Ground States of two-component Bose-Einstein Condensates with an External Driving Field

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Abstract. In the paper, we prove existence and uniqueness results for the ground states of the coupled Gross-Pitaevskii equations for describing two-component Bose-Einstein condensates with an external driving field and obtain the limiting behavior of the ground states with large parameters. Efficient and accurate numerical methods based on continuous normalized gradient flow and gradient flow with discrete normalization are presented for computing the ground states numerically. A modified backward Euler finite difference scheme is proposed to discretize the gradient flows. Numerical results are reported to demonstrate the efficiency and accuracy of the numerical methods and show the rich phenomena of the ground sates in the problem.

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Key words: Bose-Einstein condensate, coupled Gross-Pitaevskii equations, two-component, ground state, normalized gradient flow, external driving field, energy.

1. Introduction

Since the first realization of Bose-Einstein condensates (BEC) in dilute bosonic gas in 1995 [1,8,15], theoretical studies and numerical methods have been extensively developed for the single-component BEC [4,5,23,29]. Recently, BEC with multiple species have been realized in experiments [17,18,25,26,28,32,34] and some interesting phenomenon absent in single-component BEC were observed in experiments and studied in theory [2,6,7,9,16,20,24]. The simplest multi-component BEC is the binary mixture, which can be used as a model for producing coherent atomic beams (also called as atomic laser) [30,31]. The first experiment of two-component BEC was performed in JILA with \( |F = 2, m_F = 2 \rangle \) and \( |1, −1 \rangle \) spin states of \(^{87}\text{Rb} \) [28]. Since then, extensive experimental and theoretical studies of two-component BEC have been carried out in the last several years [3,10,19,27,35,38].

At temperature \( T \) much smaller than the critical temperature \( T_c \) and after proper nondimensionalization and dimension reduction [29,38], a two-component BEC with an external driving field can be well described by the following coupled Gross-Pitaevskii equations.

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(CGPEs) in the dimensionless form [29,37,38]:

\[
i\partial_t \psi_1 = \left[ -\frac{1}{2} \nabla^2 + V(x) + \delta + (\beta_{11} |\psi_1|^2 + \beta_{12} |\psi_2|^2) \right] \psi_1 + \lambda \psi_2, \tag{1.1}\n\]

\[
i\partial_t \psi_2 = \left[ -\frac{1}{2} \nabla^2 + V(x) + (\beta_{12} |\psi_1|^2 + \beta_{22} |\psi_2|^2) \right] \psi_2 + \lambda \psi_1, \quad x \in \mathbb{R}^d.
\]

Here, \( t \) is time, \( x \in \mathbb{R}^d \) \((d = 1,2,3)\) is the Cartesian coordinate vector, \( \Psi(x,t) := (\psi_1(x,t), \psi_2(x,t))^T \) is the complex-valued macroscopic wave function, \( V(x) \) is the real-valued external trapping potential, \( \lambda \) is the effective Rabi frequency describing the strength of the external driving field, \( \delta \) is the Raman transition constant, and \( \beta_{jl} = \beta_{lj} = \frac{4\pi \alpha_0 N}{a} \) \((j,l = 1,2)\) are interaction constants with \( N \) being the total number of particle in the two-component BEC, \( a_0 \) being the dimensionless spatial unit and \( a_{jl} = a_{lj} \) \((j,l = 1,2)\) being the \( s \)-wave scattering lengths between the \( j \)th and \( l \)th component (positive for repulsive interaction and negative for attractive interaction). It is necessary to ensure that the wave function is properly normalized. Especially, we require

\[
|\Psi|^2 := |\Psi|_2^2 = \int_{\mathbb{R}^d} \left[ |\psi_1(x,t)|^2 + |\psi_2(x,t)|^2 \right] \, dx = 1. \tag{1.2}
\]

The dimensionless CGPEs (1.1) conserves the total mass or normalization, i.e.

\[
N(t) := ||\Psi(\cdot,t)||^2 = N_1(t) + N_2(t) \equiv ||\Psi(\cdot,0)||^2 = 1, \quad t \geq 0, \tag{1.3}
\]

with

\[
N_j(t) = ||\psi_j(x,t)||^2 = ||\psi_j(x,t)||_2^2 = \int_{\mathbb{R}^d} |\psi_j(x,t)|^2 \, dx, \quad t \geq 0, \quad j = 1,2, \tag{1.4}
\]

and the energy

\[
E(\Psi) = \int_{\mathbb{R}^d} \left[\frac{1}{2} (|\nabla \psi_1|^2 + |\nabla \psi_2|^2) + V(x)(|\psi_1|^2 + |\psi_2|^2) + \delta |\psi_1|^2 + \frac{1}{2} \beta_{11} |\psi_1|^4 \\
+ \frac{1}{2} \beta_{22} |\psi_2|^4 + \beta_{12} |\psi_1|^2 |\psi_2|^2 + 2\lambda \cdot \Re(\psi_1 \bar{\psi}_2) \right] \, dx, \tag{1.5}
\]

with \( \bar{f} \) and \( \Re(f) \) denoting the conjugate and real part of a function \( f \), respectively. In addition, if there is no external driving field in (1.1), i.e. \( \lambda = 0 \), the mass of each component is also conserved, i.e.

