the constant functions.

b) *For a point x in X, let $\mathbb{Z}_x^{(X,\geq)}$ be the subcone of functions in $\mathbb{Z}^{(X,\geq)}$ that vanish at x. Then ν is an isomorphism from $\mathbb{Z}_x^{(X,\geq)}$ to the Hibi cone $(\mathbb{Z}^+)^{(X-\{x_o\},\geq)}$.*

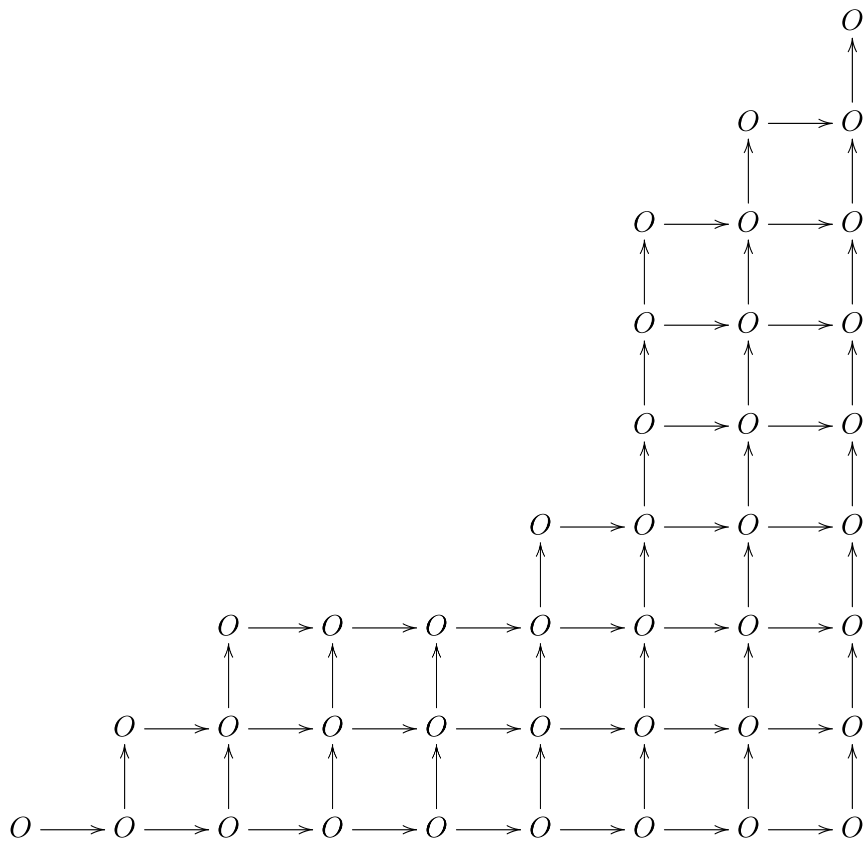

FIGURE 12. The poset $\Gamma''_{9,3,3}$.

Proof. It is straightforward to check that ν takes sums to sums. Since x_0 is the minimum element of X, any function f in $\mathbb{Z}^{(X,\geq)}$ will satisfy $f(x) \geq f(x_o)$ for any x in X, so that $\nu(f)$ will take on non-negative values, and thus will belong to the Hibi cone $(\mathbb{Z}^+)^{(X-\{x_o\},\geq)}$.

Also, it is clear that the constant functions will be in the kernel of ν. Since the rank of $\mathbb{Z}^{(X-\{x_o\},\geq)}$ is just one less than the rank of $\mathbb{Z}^{(X,\geq)}$, it follows that the kernel of ν must be exactly the constant functions; this can also be verified directly from the definition of ν.

Since x_o is the minimum element of X, if f is any element of the Hibi cone $(\mathbb{Z}^+)^{(X-\{x_o\},\geq)}$, then the extension $\tilde{f}$ of f to X such that $\tilde{f}(x_o) = 0$ will belong to the Hibi cone of X, which of course is a subcone of $\mathbb{Z}^{(X,\geq)}$. Another simple check gives $\nu(\tilde{f}) = f$, so that ν is surjective.

