MARTINGALE OPTIMAL TRANSPORT
IN THE SKOROKHOD SPACE

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Abstract. The dual representation of the martingale optimal transport problem in the Skorokhod space of multi dimensional càdlàg processes is proved. The dual is a minimization problem with constraints involving stochastic integrals and is similar to the Kantorovich dual of the standard optimal transport problem. The constraints are required to hold for very path in the Skorokhod space. This problem has the financial interpretation as the robust hedging of path dependent European options.

1. Introduction

Model independent approach to financial markets provides hedges without referring to a particular probabilistic structure. It is also shown to be closely connected to the classical Monge-Kantorovich optimal transportation problem. In this paper, we prove this connection for quite general financial markets that offer multi risky assets with càdlàg (right continuous with left hand limits) trajectories. This generality is strongly motivated by the fact that investors use several assets in their portfolios and the observed stock price processes contain jump components [3, 4]. The main result is a Kantorovich type duality for the super-replication cost of an exotic option \( G \), which is simply a nonlinear function of the whole stock trajectory. It is well documented that this duality is central to understanding the financial markets. In particular, several other important results including the fundamental theorem of asset pricing follow from it.

As it is standard in these problems, following [18] we assume that a linear set of options \( \mathcal{H} \) is available for static investment with a known price \( \mathcal{L}(h) \) for \( h \in \mathcal{H} \). In addition to this static investment, the investor can dynamically use stocks in her portfolio. Let an admissible predictable process \( \gamma \) represent this dynamic position in the stock whose price process is denoted by \( S \) with values in the positive orthant \( \mathbb{R}^d_+ \). An investment strategy \( (h, \gamma) \) super-replicates a exotic option if its final value

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at maturity $T$ dominates $G$ in all possible cases, i.e.,

\[ h(S) + \int_0^T \gamma_u(S) \, dS_u \geq G(S), \quad \forall S \in \mathbb{D}, \]

where $\mathbb{D}$ is the set of all stock process $S$ that are càdlàg, $S_0 = (1, \ldots, 1)$ and continuous at maturity $T$. Technical issues related to the stochastic integral and admissible strategies are discussed in Section 2, Definition 2.5. The minimal super-replicating cost is then given by

\[ V(G) := \inf \{ \mathcal{L}(h) : \text{there exists an admissible predicable process } \gamma \text{ so that } (h, \gamma) \text{ super-replicates } G \}. \]

As usual, the dual elements are martingale measures $Q$ that are consistent with the given option data. Namely, let $\mathcal{M}_\mathcal{L}$ be the set of all measures on $\mathbb{D}$ so that the canonical process $S$ is a martingale with the canonical filtration $\mathcal{F}$ and $E_Q[g] \leq L(h)$, $h \in \mathcal{H}$.

We then have the following duality result,

\[ V(G) = \sup_{Q \in \mathcal{M}_\mathcal{L}} E_Q[G]. \]

The above result is proved in Theorem 2.9 for a bounded $G$ that is uniformly continuous in the Skorokhod topology. In this paper, the pair $\mathcal{H}, \mathcal{L}$ is defined through a given probability measure $\mu$. Then, we take $\mathcal{H}$ to be the set of all functions of the type $g(S_T)$ with $g \in L^1(\mathbb{R}_+^{d}, \mu)$ and $\mathcal{L}(g) = \int g \, d\mu$. The only assumption on $\mu$ is that $\int x \, d\mu(x) = S_0 = (1, \ldots, 1)$. Assumptions on $G$ are relaxed in Sections 5.1 and 5.2.

Our approach, as in [14, 15], relies on a discretization procedure. We then use a classical min-max theorem for the discrete approximation and a classical constrained duality result of Föllmer and Kramkov [16]. The technical steps are to prove that the approximations on both side of the dual formula converge. The multi-dimensionality and the discontinuous behavior of the stock process introduce several technical difficulties. In particular, we introduce appropriate portfolio constraints in the approximate discrete markets. This new feature of the discretization is essential and enables us to control the error terms due to the multi-dimensionality and the possible discontinuities of the stock process.

Another technical difficulty originates from the fact the set of martingale measures $\mathcal{M}_\mu$ is not compact. Therefore, passage to the limit in the dual side requires probabilistic constructions. In particular, we prove that the dual problem as seen as a function of the probability measure $\mu$ (with fixed $G$) has some continuity properties. This is proved in Section 4, Theorem 4.1.

The structure studied in this paper is similar to that of [18] and also of [6, 8, 9, 10, 12, 13, 14, 17, 20, 21, 22, 23, 24, 26]. We refer the reader to the excellent survey of Hobson [19] and to our previous papers [14, 15] and to the references therein. A related issue is the fundamental theorem of asset pricing (FTAP) in these markets. This problem in the robust setting in discrete time is studied in [2], [7] and [15]. [2] proves FTAP in the model independent framework with a general $\mathcal{H}$ containing a power option. [7] considers a discrete time market in which a set of probability measures $\mathcal{P}$ is assumed. The super replication is defined by demanding (1.1) not for every path $S$ but $\mathbb{P}$ almost surely for every $\mathbb{P} \in \mathcal{P}$ (i.e., $\mathcal{P}$-quasi-surely). FTAP and duality (under the assumption of no-arbitrage) is proved for a finite dimensional $\mathcal{H}$.
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but possibly with no power option. The notions of no-arbitrage considered in [2] and [7] are different. In our earlier work [15] we prove model-independent duality for a discrete time market with proportional costs. FTAP follows as a consequence of the duality. However, the form of FTAP depends on the particular notion of no-arbitrage. A discussion of different notions is also provided in [15]. In continuous time, the desirable extension to the general quasi-sure setting remains open with the exception of [17] in which a certain class $P$ is considered.

The paper is organized as follows. The main results are formulated in the next section. In Section 3, Theorem 2.9 is proved. In section 4, we prove a continuity result for the dependence of the dual problem on the measure $\mu$. The final section, is devoted to extensions.

Notation. We close this introduction with a list of some of the notation used in this paper.

- $\mathbb{R}_+ := (0, \infty)$ is the set of all positive real numbers.
- $\mathbb{N} := \{1, 2, \ldots\}$ is the set of positive integers.
- $\mathbb{D}$ is the set of all $\mathbb{R}_d^+$ valued càdlàg processes $S$ that are continuous at $t = T$ and also satisfy $S_0 = (1, \ldots, 1)$; Section 2.
- The similar set $\mathbb{D}([0, T]; \mathbb{R}_d)$ for $\mathbb{R}_d$ valued processes is defined in Section 2.
- $\mathcal{F}$ is the canonical process and $\mathcal{F}$ is the canonical filtration on $\mathbb{D}$; Section 2.
- $\|S\| = \sup\{|S_t| : t \in [0, T]\}$.
- $\mathcal{H}$ is set of statically tradable options. In this paper, it is the set of all functions of the form $h(S) = g(S_T)$, where $g \in L^1(\mathbb{R}_d^+, \mu)$ for some probability measure $\mu$; see subsection 2.1.
- $d$ is the Skorokhod metric on $\mathbb{D}$, see Section 2.4.
- For a positive integer $n$ and $S \in \mathbb{D}$, stopping times $\tau_k = \tau_k(n)(S)$'s and the random integer $M = M(n)(S)$ are defined in subsection 3.1.
- For a positive integer $n$ and $S \in \mathbb{D}$, random times $\hat{\tau}_k = \hat{\tau}_k(n)(S)$'s are defined in subsection 3.3 as a function of the stopping times $\tau_k$'s.
- Maps $\hat{\Pi} : \mathbb{D} \to \mathbb{D}$ and $\hat{\Pi}, \Pi : \mathbb{D} \to \mathbb{D}$ are constructed in subsection 3.3.

When possible we followed the convention that the notation $\hat{\cdot}$ is reserved for objects on the countable space $\hat{\mathbb{D}}$, such as $\hat{S}$ is a generic point in $\hat{\mathbb{D}}$ and $\hat{\tau}_k$'s are its jump times.

2. Preliminaries and main results

The financial market consists of a savings account which is normalized to unity $B_t \equiv 1$ by discounting and of $d$ risky assets with price process $S_t \in \mathbb{R}_d^+$, $t \in [0, T]$, where $T < \infty$ is the maturity date. Without loss of generality we set the initial stock values to one, i.e., $S_0 = (1, \ldots, 1)$. We assume that each component of the price process is right continuous with left hand limits (i.e., a càdlàg process) which is also continuous at maturity $t = T$. $\mathbb{D}$ denotes the set of all càdlàg functions

$$S = (S^{(1)}, \ldots, S^{(d)}) : [0, T] \to \mathbb{R}_d^+,$$

that are continuous at $t = T$ and also satisfy $S_0 = (1, \ldots, 1)$. Then, any element of $\mathbb{D}$ can be a possible path for the stock price process. This is the only assumption that we make on our financial market.
We set $D([0, T]; \mathbb{R}^d)$ be the set of all càdlàg processes that take values in $\mathbb{R}^d$ (rather than $\mathbb{R}^d_+$ as in the case of $\mathbb{D}$) that start from $S_0 = (1, \ldots , 1)$ and are continuous at $T$.

Consider a European path dependent option with the payoff $X = G(S)$ where $G : D([0, T]; \mathbb{R}^d) \to \mathbb{R}$.

Although only the values of $G$ on $D$ are needed to define the problem, technically we require $G$ to be defined on the larger space $D([0, T]; \mathbb{R}^d)$. However, in almost all cases extension of a function defined on $D$ to $D([0, T]; \mathbb{R}^d)$ is straightforward; see Remark 2.8 below.

In probability theory, most processes are required to be either progressively measurable or predictable with respect to a filtration. In the context of this paper, the natural filtration is canonical filtration generated by the canonical process. Then, we have the following equivalent definition of progressive measurability.

**Definition 2.1.** We say that a process $\gamma : [0, T] \times D \to \mathbb{R}^d$ is progressively measurable if for any $v, \tilde{v} \in D$ and $t \in [0, T],$

\[
S_u = \tilde{S}_u, \quad \forall u \in [0, t] \Rightarrow \gamma_t(S) = \gamma_t(\tilde{S}).
\]

It is well known that if $\gamma$ is left continuous and progressively measurable, then it is predictable with respect to the canonical filtration. Hence, in the sequel we check the predictability of any left continuous process by verifying (2.1).

### 2.1. Tradable Options.

$\mathcal{H}$ represents the set of all options available for trading. Although in this paper we use a specific class, in general it is assumed to be a linear subset of real-valued functions on $\mathbb{D}$. It is always assumed that

$h_{\text{cash}}, h_1, \ldots , h_d \in \mathcal{H},$ where $h_{\text{cash}} \equiv 1,$ $h_k(S) := S^{(k)}_T, \quad \forall k = 1, \ldots , d.$

The price of these options are given through an operator

$\mathcal{L} : \mathcal{H} \to \mathbb{R}.$

The essential assumptions on $\mathcal{L}$ are the convexity, an appropriate continuity and

$\mathcal{L}(h_{\text{cash}}) = 1,$ $\mathcal{L}(h_k) = S_0^{(k)} = 1, \quad \forall k = 1, \ldots , d.$

The last condition implies that the dual elements are martingale measures. So it might be interesting to relax it so as to allow for local martingale measures.

**Example 2.2.** In this example, we discuss possible examples of $(\mathcal{H}, \mathcal{L})$.

1. For a $\mu$ be a probability measure, let

\[
\mathcal{H} = \{ h(S) = g(S_T) : g \in L^1(\mathbb{R}^d_+, \mu) \}.
\]

The pricing operator given through the probability measure $\mu$ by,

\[
\mathcal{L}(g) = \int_{\mathbb{R}^d_+} g \, d\mu.
\]

We assume that $\mu$ satisfies

\[
\int h_k d\mu = \int x_k d\mu(x) = S_0^{(k)} = 1, \quad \forall k = 1, \ldots , d.
\]
The above linear pricing rule is equivalent to assume that the distribution of $S_T$ is known and equal to $\mu$.

2. It is sometimes technically convenient to assume that a power option is also available in the market. So assume the market structure as in the first example and strengthen (2.4) by adding a moment condition for a fixed $p > 1$,

$$\int |x|^p d\mu < \infty, \quad \int h_k d\mu = \int x_k d\mu(x) = S_0^{(k)} = 1, \quad \forall k = 1, \ldots, d.$$ (2.5)

3. Let $\mu_1, \ldots, \mu_d$ be a probability measures on $\mathbb{R}_+$. $\mathcal{H}$ is the vector space which is given by

$$\mathcal{H} = \{ h(S) = g(S_T)|g(x_1, \ldots, x_d) = \sum_{i=1}^d g_i(x_i), \quad g_i \in L^1(\mathbb{R}_+, \mu_i) \}.$$ The pricing operator is given by

$$\mathcal{L}(g) = \sum_{i=1}^d \int g_i d\mu_i.$$ (4)

4. $\mathcal{H}$ is set of all functions of $S_{t_k}$ for some time points $0 < t_1 < \ldots t_N \leq T$. The functional $\mathcal{L}$ is given through $N$ probability measures that are in convex order. Analysis of this example is essentially the same as the first one.