\[
N_1(t) \equiv \int_{\mathbb{R}^d} |\psi_1(x,0)|^2 \, dx := \alpha, \quad N_2(t) \equiv \int_{\mathbb{R}^d} |\psi_2(x,0)|^2 \, dx := 1 - \alpha, \quad t \geq 0, \tag{1.6}
\]

with \( 0 \leq \alpha \leq 1 \) a given constant.

The ground state \( \Phi_\lambda(x) := (\phi_1^\lambda(x), \phi_2^\lambda(x))^T \) of the two-component BEC with an external driving field (1.1) is defined as the minimizer of the following nonconvex minimization problem:
Ground states of two-component BEC with external driving field

Find \( \Phi_g \in S \), such that

\[
E_g := E(\Phi_g) = \min_{\Phi \in S} E(\Phi),
\]

(1.7)

where \( S \) is a nonconvex set defined as

\[
S := \left\{ \Phi = (\phi_1, \phi_2)^T \mid ||\Phi||^2 = \int_{\mathbb{R}^d} (|\phi_1(x)|^2 + |\phi_2(x)|^2) \, dx = 1, \, E(\Phi) < \infty \right\}.
\]

(1.8)

It is easy to see that the ground state \( \Phi_g \) satisfies the following Euler-Lagrange equations,

\[
\mu \phi_1 = \left[ -\frac{1}{2} \nabla^2 + V(x) + \delta + (\beta_{11} |\phi_1|^2 + \beta_{12} |\phi_2|^2) \right] \phi_1 + \lambda \phi_2,
\]

\[
\mu \phi_2 = \left[ -\frac{1}{2} \nabla^2 + V(x) + (\beta_{12} |\phi_1|^2 + \beta_{22} |\phi_2|^2) \right] \phi_2 + \lambda \phi_1,
\]

(1.9)

under the constraint

\[
||\Phi||^2 := ||\Phi||_2^2 = \int_{\mathbb{R}^d} \left[ |\phi_1(x)|^2 + |\phi_2(x)|^2 \right] \, dx = 1,
\]

(1.10)

with the eigenvalue \( \mu \) being the Lagrange multiplier or chemical potential corresponding to the constraint (1.10), which can be computed as

\[
\mu = \mu(\Phi) = \int_{\mathbb{R}^d} \left[ \frac{1}{2} \left( |\nabla \phi_1|^2 + |\nabla \phi_2|^2 \right) + V(x)(|\phi_1|^2 + |\phi_2|^2) + \delta |\phi_1|^2 + \beta_{11} |\phi_1|^4 \\
+ \beta_{22} |\phi_2|^4 + 2\beta_{12} |\phi_1|^2 |\phi_2|^2 + 2\lambda \cdot \text{Re}(\phi_1 \phi_2^*) \right] \, dx.
\]

(1.11)

In fact, the above time-independent CGPEs (1.9) can also be obtained from the CGPEs (1.1) by substituting the ansatz

\[
\psi_1(x, t) = e^{-i\mu t} \phi_1(x), \quad \psi_2(x, t) = e^{-i\mu t} \phi_2(x).
\]

(1.12)

The eigenfunctions of the nonlinear eigenvalue problem (1.9) under the normalization (1.10) are usually called as stationary states of the two-component BEC (1.1). Among them, the eigenfunction with minimum energy is the ground state and those whose energy are larger than that of the ground state are usually called as excited states.

If there is no external driving field in (1.1), i.e. \( \lambda = 0 \), for any given \( \alpha \in [0, 1] \), another type ground state \( \Phi_{\alpha g}(x) = (\phi_{1g}(x), \phi_{2g}(x))^T \) of the two-component BEC is defined as the minimizer of the following nonconvex minimization problem:

Find \( \left( \Phi_{\alpha g} \in S_\alpha \right) \), such that
\[ E^a_g := E_0 \left( \Phi^a_g \right) = \min_{\Phi \in S_a} E_0 (\Phi), \] (1.13)

where \( S_a \) is a nonconvex set defined as

\[ S_a := \{ \Phi = (\phi_1, \phi_2)^T | ||\phi_1||^2 = \alpha, ||\phi_2||^2 = 1 - \alpha, E_0(\Phi) < \infty \}, \] (1.14)

and the energy functional \( E_0(\Phi) \) is defined as

\[
E_0(\Phi) = \int_{\mathbb{R}^d} \left[ \frac{1}{2} \left( |\nabla \phi_1|^2 + |\nabla \phi_2|^2 \right) + V(x)(|\phi_1|^2 + |\phi_2|^2) + \delta |\phi_1|^2 + \frac{1}{2} \beta_{11} |\phi_1|^4 + \frac{1}{2} \beta_{22} |\phi_2|^4 + \beta_{12} |\phi_1|^2 |\phi_2|^2 \right] dx. \] (1.15)