Let $\mathbf{1}$ denote the function on X that takes the value 1 at every point. Fix an element x of X. Then any element f of $\mathbb{Z}^{(X,\geq)}$ can be written as $f = (f - f(x)\mathbf{1}) + f(x)\mathbf{1}$; that is, as a sum of an element in $\mathbb{Z}_x^{(X,\geq)}$ and a constant function. This allows us to conclude that ν will be an isomorphism from $\mathbb{Z}_x^{(X,\geq)}$ to $(\mathbb{Z}^+)^{(X-\{x_o\},\geq)}$, as desired for part b). $\square$

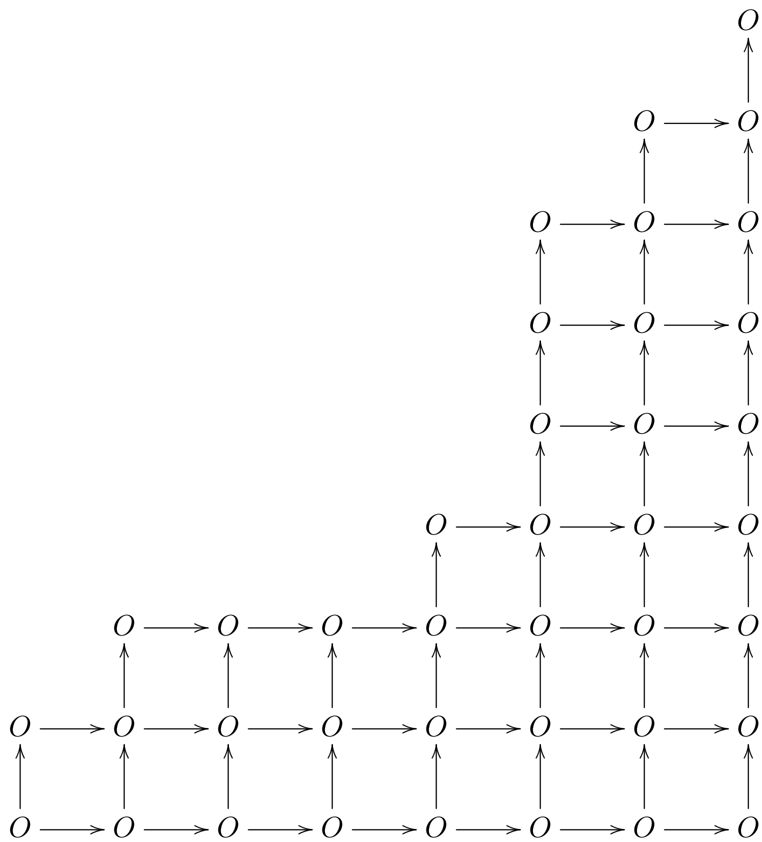

FIGURE 13. The poset $\Gamma'''_{9,3,3}$.

From Lemma 7.1, we may conclude that, if we set

$$\Gamma'''_{n,p,q} = \Gamma''_{n,p,q} - \left\{ \begin{bmatrix} -n \\ -n \end{bmatrix} \right\}, \tag{7.9}$$

then the semigroup of GT patterns with for representations $\rho_n^{D,E}$ with *depth* $(D) \leq k$ and *depth* $(E) \leq \ell$ is isomorphic to the Hibi cone on $\Gamma'''_{n,p,q}$. See Figure 13 for a picture of $\Gamma'''_{9,3,3}$.