5. $\mathcal{H}$ is a finite dimensional space spanned by finitely many Lipschitz functions and possibly the power options $h(S) = |S_T|^p$ for some $p > 1$.

6. $\mathcal{H}$ could be as in the previous examples with a possibly nonlinear pricing operator $\mathcal{L}$. Such an example is studied in [15].

□

In this paper, to simplify the presentation we mainly consider the first case. Namely, we assume that $(\mathcal{H}, \mathcal{L})$ satisfy (2.2), (2.3), (2.4). To extend our results to unbounded options $G$, we also utilize (2.5) in Section 5. Other cases discussed in detail in our forthcoming studies.

**Remark 2.3.** Again let the tradable options to be of the form $h(S) = g(S_T)$. However, now assume that $g$ is a bounded and continuous function of $\mathbb{R}_+^d$. Then, a careful analysis proof shows that the duality result Theorem 2.9 holds for this problem with the same dual problem. Hence, the super-replication cost with this class of tradable options is the equal to the one with the larger class $g \in L^1(\mathbb{R}_+^d, \mu)$. See Remark 3.8 and also Remark 2.10 below.

□

Next we discuss the importance of the power option.

**Remark 2.4.** The following example highlights the role of the power option assumed in (2.4) and it is communicated to us by Marcel Nutz.

Suppose $d = 1$. Let $h^*(S) = \chi_{[0.5, \infty)}(S_T)$ and $\mathcal{H}$ be the three dimensional space spanned by $h^*, h_1, h_{cash}$. Further let $\mathcal{L}$ be a linear functional on $\mathcal{H}$ with $\mathcal{L}(h_{cash}) = 1$ and $\mathcal{L}(h^*) = 0$. For an exotic option $G$, $V(G)$ is the super-replication cost. Let $\tilde{\mathcal{H}}$ be the extended market that also includes the power option with $\tilde{V}(G)$ as the corresponding super-replication cost.
In both markets, the investor can buy the digital option \( h^* \) with zero cost. Clearly, this implies some kind of arbitrage since \( h^* \geq 0 \) and is not identically equal to zero. However, for the market \( \mathcal{H} \) this arbitrage does not agree with the notion defined in [2] but agrees with the one given in [7]. In \( \mathcal{H} \) there is arbitrage in both senses and the super-replication cost \( \mathcal{V}(G) = -\infty \).

In the smaller market \( \mathcal{H} \) it follows directly that \( \mathcal{V}(0) = 0 \). But we claim that there is no martingale measure that is consistent with \( L \). Indeed, if there were a martingale measure \( Q \) satisfying \( \mathbb{E}_Q[h^*] = \mathcal{L}(h^*) = 0 \), then the support of the distribution \( \mu \) of \( S_T \) under \( Q \) must be a subset of \([0, 0.5)\). On the other hand, since \( Q \) is a martingale measure, \( \int x d\mu(x) = S_0 = 1 \). Hence, the set \( \mathcal{M}_\mu \) is empty. This means that the duality (1.2) does not hold in \( \mathcal{H} \) while it holds in the market \( \mathcal{H} \) that contains the power option. (Note that by convention the supremum over an empty set is defined to be minus infinity.)

Although the duality does not hold in \( \mathcal{H} \) with the dual set \( \mathcal{M}_\mu \), in this example it would hold if one relaxes the dual set of measures to include the local martingale measures as well.

2.2. Martingale Measures. Set \( \Omega := \mathbb{D} \) and let \( \mathcal{F} \) be the \( \sigma \)-algebra which is generated by the cylindrical sets. Let \( S = (S_t)_{0 \leq t \leq T} \) be the canonical process given by \( S_t(\omega) := \omega_t \), for all \( \omega \in \Omega \).

A probability measure \( Q \) on the space \((\Omega, \mathcal{F})\) is a martingale measure, if the canonical process \((S_t)_{t=0}^T\) is a martingale with respect to \( Q \) and \( S_0 = (1, \ldots, 1) \), \( Q \)-a.s.

For a probability measure \( \mu \) on \( \mathbb{R}_d^+ \), let \( \mathcal{M}_\mu \) be the set of all martingale measures \( Q \) such that the probability distribution of \( S_T \) under \( Q \) is equal to \( \mu \). Observe that condition \( \int x d\mu(x) = 1 \) in (2.4) is equivalent to \( \mathcal{M}_\mu \neq \emptyset \).

2.3. Admissible portfolios. Next, we describe the continuous time trading in the underlying asset \( S \). We essentially adopt the path-wise approach which was already used in [14]. However, the present setup is more delicate than the one in [14]. Indeed, due to the possible discontinuities of the integrator \( S \), we require that the trading strategies are left continuous of bounded variation. Then, for any left continuous function \( \gamma : [0, T] \to \mathbb{R}^d \) of bounded variation and a càdlàg function \( S \in \mathbb{D} \), we use integration by parts (see Section 1.7 in [27]) to define

\[
\int_0^t \gamma_a dS_u := \gamma_t \cdot S_t - \gamma_0 \cdot S_0 - \int_0^t S_u \cdot d\gamma_a,
\]

where for \( a, b \in \mathbb{R}^d \), \( a \cdot b \) is the usual scalar product. Furthermore, the last term in the above right hand side is the Lebesgue-Stieltjes integral and not the standard Riemann–Stieltjes integral which was used in [14].

In particular, when \( \gamma \) is also progressively measurable (c.f., (2.1)) then for any martingale measure \( Q \in \mathcal{M}_\mu \), the stochastic integral \( \int \gamma_a dS_u \) is well-defined and both the pathwise constructed integral and the stochastic integral agree \( Q \) almost surely.

In the sequel, we use this equality repeatedly.

These considerations lead us to the following definition.
Definition 2.5. A semi-static portfolio is a pair \( \phi := (g, \gamma) \), where \( g \in L^1(\mathbb{R}^d_+, \mu) \) and \( \gamma : [0, T] \times D \to \mathbb{R}^d \) is left continuous and is progressively measurable, where \( \gamma_t(S) \) denotes the number of shares in the portfolio \( \phi \) at time \( t \), before a transfer is made at this time.

A semi-static portfolio is admissible, if for every \( Q \in \mathcal{M}_\mu \) the stochastic integral \( \int_0^T \gamma_u dS_u \) is a \( Q \) super-martingale.

An admissible semi-static portfolio is called super-replicating, if
\[
g(S_T) + \int_0^T \gamma_u(S) dS_u \geq G(S), \quad \forall S \in D.
\]
The (minimal) super-hedging cost of \( G \) is defined by,
\[
V(G) := \inf \left\{ \int g d\mu : \exists \gamma \text{ such that } \phi := (g, \gamma) \text{ is super-replicating} \right\}.
\]

Remark 2.6. The condition of admissibility depends on the measure \( \mu \). Hence the set of admissible controls and the super-replication cost also have this dependence. One may remove this dependence by considering continuous and bounded \( g \)'s instead of \( L^1(\mathbb{R}^d_+, \mu) \) functions. And for admissibility, instead of requiring that the stochastic integral \( \int \gamma_u dS_u \) is a \( Q \) super-martingale for every \( Q \in \mathcal{M}_\mu \), one may impose the condition that this integral is uniformly bounded from below in \( S \). A careful analysis of the proof of Theorem 2.9 reveals that the duality (under the hypothesis of Theorem 2.9) holds with this smaller class of admissible portfolios and hence the super-replication cost is not changed. See Remarks 3.8 and 2.10 below.

In the case when \( \mu \) satisfies (2.5) with an exponent \( p > 1 \), if there exists \( C > 0 \) satisfying
\[
(2.6) \quad \int_0^T \gamma_u(S) dS_u \geq -C \left( 1 + \sup_{0 \leq u \leq t} |S_u|^p \right), \quad \forall t \in [0, T], \quad S \in D,
\]
then the stochastic integral is a \( Q \) super-martingale for each \( Q \in \mathcal{M}_\mu \) due to Doob’s inequality and (2.5).

In the sequel, we check the admissibility of \( \gamma \) by verifying either the above condition with \( p > 1 \) when (2.5) holds or again the above inequality but with \( p = 0 \) when we only have (2.4).

2.4. Martingale optimal transport on the space \( D \). We continue by stating the duality result. Since our approach relies on discretization, one requires the regularity of the exotic option. One may then relax this regularity through analytical methods as we have done in [15]. Since the emphasis of this paper is the possible discontinuity of the stock process and multi-dimensionality, we do not seek the most general condition on \( G \). We first prove the duality when \( G \) is bounded and uniformly continuous in the Skorokhod topology. We then relax this condition in Section 5 below. To state the condition on \( G \), recall the Skorokhod metric on \( D([0, T]; \mathbb{R}^d) \),
\[
d(\omega, \tilde{\omega}) := \inf_{\lambda \in \Lambda[0, T]} \sup_{t \in [0, T]} (|\omega(t) - \tilde{\omega}(\lambda(t))| + |\lambda(t) - t|),
\]
where \( \Lambda[0, T] \) is the set of all strictly increasing onto functions \( \lambda : [0, T] \to [0, T] \).
Assumption 2.7. We assume that the exotic option
\[ G : \mathbb{D}([0,T]; \mathbb{R}^d) \to \mathbb{R}, \]
is bounded and uniformly continuous, i.e., there exists a continuous bounded function (modulus of continuity) \( m_G : \mathbb{R}_+ \to \mathbb{R}_+ \) so that
\[ |G(\omega) - G(\tilde{\omega})| \leq m_G(d(\omega, \tilde{\omega})), \quad \forall \omega, \tilde{\omega} \in \mathbb{D}([0,T]; \mathbb{R}^d). \]
\[ \square \]

Remark 2.8. For technical reasons, we assume that \( G \) defined not only on \( \mathbb{D} \) but in the larger space \( \mathbb{D}([0,T]; \mathbb{R}^d) \). However, suppose that \( G \) is given only on its natural domain \( \mathbb{D} \) rather than the whole space \( \mathbb{D}([0,T]; \mathbb{R}^d) \). Assume that \( G \) is uniformly continuous on \( \mathbb{D} \). Then, one can extend \( G \) to the larger space still satisfying the above assumption and the main duality result is independent of the particular extension chosen.

\[ \text{Indeed, a direct closure argument extends } G \text{ to a uniformly continuous function } \tilde{G} \text{ defined on } \mathbb{D}([0,T]; [0, \infty)^d). \] Then, we define \( \tilde{G}(\tilde{S}) := \tilde{G}(\tilde{S}') \) for every \( \tilde{S} \in \mathbb{D}([0,T]; \mathbb{R}^d) \), where \( \tilde{S}'(i) := [\tilde{S}(i)] \), \( i = 1, ..., d \) and \( t \in [0,T] \). \[ \square \]

The following result is an extension of Theorem 2.7 in [14] to the case of multi-dimensional stock price process with possible jumps. Its proof is completed in the subsequent sections. Relaxations of the Assumption 2.7 are provided in Section 5.

Theorem 2.9. We assume that \((\mathcal{H}, L)\) is as in (2.2), (2.3) and the probability measure \( \mu \) satisfies (2.4). Then for any exotic option satisfying Assumption 2.7, we have the dual representation for the minimal super-replication cost defined in Definition 2.5,
\[ V(G) = \sup_{Q \in \mathcal{M}_\mu} \mathbb{E}_Q [G(S)], \]
where \( \mathbb{E}_Q \) denotes the expectation with respect to the probability measure \( Q \).
\[ \text{Proof. Let } Q \in \mathcal{M}_\mu. \text{ Then, for any admissible strategy } \gamma, \text{ the path-wise integral } \int \gamma_u dS_u \text{ agrees with stochastic integral } Q\text{-almost surely and in view of Definition 2.5 this integral is a } Q \text{ super-martingale. Now suppose that } (g, \gamma) \text{ is an admissible super-replicating semi-static portfolio. Then,} \]
\[ \mathbb{E}_Q \left[ \int_0^T \gamma_u(S) dS_u \right] \leq 0, \quad \text{and} \quad \mathbb{E}_Q[g(S_T)] = \int g d\mu. \]
We take the expected value with respect to \( Q \) in the super-replication inequality and use the above observations to arrive at,
\[ V(G) \geq \sup_{Q \in \mathcal{M}_\mu} \mathbb{E}_Q [G(S)]. \]
The opposite inequality is proved through an approximation argument. In subsection 3.2, we define a a sequence of problems \( V^{(n)}(G) \) by considering super-replication on a countable subset of \( \mathbb{D} \) with bounded portfolio processes. Then, in Corollary 3.7, we prove that
\[ V(G) \leq \limsup_{n \to \infty} V^{(n)}(G) \leq \sup_{Q \in \mathcal{M}_\mu} \mathbb{E}_Q [G(S)]. \]
\[ \square \]
Remark 2.10. In the above proof, the lower bound for $V(G)$ follows from a classical direct argument. For this argument the minimal conditions for $(g, \gamma)$ are the ones assumed in Definition 2.5. Namely, the integrability of $g$ with respect to $\mu$ and the super-martingality of the stochastic integral. Therefore, any smaller class of semi-static portfolios would also satisfy the lower bound trivially. \hfill \Box

3. Proof of (2.7) under Assumption 2.7.

3.1. Discretization of $\mathbb{R}^d_+$ and stopping times. In this subsection, we construct a sequence of stopping times that will be central to our discretization procedure.

