Long Sequences of Descending Theories and other Miscellenia on Slow Consistency*

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Abstract

For a provably recursive function $f : \mathbb{N} \to \mathbb{N}$ of **PA** one can consider the notion of f-consistency for **PA**, $\mathrm{Con}_f(\mathbf{PA}) := \forall x\, \mathrm{Con}(\mathbf{PA} \restriction_{f(x)})$, where $\mathbf{PA} \restriction_k$ denotes the fragment of **PA** with induction restricted to Σ_k formulae. It was shown in [8] that for a certain slow growing function f the strength of $\mathbf{PA} + \mathrm{Con}_f(\mathbf{PA})$ lies strictly between **PA** and $\mathbf{PA} + \mathrm{Con}(\mathbf{PA})$. Letting τ_{BH} be the Bachmann-Howard ordinal, this paper exhibits for every $\alpha < \tau_{\mathrm{BH}}$, a hierarchy of ever slower functions $(f_\xi)_{\xi<\alpha}$ of length α such that for $\zeta < \xi < \alpha$ one has $\mathbf{PA} \lhd \mathbf{PA} + \mathrm{Con}_{f_\xi}(\mathbf{PA}) \lhd \mathbf{PA} + \mathrm{Con}_{f_\zeta}(\mathbf{PA}) \lhd \mathbf{PA} + \mathrm{Con}(\mathbf{PA})$, where $T_1 \lhd T_2$ conveys that T_2 interprets T_1 but T_1 does not interpret T_2. This confirms a conjecture stated in [8, 3.2.2].

It is also observed that the axioms of Gödel-Löb logic, **GL**, hold for any provability interpretation embodying slow provability. As a result, one obtains the equivalent of Solovay's completeness theorem for **GL** for all of these slow provability notions.

Keywords: Peano arithmetic, Bachmann-Howard ordinal, consistency strength, interpretation, fast growing function, slow consistency, slow provability interpretations
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1 Introduction

The question whether there are "natural" theories lying strictly between Peano arithmetic, **PA**, and **PA** augmented by the standard consistency statement Con(**PA**) informed the paper [8] and led to a notion of slow consistency. It was also shown ([8, 4.2]) that iterating slow consistency n-times gives rise to string of n theories of increasing strength lying between **PA** and **PA** + Con(**PA**). In the meantime, it has been shown by A. Freund [6] as well as P. Henk and F. Pakhomov [11] that iterating this operation through all the ordinals $< \varepsilon_0$ yields a hierarchy of theories strictly residing between **PA** and **PA** + Con(**PA**), confirming a conjecture stated in [8, 4.4]. The hierarchy also reaches Con(**PA**) at level ε_0 (see e.g. [6, 3.6]) and thus cannot be extended below **PA** + Con(**PA**). The current paper will show that there are much longer **descending** sequences of theories between **PA** + Con(**PA**) and **PA**.

For a provably recursive function $f : \mathbb{N} \to \mathbb{N}$ of **PA** one can consider the notion of f-consistency for **PA**, $\mathrm{Con}_f(\mathbf{PA}) := \forall x \, \mathrm{Con}(\mathbf{PA}\restriction_{f(x)})$, where $\mathbf{PA}\restriction_k$ denotes the fragment of **PA** with induction restricted to Σ_k formulae. It was shown in [8] that for a certain slow growing function f the strength of $\mathbf{PA} + \mathrm{Con}_f(\mathbf{PA})$ lies strictly between **PA** and **PA** + Con(**PA**). Letting τ_{BH} be the Bachmann-Howard ordinal, this paper exhibits for every $\alpha < \tau_{\text{BH}}$, a hierarchy of ever slower functions $(f_\xi)_{\xi<\alpha}$ of length α such that for $\zeta < \xi < \alpha$ one has $\mathbf{PA} \lhd \mathbf{PA} + \mathrm{Con}_{f_\xi}(\mathbf{PA}) \lhd \mathbf{PA} + \mathrm{Con}_{f_\zeta}(\mathbf{PA}) \lhd \mathbf{PA} + \mathrm{Con}(\mathbf{PA})$, where $T_1 \lhd T_2$ conveys that T_2 interprets T_1 but T_1 does not interpret T_2.[1] This confirms ruminations stated in 3.2.2 in [8].

The paper is organized as follows. Section 2 introduces an ordinal representation system for the Bachmann-Howard ordinal together with an assignment of fundamental sequences. This gives rise to the hierarchies of slow and fast growing functions along this ordinal. Some properties about these hierarchies that can be proved in **PA** are considered in section 3. Section 4 presents the heart of the paper as described above. The final section 5 contains the miscellaneous observation that the axioms of Gödel-Löb logic, **GL**, hold for any provability interpretation embodying slow provability. As a consequence, one obtains the equivalent of Solovay's completeness theorem

[1]Let S and S' be arbitrary theories. S' is *interpretable in* S or S *interprets* S' (in symbols $S' \unlhd S$) "if roughly speaking, the primitive concepts and the range of the variables of S' are defined in such a way as to turn every theorem of S' into a theorem of S" (quoted from [15] p. 96; for details see [15, section 6]).

For theories S and S' such that $\mathbf{PA} \subseteq S, S'$ having the same language as **PA**, $S' \unlhd S$ is actually equivalent to saying that every Π_1 statement provable in S is also provable in S'. This is due to Guaspari [10] and Lindström [14] (see also [15, Theorem 6]).

for **GL** for all of these slow provability notions.

2 An ordinal representation system for the Bachmann-Howard ordinal

There are many articles featuring ordinal representation systems for the Bachmann-Howard ordinal. Here we shall use a syntactic approach that suits our purposes. We first define a set of terms $\mathcal{T}$ and a linear ordering $\prec$ on $\mathcal{T}$. The desired ordinal representation system $\mathcal{T}(\Omega)$ will then arise as a proper subset of $\mathcal{T}$. $\mathcal{T}(\Omega)$ will also be equipped with an assignment of fundamental sequences, giving rise to hierarchies of fast and slow growing functions.