Again, it is easy to see that the ground state \( \Phi^a_g \) satisfies the following Euler-Lagrange equations,

\[
\mu_1 \phi_1 = \left[ -\frac{1}{2} \nabla^2 + V(x) + \delta + (\beta_{11} |\phi_1|^2 + \beta_{12} |\phi_2|^2) \right] \phi_1, \quad x \in \mathbb{R}^d, \] \[
\mu_2 \phi_2 = \left[ -\frac{1}{2} \nabla^2 + V(x) + (\beta_{12} |\phi_1|^2 + \beta_{22} |\phi_2|^2) \right] \phi_2, \quad x \in \mathbb{R}^d, \] (1.16)

under the two constraints

\[
||\phi_1||^2 := \int_{\mathbb{R}^d} |\phi_1(x)|^2 \, dx = \alpha, \quad ||\phi_2||^2 := \int_{\mathbb{R}^d} |\phi_2(x)|^2 \, dx = 1 - \alpha, \] (1.17)

with \( \mu_1 \) and \( \mu_2 \) being the Lagrange multipliers or chemical potentials corresponding to the two constraints (1.17). Again, the above time-independent CGPEs (1.16) can also be obtained from the CGPEs (1.1) with \( \lambda = 0 \) by substituting the ansatz

\[
\psi_1(x, t) = e^{-i\mu_1 t} \phi_1(x), \quad \psi_2(x, t) = e^{-i\mu_2 t} \phi_2(x). \] (1.18)

It is easy to see that the ground state \( \Phi^a_g \) defined in (1.7) is equivalent to

\[
\text{Find } \left( \Phi^a_g \in S \right), \text{ such that } \] \[
E \left( \Phi^a_g \right) = \min_{\Phi \in S} E(\Phi) = \min_{\alpha \in [0,1]} E(\alpha), \quad E(\alpha) = \min_{\Phi \in S_a} \left( \Phi^{\alpha}_g \right). \] (1.19)

There are some analytical and numerical studies for the ground states of two-component BEC without external driving field, i.e. based on the definition of (1.13), in the literatures [2, 12, 13, 24]. To our knowledge, there is no analytical and numerical results for the ground states of two-component BEC with an external driving field, i.e. based on the definition of (1.7). The main aim of this paper is to establish existence and uniqueness
results for the ground states of two-component BEC with an external driving field and to propose efficient and accurate numerical methods for computing these ground states.

The paper is organized as follows. In Section 2, we prove existence and uniqueness results for the ground states. In Section 3, some limiting behavior of the ground states are established when the parameters $\lambda$ and/or $\delta$ go to infinity. Efficient and accurate numerical methods for computing the ground states are proposed and analyzed in Section 4, and numerical results are reported in Section 5. Finally, some concluding remarks are drawn in Section 6. Throughout this paper, the constant $C$ denotes a generic constant and we adopt the standard notation of Sobolev spaces.

2. Existence and uniqueness results for the ground states

In this section, we will establish existence and uniqueness results for the ground states of two-component BEC with and without an external driving field, i.e. the nonconvex minimization problems (1.7) and (1.13), respectively. Let

$$\mathbf{B} = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{12} & \beta_{22} \end{pmatrix},$$

we say $\mathbf{B}$ is positive semi-definite iff $\beta_{11} \geq 0$ and $\beta_{11}\beta_{22} - \beta_{12}^2 \geq 0$; and $\mathbf{B}$ is nonnegative iff $\beta_{11} \geq 0$, $\beta_{12} \geq 0$ and $\beta_{22} \geq 0$. Without loss of generality, throughout the paper, we assume $\beta_{11} \geq \beta_{22}$. In two dimensions (2D), i.e. $d = 2$, let $C_b$ be the best constant in the following inequality

$$\|f\|_{L^4(\mathbb{R}^2)}^4 \leq \frac{1}{C_b} \|\nabla f\|_{L^2(\mathbb{R}^2)}^2 \|f\|_{L^2(\mathbb{R}^2)}^2, \quad \forall f \in H^1(\mathbb{R}^2).$$

(2.2)

The best constant $C_b$ can be attained at some $H^1$ function [36] and it is crucial in considering the existence of ground states in 2D.