For $n \in \mathbb{N}$ and $x \in \mathbb{R}^d_+$ define an open set by,

$$O(x, n) := \left\{ y \in \mathbb{R}^d_+ : |y - x| < \sqrt{d} \cdot 2^{-n} \right\}.$$ 

For $S \in \mathcal{D}$, set $\tau_0 = 0$ and define $\tau_{k+1} = \tau_{k+1}^{(n)}(S)$ by,

$$\tau_{k+1} := T \wedge \left( \tau_k + \sqrt{d} \cdot 2^{-n} \right) \wedge \inf \{ t > \tau_k : \mathbb{S}_t \notin O(S_{\tau_k}, n) \}, \quad k = 0, 1, \ldots ,$$

where we set $\tau_{k+1} = T \wedge (\tau_k + \sqrt{d} \cdot 2^{-n})$ if the above set is empty. To ease the notation we suppress the dependence on $n$ and $S$ when this dependence is clear. Set

$$M = M(n)(S) := \min \{ k \in \mathbb{N} : \tau_k = T \}.$$ 

Since $S$ is càdlàg and $S \in \mathbb{R}^d_+$, $M < \infty$. It is also clear that

$$0 = \tau_0 < \tau_1 < \ldots < \tau_M = T$$

are stopping times with respect to the filtration which is generated by $S$. Moreover, for $k = 0, 1, \ldots , M - 1$,

$$|\tau_{k+1} - \tau_k|, |S_t - S_{\tau_k}| \leq \sqrt{d} \cdot 2^{-n}, \quad \forall t \in [\tau_k, \tau_{k+1}).$$

Also, by continuity of $S$ at $T$, the above holds in the closed interval $[\tau_{M-1}, T]$.

3.2. Approximation. In this section, we define a sequence of super-replication problems defined on a countable probability space. In the later sections, we show that this sequence approximates the original problem. Since the probability space is countable, robust (or equivalently point wise) and the probabilistic super-replications agree with a properly chosen probability measure. This allows us to use classical techniques to analyze the approximating problem.

We fix $n \in \mathbb{N}$ and define a sequence of probability spaces $\hat{\mathcal{D}} = \hat{\mathcal{D}}^{(n)}[0, T]$. Set

$$A^{(n)} := \left\{ 2^{-n}m : m = (m_1, \ldots , m_d) \in \mathbb{N}^d \right\},$$

$$B^{(n)} := \left\{ k\sqrt{d} \cdot 2^{-n} : k \in \mathbb{N} \right\} \cup \left\{ \sqrt{d} \cdot 2^{-n}/k : k \in \mathbb{N} \right\}.$$ 

Definition 3.1. A process $\hat{S} \in \mathcal{D}$ belongs to $\hat{\mathcal{D}}$, if there exists a nonnegative integer $M$ and a partition $0 = t_0 < t_1 < \ldots < t_M < T$ such that

$$\hat{S}_t = \sum_{k=0}^{M-1} \hat{S}_{t_k} \chi_{[t_k, t_{k+1})}(t) + \hat{S}_{t_M} \chi_{[t_M, T]}(t)$$

where $\hat{S}_0 = (1, \ldots , 1)$, $\hat{S}_T = \hat{S}_{t_M} \in A^{(n)}$ and

$$\hat{S}_{t_k} \in A^{(n+k)}, \quad \forall \, k = 1, \ldots , M - 1, \quad t_k - t_{k-1} \in B^{(n+k)}, \quad \forall \, k = 1, \ldots , M.$$
Since the set \( \hat{D} \) is countable, there exists a probability measure \( \mathbb{P} = \mathbb{P}^{(n)} \) on \( D \) supported on \( \hat{D} \) which gives positive weight to every element of \( \hat{D} \).

Let the probability structure \( \Omega := \hat{D} \), the canonical map \( \hat{S} \) and the filtration \( \mathcal{F} \) be as in subsection 2.4. Introduce a new filtration \( \hat{\mathcal{F}} = (\hat{\mathcal{F}}_t)_{t \in [0,T]} \) by completing \( \mathcal{F} \) by the null sets of \( \mathbb{P} \). Note that all of this structure depends on \( n \) but this dependence is suppressed in our notation. Under the measure \( \mathbb{P} \), the canonical map \( \hat{S} \) has finitely many jumps. Let \( M = M(\hat{S}) \) be number of jumps and

\[
0 < \hat{\tau}_1 < \ldots < \hat{\tau}_M < T
\]

be the jump times of \( \hat{S} \). We set \( \hat{\tau}_0 = 0, \hat{\tau}_{M+1} = T \). We recall that the canonical process \( \hat{S} \) is continuous at \( T \).

A trading strategy on the filtered probability space \((\Omega, \{\hat{\mathcal{F}}_t\}_{t=0}^T, \mathbb{P})\) is simply a predictable stochastic process \( \hat{\gamma} \) with respect to the filtration \( \hat{\mathcal{F}} \). Next, consider a constrained financial market, in which the trading strategy satisfies the bound

\[
\hat{\gamma} : [0,T] \times \hat{D} \to [-n, n].
\]

The statically tradable options are bounded real valued functions of \( A^{(n)} \).

We also define a probability measure \( \hat{\mu} \) on \( A^{(n)} \) by,

\[
\hat{\mu}(\{m2^{-n}\}) := \mu \left( \left\{ x \in \mathbb{R}^d_+ : \pi^{(n)}(x) = m2^{-n} \right\} \right), \quad m \in \mathbb{N}_d,
\]

where \( \mu \) is the probability measure defining the operator \( \mathcal{L} \) in subsection 2.1 and

\[
(3.2) \quad \pi^{(n)} : \mathbb{R}^d_+ \to A^{(n)} := \{ 2^{-n} k : k = (k_1, \ldots, k_d) \in \mathbb{N}_d \}
\]

is given by

\[
\pi^{(n)}(x)_i := 2^{-n} \lfloor 2^n x_i \rfloor, \quad i = 1, \ldots, d,
\]

and for \( a \in \mathbb{R}_+, [a] \in \mathbb{N} \) is smallest integer greater or equal to \( a \).

We summarize this in the following by defining the probabilistic super-replication problem on the set \( \hat{D} \).

**Definition 3.2.** A (probabilistic) semi-static portfolio is a pair \((\hat{g}, \hat{\gamma})\) such that \( \hat{g} : A^{(n)} \to \mathbb{R} \) is a bounded function, \( \hat{\gamma} : [0,T] \times \hat{D} \to [-n, n] \) is predictable and the stochastic integral \( \int \hat{\gamma}_a d\hat{\mathcal{E}}_a \) exists.

A semi-static portfolio is admissible if there exists \( C > 0 \) such that

\[
\int_0^t \hat{\gamma}_a d\hat{\mathcal{E}}_a \geq -C, \quad \mathbb{P} - a.s., \quad t \in [0,T].
\]

A semi-static portfolio is \( \mathbb{P} \)-super-replicating, if

\[
(3.3) \quad \hat{g}(S_T) + \int_0^T \hat{\gamma}_a d\hat{\mathcal{E}}_a \geq G(S), \quad \mathbb{P} - a.s.
\]

The (minimal) super-hedging cost of \( G \) is defined by,

\[
V^{(n)}(G) := \inf \left\{ \int \hat{g} d\hat{\mu} : \exists \gamma \text{ such that } \hat{\phi} := (\hat{g}, \hat{\gamma}) \right. \text{ is admissible and super-replicating } \}.
\]

\( \square \)

We note that (3.3) is equivalent to having the same inequality for every \( \hat{S} \in \hat{D} \).
Remark 3.3. The bound \( n \) that we place on the \( \gamma \) is somehow arbitrary. Indeed, any bound that converges to infinity with \( n \) and goes to zero when multiplied by \( 2^{-n} \) would suffice. This flexibility might be useful in possible future extensions. \( \Box \)

3.3. Lifting. An important step in our approach is to “lift” a given probabilistic semi-static portfolio \( \phi = (h, \hat{\gamma}) \) to an admissible portfolio \( \phi \) for the original financial market.

We start the construction of this lift by defining an approximation of the the stopping times \( \tau_k = \tau_k^{(n)}(S) \) defined in subsection 3.1. Recall also the random integer \( M = M^{(n)}(S) \) defined in subsection 3.1 and the set \( B^{(i)} \) defined in Definition 3.1. Set

\[
\hat{\tau}_0 := 0, \quad \hat{\tau}_1 = \sqrt{d} \, 2^{-n}, \quad \hat{\tau}_{M+1} := T.
\]

For \( k = 2, \ldots, M \) recursively define,

\[
\hat{\tau}_k := \hat{\tau}_{k-1} + (1 - \sqrt{d} \, 2^{-n}/T) \sup \{ \Delta t > 0 \mid \Delta t \in B^{(n+k)} \} \tau_k < \tau_{k-1} - \tau_{k-2} \}
\]

We note that due to the definition of \( B^{(i)} \) the above set is always non-empty. We collect some properties of these random times in the following lemma.

Lemma 3.4. Random times \( \hat{\tau}_k \)’s satisfy,

\[
0 = \hat{\tau}_0 < \sqrt{d} \, 2^{-n} = \hat{\tau}_1 < \ldots < \hat{\tau}_M < \hat{\tau}_{M+1} = T,
\]

and

\[
|\hat{\tau}_k - \tau_k| \leq \sqrt{d} \, 2^{-n+1}, \quad \forall \ k = 0, \ldots, M.
\]

Proof. The above definitions yield,

\[
\hat{\tau}_M = \hat{\tau}_1 + \sum_{k=2}^{M} (\hat{\tau}_k - \hat{\tau}_{k-1})
\]

\[
< \sqrt{d} \, 2^{-n} + (1 - \sqrt{d} \, 2^{-n}/T) \sum_{k=2}^{M} (\tau_{k-1} - \tau_{k-2})
\]

\[
= \sqrt{d} \, 2^{-n} + (1 - \sqrt{d} \, 2^{-n}/T)(\tau_{M-1} - \tau_0)
\]

\[
< \sqrt{d} \, 2^{-n} + (1 - \sqrt{d} \, 2^{-n}/T)T = T.
\]

This proves that

\[
0 = \hat{\tau}_0 < \sqrt{d} \, 2^{-n} = \hat{\tau}_1 < \ldots < \hat{\tau}_M < \hat{\tau}_{M+1} = T.
\]

Moreover, for any \( k = 2, \ldots, M \),

\[
\hat{\tau}_k = \hat{\tau}_1 + \sum_{j=2}^{k} (\hat{\tau}_j - \hat{\tau}_{j-1})
\]

\[
< \sqrt{d} \, 2^{-n} + (1 - \sqrt{d} \, 2^{-n}/T) \sum_{j=2}^{k} (\tau_{j-1} - \tau_{j-2})
\]

\[
= \sqrt{d} \, 2^{-n} + (1 - \sqrt{d} \, 2^{-n}/T)(\tau_{k-1} - \tau_0) = \tau_{k-1} + \sqrt{d} \, 2^{-n}(1 - \tau_{k-1}/T)
\]

\[
< \tau_{k-1} + \sqrt{d} \, 2^{-n}.
\]

The definition of \( \hat{\tau}_k \) and the set \( B^{(i)} \), imply that for any \( j = 2, \ldots, M \),

\[
\hat{\tau}_j - \hat{\tau}_{j-1} \geq \tau_{j-1} - \tau_{j-2} - \sqrt{d} \, 2^{-(n+j)}.
\]
We use this to estimate \( \hat{\tau}_k \) with \( k = 2, \ldots, M \), from below as follows.

\[
\hat{\tau}_k = \hat{\tau}_1 + \sum_{j=2}^{k} [\hat{\tau}_j - \hat{\tau}_{j-1}]
\]

\[
\geq \sqrt{d} \ 2^{-n} + (1 - \sqrt{d} \ 2^{-n}/T) \sum_{j=2}^{k} [\tau_{j-1} - \tau_{j-2} - \sqrt{d} \ 2^{-(n+j)}]
\]

\[
\geq \sqrt{d} \ 2^{-n} + (1 - \sqrt{d} \ 2^{-n}/T)[\tau_{k-1} - \tau_0] - \sqrt{d} \ 2^{-n}
\]

\[
= \tau_{k-1} - \sqrt{d} \ 2^{-n} \tau_{k-1}/T
\]

\[
> \tau_{k-1} - \sqrt{d} \ 2^{-n}.
\]

Since \( \hat{\tau}_{M+1} = \tau_M = T \), \( \hat{\tau}_1 = \sqrt{d} \ 2^{-n}, \tau_0 = 0 \), this proves that

\[
|\hat{\tau}_k - \tau_{k-1}| \leq \sqrt{d} \ 2^{-n}, \quad \forall \ k = 1, \ldots, M + 1.
\]

Also, by construction \( |\tau_{k+1} - \tau_k| \leq \sqrt{d} \ 2^{-n} \) for all \( k = 0, \ldots, M - 1 \). These inequalities complete the proof of the lemma. \( \square \)

We now define a map \( \hat{\Pi} = \hat{\Pi}^{(n)} : \mathbb{D} \to \mathbb{D} \) by,

\[
(3.4) \quad \hat{\Pi}_t(S) := \sum_{k=0}^{M-1} \pi^{(n+k)}[S_{\tau_k}] \chi_{[\hat{\tau}_k, \hat{\tau}_{k+1})}(t) + \pi^{(n)}[S_{\tau_M}] \chi_{[\hat{\tau}_M, T]}(t),
\]

where \( \pi^{(n)} \) is defined in (3.2).