Definition 2.1. The set of terms $\mathcal{T}$, its *principal terms*, and the relation $\prec$ are defined inductively by the following clauses. Below $a \preccurlyeq b$ stands for $a \prec b \vee a = b$.

1. $0 \in \mathcal{T}$ and $1 \in \mathcal{T}$. 1 is a principal term.
2. If $a \in \mathcal{T}$, then $\psi a \in \mathcal{T}$ and ψa is a principal term.
3. If $a_0, \ldots, a_n \in \mathcal{T}$ are principal terms, $n \geq 1$, and $a_n \preccurlyeq \ldots \preccurlyeq a_0$, then $(a_0, \ldots, a_n) \in \mathcal{T}$.
4. If $a, b \in \mathcal{T}$, $0 \prec a$ and b is a principal term of either form 1 or ψc, then $\Omega^a b \in \mathcal{T}$ is a principal term.
5. If $a \in \mathcal{T}$ and $a \neq 0$ then $0 \prec a$.
6. $1 \prec a$ for any $a \in \mathcal{T}$ such that $a \neq 0$ and $a \neq 1$.
7. If $b \prec a$ then $\psi b \prec \psi a$.
8. $\psi a \prec \Omega^c b$ whenever $\psi a, \Omega^c b \in \mathcal{T}$.
9. If $\Omega^a b, \Omega^c d \in \mathcal{T}$, then $\Omega^a b \prec \Omega^c d$ whenever $a \prec c$ or $a = c$ and $b \prec d$.
10. If $a = (a_0, \ldots, a_n), b \in \mathcal{T}$, $a_0 \prec b$ and b is a principal term, then $a \prec b$.
11. If $a = (a_0, \ldots, a_n), b \in \mathcal{T}$ and b is a principal term $\preccurlyeq a_0$ then $b \prec a$.
12. If $a = (a_0, \ldots, a_n)$, $b = (b_0, \ldots, b_m)$, $a, b \in \mathcal{T}$, and there exists $i \leq \min(m, n)$ such that $a_i \prec b_i$ and $\forall j < i\, a_j = b_j$, then $a \prec b$.

13. If $a = (a_0, \ldots, a_n)$, $a' = (a_0, \ldots, a_n, a_{n+1}, \ldots, a_m)$ and $a, a' \in \mathcal{T}$, then $a \prec a'$.

From now on, we will use the shorthand Ω for $\Omega^1 1$.

Corollary 2.2. *$\prec$ furnishes $\mathcal{T}$ with a linear ordering. However, it is not a well-ordering since e.g.* $\psi\Omega \succ \psi(\psi\Omega) \succ \psi(\psi\psi(\Omega)) \succ \psi(\psi(\psi(\psi\Omega))) \succ \ldots$.

Definition 2.3. The terms $0, 1, (1,1), (1,1,1), \ldots$ will be identified with the natural numbers.

We define $\omega := \psi 0$. One then has $a \prec \omega$ iff a is a natural number in this sense.

A term $(a_0, \ldots, a_n)$ of $\mathcal{T}$ can be viewed as a sum of $a_0, \ldots, a_n$ (in the ordinal sense). We extend this to all terms of $\mathcal{T}$ as follows.

1. $a + 0 := 0 + a := a$.

2. Identifying a principal term b with (b), we can write any non-zero term $a \in \mathcal{T}$ as $(a_0, \ldots, a_n)$, where $a_0, \ldots, a_n$ are principal terms and $a_n \preccurlyeq \ldots a_0$, of course allowing the case $n = 0$. We then define

$$(a_0, \ldots, a_n) + (b_0, \ldots, b_m) := (a_0, \ldots, a_{k-1}, b_0, \ldots, b_m)$$

   where $k := \max\{l \leq n+1 \mid \forall i < l \, b_0 \preccurlyeq a_i\}$.

The operation $+$ is obviously associative on $\mathcal{T}$. Clearly for $(a_0, \ldots, a_n) \in \mathcal{T}$, we then have $(a_0, \ldots, a_n) = a_0 + \ldots + a_n$.

For principal terms a put $a \cdot 0 := 0$ and $a \cdot (n+1) := (a \cdot n) + a$, where n is a natural number.

We also define $\Omega^a c$ for arbitrary $a \in \mathcal{T}$ and $c \prec \Omega$. Let $\Omega^0 c := c$ and $\Omega^a 0 := 0$. For $c = (c_0, \ldots, c_n)$ put $\Omega^a c := \Omega^a c_0 + \ldots + \Omega^a c_n$.

Henceforth we shall often write Ω^a for $\Omega^a 1$.

Next we define for each $c \in \mathcal{T}$ a *distinguished fundamental sequence* $(c[x])_{x \prec \mathsf{tp}(c)}$.

Definition 2.4. 1. $\mathsf{tp}(0) := 0$, $\mathsf{tp}(1) := 1$ and $1[0] := 0$.

2. If $(a_0, \ldots, a_n) \in \mathcal{T}$ with $n > 0$, then $\mathsf{tp}(a_0, \ldots, a_n) = \mathsf{tp}(a_n)$ and $(a_0, \ldots, a_n)[x] := a_0 + \ldots + a_{n-1} + (a_n[x])$ for $x \prec \mathsf{tp}(a_n)$.

3. If $\Omega^a b \in \mathcal{T}$ and b is a principal term other than 1, then $\mathsf{tp}(\Omega^a b) := \mathsf{tp}(b)$ and $(\Omega^a b)[x] := \Omega^a b[x]$ for $x \prec \mathsf{tp}(b)$.

