In this brief note, we compare two frameworks for characterizing possible operations in quantum thermodynamics. One framework considers Thermal Operations—unitaries which conserve energy. The other framework considers all maps which preserve the Gibbs state at a given temperature. Thermal Operations preserve the Gibbs state; hence a natural question which arises is whether the two frameworks are equivalent. Classically, this is true—Gibbs-Preserving Maps are no more powerful than Thermal Operations. Here, we show that this no longer holds in the quantum regime: a Gibbs-Preserving Map can generate coherent superpositions of energy levels while Thermal Operations cannot. This gap has an impact on clarifying a mathematical framework for quantum thermodynamics.

In recent approaches to thermodynamics [1–13], one specifies a set of state transformations which an experimenter is allowed to perform “for free”, i.e. at no work cost—such a framework is called a resource theory. Among the various mathematical frameworks proposed to model thermodynamical operations, two have proven particularly useful, namely the resource theory of Thermal Operations and the Gibbs-Preserving Maps. Classically, these two frameworks are equivalent. If a transition between initial and final states block diagonal in their energy eigenbasis is possible by Gibbs-Preserving Maps, then it is also possible via Thermal Operations [7]. One might suppose that this equivalence holds for arbitrary quantum states. In this short note, we show that this is not the case: Gibbs-Preserving Maps can perform transitions which Thermal Operations are incapable of.

**Thermal Operations**.—The resource theory of Thermal Operations has been extensively exploited to understand thermodynamics at the quantum level [4, 7, 14–16]. One is allowed to perform any arbitrary joint unitary operation, on a system and a bath, which conserves the total energy on any state consistent with the initial density matrix of the input. Thermal Operations also include bringing in arbitrary systems which are in the Gibbs state (with arbitrary Hamiltonians). Finally, Thermal Operations allow subsystems to be discarded for free, regardless of their state. Observe that Thermal Operations cannot change the Gibbs state into any other state [4, 7, 14]. What’s more, the Gibbs state is the only state which has this property [13]. Gibbs states are thus the only state which can be allowed for free—if any other state were allowed, arbitrary state transformations would be possible.

Crucially, Thermal Operations are not capable of generating coherent superpositions of energy levels: a Thermal Operation must, by definition, commute with the total Hamiltonian, and thus cannot generate such a superposition starting from an energy eigenstate.

**Gibbs-Preserving Maps**.—In the framework of Gibbs-Preserving Maps, one allows to carry out any completely positive, trace-preserving map on a system which preserves the Gibbs state (or “Gibbs-Preserving Map”, for short). These maps are a natural quantum-mechanical generalization of the stochastic matrices used to characterize the so-called d-majorization or mixing character [17–21]. In any reasonable thermodynamical framework, a map that does not preserve the Gibbs state must cost work; this fact makes Gibbs-Preserving Maps a conservative choice of framework for proving fundamental limits.

Since a Thermal Operation preserves the Gibbs state, the state transformations possible with Thermal Operations are necessarily included in those achievable with Gibbs-Preserving Maps. Is the converse true? It is for states which are block diagonal in their energy eigenbasis. This can be seen as follows. A necessary and sufficient condition for transitions via Thermal Operations is thermo-majorization [7], a partial order which is a generalization of majorization [20, 22–24]. More precisely, transformations are completely characterized in terms of thermo-majorization of the initial and final states’ spectrum with respect to the Gibbs state. Classic results about majorization ensure the existence of a stochastic matrix mapping the vector of eigenvalues of the initial state to the final state’s eigenvalues, while preserving the Gibbs state’s spectrum. Written out in full as a channel on quantum states, it is a Gibbs-Preserving Map [25].

We now address the question of whether Gibbs-Preserving Maps are strictly more powerful than Thermal Operations, on arbitrary input states. We show that this is the case, by exhibiting an example of a Gibbs-Preserving Map that performs a transformation forbidden by Thermal Operations.

**The Example**.—Consider a two-level system with an energy gap $\Delta E$. We denote the ground state by $|0\rangle$ and the excited state by $|1\rangle$. Consider now the transformation:

$$|1\rangle \rightarrow \rho ,$$

(1)
where \( \rho \) is any pure or mixed state. Depending on \( \rho \), (in particular, in case \( |\rho\rangle = |+\rangle := \frac{1}{2}(|0\rangle + |1\rangle) \)), this transformation needs to “build” coherence between the energy levels, which, as noted above, cannot be achieved with Thermal Operations. We now argue that, for any \( \rho \), there exists nevertheless a Gibbs-Preserving Map performing this transition. Let \( \beta \) be a fixed inverse temperature, and denote the Gibbs state on the system by

\[
\gamma = p_0|0\rangle\langle 0| + p_1|1\rangle\langle 1| \quad \text{with} \quad p_0 = 1/Z, \quad p_1 = e^{-\beta \Delta E}/Z \quad \text{and} \quad Z = 1 + e^{-\beta \Delta E}.
\]