It is clear by the definition of \( \pi^{(n)} \), \( \hat{\tau}_k \)'s and Definition 3.1, that \( \hat{\Pi}(S) \in \hat{\mathbb{D}} \) for every \( S \in \mathbb{D} \). We also note that \( S_{\tau_M} = S_T \) and that \( S \) is continuous at \( T \). For comparison, we also define

\[
\check{\Pi}_t(S) := \sum_{k=0}^{M-2} \pi^{(n+k)}[S_{\tau_k}] \chi_{[\tau_k, \tau_{k+1})}(t) + \pi^{(n)}[S_{\tau_{M-1}}] \chi_{[\tau_{M-1}, T]}(t),
\]

\[
\Pi_t(S) := \sum_{k=0}^{M-2} S_{\tau_k} \chi_{[\tau_k, \tau_{k+1})}(t) + S_{\tau_{M-1}} \chi_{[\tau_{M-1}, T]}(t).
\]

**Lemma 3.5.** Let \( d \) be the Skorokhod metric. Then, for every \( S \in \mathbb{D} \),

\[
d(S, \Pi(S)), \ d(\Pi(S), \hat{\Pi}(S)) \leq \sqrt{d} \ 2^{-n}, \quad d(\Pi(S), \hat{\Pi}(S)) \leq 3\sqrt{d} \ 2^{-n}.
\]

**Suppose \( G \) satisfies Assumption (2.7). Then,**

\[
|G(S) - G(\hat{\Pi}(S))| \leq 3m_G(3\sqrt{d} \ 2^{-n}).
\]

**Proof.** In view of (3.1), we have,

\[
d(S, \Pi(S)) \leq \| S - \Pi(S) \|_{\infty}
\]

\[
= \max_{k=0, \ldots, M-1} \sup \{ |S_t - S_{\tau_k}| : t \in [\tau_k, \tau_{k+1}) \} \vee |S_T - S_{\tau_{M-1}}|
\]

\[
\leq \sqrt{d} 2^{-n}.
\]

Next we estimate directly that

\[
d(\Pi(S), \hat{\Pi}(S)) \leq \| \Pi(S) - \hat{\Pi}(S) \|_{\infty} \leq \sup_{x \in \mathbb{R}^d, k \geq 0} |\pi^{(n+k)}(x) - x| \leq \sqrt{d} \ 2^{-n}.
\]
Define $\Lambda : [0, T] \to [0, T]$ by $\Lambda(0) = 0$, $\Lambda(\hat{\tau}_k) = \tau_k$ for $k = 1, \ldots, M - 1$, 
$$
\Lambda(\hat{\tau}_M) = [\tau_{M-1} + T]/2, \quad \Lambda(M) = \Lambda(T) = \tau_M = T,
$$
and to be piecewise linear at other points. Then, it is clear that $\Lambda$ is an increasing function and 
$$
\hat{\Pi}_{\Lambda(t)}(S) = \hat{\Pi}_t(S), \quad \forall \ t \in [0, \hat{\tau}_{M-1}].
$$
Moreover, for $t \in [\hat{\tau}_{M-1}, T]$,
$$
\hat{\Pi}_{\Lambda(t)}(S) = \pi^{(n)}(S_{\tau_{M-1}}).
$$
Hence, by (3.1) and the continuity of $\mathcal{S}$ at $t$,
$$
\sup_{t \in [0, T]} \{ |\hat{\Pi}_{\Lambda(t)}(S) - \hat{\Pi}_t(S)| + |\Lambda(t) - t| \} = \sqrt{d} 2^{-n}.
$$

We now use the above estimate together with Lemma 3.4 and the above $\Lambda$ in the definition of the Skorokhod metric. The result is
$$
d(\Pi(S), \hat{\Pi}(S)) \leq \sup_{t \in [0, T]} \left\{ \left| \Pi_{\Lambda(t)}(S) - \Pi_t(S) \right| + \left| \Lambda(t) - t \right| \right\} 
= \sqrt{d} 2^{-n} + \max_{k=1,\ldots,M-1} \left\{ |\hat{\tau}_{k+1} - \tau_k| \right\} \leq \sqrt{d} 2^{-n} + \sqrt{d} 2^{-n+1}.
$$

Suppose $G$ satisfies Assumption 2.7. We now use the above estimates to obtain
$$
|G(S) - G(\hat{\Pi}(S))| \leq |G(S) - G(\Pi(S))| + |G(\Pi(S)) - G(\hat{\Pi}(S))| 
+ |G(\hat{\Pi}(S)) - G(\Pi(S))| 
\leq 2m_G(\sqrt{d} 2^{-n}) + m_G(3\sqrt{d} 2^{-n}) 
\leq 3m_G(3\sqrt{d} 2^{-n}).
$$

We are ready to define the lift. Let $\hat{\phi} = (\hat{g}, \hat{\gamma})$ be a semi-static portfolio in the sense of Definition 3.2. Define a portfolio $\phi := \Psi(\hat{\phi}) := (g, \gamma)$ for the original problem by
$$
g(x) := \hat{g}\left(\pi^{(n)}(x)\right), \quad x \in \mathbb{R}^d_+,
$$
(3.5)
$$
\gamma_t(S) := \sum_{k=0}^{M-1} \hat{\gamma}_{\tau_{k+1}}(S)(\hat{\Pi}(S)) \chi_{(\tau_k(S), \tau_{k+1}(S))}(t), \quad t \in [0, T].
$$

Observe that by definition $\gamma_0(S) = 0$.

The following lemma provides the important properties of the above mapping.

**Lemma 3.6.** For a semi-static portfolio $\hat{\phi} = (\hat{g}, \hat{\gamma})$ in the sense of Definition 3.2 and let $\phi = (g, \gamma)$ be defined as in (3.5). Then, $\phi$ is admissible in sense defined in Definition 2.5 and has the following properties,
$$
\int_{\mathbb{R}^d_+} g d\mu = \int_{\mathbb{R}^d_+} \hat{g} d\hat{\mu},
$$
$$
\left| \int_0^t \gamma_u(\hat{\Pi}(S)) d\hat{\Pi}_u(S) - \int_0^t \gamma_u(S) d\mathcal{S}_u \right| \leq \sqrt{d} n 2^{-n+1}, \quad \forall \ S \in \mathcal{D}, \ t \in [0, T].
$$
Proof. Using the definition of \( \hat{\mu} \) and \( g \), we directly calculate that
\[
\int_{\mathbb{R}_+} gd\mu = \sum_{m \in \mathbb{N}_d} \hat{g}(m2^{-n}) \mu \left( \{ x : \pi^{(n)}(x) = m2^{-n} \} \right)
\]
\[
= \sum_{m \in \mathbb{N}_d} \hat{g}(m2^{-n}) \mu \left( \{ m2^{-n} \} \right) = \int_{\mathcal{A}^{(n)}} \hat{g}d\hat{\mu}.
\]

Since \( \hat{\phi} \) is bounded by definition, the admissibility of \( \phi \) would follow if \( \gamma \) is progressively measurable. We show this by verifying (2.1). Towards this goal, let \( S, \hat{S} \in \mathcal{D} \) and \( t \in [0, T] \) be such that \( S_u = \hat{S}_u \) for all \( u \leq t \). We have to show that \( \gamma_t(S) = \gamma_t(\hat{S}) \).

Since \( \gamma_0(S) = \gamma_0(\hat{S}) = 0 \), we may assume that \( t > 0 \). Let \( 0 \leq k_t(S) \) be the integer such that \( t \in (\tau_{k_t}(S), \tau_{k_t+1}(S)) \). Since by hypothesis \( S \) and \( \hat{S} \) agree on \([0, t]\), their jump times up to time \( t \) also agree. In particular, \( k_t(S) = k_t(\hat{S}) =: k_t \) and
\[
\tau_i(S) = \tau_i(\hat{S}) < t \quad \text{and} \quad S_{\tau_i(S)} = \hat{S}_{\tau_i(\hat{S})}, \quad \forall \, i = 1, \ldots, k_t.
\]
Since for any \( k \geq 0 \), \( \tau_{k+1} \) is defined directly by \( \tau_1, \ldots, \tau_k \), we also conclude that \( \hat{\tau}_i(S) = \hat{\tau}_i(\hat{S}) \), \( \forall \, i = 0, 1, \ldots, k_t + 1 \).

Set \( \theta := \tau_{k_t+1}(S) = \hat{\tau}_{k_t+1}(\hat{S}) \) so that
\[
\gamma_t(S) = \hat{\gamma}_\theta(\hat{\mathcal{P}}(\hat{S})) \quad \text{and} \quad \gamma_t(\hat{S}) = \hat{\gamma}_\theta(\hat{\mathcal{P}}(\hat{S})).
\]
Since, \( \hat{\gamma} \) is predictable, to prove \( \gamma_t(S) = \gamma_t(\hat{S}) \) it suffices to show that
\[
\hat{\mathcal{P}}_u(S) = \hat{\mathcal{P}}_u(\hat{S}), \quad \forall \, u < \theta.
\]

By the definition of \( \hat{\mathcal{P}} \), for any \( u < \theta \) there exists an integer \( k \leq k_t \) (same for both \( S \) and \( \hat{S} \)) so that
\[
\hat{\mathcal{P}}_u(S) = \pi(S_{\tau_k}) \quad \text{and} \quad \hat{\mathcal{P}}_u(\hat{S}) = \pi(\hat{S}_{\tau_k}).
\]

Now recall that \( S \) and \( \hat{S} \) agree on \([0, t]\) and \( \tau_k \leq \tau_{k_t} < t \). Hence, \( S_{\tau_k} = \hat{S}_{\tau_k} \) and consequently \( \hat{\mathcal{P}}_u(S) = \hat{\mathcal{P}}_u(\hat{S}) \). This proves that \( \gamma \) is progressively measurable.

We continue by estimating the difference of the two integrals. In view of the definitions, we have the following representations for the stochastic integrals,
\[
\int_0^T \gamma_u(S)dS_u = \sum_{k=1}^M \hat{\gamma}_{\tau_k}(S)(\hat{\mathcal{P}}(\hat{S})) \left( S_{\tau_k}(S) - S_{\tau_{k-1}}(S) \right)
\]
and
\[
\int_0^T \hat{\gamma}_u(\hat{\mathcal{P}}(\hat{S}))d\hat{\mathcal{P}}_u(\hat{S}) = \sum_{k=1}^M \hat{\gamma}_{\tau_k}(\hat{S})(\hat{\mathcal{P}}(\hat{S})) \left( \pi^{(n+k)}(S_{\tau_k}(S)) - \pi^{(n+k-1)}(S_{\tau_{k-1}}(S)) \right).
\]
Set
\[
\mathcal{I} := \int_0^T \hat{\gamma}_u(\hat{\mathcal{P}}(\hat{S}))d\hat{\mathcal{P}}_u(\hat{S}) - \int_0^T \gamma_u(S)dS_u.
\]
Since the portfolio \( \hat{\gamma} \) is bounded by \( n \), we have the following estimate,
\[
|\mathcal{I}| \leq 2\|\hat{\gamma}\|_{\infty} \sum_{k=1}^M \left| \pi^{(n+k)}(S_{\tau_k}(S)) - S_{\tau_k}(S) \right| \leq 2n \sum_{k=1}^M \sqrt{d} \, 2^{-(n+k)} \leq \sqrt{d} \, n2^{-n+1}.
\]
In view of the above results and the construction, \( g \) is bounded and therefore, \( g \in L^1(\mathbb{R}_+^d; \mu) \). Moreover, \( \gamma \) is shown to be progressively measurable and for \( t \in [\tau_k, \tau_{k+1}) \)

\[
\int_0^t \gamma_u(S)dS_u \geq \int_0^t \hat{\gamma}_u(\hat{\Pi}(S))d\hat{\Pi}_u(S) - 2\|\hat{\gamma}\|_\infty \sum_{k=1}^M |\pi^{(n+k)}(S_{\tau_k}(S)) - S_{\tau_k}(S)| - \frac{2}{\tau_n} \geq -C - \sqrt{d} n 2^{-n+1} - \frac{2}{\tau_n},
\]

where the last inequality follows from the fact that \( \hat{\gamma} \) is admissible in the sense of Definition 3.2. Hence, the stochastic integral is bounded from below and consequently is a \( \mathbb{Q} \) super-martingale for every \( \mathbb{Q} \in \mathcal{M}_\mu \). These arguments imply that the lifted portfolio \((g, \gamma)\) is admissible. \( \square \)

The above lifting result provides an immediate connection between \( V(G) \) and \( V^{(n)}(G) \).