4. If $\Omega^a \in \mathcal{T}$, $a \neq 0$ and $\mathsf{tp}(a) \neq 1$, then $\mathsf{tp}(\Omega^a) = \mathsf{tp}(a)$ and $\Omega^a[x] := \Omega^{a[x]}$ for $x \prec \mathsf{tp}(a)$.

5. If $\Omega^a \in \mathcal{T}$ and $\mathsf{tp}(a) = 1$, then $\mathsf{tp}(\Omega^a) = \Omega$ and $\Omega^a[x] := \Omega^{a[0]}x$ for $x \prec \Omega$.

6. Recall that $\omega = \psi 0$. $\mathsf{tp}(\omega) = \omega$ and $\omega[n] := n + 1$ for $n \prec \omega$.

7. If $\psi a \in \mathcal{T}$ and $\mathsf{tp}(a) = \omega$, then $\mathsf{tp}(\psi a) = \omega$ and $(\psi a)[n] := \psi a[n]$ for $n \prec \omega$.

8. If $\psi a \in \mathcal{T}$ and $\mathsf{tp}(a) = \Omega$, then $\mathsf{tp}(\psi a) = \omega$ and $(\psi a)[0] := \psi a[0]$ and $(\psi a)[n+1] := \psi a[(\psi a)[n]]$.

9. If $\psi a \in \mathcal{T}$ and $\mathsf{tp}(a) = 1$, then $\mathsf{tp}(\psi a) := \omega$, $(\psi a)[n] := \psi a[0] \cdot (n + 1)$.

Note that clause 5 above entails that $\mathsf{tp}(\Omega) = \Omega$ and $\Omega[x] = x$ for $x \prec \Omega$.

Corollary 2.5. *(i) For $a \in \mathcal{T}$, $\mathsf{tp}(a)$ is either 0,1, ω or Ω.*

(ii) For $a \in \mathcal{T}$ with $a \prec \Omega$, $\mathsf{tp}(a)$ is either 0,1, or ω.

Lemma 2.6. *1. $\mathsf{tp}(a) = 0$ iff $a = 0$.*

2. $\mathsf{tp}(a) = 1$ iff $a = a[0] + 1$.

3. If a is not a term of either form $0, 1, \omega, \Omega$, then $\mathsf{tp}(a) \prec a$.

4. If $x \prec \mathsf{tp}(a)$ then $a[x] \prec a$.

5. If $x \prec y \prec \mathsf{tp}(a)$ then $a[x] \prec a[y]$.

Definition 2.7. The *norm* of a term $a \in \mathcal{T}$, $\mathsf{N}(a)$ measures its syntactic complexity.

1. $\mathsf{N}(0) := 0$, $\mathsf{N}(1) := 1$.

2. $\mathsf{N}(\psi a) := \mathsf{N}(a) + 1$.

3. $\mathsf{N}(a_0, \ldots, a_n) := \mathsf{N}(a_0) + \cdots + \mathsf{N}(a_n)$.

4. $\mathsf{N}(\Omega^a b) := \mathsf{N}(a) + \mathsf{N}(b)$.

As $\mathcal{T}$ is not well-ordered we need to single out a subsystem that is. This will be achieved by collecting the subterms c of a that appear in the shape ψc in a.

Definition 2.8. 1. $\mathsf{K}\,0 := \mathsf{K}\,1 := \emptyset$.

2. $\mathsf{K}(a_0, \ldots, a_n) := \mathsf{K}\,a_0 \cup \ldots \cup \mathsf{K}\,a_n$.

3. $\mathsf{K}\,\psi a := \{a\} \cup \mathsf{K}\,a$.

4. $\mathsf{K}\,\Omega^a b := \mathsf{K}\,a \cup \mathsf{K}\,b$.

Definition 2.9. We give an inductive definition of the subset $\mathcal{T}(\Omega)$ of $\mathcal{T}$. We write $\mathsf{K}\,a \prec b$ to convey that for all $x \in \mathsf{K}\,a$, $x \prec b$.

1. $0, 1, \Omega \in \mathcal{T}(\Omega)$.

2. If $(a_0, \ldots, a_n) \in \mathcal{T}$ and $a_0, \ldots, a_n \in \mathcal{T}(\Omega)$, then $(a_0, \ldots, a_n) \in \mathcal{T}(\Omega)$.

3. $a \in \mathcal{T}(\Omega)$ and $\mathsf{K}\,a \prec a$ then $\psi a \in \mathcal{T}(\Omega)$.

4. If $\Omega^a b \in \mathcal{T}$ and $a, b \in \mathcal{T}(\Omega)$ then $\Omega^a b \in \mathcal{T}(\Omega)$.

Lemma 2.10. *1. If $a \in \mathcal{T}(\Omega)$, then $\mathsf{tp}(a) \in \mathcal{T}(\Omega)$.*

2. If $a, x \in \mathcal{T}(\Omega)$ and $x \prec \mathsf{tp}(a)$, then $a[x] \in \mathcal{T}(\Omega)$.

Proof: These results can be proved by induction on $\mathsf{N}\,(\alpha) + \mathsf{N}\,(x)$ within a weak fragment of **PA** (e.g. in the fragment with just Σ_1 induction). □

$\mathcal{T}(\Omega)$ is well-ordered by $\prec$, however, transfinite induction on $\prec$ cannot be proved in **PA**.

Convention. For the reminder of this article, lower case Greek letters $\alpha, \beta, \gamma, \ldots$ are always supposed to range over elements of $\mathcal{T}(\Omega)$.

In the representation system $\mathcal{T}(\Omega)$, the role of the first fixed point of the ordinal function $\xi \mapsto \omega^\xi$, known as ε_0, is played by $\psi\Omega$.