2.1. For the case with an external driving field, i.e. problem (1.7)

Denote

$$\mathcal{D} = \left\{ \Phi = (\phi_1, \phi_2)^T \mid V |\phi_j|^2 \in L^1(\mathbb{R}^d), \phi_j \in H^1(\mathbb{R}^d) \cap L^4(\mathbb{R}^d), \ j = 1, 2 \right\},$$

(2.3)

then the ground state $\Phi_g$ of (1.7) is also given by

Find $\left( \Phi_g \in \mathcal{D}_1 \right)$, such that

$$E_g := E(\Phi_g) = \min_{\Phi \in \mathcal{D}_1} E(\Phi),$$

(2.4)

where

$$\mathcal{D}_1 = \mathcal{D} \cap \left\{ \Phi = (\phi_1, \phi_2)^T \mid \|\Phi\|^2 = \int_{\mathbb{R}^d} (|\phi_1(x)|^2 + |\phi_2(x)|^2) \, dx = 1 \right\}.$$
In addition, we introduce the auxiliary energy functional

$$
\overline{E}(\Phi) = \int_{\mathbb{R}^d} \left\{ \frac{1}{2} \left( |\nabla \phi_1|^2 + |\nabla \phi_2|^2 \right) + \left[ V(x) \left( |\phi_1|^2 + |\phi_2|^2 \right) + \delta |\phi_1|^2 \right] \\
+ \left( \frac{1}{2} \beta_{21} |\phi_1|^4 + \frac{1}{2} \beta_{22} |\phi_2|^4 + \beta_{12} |\phi_1|^2 |\phi_2|^2 \right) - 2 |\lambda| \cdot |\phi_1| \cdot |\phi_2| \right\} \, dx,
$$

(2.6)

and the auxiliary nonconvex minimization problem

Find \( \Phi \in \mathcal{D}_1 \), such that

$$
\overline{E}(\Phi) = \min_{\Phi \in \mathcal{D}_1} \overline{E}(\Phi).
$$

(2.7)

For \( \Phi = (\phi_1, \phi_2)^T \), we write \( E(\phi_1, \phi_2) = E(\Phi) \) and \( \overline{E}(\phi_1, \phi_2) = \overline{E}(\Phi) \). Then we have the following lemmas:

**Lemma 2.1.** For the minimizers \( \Phi^g(x) = (\phi_1^g(x), \phi_2^g(x))^T \) of the nonconvex minimization problems (2.4) and (2.7), we have

i). If \( \Phi^g \) is a minimizer of (2.4), then \( \phi_1^g(x) = e^{i\theta_1} |\phi_1(x)| \) and \( \phi_2^g(x) = e^{i\theta_2} |\phi_2(x)| \) with \( \theta_1 \) and \( \theta_2 \) two constants satisfying \( \theta_1 = \theta_2 \) if \( \lambda < 0 \); and \( \theta_1 = \theta_2 \pm \pi \) if \( \lambda > 0 \). In addition, \( \Phi^g = (e^{i\theta_1} \phi_1^g, e^{i\theta_2} \phi_2^g)^T \) with \( \theta_3 \) and \( \theta_4 \) two constants satisfying \( \theta_3 = \theta_4 \) if \( \lambda < 0 \); and \( \theta_3 = \theta_4 \pm \pi \) if \( \lambda > 0 \) is also a minimizer of (2.4).

ii). If \( \Phi^g \) is a minimizer of (2.7), then \( \phi_1^g(x) = e^{i\theta_1} |\phi_1(x)| \) and \( \phi_2^g(x) = e^{i\theta_2} |\phi_2(x)| \) with \( \theta_1 \) and \( \theta_2 \) two constants. In addition, \( \Phi^g = (e^{i\theta_1} \phi_1^g, e^{-i\theta_2} \phi_2^g)^T \) with \( \theta_3 \) and \( \theta_4 \) two constants is also a minimizer of (2.7).

iii). If \( \Phi^g \) is a minimizer of (2.4), then \( \Phi^g \) is also a minimizer of (2.7).

iv). If \( \Phi^g \) is a minimizer of (2.7), then \( \Phi^g = (|\phi_1^g|, -\text{sign}(\lambda)|\phi_2^g|)^T \) is a minimizer of (2.4).

**Proof.** For any \( \Phi(x) = (\phi_1(x), \phi_2(x))^T \in \mathcal{D}_1 \), we write it as

$$
\phi_1(x) = e^{i\theta_1(x)} |\phi_1(x)|, \quad \phi_2(x) = e^{i\theta_2(x)} |\phi_2(x)|, \quad x \in \mathbb{R}^d.
$$

(2.8)

Then we have

$$
\nabla \phi_1(x) = e^{i\theta_1(x)} \left[ |\nabla \phi_1(x)| + i |\phi_1(x)| \nabla \theta_1(x) \right], \\
\nabla \phi_2(x) = e^{i\theta_2(x)} \left[ |\nabla \phi_2(x)| + i |\phi_2(x)| \nabla \theta_2(x) \right].
$$