**Corollary 3.7.** Under the hypothesis of Theorem 2.9, the minimal super-replication costs satisfy

\[
V(G) \leq V^{(n)}(G) + \sqrt{d} n 2^{-n+1} + 3m_G(3\sqrt{d} 2^{-n}).
\]

In particular,

\[
V(G) \leq \liminf_{n \to \infty} V^{(n)}(G).
\]

**Proof.** Let \( \hat{\phi} \) and \( \phi \) be as in Lemma 3.6. Further assume that \( \hat{\phi} \) is super-replicating \( G \) on \( \hat{\mathbb{D}} \). Let \( S \in \mathbb{D} \). Then, \( \hat{\Pi}(S) \in \hat{\mathbb{D}} \) and

\[
\hat{g}(\hat{\Pi}(S)) + \int_0^T \hat{\gamma}_t(\hat{\Pi}(S))d\hat{\Pi}(S)_t \geq G(\hat{\Pi}(S)).
\]

By definition of \( g \) and \( \hat{\Pi} \),

\[
g(S_T) = \hat{g}(\pi^{(n)}(S_T)) = \hat{g}(\hat{\Pi}(S)).
\]

Then, in view of Lemma 3.6,

\[
g(S_T) + \int_0^T \gamma_t(S)dS_t \geq \hat{g}(\hat{\Pi}(S)) + \int_0^T \hat{\gamma}_t(\hat{\Pi}(S))d\hat{\Pi}(S)_t - \sqrt{d} n 2^{-n+1}
\]

\[
\geq G(\hat{\Pi}(S)) - \sqrt{d} n 2^{-n+1}
\]

\[
\geq \hat{G}(S) := G(S) - \sqrt{d} n 2^{-n+1} - 3m_G(3\sqrt{d} 2^{-n}).
\]

Hence, \( \phi \) super-replicates \( \hat{G} \). This implies that \( \int gd\mu \geq V(G) \). Since by construction \( \int gd\mu = \int \hat{g}d\hat{\mu} \), and since above inequality holds for every super-replicating \( \hat{\phi} \), we conclude that \( V(G) \leq V^{(n)}(G) \). It is also clear that

\[
V(G) = V(\hat{G}) + \sqrt{d} n 2^{-n+1} + 3m_G(3\sqrt{d} 2^{-n})
\]

\[
\leq V^{(n)}(G) + \sqrt{d} n 2^{-n+1} + 3m_G(3\sqrt{d} 2^{-n}).
\]

\( \square \)

**Remark 3.8.** Observe that since \( \hat{g} \) is bounded, so is the lifted static hedge \( g \). Hence in the Definition 2.5, one may use the class

\[
\mathcal{H} := \{ h(S) = g(S_T) : g \in L^\infty(\mathbb{R}_+^d; \mu) \}.
\]
Moreover, it is not difficult to construct \( g \) so that it agrees with \( \hat{g} \) on \( A^{(n)} \) and is continuous. This construction would enable us to consider the even smaller class \( \mathcal{H} \) with bounded and continuous \( g \)'s.

Moreover, in Definition 3.2 the stochastic integral \( \hat{\gamma}_u dS_u \) is assumed to be bounded from below by a constant \( C \). In view of the above Lemma, also the lifted portfolio satisfies that the path wise integral \( \int \gamma_u S_u \) is also bounded from below, possibly with a slightly larger constant. This shows that in Definition 2.5 it would be sufficient to consider \( \gamma \)'s so that the integrals are bounded from below, instead of assuming that their stochastic equivalents are \( Q \) super-martingales for every \( Q \in M_{\mu} \).

The above Corollary is the only place in the proof of the upper bound (under the hypothesis of Theorem 2.9) where the exact definition of admissibility is important. Hence the above discussions and Remark 2.10 show that for a bounded class of integrands \((g, \gamma)\), the super-replication cost of \( G \) would be same if we one considers the described smaller class of admissible strategies \((g, \gamma)\).

For exotic options \( G \) discussed in Section 5, in view of Lemma 5.2 one needs to consider \( g \)'s so that \( |g(x)| \leq c(1 + |x|) \) for every \( x \in \mathbb{R}^d_+ \) for some constant \( c \). Since, in that section \( \mu \) is assumed to satisfy (2.5), such a \( g \) is also in \( L^1(\mathbb{R}^d_+; \mu) \). For the integrand \( \gamma \), again due to the proof of Lemma 5.2, one needs to assume at least the condition (2.6). Then, in view of Doob's inequality and the assumption (2.5), one can conclude that the stochastic integral is a \( Q \) super-martingale for every \( Q \in M_{\mu} \). Hence, the duality in Section 5 would also hold for this slightly smaller class of integrands.

\[ \square \]

### 3.4. Analysis of \( V^{(n)}(G) \)

In view of the previous Corollary, to complete the proof of (2.7), we need to show the following inequality,

\[
\lim_{n \to \infty} \sup \mathbb{E}_{Q} V^{(n)}(G) \leq \sup_{Q \in M_{\mu}} \mathbb{E}_{Q} |G(S)|.
\]

This is done in two steps. We first use a standard min-max theorem and the constrained duality result of [16] to get a dual representation for \( V^{(n)}(G) \) (in fact we obtain an upper bound). We then analyze this dual by probabilistic techniques.

We start with a definition.

**Definition 3.9.** Let \( \mathcal{P} \) be the set of all probability measures \( Q \) which are supported on \( \tilde{\Omega} = \mathbb{D}^{(n)}[0,T] \). For \( c > 0 \) let \( M(n,c) \subset \mathcal{P} \) be the set of all probability measures that has the following properties,

\[
\sum_{m \in \mathbb{N}^d} \mathbb{E}_{Q} \left( \hat{S}_T = m2^{-n} \right) - \hat{\mu} \left( \{m2^{-n}\} \right) \leq \frac{c}{n}
\]

and

\[
\mathbb{E}_{Q} \left[ \sum_{k=1}^{M+1} \mathbb{E}_{Q} (\hat{S}_{\tau_k} | \mathcal{F}_{\tau_{k-1}}) - \hat{S}_{\tau_{k-1}} \right] \leq \frac{c}{n},
\]

where as defined before, \( \hat{\tau}_1(\hat{S}) < \ldots < \hat{\tau}_M(\hat{S}) \) are the jump times of the piecewise constant process \( \hat{S} \in \tilde{\Omega} \) and \( \hat{\tau}_0 = 0, \hat{\tau}_{M+1} = T \).

We refer the reader to page 105 in [27] for the definition of the \( \sigma \)-algebra \( \mathcal{F}_{\tau_t} \). Indeed, for any stopping time \( \tau \in [0,T] \), \( \mathcal{F}_{\tau_t} \) is defined to be the smallest \( \sigma \)-algebra that contains \( \mathcal{F}_0 \) and all sets of the from \( A \cap \{ \tau > t \} \) for all \( t \in (0,T] \) and \( A \in \mathcal{F}_t \).
Clearly, $F_\tau \subset \mathcal{F}_\tau$ and $\tau$ is $\mathcal{F}_\tau$-measurable. Moreover, if $X$ is a predictable process, then $X_\tau$ is $\mathcal{F}_\tau$-measurable (Theorem 8, page 106 [27]).

The following lemma is proved by using the results of [16] on hedging under constraints, and applying a classical min-max theorem.

**Lemma 3.10.** Suppose that $0 \leq G \leq c$ for some constant $c > 0$. Then,

$$V^{(n)}(G) \leq \sup_{Q \in \mathcal{M}(c,n)} \mathbb{E}_Q [G(S)]^+,$$

where we set the right hand side is equals to zero if $\mathcal{M}(c,n)$ is empty.

**Proof.** We proceed in several steps.

**Step 1.** In view of its definition, for any bounded function $\hat{g}$ on $A^{(n)}$, we have

$$V^{(n)}(G) \leq \psi^{(n)}(G \circ \hat{g}) + \int g d\hat{\mu},$$

where $G \circ \hat{g}(S) := G(S) - \hat{g}(S_T)$ and for any bounded measurable real valued function $\xi$ on $\mathbb{D}$,

$$\psi^{(n)}(\xi) = \inf \left\{ z \in \mathbb{R} : \exists \gamma \text{ such that } |\gamma| \leq n, \ z + \int_0^T \gamma_u dS_u \geq \xi, \ \mathbb{P} - a.s. \right\}$$

to be the "classical" super–hedging price of the European claim $\xi$ under the constraint that absolute value of the number of the stocks in the portfolio is bounded by $n$. Furthermore, (as usual) we require that there exists $M > 0$ such that $\int_0^T \gamma_u dS_u \geq -M$, for every $t \in [0, T]$.

**Step 2.** Under any measure $Q \in \mathcal{P}$ the canonical process $S$ on $\mathbb{D}$ is piecewise constant with jump times $0 < \tau_1 < \ldots < \tau_M < T$. So it is clear that the canonical process is a $Q$ semi-martingale. Moreover, it has the following decomposition, $S = M^Q - A^Q$ where

$$A^Q_t = \sum_{k=1}^M \chi_{[\tau_k, \tau_{k+1})}(t) \sum_{j=1}^k |S_{\tau_{j-1}} - \mathbb{E}_Q(S_{\tau_j} | \mathcal{F}_{\tau_{j-1}})|, \ \forall t \in [0, T)$$

$$A^Q_T := \lim_{t \uparrow T} A^Q_t,$$

a predictable process of bounded variation and $M^Q_t = A^Q_t + S_t, \ t \in [0, T]$, is a $Q$ martingale. Then, from Example 2.3 and Proposition 4.1 in [16] it follows that

$$\psi^{(n)}(\xi) = \sup_{Q \in \mathcal{P}} \mathbb{E}_Q \left[ \xi - n \sum_{k=1}^M |S_{\tau_{k-1}} - \mathbb{E}_Q(S_{\tau_k} | \mathcal{F}_{\tau_{k-1}})| \right].$$

**Step 3.** Set

$$\mathcal{Z} := \{ \hat{g} : A^{(n)} \to \mathbb{R} : ||\hat{g}||_\infty \leq n \}.$$

In view of the previous steps,

$$V^{(n)}(G) \leq \inf_{\hat{g} \in \mathcal{Z}} \sup_{Q \in \mathcal{P}} \mathcal{G}(\hat{g}, Q),$$

where $\mathcal{G} : \mathcal{Z} \times \mathcal{P} \to \mathbb{R}$ is given by

$$\mathcal{G}(\hat{g}, Q) := \mathbb{E}_Q \left[ G - n \sum_{k=1}^M |\mathbb{E}_Q(S_{\tau_k} | \mathcal{F}_{\tau_{k-1}}) - S_{\tau_{k-1}}| \right] + \int \hat{g} d\hat{\mu} - \mathbb{E}_Q(\hat{g}(S_T)).$$
Step 4. In this step is to interchange the order of the infimum and supremum by applying a standard min-max theorem. Indeed, consider the vector space \( \mathbb{R}^{A^{(n)}} \) of all functions \( \hat{g} : A^{(n)} \to \mathbb{R} \) equipped with the topology of point-wise convergence. Clearly, this space is locally convex. Also, since \( A^{(n)} \) is countable, \( Z \) is a compact subset of \( \mathbb{R}^{A^{(n)}} \). The set \( \mathcal{P}_N \) can be naturally considered as a convex subspace of the vector space \( \mathbb{R}^{\mathcal{P}}_N \). In order to apply a min-max theorem, we also need to show continuity and concavity.