The Bachmann-Howard ordinal, τ_{BH}, is the first ordinal which is larger than any ordinal in $\mathcal{T}(\Omega) \cap \Omega := \{\alpha \in \mathcal{T}(\Omega) \mid \alpha \prec \Omega\}$. We shall often write $\alpha \prec \tau_{\text{BH}}$ rather than $\alpha \in \mathcal{T}(\Omega) \cap \Omega$.

Definition 2.11. For functions $f : \mathbb{N} \to \mathbb{N}$ we use exponential notation $f^0(x) = x$ and $f^{k+1}(x) = f(f^k(x))$ to denote repeated compositions of f.

We define two hierarchies of functions $G_\alpha, F_\alpha : \mathbb{N} \to \mathbb{N}$ for $\alpha \in \mathcal{T}(\Omega) \cap \Omega$.

$$\begin{aligned} G_0(n) &:= 0 \\ G_{\alpha+1}(n) &:= G_\alpha(n) + 1 \\ G_\alpha(n) &:= G_{\alpha[n]}(n) \text{ if } \mathsf{tp}(\alpha) = \omega \end{aligned}$$

$$\begin{aligned} F_0(n) &:= n + 1 \\ F_{\alpha+1}(n) &:= F_\alpha^{n+1}(n) \\ F_\alpha(n) &:= F_{\alpha[n]}(n) \text{ if } \mathsf{tp}(\alpha) = \omega. \end{aligned}$$

It is well known result (due to Girard [9]) that every function F_α in the fast growing hierarchy with $\alpha \prec \varepsilon_0$ is eventually majorized by a function G_β in the slow growing hierarchy for some $\beta \prec \tau_{\rm BH}$. Moreover, the "catching-up" doesn't happen earlier than $\tau_{\rm BH}$. Proofs of this hierarchy comparison theorem can be found in [32, 1, 33]. For the particular assignment of fundamental sequences used in the current paper, this is done in [1].

3 Capturing the F_α's and G_α's in PA

The definition of the functions F_α employs transfinite recursion on α. It is therefore not immediately clear how we can speak about these functions in arithmetic. Later on we shall need to refer to a definition of $F_\alpha(x) = y$ which works in an arbitrary model of **PA**. In [12] many facts about the functions F_α for $\alpha \leq \varepsilon_0$, as befits their definition, are proved by transfinite induction on the ordinals $\leq \varepsilon_0$. In [12] there is no attempt to determine whether they are provable in **PA** (let alone in weaker theories). In what follows we will have to assume that some of the properties of the F_α's even for $\alpha \prec \tau_{\rm BH}$ hold in all models of **PA**. As a consequence, we will have to establish these results in **PA**. As it turns out, this can be done via a formula of low complexity.

Lemma 3.1. *There is a Δ_0-formula expressing $F_\alpha(x) = y$ (as a predicate of α, x, y).*

Proof: This is shown in [29, 5.2] for the hierarchy up to ε_0 but by basically the same proofs it can be extended up to $\tau_{\rm BH}$. The main idea is that the computation of $F_\alpha(x)$ can be described as a rewrite systems, that is, as a sequence of manipulations of expressions of the form

$$F_{\alpha_1}^{n_1}(F_{\alpha_2}^{n_2}(\ldots(F_{\alpha_k}^{n_k}(n))\ldots)),$$

where $n_1, \ldots, n_k \in \omega - \{0\}$ and $\alpha_1 > \ldots > \alpha_k \geq 0$. □

Definition 3.2. The computation of $G_\alpha(x)$ is closely connected with the step-down relations of [12] and [24]. For convenience we define $(\alpha+1)[n] := \alpha$ and $0[n] :=$ for all $n \prec \omega$.

For $\alpha < \beta$ we write $\beta \underset{n}{\longrightarrow} \alpha$ if for some sequence of ordinals $\gamma_0, \ldots, \gamma_r$ we have $\gamma_0 = \beta$, $\gamma_{i+1} = \gamma_i[n]$, for $0 \leq i < r$, and $\gamma_r = \alpha$. If we also want to record the number of steps r, we shall write $\alpha \overset{r}{\underset{n}{\longrightarrow}} \beta$.

Note that if r is the smallest number such that $\alpha \overset{r}{\underset{n}{\longrightarrow}} 0$ then $r \geq G_\alpha(n)$. With a bit more effort one can also prove that

$$G_\alpha(n+1) \geq r \geq G_\alpha(n).$$

Lemma 3.3. *(i) Let $\alpha \underset{n}{\longrightarrow} \beta$, $\alpha \underset{n}{\longrightarrow} \gamma$, $\beta > \gamma$. Then $\beta \underset{n}{\longrightarrow} \gamma$.*

(ii) Let $\alpha \underset{n}{\longrightarrow} \beta$, $\beta \underset{n}{\longrightarrow} \gamma$. Then $\alpha \underset{n}{\longrightarrow} \gamma$.

Proof: This is evident from the definition. □

It is well known that for $\alpha \prec \varepsilon_0$, **PA** $\vdash \forall x\, \exists y\, F_\alpha(x) = y$. Owing to Gentzen, **PA** proves transfinite induction up to α. As a result, it is easy to prove in **PA** that $\forall \xi \preccurlyeq \alpha\, \forall x\, \exists y\, F_\xi(x) = y$ by induction on ξ.

The main technical tool for proving properties about the G_α's and F_α's for larger α is the following.

Theorem 3.4. *(i) For all $\alpha \in \mathcal{T}(\Omega)$, if $\alpha \prec \Omega$, then* **PA** $\vdash \forall x\, \exists y\, \alpha \overset{y}{\underset{x}{\longrightarrow}} 0$.