We now want to bring the factors ρ_p^D and ρ_q^E to the story. Since D and E vary independently, it might seem that we should just be dealing with the tensor product $\mathbf{F}^+(GL_p) \otimes \mathbf{F}^+(GL_q)$ of polynomial flag algebras for GL_p and GL_q. However, we see that E is actually entering into the description of GL_n representations via its negative transpose, so for consistency's sake, we should replace ρ_q^E by its dual $(\rho_q^E)^*$, which we can do by dualizing the whole action of GL_q (which can be done by applying the automorphism $g \to (g^t)^{-1}$ before acting by GL_q). If we do this, we are looking at the summands $\rho_p^D \otimes (\rho_q^E)^*$ of the flag algebra $\mathbf{F}(GL_p \times GL_q)$. We note that for these representations, the entries of the GT diagrams for the representations ρ_p^D will always be non-negative, while the entries of the G-T diagrams for $(\rho_q^E)^*$ will be non-positive. In particular, all the entries in Γ_{GL_p} will be larger than all the entries Γ_{GL_q}. Thus, if we form the ordinal sum [Sta2, §3.2] (see also Proposition 2.2) $\Gamma_{GL_p} \oplus > \Gamma_{GL_q}$, then the semigroup of GT patterns that are non-negative on Γ_{GL_p} and non-positive on Γ_{GL_q} will combine to define a $\mathbb{Z}$-valued order-preserving function on $\Gamma_{GL_p}\oplus > \Gamma_{GL_q}$. However, there is nothing preventing a $\mathbb{Z}$-valued order-preserving function on $\Gamma_{GL_p}\oplus > \Gamma_{GL_q}$ from taking negative values on $\Gamma_{GL_p}\cup$ or positive-values on Γ_{GL_q}. To ensure that this does not happen, we introduce an extra point γ such that $\Gamma_{GL_p} \geq \gamma \geq \Gamma_{GL_q}$, and require our functions to take the value 0 at γ. This will then ensure the appropriate sign behavior. Summarizing, we see that the subalgebra of $\mathbf{F}^+(GL_p) \otimes \mathbf{F}^+(GL_q)$ spanned by representations of the form ρ_p^D and ρ_q^E has a standard monomial theory based on the semigroup of $\mathbb{Z}$-valued, order preserving functions on $\Gamma_{GL_p}\oplus > \Gamma_{GL_q} \cup \{\gamma\}$, that take the value 0 at γ.

We would like to have a description based on a Hibi cone. We can arrange this by a construction parallel to what we did above for GL_n. Indeed, $\Gamma_{GL_p}\oplus > \Gamma_{GL_q} \cup \{\gamma\}$ has a unique minimum element, which is just the minimum element of Γ_{GL_q}. To be precise, let's use the notation

$$\begin{aligned} \Gamma^{+}_{GL_p,GL_q} &= \Gamma_{GL_p}\oplus > \Gamma_{GL_q}, \\ \Gamma^{+'}_{GL_p.GL_q} &= \Gamma^{+}_{GL_p,GL_q} \cup \{\gamma\}, \\ \Gamma^{+''}_{GL_p,GL_q} &= \Gamma^{+'}_{GL_p.GL_q} - \{z_o\}, \end{aligned} \tag{7.10}$$

where z_o is the minimal element of Γ_{GL_q}, equivalently, the minimal element of $\Gamma^{+}_{GL_p,GL_q}$. We remark that the notation $\Gamma^{+}_{GL_p,GL_q}$ is similar to the one without superscript + used at the end of §6. We hope that the reader will not be confused.

We have just seen that the Hibi cone on the set $\Gamma^{+''}_{GL_p,GL_q}$ describes the GT patterns for the subalgebra of $\mathbf{F}(GL_p \times GL_q)$ spanned by the representations $\rho_p^D \otimes (\rho_q^E)^*$. Above, we saw that the GT patterns for the subalgebra of $\mathbf{F}(GL_n)$ spanned by representations $\rho_n^{D,E}$ can be described by the Hibi cone for $\Gamma'''_{n,p,q}$. Now we would like to put them together. As in the case of the action of GL_n on $P(M_{nm})$, the conditions on the G-T posets again match the top diagonals with each other, and it is not hard to check that if we add the "null point" of each set - the point that was assigned the value zero before shifting by the value at the minimum element, then we can identify these two points, and the identification of diagonal elements extends to an order isomorphism of the diagonals with the null points adjoined. We will call these the *extended top diagonals* of $\Gamma'''_{n,p,q}$ and $\Gamma^{+''}_{GL_p,GL_q}$ respectively. Also, the mappings described by Lemma 7.1 to produce Hibi cones are consistent with each other: they match the respective minimum points, and they match null points. Taking all these facts into account, we arrive at the following statement:

Theorem 7.2. *When $p + q < n$, the algebra $\overline{H}(V_{n,p,q}, GL_n)$ has a flat deformation to a Hibi ring based on the poset formed by the minimal extension of $\Gamma'''_{n,p,q}$ and $\Gamma^{+''}_{GL_p,GL_q}$ amalgamated along their extended top diagonals.*

Remark 7.3. *We can use the same trick as in the case of $P(M_{nm})$ and flip the poset $\Gamma^{+''}_{GL_p,GL_q}$ across the diagonal, to represent the poset for $\overline{H}(V_{n,p,q}, GL_n)$ as a nice subset of $\mathbb{Z}^2$. For example, the case of $n = 9, p = 3 = q$ produces the diagram in Figure 14. Again the Θs mark the points of the subset of amalgamation.*

7.2. **Beyond stable range ($p + q \geq n$).** We would like to extend the above discussion to include cases when $p + q \geq n$. First, we should remark that the case of $p + q = n$ is in

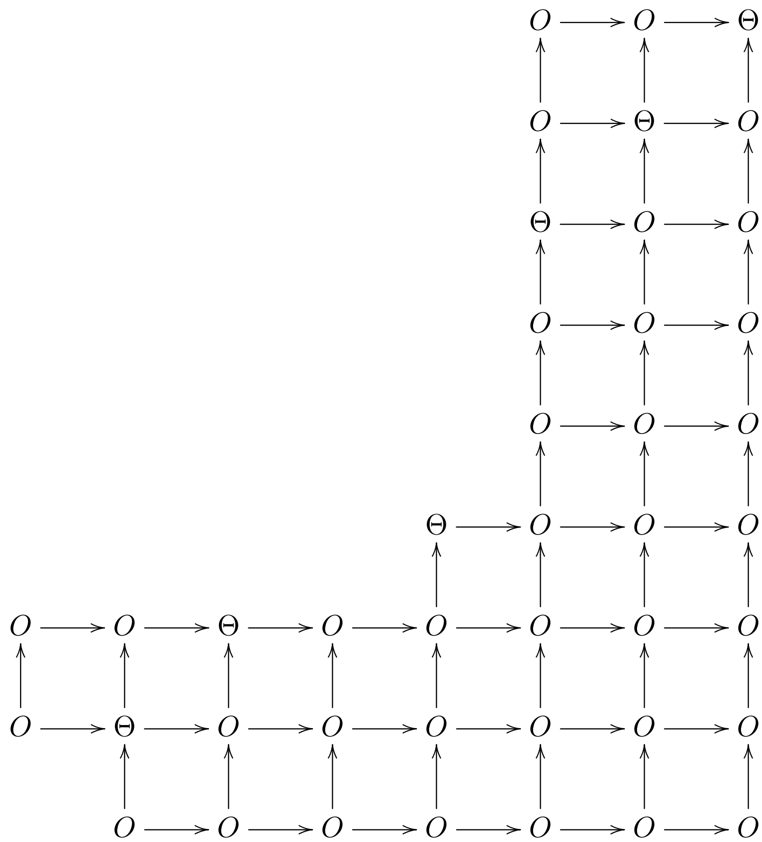

FIGURE 14. The poset for $\overline{H}(V_{9,3,3}, GL_n)$.

fact essentially the same as $p+q < n$. The difference is that the triangle $T(-n+q, -p-1)$ is empty, so there are no points to eliminate. However, to enforce the condition that the highest weight turn from non-negative to non-positive between $\begin{bmatrix} -p \\ -p \end{bmatrix}$ and $\begin{bmatrix} -n-1+q \\ -n-1+q \end{bmatrix}$, we adjoin the point

$$\gamma_1 = \begin{bmatrix} -p-1 \\ -n+q \end{bmatrix}, \tag{7.11}$$

which is just above the main diagonal, and sits between $\begin{bmatrix} -n-1+q \\ -n-1+q \end{bmatrix}$ and $\begin{bmatrix} -p \\ -p \end{bmatrix}$ in the order on $\mathbb{Z}^2$; and we assign the value 0 to γ_1. With this understanding of $\Gamma''_{n,p,q}$, the discussion above extends to the case of $p+q = n$.