\( \mathcal{G} \) is affine in the first variable, and by the bounded convergence theorem, it is continuous in this variable. We claim that \( \mathcal{G} \) is concave in the second variable. To this purpose, it is sufficient to show that for any \( k \geq 1 \) the map
\[
\mathbb{Q} \to \mathbb{E}_\mathbb{Q} |\mathbb{E}_\mathbb{Q}(\mathbb{S}_{\tau_k}|\mathcal{F}_{\tau_{k-1}}) - \mathbb{S}_{\tau_{k-1}}|
\]
is convex. Set \( X = \mathbb{S}_{\tau_k} - S_{\tau_{k-1}}, \mathcal{F} := \mathcal{F}_{\tau_k} - \mathcal{F}_{\tau_{k-1}} \) and \( Y = \mathbb{E}_\mathbb{Q}(X|\mathcal{F}) \). For probability measures \( \mathbb{Q}_1, \mathbb{Q}_2 \) and \( \lambda \in (0,1) \), set \( Y_1 = \mathbb{E}_\mathbb{Q}_1(X|\mathcal{F}) \) and \( \mathbb{Q} = \lambda \mathbb{Q}_1 + (1 - \lambda) \mathbb{Q}_2 \). Then,
\[
\mathbb{E}_\mathbb{Q}|Y| = \mathbb{E}_\mathbb{Q}(Y\mathbb{1}_{\{Y>0\}}) - \mathbb{E}_\mathbb{Q}(Y\mathbb{1}_{\{Y<0\}})
= \mathbb{E}_\mathbb{Q}(X\mathbb{1}_{\{Y>0\}}) - \mathbb{E}_\mathbb{Q}(X\mathbb{1}_{\{Y<0\}})
= \lambda (\mathbb{E}_\mathbb{Q}_1(X\mathbb{1}_{\{Y>0\}}) - \mathbb{E}_\mathbb{Q}_2(X\mathbb{1}_{\{Y<0\}}))
+ (1 - \lambda) (\mathbb{E}_\mathbb{Q}_2(X\mathbb{1}_{\{Y>0\}}) - \mathbb{E}_\mathbb{Q}_2(X\mathbb{1}_{\{Y<0\}}))
= \lambda (\mathbb{E}_\mathbb{Q}_1(Y_1\mathbb{1}_{\{Y>0\}}) - \mathbb{E}_\mathbb{Q}_2(Y_1\mathbb{1}_{\{Y<0\}}))
+ (1 - \lambda) (\mathbb{E}_\mathbb{Q}_2(Y_2\mathbb{1}_{\{Y>0\}}) - \mathbb{E}_\mathbb{Q}_2(Y_2\mathbb{1}_{\{Y<0\}}))
\leq \lambda \mathbb{E}_\mathbb{Q}_1|Y_1| + (1 - \lambda) \mathbb{E}_\mathbb{Q}_2|Y_2|.
\]
This yields the convexity of \( \mathcal{G} \) in the \( \mathbb{Q} \)-variable.

Step 5. Next, we apply the min-max theorem, Theorem 45.8 in [30] to \( \mathcal{G} \). The result is,
\[
\inf_{\hat{g} \in \mathcal{Z}} \sup_{\mathbb{Q} \in \mathcal{P}} \mathcal{G}(\hat{g}, \mathbb{Q}) = \sup_{\mathbb{Q} \in \mathcal{P}} \inf_{\hat{g} \in \mathcal{Z}} \mathcal{G}(\hat{g}, \mathbb{Q}).
\]
Together with Step 3, we conclude that
\[
V^{(n)}(G) \leq \sup_{\mathbb{Q} \in \mathcal{P}} \inf_{h \in \mathcal{Z}} \mathcal{G}(h, \mathbb{Q}).
\]

Finally, for any measure \( \mathbb{Q} \in \mathcal{P} \), define \( h^{\mathbb{Q}} \in \mathcal{Z} \) by
\[
h^{\mathbb{Q}}(m2^{-n}) = c \text{ sign } \left[ \mathbb{Q} \left( \{ \mathbb{S}_T = m2^{-n} \} \right) - \hat{\mu} \left( \{ \mathbb{S}_T = m2^{-n} \} \right) \right], \ m \in \mathbb{N}^d.
\]
Then, by choosing \( h^{\mathbb{Q}} \) in the min-max formula, we arrive at,
\[
V^{(n)}(G) \leq \sup_{\mathbb{Q} \in \mathcal{P}} \mathcal{G}(h^{\mathbb{Q}}, \mathbb{Q}).
\]
Moreover,
\[
\int h^{\mathbb{Q}} d\hat{\mu} - \mathbb{E}_\mathbb{Q} h^{\mathbb{Q}}(\mathbb{S}_T) = -n \sum_{m \in \mathbb{N}^d} \mathbb{Q} \left( \hat{\mathbb{S}}_T = m2^{-n} \right) - \hat{\mu} \left( \{ m2^{-n} \} \right).
\]
Hence if \( \mathbb{Q} \) does not belong to the set \( \mathcal{M}(c,n) \), then
\[
\mathcal{G}(h^{\mathbb{Q}}, \mathbb{Q}) \leq \mathbb{E}_\mathbb{Q} |\mathcal{G}(\mathbb{S})| - c.
\]
By hypothesis, \( 0 \leq G \leq c \) and therefore, \( \mathbb{E}_\mathbb{Q} |\mathcal{G}(\mathbb{S})| \leq c \) and \( V^{(n)}(G) \geq 0 \). Hence, we may restrict the maximization to \( \mathbb{Q} \in \mathcal{M}(c,n) \). Moreover, if \( \mathcal{M}(c,n) \) is empty, then we can conclude that \( V^{(n)}(G) \leq 0 \). \( \square \)
3.5. Proof of (2.7) completed. We first prove the result for nonnegative claims.

Lemma 3.11. Suppose that $0 \leq G \leq c$ and satisfies the Assumption 2.7. Then

\begin{equation}
\limsup_{n \to \infty} \sup_{Q \in \mathcal{M}(c,n)} \mathbb{E}_Q[G(S)]^+ \leq \sup_{Q \in \mathcal{M}_\mu} \mathbb{E}_Q[G(S)].
\end{equation}

Proof. Without loss of generality (by passing to a subsequence), we may assume that the sequence on the left hand side of (3.8) is convergent. Moreover, we may assume that for sufficiently large $n$ the set $\mathcal{M}(c,n)$ is not empty, otherwise (3.8) is trivially satisfied.

**Step 1.** Choose $Q_n \in \mathcal{M}(c,n)$ such that

\[
\left[ \sup_{Q \in \mathcal{M}(c,n)} \mathbb{E}_Q[G(S)] \right]^+ \leq 2^{-n} + \mathbb{E}_Q[G(S)].
\]

Hence,

\[
\lim_{n \to \infty} \mathbb{E}_{Q_n}[G(S)] = \limsup_{n \to \infty} \left[ \sup_{Q \in \mathcal{M}(c,n)} \mathbb{E}_Q[G(S)] \right]^+.
\]

Recall the decomposition given in the second step of the proof of Lemma 3.10. Set $\mathcal{M}^n := \mathcal{M}^{Q_n}$, $A^n := A^{Q_n}$. Since $G$ is uniformly continuous in the Skorokhod metric,

\[
|G(S) - G(M^n(S))| \leq m_G(n^{-1/2}), \quad \text{whenever} \quad \sup_{t \in [0,T]} A^n_t(S) \leq n^{-1/2}.
\]

Therefore, since $|G(S) - G(M^n(S))| \leq c$,

\[
|\mathbb{E}_{Q_n}[G(S) - G(M^n(S))]| \leq m_G(n^{-1/2}) + c \sup_{t \in [0,T]} A^n_t(S) \leq n^{-1/2}.
\]

We now use the representation (3.7) of $A^n$ together with the Markov inequality. The result is,

\[
Q_n(\sup_{t \in [0,T]} A^n_t \geq n^{-1/2}) \leq n^{1/2} \mathbb{E}_{Q_n} \sum_{k=1}^{M} |\mathbb{E}_{Q_n}(S_{\tau_k} \mid \mathcal{F}_{\tau_k}) - S_{\tau_k-1}| \leq cn^{-1/2},
\]

where the last inequality follows from the fact that $Q_n \in \mathcal{M}(c,n)$ and (3.6). Therefore, we have concluded that

\[
\limsup_{n \to \infty} \left[ \sup_{Q \in \mathcal{M}(c,n)} \mathbb{E}_Q[G(S)] \right]^+ = \lim_{n \to \infty} \mathbb{E}_{Q_n}[G(M^n(S))].
\]

**Step 2.** As in subsection 2.2, let $\tilde{\Omega} := \mathcal{D}([0,T]; \mathbb{R}^d)$, $\tilde{\mathcal{F}}$ be the filtration generated by the canonical process $\tilde{S}$. For a probability measure $\tilde{\mu}$ on $\mathbb{R}^d$, set $\tilde{M}_{\tilde{\mu}}$ be set of measures $\tilde{Q}$ on $\mathcal{D}([0,T]; \mathbb{R}^d)$ such that the canonical process is a martingale that starts at $\tilde{S}_0 = (1, \ldots, 1)$ and the distribution of $\tilde{S}_T$ under $\tilde{Q}$ is equal to $\tilde{\mu}$. Note that when the support of $\tilde{\mu}$ is on $\mathbb{R}^d_+$, then the support of any measure $\tilde{Q} \in \tilde{M}_{\tilde{\mu}}$ is included in $\mathbb{D}$. Hence, in that case $\tilde{M}_{\tilde{\mu}}$ is the same as $\tilde{M}_{\tilde{\mu}}$ defined earlier.

We set

\begin{equation}
v(\tilde{\mu}) := \sup_{\tilde{Q} \in \tilde{M}_{\tilde{\mu}}} \mathbb{E}_{\tilde{Q}}[G(\tilde{S})].
\end{equation}
Let $\tilde{Q}_n$ be the measure on $D([0,T];\mathbb{R}^d)$ induced by $M^n$ under $Q_n$, i.e., for any Borel subset $C \subset D([0,T];\mathbb{R}^d)$,

$$\tilde{Q}_n(C) := Q_n \{ S \in D : M^n(S) \in C \}.$$

Further, let $\nu_n$ be the distribution of $M^n_T$ under the measure $Q_n$. Since $M^n$ is a martingale, it is clear that $\tilde{Q}_n \in \tilde{M}\nu_n$. Then, the previous step implies that

$$\lim sup_{n \to \infty} \left[ \sup_{Q \in M(c,n)} \mathbb{E}_Q [G(S)] \right]^+ \leq \lim_{n \to \infty} \nu_n.$$

**Step 3.** Since $Q_n \in M(c,n)$, (3.6) implies that

$$\mathbb{E}_{Q_n} |S_T - M^n_T(S)| = \mathbb{E}_{Q_n} |A^n_T| \leq \frac{c}{n}.$$

Let $\mu_n$ be the distribution of $S_T$ under $Q_n$. Then, by the definition of $M(c,n)$, $\mu_n$ converges weakly to $\mu$. Then, above inequalities imply that $\nu_n$ also converges weakly to $\mu$.

Since each component $S_t^{(k)} > 0$, for all $t \in [0,T]$ and $k = 1, \ldots, d$,

$$\mathbb{E}_{Q_n} [(M^n_t)^{(k)}(S)]^- = \mathbb{E}_{Q_n} [(- (M^n_T)^{(k)}(S)) \chi_{\{(M^n_T)^{(k)}(S) \leq 0\}}] \leq \mathbb{E}_{Q_n} [(S_T^{(k)} - (M^n_T)^{(k)}(S)) \chi_{\{(M^n_T)^{(k)}(S) \leq 0\}}] \leq \mathbb{E}_{Q_n} |S_T - M^n_T(S)| = \mathbb{E}_{Q_n} |A^n_T| \leq \frac{c}{n}.$$

Hence, for each $k = 1, \ldots, d$,

$$\lim_{n \to \infty} \int_{\mathbb{R}} (x_k)^{-} d\nu_n(x) = 0.$$

Hence we are in a position to use the continuity result, Theorem 4.1 proved in the next section. This implies that

$$\lim_{n \to \infty} v(\nu_n) = v(\mu).$$

Since $\mu$ is supported on $\mathbb{R}^d_+$, as remarked before, $\tilde{M}_\mu = M_\mu$. We now combine all the steps of this proof to arrive at,

$$\lim sup_{n \to \infty} \left[ \sup_{Q \in M(c,n)} \mathbb{E}_Q [G(S)] \right]^+ = \lim_{n \to \infty} \mathbb{E}_{Q_n} [G(M^n(S))] \leq \lim_{n \to \infty} v(\nu_n) = v(\mu) = \sup_{Q \in \tilde{M}_\mu} \mathbb{E}_Q [G(S)].$$

\qed

**Corollary 3.12.** Suppose $G$ satisfies the Assumption 2.7. Then, (2.7) holds.

**Proof.** In view of Corollary 3.7, Lemmas 3.10 and 3.11, (2.7) holds for all $G$ satisfying Assumption 2.7 and is also nonnegative. Hence, for any bounded $G$ satisfying Assumption 2.7, we conclude that (2.7) holds for $\tilde{G} := G + \|G\|_{\infty} \geq 0$. Since $V(G) = V(\tilde{G}) - \|G\|_{\infty}$, (2.7) follows for $G$ as well.
4. Continuity of the dual with respect to $\mu$

In this section, we prove a continuity result for a martingale optimal transport problem on the space $D$. Recall the functional $v(\tilde{\mu})$ defined in (3.9) and the set of martingale measures $\tilde{M}_\nu$ again defined in (3.9).