(ii) For $\beta \prec \tau_{BH}$, **PA** $\vdash \forall x\, \exists y\, G_\beta(x) = y$.

It is well-known that G_β is majorized by some F_α with $\alpha \prec \varepsilon_0$, however, most proofs (e.g. [9, 32, 4]) make use of transfinite induction up to τ_{BH}, a principle that is not available in **PA**. A proof of (i) can be obtained from the results in [33]. But very explicitly (i) is stated in Ulf Schmerl's paper [23, Theorem 18].

(ii) is an immediate consequence of (i). □

Lemma 3.5. *Let $\alpha \prec \tau_{BH}$ be a limit. Then* **PA** *proves the following statements:*

1. *If $x < y$ then $\alpha[y] \underset{1}{\longrightarrow} \alpha[x]$.*
2. *For all $\beta \prec \alpha$, if $x < y$ and $\alpha \underset{x}{\longrightarrow} \beta$, then $\alpha \underset{y}{\longrightarrow} \beta$.*
3. *For all $\beta \prec \alpha$ there exists x such that $\alpha \underset{x}{\longrightarrow} \beta$.*

Proof: These results are usually proved by transfinite induction on α (e.g. [12]). However, a careful analysis shows that it is enough to know that for all x and all $\beta \preccurlyeq \alpha$ there exists r such that $\beta \overset{r}{\underset{x}{\longrightarrow}} 0$, which is guaranteed by Theorem 3.4. Transfinite induction can then be replaced by ordinary induction on r. □

Lemma 3.6. *We use $F_\alpha(x)\downarrow$ to denote $\exists y\, F_\alpha(x) = y$. $F_\alpha \downarrow$ stands for $\forall x\, F_\alpha(x)\downarrow$.*

Fix $\alpha \prec \tau_{BH}$. The following are provable in **PA**:

(i) For any β and x, if $\alpha \underset{x}{\longrightarrow} \beta$ and $F_\alpha(x)\downarrow$, then $F_\beta(x)\downarrow$ and $F_\alpha(x) \geq F_\beta(x)$.

(ii) *For any* $\beta \prec \alpha$ *and* $x > 3$*, if* $\alpha \underset{x}{\rightarrow} \beta$ *and* $F_\alpha(x)\downarrow$*, then* $F_\beta(x+1)\downarrow$ *and* $F_\alpha(x) > F_\beta(x+1)$*.*

(iii) *For any* $\beta \preccurlyeq \alpha$*, if* $F_\beta(x)\downarrow$ *and* $x > y$*, then* $F_\beta(y)\downarrow$ *and* $F_\beta(x) \geq F_\beta(y)$*.*

(iv) *If* $\beta \preccurlyeq \alpha$ *and* $F_\alpha\downarrow$*, then* $F_\beta\downarrow$*.*

(v) *If* $i > 0$ *and* $F^i_\alpha(x)\downarrow$ *then* $x < F^i_\alpha(x)$*.*

Proof: Similar properties are stated in [29, Proposition 5.4].
(i): Use induction on r, where $\alpha \overset{r}{\underset{x}{\rightarrow}} \beta$.
(ii): Similar to (i).
(iii) follows from (i) and Lemma 3.5 since $\beta[x] \underset{1}{\rightarrow} \beta[y]$.
(iv) is a consequence of (ii) and Lemma 3.5.
(v): See [29, Proposition 5.4(i)] for a similar result. □

There is an additional piece of information that is provided by the particular coding and Δ_0 formula denoting $F_\alpha(x) = y$ used in [29, 5.2], namely that there is a fixed polynomial P in one variable such that for all $\alpha \prec \tau_{\text{BH}}$, the number of steps it takes to compute $F_\alpha(x)$ is always bounded by $P(F_\alpha(x))$.

4 The hierarchy

Definition 4.1. For each $\alpha \prec \tau_{\text{BH}}$ we shall define a hierarchy of functions $(F^*_\beta)_{\beta\prec\alpha}$. First let $\alpha^* := \max(\mathsf{K}\,\alpha \cup \{0\}) + \Omega^\alpha$. note that $\mathsf{K}\,\alpha^* \cup \mathsf{K}\,\alpha \prec \alpha^*$, thus $\mathsf{K}\,(\alpha^* + \Omega \cdot \alpha) \prec \alpha^*$. Moreover, for $0 \prec \beta \prec \alpha$ one shows by induction on the buildup of β that $\max(\mathsf{K}\,\beta \cup \{0\}) \preccurlyeq \max(\mathsf{K}\,\alpha \cup \{0\})$ and hence $\mathsf{K}\,(\alpha^* + \Omega^\beta) \prec \alpha^*$. As a result, for all $0 \prec \beta \prec \alpha$, $\psi(\alpha^* + \Omega^\beta) \in \mathcal{T}(\Omega)$ and whenever $0 \prec \zeta \prec \xi \prec \alpha$, then

$$\psi(\alpha^* + \Omega^\zeta) \prec \psi(\alpha^* + \Omega^\xi)\,.$$

Now put

$$F^*_\beta \;:=\; F_{\psi(\alpha^*+\Omega^{1+\beta})}$$

for $\beta \prec \alpha$.

We define f_β to be the 'logarithm' of F^*_β, i.e.,

$$f_\beta(n) \;:=\; \max(\{k < n \mid \exists y \leq n\, F^*_\beta(k) = y\} \cup \{0\})\,.$$

Note that f_β is a provably recursive function of **PA**. Also note that $\text{Con}_{f_\beta}(\mathbf{PA})$ is equivalent (in **PA**) to the statement

$$\forall x\,(F^*_\beta(x)\downarrow \rightarrow\; \text{Con}(\mathbf{PA}\restriction_x))\,.$$

Let T^α_β be the theory $\mathbf{PA} + \text{Con}_{f_\beta}(\mathbf{PA})$.