(2.9)
Plugging (2.9) into (1.5) with $\Psi = \Phi$ and (2.6), we obtain

\[
E(\phi_1, \phi_2) = E(|\phi_1|, -\text{sign}(\lambda)|\phi_2|) + \int_{\mathbb{R}^d} \frac{1}{2} \left[ |\phi_1|^2|\nabla \theta_1|^2 + |\phi_2|^2|\nabla \theta_2|^2 + 4|\lambda| \left( 1 + \text{sign}(\lambda) \cos(\theta_1 - \theta_2) \right) |\phi_1||\phi_2| \right] dx,
\]

(2.10)

\[
\widetilde{E}(\phi_1, \phi_2) = \widetilde{E}(|\phi_1|, |\phi_2|) + \int_{\mathbb{R}^d} \frac{1}{2} \left[ |\phi_1|^2|\nabla \theta_1|^2 + |\phi_2|^2|\nabla \theta_2|^2 \right] dx,
\]

(2.11)

\[
E(|\phi_1|, -\text{sign}(\lambda)|\phi_2|) = \widetilde{E}(|\phi_1|, |\phi_2|) \leq \widetilde{E}(\phi_1, \phi_2),
\]

(2.12)

\[
\widetilde{E}(\phi_1, \phi_2) \leq E(\phi_1, \phi_2), \quad \Phi \in \mathcal{D}_1.
\]

(2.13)

i). If $\Phi_g$ is a minimizer of (2.4), then we have

\[
E(\phi_1^g, \phi_2^g) = E(|\phi_1^g|, -\text{sign}(\lambda)|\phi_2^g|).
\]

(2.14)

Plugging (2.14) into (2.10) with $\Phi = \Phi_g$, we get

\[
\int_{\mathbb{R}^d} \frac{1}{2} \left[ |\phi_1^g|^2|\nabla \theta_1^g|^2 + |\phi_2^g|^2|\nabla \theta_2^g|^2 + 4|\lambda| \left( 1 + \text{sign}(\lambda) \cos(\theta_1^g - \theta_2^g) \right) |\phi_1^g||\phi_2^g| \right] dx = 0.
\]

This immediately implies that

\[
\nabla \theta_1^g = 0, \quad \nabla \theta_2^g = 0, \quad 1 + \text{sign}(\lambda) \cos(\theta_1^g - \theta_2^g) = 0,
\]

(2.15)

and thus

\[
\theta_1^g(x) \equiv \theta_1, \quad \theta_2^g(x) \equiv \theta_2, \quad \theta_1 = \begin{cases} \theta_2, & \lambda > 0, \\ \theta_2 \pm \pi, & \lambda < 0. \end{cases}
\]

(2.16)

In addition, we have

\[
E(\Phi_g) = E(|\phi_1^g|, -\text{sign}(\lambda)|\phi_2^g|) = E(\Phi_g),
\]

(2.17)

which immediately implies that $\Phi_g$ is also a minimizer of (2.4).

ii). Follows the analogue proof as those in part i) and we omitted the details here.

iii). If $\Phi_g$ is a minimizer of (2.4), noticing (2.10)-(2.12), we have

\[
\widetilde{E}(\phi_1^g, \phi_2^g) = \widetilde{E}(|\phi_1^g|, |\phi_2^g|) = E(|\phi_1^g|, -\text{sign}(\lambda)|\phi_2^g|) = E(\phi_1^g, \phi_2^g)
\]

\[
\leq E(|\phi_1|, -\text{sign}(\lambda)|\phi_2|) \leq \widetilde{E}(\phi_1, \phi_2) = \widetilde{E}(\Phi), \quad \Phi \in \mathcal{D}_1,
\]

(2.18)

which immediately implies that $\Phi_g$ is a minimizer of (2.7).

iv). If $\Phi_g$ is a minimizer of (2.7), noticing (2.11) and (2.13), we have

\[
E(\Phi_g) = E(|\phi_1^g|, -\text{sign}(\lambda)|\phi_2^g|) = \widetilde{E}(|\phi_1^g|, |\phi_2^g|) = \widetilde{E}(\phi_1^g, \phi_2^g)
\]

\[
\leq \widetilde{E}(\phi_1, \phi_2) \leq E(\phi_1, \phi_2) = E(\Phi), \quad \Phi \in \mathcal{D}_1,
\]

(2.19)

which immediately implies that $\Phi_g$ is a minimizer of (2.4).
Lemma 2.2. (strict convexity of $E$) Assume that the matrix $B$ is positive semi-definite and at least one of the parameters $\lambda, \gamma_1 := \beta_{11} - \beta_{22}$ and $\gamma_2 := \beta_{11} - \beta_{12}$ is nonzero, for $(\rho_1, \rho_2)^T$ with $\rho_1, \rho_2 \geq 0$, $\sqrt{\rho_1}, \sqrt{\rho_2} \in \mathcal{D}_1$, then $E[\sqrt{\rho_1}, \sqrt{\rho_2}]$ is strictly convex in $(\rho_1, \rho_2)$.