Let \( \Phi \) be defined as

\[
\Phi (\cdot) = (0| \cdot |0) \sigma + (1| \cdot |1) \rho ,
\]

for some state \( \sigma \) which we have not yet fixed. Note that \( \Phi \) is completely positive and trace-preserving. We also have \( \Phi (|1\rangle\langle 1|) = \rho \) by construction. The condition that \( \Phi \) be Gibbs-preserving, \( \Phi (\gamma) = \gamma \), gives us

\[
p_0 \sigma + p_1 \rho = \gamma ,
\]

which implies

\[
\sigma = p_0^{-1} (\gamma - p_1 \rho) .
\]

This choice of \( \sigma \) has unit trace, and is positive semidefinite; indeed, as \( \gamma \geq p_1 \mathbb{1} \) (since \( p_1 \) is the smallest eigenvalue of \( \gamma \)) and \( p_1 \leq 1 \), we have \( \gamma - p_1 \rho \geq 0 \). This means that, with this choice of \( \sigma \), \( \Phi \) is precisely a completely positive, trace-preserving, Gibbs-preserving channel which maps \( |1\rangle \) to \( \rho \). This map is forbidden by Thermal Operations if \( \rho \) contains a coherent superposition over energy levels, and we have the desired counterexample.

This example can easily be generalized to a system of \( n \) arbitrary energy levels: if \( |n\rangle \), of energy \( E_n \), is such that no other state has higher energy, a Gibbs-Preserving Map \( \Phi \) transforming \( |n\rangle \) into any \( \rho \) is given by

\[
\Phi (\cdot) = \text{tr}[|n\rangle\langle n|] \sigma + \text{tr}[|n\rangle\langle n| \cdot] \rho ,
\]

where \( \sigma = (\gamma - p_n \rho)/(1 - p_n) \) and where the Gibbs state is \( \gamma = \sum p_i |i\rangle\langle i| \) with \( p_i = e^{-\beta E_i}/Z \) and \( Z = \sum e^{-\beta E_i} \).

Discussion.—This observation leaves open the question which of the two classes of operations capture the actual physical situation. One could have argued from the start that the transition (1) should have been possible for any \( \rho \); indeed, the initial state has both maximal purity and highest possible energy. One might then conclude that the resource theory of Thermal Operations is too restrictive, and that some form of “coherence-manipulating” operation should be allowed, as done in Refs. [4, 12]. However, it is unclear if the resources necessary to create superpositions are ones which require work to produce. Adopting the point of view of the resource theory of reference frames, Gibbs-preserving operations do not commute with time translation, and thus require the agent to possess a time reference frame [28, 29] which gets degraded and costs work to produce. Whether one can assume such a resource or not is expected to be a matter for the particular setting studied. (Additionally, constraints on transformation of coherent superpositions of energy eigenstates with Thermal Operations are discussed in Ref. [30].)

Finally, it is worth noting that for Thermal Operations, there exists a set of conditions which act as second laws, restricting which state transformations are allowed [13]. These take the form of a distance measure to the Gibbs state, and are thus also a set of restrictions for Gibbs-Preserving Maps (due to the data processing inequality for the Rényi relative entropies). As we now see that the two frameworks are inequivalent, this implies that a complete set of second laws will necessarily involve functions which cannot be expressed in such a form.

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[25] On a side note, this does not imply that Gibbs-Preserving Maps are equivalent to Thermal Operations as channels even when acting on block diagonal states. Rather, they are only equivalent in terms of state transitions. In other words, while the same pair *(input state, output state)* can be achieved in both frameworks for block-diagonal states, the actual logical processes, i.e. trace-preserving completely positive maps or channels, that one can perform, differ. Note also that even for a given fixed input state the actual channel performed is in general relevant, and not only the input and output state, as the full information about the channel can be obtained by keeping a purification of the input. Additionally, a classic example (for the trivial Hamiltonian $H = 0$) of a map preserving the fully mixed state but which is not a Thermal Operation is the Choi-Jamiołkowski map of the two-party reduced state of the Aharonov [26, Note [9]] or determinant state $|A⟩_{ABC} = \frac{1}{\sqrt{6}} (|012⟩ + |120⟩ + |201⟩ - |210⟩ - |102⟩ - |021⟩)$, which is up to a local unitary the same example as in [27].