A result we shall draw on is that the ordinals $\psi(\alpha^*+\Omega^{1+\beta})$ are ε-numbers, i.e. a fixed point of the enumeration function of the additive principal numbers, $\xi \mapsto \omega^\xi$.

Lemma 4.2. *Let α, α^*, β be as in Definition 4.1. For $\delta \prec \psi(\alpha^*+\Omega^{1+\beta})$ define $h(\delta) := \max(K\delta \cup \{0\})+1$. Now $h(\delta) \prec \alpha^*$ or there exists a unique $\delta' \prec \Omega^{1+\beta}$ such that $h(\delta) = \alpha^* + \delta'$. Set $\delta_0 := 0$ in the former case and $\delta_0 := \delta'$ in the latter case. Now define $\ell(\delta) := \psi(\alpha^*+\delta_0+\delta)$. Since $\mathsf{K}\,(\alpha^*+\delta_0+\delta) \prec \alpha^*+\delta_0+\delta$ we have $\psi(\alpha^* + \delta_0 + \delta) \in \mathcal{T}(\Omega)$ and $\delta \prec \psi(\alpha^* + \delta_0 + \delta)$. Moreover, if $\eta \prec \delta \prec \psi(\alpha^* + \Omega^{1+\beta})$, then*

$$\ell(\eta) \prec \ell(\delta) \prec \psi(\alpha^* + \Omega^{1+\beta})\,.$$

Since $\ell(\delta)$ is an additive principal number, this entails that $\psi(\alpha^* + \Omega^{1+\beta})$ is an ε-number.

Definition 4.3. Let E denote the "stack of two's" function, i.e. $E(0) = 0$ and $E(n+1) = 2^{E(n)}$.

Given two elements a and b of a non-standard model $\mathfrak{M}$ of **PA**, we say that 'b **is much larger than** a' if for every standard integer k we have $E^k(a) < b$.

If $\mathfrak{M}$ is a model of **PA** and $\mathfrak{I}$ is a substructure of $\mathfrak{M}$ we say that $\mathfrak{I}$ is an **initial segment** of $\mathfrak{M}$, if for all $a \in |\mathfrak{I}|$ and $x \in |\mathfrak{M}|$, $\mathfrak{M} \models x < a$ implies $x \in |\mathfrak{I}|$. We will write $\mathfrak{I} < b$ to mean $b \in |\mathfrak{M}| \setminus |\mathfrak{I}|$. Sometimes we write $a < \mathfrak{I}$ to indicate $a \in |\mathfrak{I}|$.

Theorem 4.4. *Fix $\alpha \prec \tau_{BH}$. For all $0 \prec \gamma \prec \beta \prec \alpha$,*

$$\mathbf{PA} \lhd T^\alpha_\beta \lhd T^\alpha_\gamma \lhd \mathbf{PA} + \mathrm{Con}(\mathbf{PA}).$$

Proof: First we want to show that $T^\alpha_\beta \unlhd T^\alpha_\gamma$. We know that there exists a number n such that $\beta \underset{n}{\longrightarrow} \gamma$. Provably in **PA** we therefore have that $F^*_\beta(x)\!\downarrow$ implies $F^*_\gamma(x)\!\downarrow$, and hence $\mathrm{Con}_{f_\gamma}(\mathbf{PA})$ yields $\mathrm{Con}_{f_\beta}(\mathbf{PA})$.

It remains to find a model of T^α_γ that is not a model of T^α_β.

We shall employ the method of injecting inconsistency from [8, Theorem 4.10].

Let $\mathfrak{M}$ be a countable non-standard model of $\mathbf{PA} + F^*_\beta$ *is total*. Let M be the domain of $\mathfrak{M}$ and $a \in M$ be non-standard. Moreover, let $e = (F^*_\beta)^{\mathfrak{M}}(a)$. As a result of the standing assumption, $\mathfrak{M} \models \mathrm{Con}(\mathbf{PA}\restriction_a)$. Owing to a result of Solovay's [27, Theorem 1.1] (or similar results in [13]), there exists a countable model $\mathfrak{N}$ of **PA** such that:

(i) $\mathfrak{M}$ and $\mathfrak{N}$ agree up to e (in the sense of [8, Definition 3.9]).

(ii) $\mathfrak{N}$ thinks that $\mathbf{PA}\lceil_a$ is consistent.

(iii) $\mathfrak{N}$ thinks that $\mathbf{PA}\lceil_{a+1}$ is inconsistent. In fact there is a proof of $0=1$ from $\mathbf{PA}\lceil_{a+1}$ whose Gödel number is less than 2^{2^e} (as computed in $\mathfrak{N}$).

In actuality, to be able to apply [27, Theorem 1.1] we have to ensure that e is much larger than a, i.e., $E^k(a) < e$ for every standard number k (recall that E denotes the It is a standard fact (provable in $\mathbf{PA}$) that $E(x) \leq F_3(x)$ holds for all sufficiently large x (cf. [12, p. 269]). In particular this holds for all non-standard elements s of $\mathfrak{M}$ and hence

$$E^k(s) \leq F_3^k(s) \leq F_3^s(s) \leq F_4(s) < F_\beta^*(s),$$

so that $E^k(a) < e$ holds for all standard k, yielding that e is much larger than a.

We will now distinguish two cases.

Case 1: $\mathfrak{N} \models F_\beta^*(a+1)\uparrow$. Then also $\mathfrak{N} \models F_\beta^*(d)\uparrow$ for all $d > a$ by Lemma 3.6, (iii). Hence, in light of (ii), $\mathfrak{N} \models \mathbf{PA} + \mathrm{Con}_{F_\beta}(\mathbf{PA})$.