Proof. Similar to [23] for single-component BEC, the first term in $E$ is convex. The second and third terms in $E$ are linear and quadratic forms, respectively, since we assume that $B$ is positive semi-definite, thus these two terms are convex. Now we just need to verify the convexity of the last term. Let $\Phi_1 = (\sqrt{\rho_1}, \sqrt{\rho_2})^T \in \mathcal{D}_1$ and $\Phi_2 = (\sqrt{\rho_1'}, \sqrt{\rho_2'})^T \in \mathcal{D}_1$, for $\alpha \in (0, 1)$, then $E = ([\alpha \rho_1 + (1 - \alpha)\rho_1']^{1/2}, [\alpha \rho_2 + (1 - \alpha)\rho_2']^{1/2})^T \in \mathcal{D}_1$. By Cauchy inequality, we have

$$\alpha \sqrt{\rho_1} \sqrt{\rho_2} + (1 - \alpha) \sqrt{\rho_1'} \sqrt{\rho_2'} \leq \sqrt{\alpha \rho_1 + (1 - \alpha)\rho_1'} \times \sqrt{\alpha \rho_2 + (1 - \alpha)\rho_2'}.$$  \hspace{1cm} (2.20)

Thus the last term is also convex.

Theorem 2.1. (Existence and uniqueness of (2.7)) Suppose $V(x) \geq 0$ satisfying $\lim\limits_{|x| \to \infty} V(x) = \infty$, then there exists a minimizer $\Phi^\infty = (\phi_1^\infty, \phi_2^\infty)^T \in \mathcal{D}_1$ of (2.7) if one of the following conditions holds,

(i) $d = 1$;

(ii) $d = 2$ and $\beta_{11} \geq -C_b, \beta_{22} \geq -C_b, \beta_{12} \geq -C_b - \sqrt{C_b + \beta_{11}^2 + \beta_{22}^2}$;

(iii) $d = 3$ and $B$ is either positive semi-definite or nonnegative.

In addition, if the matrix $B$ is positive semi-definite and at least one of the parameters $\delta, \lambda, \gamma_1$ and $\gamma_2$ are nonzero, then the minimizer $(|\phi_1^\infty|, |\phi_2^\infty|)^T$ is unique.

Proof. First, we claim that $E$ is bounded below under the assumption. Case (iii) is clear. For case (i), using the constraint $||\Phi||_2 = 1$ and Sobolev inequality, for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$||\phi_j||_2^4 \leq ||\phi_j||_2^2 ||\phi_j||_2^2 \leq ||\phi_j||_2^2 \leq ||\nabla \phi_j||_2 ||\phi_j||_2 \leq \epsilon ||\nabla \phi||_2^2 + C_\epsilon, \hspace{1cm} j = 1, 2,$$

which yields the claim. For case (ii), using Cauchy inequality and Gagliardo-Nirenberg inequality, we have

$$\int_{\mathbb{R}^2} (\beta_{11} |\phi_1|^4 + \beta_{22} |\phi_2|^4 + 2\beta_{12} |\phi_1|^2 |\phi_2|^2) \ dx \geq -C_b \int_{\mathbb{R}^2} \left( |\phi_1|^2 + |\phi_2|^2 \right)^4 \ dx$$

$$\geq -\int_{\mathbb{R}^2} \left( |\phi_1|^2 + |\phi_2|^2 \right)^2 \ dx \int_{\mathbb{R}^2} \left( \nabla |\phi_1|^2 + |\phi_2|^2 \right)^2 \ dx$$

$$\geq -\int_{\mathbb{R}^2} \left( |\nabla \phi_1|^2 + |\nabla \phi_2|^2 \right) \ dx,$$

which also leads to the claim. Thus, in all the cases, we can take a minimizing sequence $\Phi^n = (\phi_1^n, \phi_2^n)^T$ in $\mathcal{D}_1$. Then there exists a constant $C$ such that $||\nabla \phi_1^n|| + ||\nabla \phi_2^n|| < C$,
\[ \| \phi_1^n \|_4 + \| \phi_2^n \|_4 < C \text{ and } \int_{\mathbb{R}^d} V(x)(|\phi_1^n(x)|^2 + |\phi_2^n(x)|^2)dx < C \text{ for all } n \geq 0. \]

Therefore \( \phi_1^n \) and \( \phi_2^n \) belong to a weakly compact set in \( L^4 \), \( H^1 = \{ \phi \mid \| \phi \|^2 + \| \nabla \phi \|^2 < \infty \} \), and \( L^4_\phi = \{ \phi \mid \int_{\mathbb{R}^d} V(x)|\phi(x)|^2 \, dx < \infty \} \) with a weighted \( L^2 \)-norm given by \( \| \phi \|_V = (\int_{\mathbb{R}^d} |\phi(x)|^2 V(x)dx)^{1/2} \). Thus, there exists a \( \Phi^\infty = (\phi_1^\infty, \phi_2^\infty)^T \in \mathcal{D} \) and a subsequence (which we denote as the original sequence for simplicity), such that

\[
\phi_1^n \to \phi_1^\infty, \quad \phi_2^n \to \phi_2^\infty, \quad \nabla \phi_1^n \to \nabla \phi_1^\infty, \quad \nabla \phi_2^n \to \nabla \phi_2^\infty, \quad \text{in } L^2. \tag{2.21}
\]