**Theorem 4.1.** Suppose $G$ satisfies the Assumption 2.7. Let $\nu_n$ be a sequence of probability measures on $\mathbb{R}^d$. Assume that $\nu_n$ converges weakly to a probability measure $\mu$ supported on $\mathbb{R}^d_+$. Further assume that for each component $k = 1, \ldots, d$,

$$\lim_{n \to \infty} \int x^{(k)} d\nu_n(x) = \int x^{(k)} d\mu(x) < \infty, \quad \text{and} \quad \lim_{n \to \infty} \int (x^{(k)})^- d\nu_n(x) = 0.$$

Then,

$$\lim_{n \to \infty} v(\nu_n) = v(\mu).$$

**Proof.** To ease the notation, we take $d = 1$. First we prove that

$$\limsup_{n \to \infty} v(\nu_n) \leq v(\mu).$$

In fact this the inequality that we used in the proof of Lemma 3.11. For each $n \in \mathbb{N}$ choose $\tilde{Q}_n \in \tilde{M}_\nu$ such that

$$v(\nu_n) \leq 2^{-n} + E_{\tilde{Q}_n}[G(\tilde{S})].$$

**Step 1.** In the first step, we construct a martingale measure in $M_\mu$ that is “close” to $\tilde{Q}_n$. This construction uses the Prokhorov’s metric which we now recall. For any two probability measures $\nu, \rho$ on $\mathbb{R}$, the Prokhorov distance $\hat{d}(\nu, \rho)$ is defined to be the smallest $\delta > 0$ so that

$$\nu(C) \leq \rho(C_\delta ) + \delta, \quad \text{and} \quad \rho(C) \leq \nu(C_\delta ) + \delta,$$

for every Borel subset $C \subset \mathbb{R}$, where

$$C_\delta := \bigcup_{x \in C} (x - \delta, x + \delta).$$

It is well known that convergence in the Prokhorov metric is equivalent to weak convergence, (for more details on Prokhorov’s metric we refer the reader to [28], Chapter 3, Section 7).

We now follow Theorem 4 on page 358 in [28] and Theorem 1 in [29] to construct a random variable $\Lambda^{(n)}$ as follows. First construct a probability space $(\tilde{\Omega}_n, \tilde{\mathcal{F}}_n, \tilde{\mathbb{P}}_n)$ and a martingale $M^{(n)}$ and a random variable $\xi^{(n)}$ uniformly distributed distributed on $[0, T]$ such that:

a. $\xi^{(n)}$ and $M^{(n)}$ are independent;

b. distribution of $M^{(n)}$ under $\mathbb{P}_n$ is equal to the measure $\tilde{Q}_n$ on $D([0, T; \mathbb{R})$. In particular,

$$E_{\tilde{\mathbb{P}}_n} [G(M^{(n)})] = E_{\tilde{Q}_n} [G(\tilde{S})].$$

We may choose the filtration $\tilde{\mathcal{F}}$ to be the smallest right-continuous filtration that is generated by the processes $M^{(n)}$ and $\xi^{(n)} := \xi^{(n)} \wedge t$. Recall that $\xi^{(n)}$ is uniformly distributed on $[0, T]$ and is independent of $M^{(n)}$.

Moreover, in view of [28, 29] there exists a measurable function $\psi^{(n)} : \mathbb{R}^2 \to \mathbb{R}$ such that the distribution of

$$\Lambda^{(n)} := \psi^{(n)}(M^{(n)} T, \xi^{(n)}),$$
on $\mathbb{R}$ is equal to $\mu$ and
\[
\bar{P}_n \left( \left| \Lambda^{(n)} - M_T^{(n)} \right| > \hat{d}(\nu_n, \mu) \right) < \hat{d}(\nu_n, \mu).
\]
In particular, $\Lambda^{(n)} - M_T^{(n)}$ converges to zero in probability.

We set
\[
\bar{N}_t^{(n)} := \mathbb{E}_{\bar{P}_n}[\Lambda^{(n)} | \bar{F}_t], \quad t \in [0, T].
\]
Then, clearly $\bar{N}_T^{(n)} = \Lambda^{(n)}$ and hence has the distribution $\mu$. Moreover, the right-
continuity of the filtration $\bar{F}$ implies that $\bar{N}_t^{(n)}$ has a càdlàg modification (for
details see [25] Chapter 3). Therefore, the measure on $\mathbb{D}$ induced by $\bar{N}^{(n)}$ under $\bar{P}_n$
is an element in $\mathcal{M}_\nu$. In particular,
\[
\mathbb{E}_{\bar{P}_n}[G(\bar{N}^{(n)})] \leq v(\mu).
\]

Step 2. In view of the choice of $Q_n$ and the constructions in the previous step, to prove (4.1) it suffices to show that
\[
\limsup_{n \to \infty} \left( \mathbb{E}_{\bar{P}_n}[G(\bar{M}^{(n)})] - \mathbb{E}_{\bar{P}_n}[G(\bar{N}^{(n)})] \right) \leq 0.
\]
In view of Assumption 2.7,
\[
G(\bar{M}^{(n)}) - G(\bar{N}^{(n)}) \leq m_G(\epsilon), \quad \text{on the set } A^{(n)}_{\epsilon},
\]
where
\[
A^{(n)}_{\epsilon} := \left\{ \sup_{0 \leq t \leq T} \left| M_t^{(n)} - N_t^{(n)} \right| > \epsilon \right\}.
\]
Hence,
\[
\mathbb{E}_{\bar{P}_n}[G(\bar{M}^{(n)})] - \mathbb{E}_{\bar{P}_n}[G(\bar{N}^{(n)})] \leq m_G(\epsilon) + ||G||_{\infty} \bar{P}_n \left( A^{(n)}_{\epsilon} \right).
\]

Step 3. In view of Step 2, (4.1) would follow if
\[
\lim_{n \to \infty} \bar{P}_n \left( A^{(n)}_{\epsilon} \right) = 0,
\]
for each $\epsilon > 0$. Towards this goal, we first observe that both $M^{(n)}$ and $N^{(n)}$ are
$(\bar{P}_n, \bar{F})$ martingales. Hence, by Doob’s maximal inequality,
\[
\bar{P}_n \left( A^{(n)}_{\epsilon} \right) \leq \frac{1}{\epsilon} \mathbb{E}_{\bar{P}_n} \left[ M_T^{(n)} - N_T^{(n)} \right].
\]
Recall that by construction $M_T^{(n)}$ has distribution $\nu_n$ and $N_T^{(n)}$ has distribution $\mu$.
Also by hypothesis, in the limit as $n$ tends to infinity first moments of $\nu_n$ are equal
to those of $\mu$. Hence,
\[
\limsup_{n \to \infty} \mathbb{E}_{\bar{P}_n} \left| N_T^{(n)} - M_T^{(n)} \right| = \limsup_{n \to \infty} \left[ 2\mathbb{E}_{\bar{P}_n} (N_T^{(n)} - M_T^{(n)})^+ - \mathbb{E}_{\bar{P}_n} (N_T^{(n)} - M_T^{(n)}) \right]
\]
\[
= 2 \limsup_{n \to \infty} \mathbb{E}_{\bar{P}_n} (N_T^{(n)} - M_T^{(n)})^+.
\]

Step 4. In view of (4.2), $N_T^{(n)} - M_T^{(n)} = \Lambda^{(n)} - M_T^{(n)}$ converges to zero in probability. Hence the previous step gives us the final reduction of (4.1). Namely, to
prove (4.1) it suffices to show the uniform integrability of the sequence of random
variables $M_T^{(n)} - N_T^{(n)}$. 

We first briefly recall that $\mathbb{M}_T^{(n)}$ has the distribution $\nu_n$, $\mathbb{N}_T^{(n)}$ has the distribution $\mu$, $\mu$ is supported on the positive real line $\mathbb{R}_+$ and by hypothesis
\[\lim_{n \to \infty} \int_{\mathbb{R}} (x)^- d\nu_n(x) = - \lim_{n \to \infty} \int_{-\infty}^0 x d\nu_n(x) = 0.\]
For brevity, set
\[X_n := \mathbb{N}_T^{(n)}, \quad Y_n := \mathbb{M}_T^{(n)},\]
and denote by $E_n$ the expectation under the measure $\tilde{\mathbb{P}}_n$. We directly estimate that
\[E_n \left[ \chi_{\{X_n-Y_n>c\}} (X_n - Y_n)^+ \right] = E_n \left[ \chi_{\{X_n-Y_n>c\}} \chi_{\{X_n-Y_n>c\}} (X_n - Y_n)^+ \right] + E_n \left[ \chi_{\{X_n-Y_n<c\}} \chi_{\{X_n-Y_n<c\}} (X_n - Y_n)^+ \right] \leq 2E_n \left[ \chi_{\{X_n>c\}} X_n \right] + 2E_n \left[ \chi_{\{X_n<c\}} Y_n \right].\]
Therefore,
\[\lim_{c \uparrow \infty} \sup_{n \in \mathbb{N}} E_n \left[ \chi_{\{X_n-Y_n>c\}} (X_n - Y_n)^+ \right] \leq 2 \lim_{c \uparrow \infty} \int_{c/2}^\infty x d\mu(x) + 2 \lim_{c \uparrow \infty} \sup_{n \in \mathbb{N}} \int_{-\infty}^{-c/2} |x| d\nu_n(x) = 0.\]
This proves the uniform integrability of the sequence $\mathbb{N}_T^{(n)} - \mathbb{M}_T^{(n)}$. Hence, (4.1) follows.

The opposite inequality is proved similarly by replacing the roles of $\nu_n$ and $\mu$. $\square$

5. Extensions

This section discusses the relaxations of the Assumption 2.7. One desirable extension is to allow linear growth. We achieve this using the techniques developed in earlier papers [14, 15]. However, this requires the stronger assumption (2.5) on the pricing measure $\mu$. This is proved in subsection 5.1 below.

The uniform continuity assumption is not satisfied by the options that involve the integrals of the stock process. Under (2.5) we are also able to relax the uniform continuity to allow the Lipschitz functions of the integrals. This extension requires a modification of the stopping times $\tau_k$, $\tilde{\tau}_k$’s which is given in subsection 5.2.

5.1. Growth Condition. This reduction is quite similar to the one given in [14]. However, the below proof contains several non trivial modifications and for completeness we provide its proof. The first result is a simple consequence of Proposition 2.1 in [1].

Recall that $\|S\|$ is the sup-norm.

Lemma 5.1. Assume (2.2), (2.3) and (2.5) and let $p$ as in (2.5). Then,
\[V(\|S\|^p) < \infty.\]

Proof. Fix $n \in \mathbb{N}$. Let $\tau_k$ and $n$ be as in subsection 3.1. We define a portfolio $(g, \gamma)$ as follows. Set $\gamma_0 = 0$. For $k = 0, 1, \ldots, n - 1$ and $t \in (\tau_k, \tau_{k+1}]$, let
\[\gamma_t(S) = - \frac{p^2}{(p-1)} \max_{0 \leq i \leq k} \left( \max_{0 \leq i \leq k} \left( S_{\tau_i}^{(1)} \right)^{p-1}, \ldots, \max_{0 \leq i \leq k} \left( S_{\tau_i}^{(d)} \right)^{p-1} \right),\]
and
\[ g(x) = \left( \frac{p}{p-1} \right)^d \sum_{i=1}^{d} x_i^p - \frac{pd}{p-1}, \quad x \in \mathbb{R}_+^d. \]

We use Proposition 2.1 in [1] to conclude that for any \( k = 0, 1, \ldots, n-1 \) and \( t \in (\tau_k, \tau_{k+1}], \)
\[ g(S_t) + \int_0^t \gamma_u dS_u \geq \max(\|S_t\|^p, \max_{0 \leq i \leq k} |S_{\tau_i}|^p). \]

Therefore, \( \phi^{(n)} := (g, \gamma) \) is admissible in the sense of Definition 2.5. Also at \( t = T, \)
\[ g(S_T) + \int_0^T \gamma_u dS_u \geq \max_{0 \leq i \leq n} |S_{\tau_i}|^p. \]

In view of the definitions of \( \tau_k \)'s, for sufficiently large \( n, \)
\[ \max_{0 \leq i \leq n} |S_{\tau_i}|^p \geq \left( \|S\| - \sqrt{d} 2^{-n} \right)^p \geq \frac{\|S\|^p}{2^p} - 1. \]

Combining all the above, we arrive at
\[ V(\|S\|^p) \leq 2^p (1 + \int gd\mu) < \infty. \]

This completes the proof of the lemma. \( \square \)

A corollary of the above Lemma is the following reduction to claims that are bounded from above.