Now we use the fact (Lemma 3.6,(ii)) that one can show in $\mathbf{PA}$ that for sufficiently large x,

$$F_\beta^*(x)\downarrow \rightarrow\ F_\gamma^*(x+1)\downarrow \wedge F_\gamma^*(x+1) < F_\beta^*(x)$$

where sufficiently large means bigger than a (standard) number computable from the (representation) α. Since a is non-standard we certainly have $(F_\gamma^*)^{\mathfrak{N}}(a+1)\downarrow$ and

$$(F_\gamma^*)^{\mathfrak{N}}(a+1) \leq (F_\beta^*)^{\mathfrak{N}}(a).$$

But since $\neg\mathrm{Con}(\mathbf{PA}\lceil_{a+1})$ holds in $\mathfrak{N}$, $\mathfrak{N}$ is not a model of T_γ^α.

Case 2: $\mathfrak{N} \models F_\beta^*(a+1)\downarrow$. We then also have $e = (F_\beta^*)^{\mathfrak{N}}(a)$, for $\mathfrak{M}$ and $\mathfrak{N}$ agree up to e and the formula '$F_\beta^*(x) = y$' is Δ_0 by Lemma 3.1. Let $c := (F_\beta^*)^{\mathfrak{N}}(a+1)$. Now an ordinal $\psi(\alpha^* + \Omega^{1+\beta})$ is an epsilon number by Lemma 4.2. Thus by [8, Corollary 3.8], for every standard n there is an initial segment $\mathfrak{I}$ of $\mathfrak{N}$ such $e < \mathfrak{I} < c$ and $\mathfrak{I}$ is a model of Π_{n+1}-induction. Moreover, it follows from the properties of $\mathfrak{N}$ and the fact that $2^{2^e} < \mathfrak{I}$, that

1. $\mathfrak{I}$ thinks that $\mathbf{PA}\lceil_a$ is consistent.

2. $\mathfrak{I}$ thinks that $\mathbf{PA}\lceil_{a+1}$ is inconsistent.

3. $\mathfrak{I}$ thinks that $F_\beta^*(a+1)$ is not defined.

Consequently, $\mathfrak{I} \models \mathrm{Con}_{f_\beta} + \Pi_{n+1}$-induction. Moreover, by the same arguments as in case 1, $\mathfrak{I}$ does not model Con_{f_γ}. Since n was arbitrary, this shows that $\mathbf{PA} + \mathrm{Con}_{f_\beta} + \neg\mathrm{Con}_{f_\gamma}$ is a consistent theory. □

Remark 4.5. Schmerl [22] showed that $\mathbf{PA}+\mathrm{Con}(\mathbf{PA})$ can be reached from **PRA** by a consistency progression $(S_\alpha)_\alpha$ along $\varepsilon_0 \cdot 2$. It is clear from the above that "most" of the theories T^α_β do not correspond to theories in this progression.

4.1 A bit of speculation

One might ponder whether the assumption "α less than the Bachmann-Howard ordinal" could be replaced by "α less than the first non recursive ordinal" in Theorem 4.4. An (anonymous) referee of this paper believes that a more general result than 4.4 could be shown and suggests the following approach.[2] To define a decent hierarchy $(F_\alpha)_{\alpha<\tau}$ of functions, the Bachmann property is usually not needed in full for an assignment of fundamental sequences to ordinals $< \tau$ as long as one defines

$$F_\lambda(n) := \max\{F_{\lambda[y]}(n) \mid y \leq n\}.$$

Lemma 3.5 could presumably be shown for a segment of ordinals τ which exceeds the Bachmann-Howard ordinal by using slowed down fundamental sequences. For epsilon numbers λ one could define

$$\lambda[n] = \max\{\beta < \lambda : N(\beta) \leq n\}$$

where $N : \tau \to \omega$ is some suitable norm function. Such an assignment of fundamental sequences would not be canonical but might (perhaps after some additional fine tuning) be good enough for strengthening Theorem 4.4.

These are interesting ideas for which we thank the referee.

5 Using slowness for modelling Gödel-Löb provability logic GL

The language of modal logic has infinitely many propositional variables and the modal operator $\Box$. Formulas are built from propositional variables via the usual propositional connectives (e.g. $\to, \neg, \wedge, \vee$) and the stipulation that $\Box A$ is a formula if A is. The logic of provability, **GL**, is formulated in this language and has the following axioms and inference rules:

[2]For unexplained notions see [2, 21].

A0. All propositional tautologies are axioms.

A1. $\Box(A \to B) \wedge \Box A \to \Box B$.

A2. $\Box A \to \Box\Box A$.

A3. $\Box(\Box A \to A) \to \Box A$.

R1. If $\vdash A \to B$ and $\vdash A$, then $\vdash B$.

R2. If $\vdash A$, then $\vdash \Box A$.

A provability interpretation of modal logic in **PA** is determined by an assignment of a sentence p^* of **PA** to each propositional variable p of **GL**. The interpretation A^* of a modal formula A commutes with the propositional connectives in the usual way (e.g., $(B \to C)^*$ is $B^* \to C^*$) and $(\Box B)^*$ is $\mathbf{Pr}_{\mathbf{PA}}(\ulcorner B^* \urcorner)$ where $\mathbf{Pr}_{\mathbf{PA}}(\ulcorner C \urcorner)$ arithmetizes provability of C in **PA** with $\ulcorner C \urcorner$ denoting the Gödel number of C.

The main result about these interpretations is Solovay's completeness theorem ([26]).

Theorem 5.1. **GL** $\vdash A$ *if and only if* **PA** $\vdash A^*$ *holds for all provability interpretations* *.