Also, we can suppose that \( \phi_1^n \) and \( \phi_2^n \) are nonnegative, since we can replace them with \( |\phi_1^n| \) and \( |\phi_2^n| \), which also minimize the functional \( \tilde{E} \). To show that \( \tilde{E} \) attains its minimal at \( \Phi^\infty \), we recall the constraint \( \| \Phi^\infty \|^2 = 1 \), then the functional \( \tilde{E} \) can be re-written as

\[
\tilde{E}(\phi_1^n, \phi_2^n) = \int_{\mathbb{R}^d} \left[ \frac{1}{2} (|\nabla \phi_1^n|^2 + |\nabla \phi_2^n|^2) + V(x)(|\phi_1^n|^2 + |\phi_2^n|^2) + \delta |\phi_1^n|^2
\big]
\]

\[
+ \frac{1}{2} \left( \beta_{11} |\phi_1^n|^4 + \beta_{22} |\phi_2^n|^4 + 2 \beta_{12} |\phi_1^n|^2 |\phi_2^n|^2 + |\lambda| |\phi_1^n - \phi_2^n|^2 \right) \right] \, dx
\]

\[-\lambda \right]. \tag{2.22}
\]

First, we show that for any given \( \epsilon > 0 \),

\[
\int_{\mathbb{R}^d} \beta_{12} |\psi_1^n|^2 |\psi_2^n|^2 \, dx \leq \liminf_{n \to \infty} \int_{\mathbb{R}^d} \beta_{12} |\psi_1^n|^2 |\psi_2^n|^2 \, dx + \epsilon. \tag{2.23}
\]

When \( \beta_{12} \geq 0 \), this is obviously true. For general \( \beta_{12} \), we decompose \( \mathbb{R}^d \) into two parts, a bounded region \( B_R = \{ |x| \leq R \} \) and \( B_R^c := \mathbb{R}^d \setminus B \), such that \( V(x) \geq 1/\eta \) on \( B_R^c \), where \( \eta > 0 \) sufficiently small, using the assumption \( \lim_{|x| \to \infty} V(x) = \infty \). Then \( \int_{B_R^c} (|\phi_1^n|^2 + |\phi_2^n|^2) \, dx \leq C \eta \).

In \( B_R^c \), using the Sobolev-Gagliardo inequality, for \( d = 3 \) and \( 2^* = 6 \), we have

\[
\int_{B_R^c} |\phi_1^n|^4 \, dx \leq \| \phi_1^n \|_2^{12} \left( \int_{B_R^c} |\phi_1^n|^2 \, dx \right)^2 \leq M C \eta^2 \| \nabla \phi_1^n \|_2^{12} \leq M C \eta^2, \tag{2.24}
\]

where \( M \) is a constant. Thus, by choosing \( R \) sufficiently large, we have

\[
\int_{B_R^c} |\phi_1^n|^4 \, dx \leq \frac{\epsilon}{2(1 + |\beta_{12}|)}, \quad \text{for all } n. \tag{2.25}
\]

In the case of \( d = 1 \), using the Sobolev inequality

\[
\| f \|_\infty \leq \| f' \|_2 \| f \|_2, \quad \text{for all } f \in H^1(\mathbb{R}), \tag{2.26}
\]

and in the case of \( d = 2 \), using the Sobolev type inequality

\[
\| f \|_6 \leq C(\| \nabla f \|_2 + \| f \|_2^2), \quad \text{for all } f \in H^1(\mathbb{R}^2), \tag{2.27}
\]
we can get the same result.

The same conclusion holds for $\phi_2^n$. Notice that for $\phi_1^\infty$ and $\phi_2^\infty$, by the weak lower semicontinuous property of $L^4(\mathbb{R}^d)$-norm, $H^1(\mathbb{R}^d)$-norm and $L^4_\sigma(\mathbb{R}^d)$-norm, we can have $||\nabla \phi_1^\infty|| + ||\nabla \phi_2^\infty|| < C$, $||\phi_1^\infty||_4 + ||\phi_2^\infty||_4 < C$ and $\int_{\mathbb{R}^d} V(x)(|\phi_1^\infty|^2 + |\phi_2^\infty|^2)dx < C$. Following the above arguments, the same conclusion holds for $\phi_1^\infty$ and $\phi_2^\infty$, i.e., we have

$$
\int_{B_R^c} |\phi_j^n|^4 \, dx \leq \frac{\varepsilon}{2(1 + |\beta_{12}|)}, \quad \int_{B_R^c} |\phi_j^\infty|^4 \, dx \leq \frac{\varepsilon}{2(1 + |\beta_{12}|)}, \quad j = 1, 2, \quad n \geq 0. \quad (2.28)
$$