**Lemma 5.2.** Suppose that \( G \) is uniformly continuous in the Skorokhod topology and
\[ |G(S)| \leq c(\|S\| + 1), \]
for some \( c \geq 0. \) Then, duality holds.

**Proof.** It suffices to show that
\[ (5.1) \quad V(G) \leq \sup_{Q \in \mathcal{M}_\mu} \mathbb{E}_Q [G(S)]. \]

We proceed in two steps.

**Step 1.** Let \( G \) be a claim satisfying the hypothesis of this lemma that is also bounded from below. For \( K > 0 \) large, set
\[ G_K := G \wedge c(K + 1). \]

Then, \( G_K \) is bounded and Theorem 2.9 applies to \( G_K \) yielding,
\[ V(G_K) = \sup_{Q \in \mathcal{M}_\mu} \mathbb{E}_Q [G_K(S)] \leq \sup_{Q \in \mathcal{M}_\mu} \mathbb{E}_Q [G(S)]. \]

Moreover, by the upper bound on \( G, \) the set \( \{G(S) \geq c(K + 1)\} \) is included in the set \( \{\|S\| \geq K\} \). Hence,
\[
\begin{align*}
G(S) &\leq G_K(S) + c(\|S\| + 1) \chi_{\{\|S\| \geq K\}}(S) \\
&\leq G_K(S) + c \frac{(\|S\| + 1)^p}{K^{p-1}} \\
&\leq G_K(S) + \frac{c 2^p}{K^{p-1}} \|S\|^p.
\end{align*}
\]
By the linearity of the market, this inequality implies that
\[ V(G) \leq V(G_K) + \frac{c^{2p}}{K^{p-1}} V(||S||^p). \]
Thus, for any \( K > 0 \),
\[ V(G) \leq \sup_{\mathbb{Q} \in \mathcal{M}_\mu} \mathbb{E}_\mathbb{Q}[G(S)] + \frac{c^{2p}}{K^{p-1}} V(||S||^p). \]
We let \( K \) tend to infinity and apply the previous lemma to arrive at (5.1). Hence, the duality holds for all \( G \) satisfying the hypothesis of the lemma and bounded from below.

**Step 2.** Now suppose that \( G \) is a general function the hypothesis of this lemma. For \( K > 0 \) large, set
\[ \hat{G}_K := G \lor (-c[K + 1]). \]
Then, \( \hat{G}_K \) is bounded from below and duality holds. As in Step 1, the linear upper bound implies that
\[ \hat{e}_K(S) := c (||S|| + 1) \chi_{\{||S|| \geq K\}}(S) \leq \frac{c^{2p}}{K^{p-1}} ||S||^p. \]
Since \( G \leq \hat{G}_K \) and duality holds for \( \hat{G}_K \),
\[ V(G) \leq V(\hat{G}_K) \leq \sup_{\mathbb{Q} \in \mathcal{M}_\mu} \mathbb{E}_\mathbb{Q}[\hat{G}_K] \leq \sup_{\mathbb{Q} \in \mathcal{M}_\mu} \mathbb{E}_\mathbb{Q}[G + \hat{e}_K] \]
\[ \leq \sup_{\mathbb{Q} \in \mathcal{M}_\mu} \mathbb{E}_\mathbb{Q}[G] + \sup_{\mathbb{Q} \in \mathcal{M}_\mu} \mathbb{E}_\mathbb{Q}[\hat{e}_K]. \]
Moreover, using the Doob’s inequality for the \( \mathbb{Q} \in \mathcal{M}_\mu \) martingale \( S \), we obtain,
\[ \sup_{\mathbb{Q} \in \mathcal{M}_\mu} \mathbb{E}_\mathbb{Q}[\hat{e}_K(S)] \leq \frac{c^{2p}}{K^{p-1}} \sup_{\mathbb{Q} \in \mathcal{M}_\mu} \mathbb{E}_\mathbb{Q}(||S||^p) \]
\[ \leq C_p \frac{c^{2p}}{K^{p-1}} \sup_{\mathbb{Q} \in \mathcal{M}_\mu} \mathbb{E}_\mathbb{Q}(|S_T|^p) \]
\[ = C_p \frac{c^{2p}}{K^{p-1}} \int |x|^p d\mu(x), \]
where \( C_p \) is the constant in the Doob’s inequality. Once again, we let \( K \) tend to infinity to arrive at (5.1).

**5.2. Options with Integrals.** Observe that the map
\[ S \in \mathcal{D}([0, T]; \mathbb{R}^d) \to \int_0^T S_u du, \]
is not uniformly continuous in the Skorokhod metric. Therefore, options that involve integrals of the above type do not satisfy the Assumption 2.7. However, by appropriately modifying the stopping times \( \tau_k \) and \( \hat{\tau}_k \) we can relax this assumption to include integral type options. However, in this subsection we prove an extension that allows for Lipschitz functions of the integral under the stronger assumption (2.5).

First we observe that the uniform continuity is used essentially in the lifting. In particular in the proofs of Lemma 3.6 and in Corollary 3.7. It is also used in the second step of the proof of Theorem 4.1. However, there uniform continuity in the usual sup-norm would be sufficient in Theorem 4.1.
So we modify the Skorokhod metric and define
\[ \tilde{d}(S, \tilde{S}) = d(S, \tilde{S}) + \left| \int_0^T S_u du - \int_0^T \tilde{S}_u du \right| . \]
It is clear that
\[ d(S, \tilde{S}) \leq \tilde{d}(S, \tilde{S}) \leq (1 + T)\|S - \tilde{S}\|. \]

**Proposition 5.3.** Suppose that \((\mathcal{H}, \mathcal{L})\) satisfies (2.2) (2.3), (2.5) and that there is a continuous function \(m_G : [0, \infty) \to [0, \infty)\) with \(m_G(0) = 0\) that satisfies
\[ |G(S) - G(\tilde{S})| \leq m_G(\tilde{d}(S, \tilde{S})), \quad \forall S, \tilde{S} \in \mathcal{D}(\mathbb{R}^d). \]
Further assume that
\[ |G(S)| \leq m(\|S\| + 1), \]
for some \(m \geq 0\). Then, duality holds.

**Proof.** Using the proof of Lemma 5.2, we may assume, without loss of generality that \(G\) is bounded.

**Step 1. Modification of Lemma 3.5.** We modify the definition of \(\tau_k\)'s so that
\[ \sqrt{d} 2^{-n} \geq \Delta \tau_1 \geq \ldots \geq \Delta \tau_M, \]
where
\[ \Delta \tau_k := \tau_k - \tau_{k-1}, \quad \forall k = 1, \ldots, M. \]
Indeed, we simply make the following modification in the definition. As before, we set \(\tau_0 = 0\) and recursively define the stopping times by,
\[ \tau_1 := \sqrt{d} 2^{-n} \wedge \inf \{ t > 0 : S_t \notin O(S_0, n) \} , \]
and for \(k = 1, \ldots, \)
\[ \tau_{k+1} := T \wedge \left( \tau_k + \left( \sqrt{d} 2^{-n} \wedge \Delta \tau_k \right) \right) \wedge \inf \{ t > \tau_k : S_t \notin O(S_{\tau_k}, n) \} , \]
where as before we use the convention that the infimum over the empty set is plus infinity. Set \(M\) to be the smallest integer such that \(\tau_M = T\). We then define \(\hat{\tau}_k\)'s as in subsection 3.1 using the above stopping times and use both of these sequences to construct \(\Pi, \tilde{\Pi}\) again as in subsection 3.1. It is clear from the proof of Lemma 3.5 and the definition of \(\tilde{d}\) that
\[ \tilde{d}(S, \Pi(S)) \leq (1 + T)\|S - \Pi(S)\| \leq (1 + T)\sqrt{d} 2^{-n}, \]
\[ \tilde{d}(\Pi(S), \tilde{\Pi}(S)) \leq (1 + T)\|\Pi(S) - \tilde{\Pi}(S)\| \leq (1 + T)\sqrt{d} 2^{-n}. \]
In view of Lemma 3.5
\[ d(\Pi(S), \tilde{\Pi}(S)) \leq 3\sqrt{d} 2^{-n}. \]
Moreover,
\[ \left| \int_0^T \left[ \Pi_u(S) - \Pi_u(\tilde{S}) \right] du \right| \leq \sum_{k=0}^{M-2} |\pi(n+k)(S_{\tau_k})| |\Delta \tau_{k+1} - \Delta \hat{\tau}_{k+1}| \]
\[ + \|S\| \|T - \tau_{M-1} + (T - \hat{\tau}_{M-1})\| \]
By construction and Lemma 3.4,
\[ |\pi^{(n+k)}(S_{\tau_k})| \leq \|S\| + \sqrt{d} \ 2^{-n}, \]
\[ T - \tau_{M-1} = \Delta \tau_M \leq \Delta \tau_1 \leq \sqrt{d} \ 2^{-n}, \]
\[ T - \hat{\tau}_M = \tau_M - \hat{\tau}_M \leq \sqrt{d} \ 2^{-n}. \]
Also by construction, for any \( k \),
\[ \Delta \hat{\tau}_k \leq (1 - \sqrt{d} \ 2^{-n}/T)\Delta \hat{\tau}_{k-1}, \quad \Delta \hat{\tau}_1 = \sqrt{d} \ 2^{-n}. \]
Hence,
\[
\sum_{k=0}^{M-2} |\pi^{(n+k)}(S_{\tau_k})| |\Delta \tau_{k+1} - \Delta \hat{\tau}_{k+1}| \leq |\|S\| + \sqrt{d} \ 2^{-n}| |\Delta \tau_1 - \sqrt{d} \ 2^{-n}| \\
+ |\|S\| + \sqrt{d} \ 2^{-n}| \sum_{k=1}^{M-2} |\Delta \tau_{k+1} - (1 - \sqrt{d} \ 2^{-n}/T)\Delta \tau_k| \\
\leq \hat{c}_1 2^{-n}|\|S\| + 1| \\
+ |\|S\| + \sqrt{d} \ 2^{-n}| |\Delta \tau_M - \Delta \tau_1| + |\|S\| + \sqrt{d} \ 2^{-n}/T| T \\
\leq \hat{c}_1 2^{-n}|\|S\| + 1| + |\|S\| + |\|S\| + \sqrt{d} \ 2^{-n}/T| T \\
\leq c_1 2^{-n}|\|S\| + 1|,
\]
where \( \hat{c}_1, c_1 \) are appropriate constants independent of \( n \) and \( S \). The above estimates imply that
\[
d(\bar{P}(S), \bar{P}(S)) \leq d(\bar{P}(S), \bar{P}(S)) + \int_0^T \left| \bar{P}_\nu(S) - \bar{P}_\nu(S) \right| du \leq 3\sqrt{d} \ 2^{-n} + c_1 |\|S\| + 1| 2^{-n}.
\]
By the triangle inequality we conclude that
\[ d(S, \bar{P}(S)) \leq c^*|\|S\| + 1| 2^{-n}, \]
for some constant \( c^* \) independent of \( n \) and \( S \).

Since \( G \) is bounded, in view of our hypothesis,
\[ |G(S) - G(\bar{P}(S))| \leq m_G(d(S, \bar{P}(S))) \wedge 2\|G\|_{\infty}. \]
Therefore,
\[ |G(S) - G(\bar{P}(S))| \leq m_G(c^{*}2^{-n/2}) + 2\|G\|_{\infty} \chi_{\{\|S\| \geq c_n\}}(S), \]
where \( c_n = 2^{n/2} - 1. \)

**Step 2. Modification of Corollary 3.7.** The lifting Lemma 3.6 remains unchanged. Therefore, the above modification of Lemma 3.5 together with Lemma 3.6 imply that (as in Corollary 3.7),
\[ V(G) \leq V^{(n)}(G) + \sqrt{d} \ n2^{-n+1} + m_G(c^{*}2^{-n/2}) + 2\|G\|_{\infty} V(\chi_{\{\|S\| \geq c_n\}}(S)). \]
On the other hand,
\[ \chi_{\{\|S\| \geq c_n\}}(S) \leq \frac{\|S\|}{c_n} \leq \frac{\|S\|^p + 1}{c_n}. \]
Hence, in view of Lemma 5.1

\[
0 \leq \lim_{n \to \infty} V(\chi_{\{\|S\| \geq c_n\}} (S)) \leq \lim_{n \to \infty} \frac{1}{c_n} V(\|S\|^p + 1) = 0.
\]

This implies that

\[
\lim_{n \to \infty} V(G) \leq \lim_{n \to \infty} V^{(n)}(G).
\]

We then proceed as before to prove the duality. \qed

References


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