Below it is shown that completeness also obtains for a notion of slow provability. Let S be a provably recursive function of **PA** with a fixed Σ_1 definition $\varphi_S(x, y)$ in the language of **PA**, i.e. $\varphi_S(x, y)$ defines the graph of S and **PA** $\vdash \forall x \exists! y\, \varphi_S(x, y)$. Below we shall write $S(x) = y$ when we actually mean $\varphi_S(x, y)$.

Moreover, the following further standing assumption will be adopted throughout:

1. **PA** proves that $S(x) \geq 1$ for all x.
2. The range of S is unbounded, i.e., for all k there exists n such that $k < S(n)$.

However, we shall not assume that **PA** can prove the latter fact. Indeed, the whole thing is only interesting when **PA** doesn't 'know' this fact, as in the case of slow growing functions f^{α}_{β}.

Definition 5.2. Define

$$\Box_s A \ := \ \exists x\, \mathbf{Pr}_{\mathbf{PA}\restriction_{S(x)}}(\ulcorner A \urcorner)] \tag{1}$$

where $\mathbf{Pr}_T(\ulcorner A \urcorner)$ arithmetizes provability of A in a theory T and $\ulcorner A \urcorner$ denotes the Gödel number of A.

Lemma 5.3. *(i) If* $\mathbf{PA} \vdash A$ *then* $\mathbf{PA} \vdash \Box_s A$.

(ii) $\mathbf{PA} \vdash \Box_s A \to \Box_s(\Box_s A)$.

(iii) $\mathbf{PA} \vdash \Box_s(A \to B) \wedge \Box_s A \to \Box_s B$.

(iv) $\mathbf{PA} \vdash \Box_s(\Box_s A \to A) \to \Box_s A$.

Proof: (i) $\mathbf{PA} \vdash A$ implies that $\mathbf{PA}\restriction k \vdash A$ for some $k > 0$. There exists n such that $S(n) \geq k$. Thus $\mathbf{Pr}_{\mathbf{PA}\restriction_{S(n)}}(\ulcorner A \urcorner)$ is a true Σ_1 statement, and hence $\mathbf{PA} \vdash \Box_s A$.

(ii) We argue in $\mathbf{PA}$. Suppose $\Box_s A$. Then $\exists x\, \mathbf{Pr}_{\mathbf{PA}\restriction_{S(x)}}(\ulcorner A \urcorner))$. The latter being a Σ_1 statement, formalized Σ_1 completeness yields

$$\mathbf{Pr}_{\mathbf{PA}\restriction_1}(\ulcorner \mathbf{Pr}_{\mathbf{PA}\restriction_{S(x)}}(\ulcorner A \urcorner)\urcorner)$$

whence $\Box_s(\Box_s A)$ since $S(n) \geq 1$ for some n.

(iii) We argue in $\mathbf{PA}$. Suppose $\Box_s(A \to B) \wedge \Box_s A$. Spelling this out there exist x, y such that $\mathbf{Pr}_{\mathbf{PA}\restriction_{S(x)}}(\ulcorner A \to B \urcorner)$ and $\mathbf{Pr}_{\mathbf{PA}\restriction_{S(y)}}(\ulcorner A \urcorner)$. Picking $z \in \{x, y\}$ such that $S(x), S(y) \leq S(z)$ we have $\mathbf{Pr}_{\mathbf{PA}\restriction_{S(z)}}(\ulcorner A \to B \urcorner)$ and $\mathbf{Pr}_{\mathbf{PA}\restriction_{S(z)}}(\ulcorner A \urcorner)$, yielding $\mathbf{Pr}_{\mathbf{PA}\restriction_{S(z)}}(\ulcorner B \urcorner)$, and hence $\Box_s B$.

(iv) We argue in $\mathbf{PA}$. Assume that $\Box_s(\Box_s A \to A)$ holds. Then there exists x such that $\mathbf{Pr}_{\mathbf{PA}\restriction_{S(x)}}(\ulcorner \Box_s A \to A \urcorner)$. Spelling the latter out we have

$$\mathbf{Pr}_{\mathbf{PA}\restriction_{S(x)}}(\ulcorner \exists y\, \mathbf{Pr}_{\mathbf{PA}\restriction_{S(y)}}(\ulcorner A \urcorner)) \to A \urcorner). \tag{2}$$

With $\dot{x}$ denoting the x^{th} numeral, (2) implies

$$\mathbf{Pr}_{\mathbf{PA}\restriction_{S(x)}}(\ulcorner \mathbf{Pr}_{\mathbf{PA}\restriction_{S(\dot{x})}}(\ulcorner A \urcorner) \to A \urcorner). \tag{3}$$

By the formalized Löb's theorem for $\mathbf{PA}\restriction_{S(x)}$ it follows from (3) that $\mathbf{Pr}_{\mathbf{PA}\restriction_{S(x)}}(\ulcorner A \urcorner)$, whence $\Box_s A$. $\Box$

Theorem 5.4. *Let* * *be an assignment of a sentence* p^* *to every propositional variable of* $\mathbf{GL}$. * *gives rise to an interpretation* ^{*_s} *that commutes with the propositional connectives and satisfies also:*

$$(\Box A)^{*_s} \;=\; \Box_s A^{*_s}.$$

Then we have

$$\mathbf{GL} \vdash B \;\Rightarrow\; \mathbf{PA} \vdash B^{*_s}.$$

Proof: Obvious by Lemma 5.3. □

The converse also holds and thus we arrive at the following.

Theorem 5.5. **GL** $\vdash B$ *if and only if* **PA** $\vdash B^{*_s}$ *holds for all assignments* $*$.

Proof: In view of Theorem 5.4, it remains to prove the direction from right to left. It can be handled by inspection of what happens in §4 of [26]. If one replaces the provability predicate Bew there by slow provability and the consistency notion by slow consistency then the same constructions work as slow provability shares crucial properties (described in Lemma 5.3) with its standard cousin. □

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