To deal with the situation when $p+q > n$, we must take into account that the quotient algebra $\overline{H}(V_{n,p,q})$ may fail to be a domain. This can be understood in terms of the geometry of the null cone for the invariants. Again, let $J^2(V, GL_n)$ be the vector space of homogeneous, quadratic, GL_n-invariant polynomials on V. We will review the natural geometric interpretation of $J^2(V, GL_n)$. Up to this point, we have been somewhat cavalier about the distinction between $\mathbb{C}^n$ and $(\mathbb{C}^n)^*$. This is not too crucial as long as only one of the two are involved, but when both are, we should be somewhat careful. In

(7.1), we have specified that

$$V = ((\mathbb{C}^n)^* \otimes (\mathbb{C}^p)^*) \oplus (\mathbb{C}^n \otimes (\mathbb{C}^q)^*) \simeq \mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^n) \oplus \mathrm{Hom}(\mathbb{C}^n, (\mathbb{C}^p)^*). \tag{7.12}$$

However, consistent with our replacement of ρ_q^E by $(\rho_q^E)^*$ in the discussion above of the stable range, we should here replace $(\mathbb{C}^q)^*$ by $\mathbb{C}^q$. This will replace the description (7.12) with

$$V = ((\mathbb{C}^n)^* \otimes (\mathbb{C}^p)^*) \oplus (\mathbb{C}^n \otimes \mathbb{C}^q)) \simeq \mathrm{Hom}((\mathbb{C}^n)^*, \mathbb{C}^q) \oplus \mathrm{Hom}(\mathbb{C}^p, (\mathbb{C}^n)^*). \tag{7.13}$$

There is a natural quadratic mapping $Q : V \to \mathrm{Hom}(\mathbb{C}^p, \mathbb{C}^q) \simeq M_{qp}$, given by

$$Q(T, S) = S \circ T = ST, \qquad \text{for} \quad T \in \mathrm{Hom}(\mathbb{C}^p, (\mathbb{C}^n)^*) \quad \text{and} \quad S \in \mathrm{Hom}((\mathbb{C}^n)^*, \mathbb{C}^q). \tag{7.14}$$

This mapping Q is easily seen to be GL_n invariant, so pulling back polynomials from M_{qp} by Q will produce GL_n invariant polynomials on V. Weyl's FFT for the action of GL_n on V can be formulated as saying that the mapping

$$Q^* : P(M_{qp}) \to J(V_{n,p,q}, GL_n) \tag{7.15}$$

is a surjection of algebras and of $GL_p \otimes GL_q$ modules.

Remark 7.4. *The statement of* (7.15)*, combined with the double tableaux construction at the beginning of this section amount to standard monomial theory of Lakshmibai et. al.* [LRSS, LaSh] *for* $J(V_{n,p,q}, GL_n)$.

This discussion shows that the variety of elements on which all invariants with zero constant term vanish (the "null-cone" for the action) consists of the points (T, S) such that $ST = 0$. This will happen exactly when $\mathrm{im}\, T \subset \ker S$, and it implies that $\mathrm{rank}\ T = \dim \mathrm{im}\, T \leq \dim \ker S = n - \dim \mathrm{im}\, S = n - \mathrm{rank}\ S$; or in other words, $\mathrm{rank}\ T + \mathrm{rank}\ S \leq n$. Of course also, $\mathrm{rank}\ T \leq q$ and $\mathrm{rank}\ S \leq p$, so the condition $\mathrm{rank}\ T + \mathrm{rank}\ S \leq n$ is automatically satisfied if $p + q \leq n$. However, if $p + q > n$, then the requirement that $\mathrm{rank}\ T + \mathrm{rank}\ S \leq n$ puts an additional limitation on the pairs (S, T) in the null cone. Moreover, when $p + q > n$, for a generic subset of the null cone, it will be the case that $\mathrm{rank}\ T + \mathrm{rank}\ S = n$, which is equivalent to saying that $\mathrm{im}\, T = \ker S$. Evidently there will be a number of choices for pairs $(\mathrm{rank}\ T, \mathrm{rank}\ S) = (k, \ell)$ such that

$$a)\ k \leq q, \qquad b)\ \ell \leq p, \qquad \text{and} \qquad c)\ k + \ell = n. \tag{7.16}$$

The collection of (S, T) corresponding to a given pair (k, ℓ) will clearly be invariant under the action of $GL_n \times GL_p \times GL_q$. It is not hard to convince oneself that, conversely, if we fix $\mathrm{rank}\ T = k \leq q$ and $\mathrm{rank}\ S = \ell \leq p$, with $k + \ell = n$, then the collection of pairs (T, S) with these ranks and such that $ST = 0$ comprise a single $GL_n \times GL_p \times GL_q$ orbit; and in particular, these constitute a connected variety.

Thus, we see that, when $p + q > n$, the null cone of the GL_n invariants breaks up into several connected components - it is a reducible variety, and correspondingly, its coordinate ring

$$P(V)/J^2(V, GL_n) \simeq \overline{H}(V, GL_n)$$

will not be a domain. Instead, it will be the sum of several domains, corresponding to the irreducible components of the null cone.

There is a parallel structure in the harmonics $H(V_{n,p,q}, GL_n)$ as described in formula (7.2). For each pair (k, ℓ) satisfying the conditions (7.16), we can consider the subsum $\mathrm{A}_{k,\ell}$ of the sum (7.2) of $H(V, GL_n)$ for which the depths of D and E are bounded by k

and ℓ. These sums will not be quite disjoint, since we only require inequalities in (7.16); but their intersections are in some sense of lower order.

Given a choice (k,ℓ) satisfying (7.16), we can give a Hibi ring description of the algebra $\mathrm{A}_{k,\ell}$ similar to that given above in the stable range, or more precisely, its extension to the case of $p+q=n$ sketched just after that discussion. The main ingredients are these:

i) The subalgebra of $\mathbf{F}(GL_n)$ given by the sum $\sum_{D,E}\rho_n^{D,E}$ with depth $D\leq k$ and depth $E\leq\ell$ is described as a Hibi ring on the set $\Gamma''_{n,k,\ell}$ as in the stable range discussion.

ii) Similarly, the subalgebra of $\mathbf{F}(GL_p\times GL_q)$ spanned by the sum $\sum_{D,E}\rho_p^D\otimes(\rho_q^E)^*$ is described as a Hibi ring on the set

$$\Gamma^{+''}_{GL_p,k,GL_q,\ell}=(\Gamma_{GL_p,k}\oplus>(\Gamma_{GL_q,\ell})^*-\{z_o\})\cup\{\gamma\}.$$

Here we have written $(\Gamma_{GL_q,\ell})^*$ rather than $\Gamma_{GL_q,\ell}$ to show that we are dealing with duals of polynomial representations. The set $(\Gamma_{GL_q,\ell})^*$ is the order opposite of $\Gamma_{GL_q,\ell}$. Up to translation in $\mathbb{Z}^2$, it can be obtained from $\Gamma_{GL_q,\ell}$ by changing the signs of the coordinates of the points in it. Another way of describing it is, delete the rightmost $q-\ell$ columns from Γ_{GL_q}. Of course, $(\Gamma_{GL_q,q})^*=\Gamma_{GL_q,q}=\Gamma_{GL_q}$. So the distinction was not necessary in the stable range discussion.

iii) The sets $\Gamma''_{n,k,\ell}$ and $\Gamma^{+''}_{GL_p,k,GL_q,\ell}$ are then amalgamated along their top diagonals, as in the stable range discussion.

Unfortunately, however, the simple geometric picture, as illustrated in Figure 14, is no longer valid, because the reflections over the diagonal of the sets $\Gamma_{GL_p,k}$ and $(\Gamma_{GL_q,\ell})^*$ will overlap when $p>k$ and $q>\ell$, and the reflection of either set will engulf the point γ_1 defined in (7.11). We do not know a simple embedding of $\Gamma^{+''}_{GL_p,k,GL_q,\ell}$ in $\mathbb{Z}^2$. We hope to return to this question.

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