Then, by the Cauchy-Schwartz inequality, we have

$$
\left| \int_{B_R^c} \beta_{12} |\phi_1^n|^2 |\phi_2^n|^2 \, dx \right| \leq |\beta_{12}| \left( \int_{B_R^c} |\phi_1^n|^4 \, dx \right)^{1/2} \left( \int_{B_R^c} |\phi_2^n|^4 \, dx \right)^{1/2} \leq \frac{\varepsilon}{2}, \quad n \geq 0, \quad (2.29)
$$

and

$$
\left| \int_{B_R^c} \beta_{12} |\phi_1^\infty|^2 |\phi_2^\infty|^2 \, dx \right| \leq \frac{\varepsilon}{2}. \quad (2.30)
$$

Next, in the ball $B_R$, applying the Sobolev embedding theorem, the strong convergence holds,

$$
\phi_1^n \rightarrow \phi_1^\infty, \quad \phi_2^n \rightarrow \phi_2^\infty, \quad \text{in } L^2(B_R) \cap L^4(B_R). \quad (2.31)
$$

By writing

$$
\left| \int_{B_R} \beta_{12} |\phi_1^n|^2 |\phi_2^n|^2 \, dx - \int_{B_R} \beta_{12} |\phi_1^\infty|^2 |\phi_2^\infty|^2 \, dx \right|
$$

$$
\leq |\beta_{12}| \left[ \int_{B_R} (|\phi_1^n|^2 - |\phi_1^\infty|^2) |\phi_2^n|^2 \, dx + \int_{B_R} (|\phi_2^n|^2 - |\phi_2^\infty|^2) |\phi_1^n|^2 \, dx \right]
$$

$$
\leq C \left( ||\phi_1^n - \phi_1^\infty||_{L^4(B_R)} + ||\phi_2^n - \phi_2^\infty||_{L^4(B_R)} \right), \quad (2.32)
$$

we have

$$
\int_{B_R} \beta_{12} |\phi_1^\infty(x)|^2 |\phi_2^\infty(x)|^2 \, dx = \lim_{n \to \infty} \int_{B_R} \beta_{12} |\phi_1^n(x)|^2 |\phi_2^n(x)|^2 \, dx. \quad (2.33)
$$

Hence, the inequality (2.23) holds by combining the above results.

By a similar argument, we can prove that

$$
\limsup_{n \to \infty} \left| \int_{\mathbb{R}^d} (|\phi_1^n|^2 + |\phi_2^n|^2) \, dx - \int_{\mathbb{R}^d} (|\phi_1^\infty|^2 + |\phi_2^\infty|^2) \, dx \right| \leq \varepsilon. \quad (2.34)
$$
Suppose $d_d$ and at least one of the parameters $0$ is also a ground state of (1.7) with $\theta \lim_{n \to \infty}$ prove the nonexistence results.

Theorem 2.2. (Existence and uniqueness of (1.7)) Suppose $V(x) \geq 0$ satisfying $\lim_{|x| \to \infty} V(x) = \infty$ and at least one of the following condition holds,

(i) $d = 1$;

(ii) $d = 2$ and $\beta_{11} \geq -C_b$, $\beta_{22} \geq -C_b$, and $\beta_{12} \geq -C_b - \sqrt{C_b + \beta_{11}} \sqrt{C_b + \beta_{22}}$;

(iii) $d = 3$ and $B$ is either positive semi-definite or nonnegative,

there exists a ground state $\Phi = (\phi_1^\delta, \phi_2^\delta)^T$ of (1.7). In addition, $\Phi := (e^{i\theta_1} |\phi_1^\delta|, e^{i\theta_2} |\phi_2^\delta|)$ is also a ground state of (1.7) with $\theta_1$ and $\theta_2$ two constants satisfying $\theta_1 - \theta_2 = \pm \pi$ when $\lambda > 0$ and $\theta_1 - \theta_2 = 0$ when $\lambda < 0$, respectively. Furthermore, if the matrix $B$ is positive semi-definite and at least one of the parameters $\delta$, $\lambda$, $\gamma_1$ and $\gamma_2$ are nonzero, then the ground state $(|\phi_1^\delta|, -\text{sign}(\lambda) |\phi_2^\delta|)^T$ is unique. In contrast, if one of the following conditions holds,

(i) $d = 2$ and $\beta_{11} < -C_b$ or $\beta_{22} < -C_b$ or $\beta_{12} < -C_b - \sqrt{C_b + \beta_{11}} \sqrt{C_b + \beta_{22}}$;

(ii) $d = 3$ and $\beta_{11} < 0$ or $\beta_{22} < 0$ or $\beta_{12} < 0$ with $\beta_{12}^2 > \beta_{11} \beta_{22}$.

there exists no ground states of (1.7).

Proof. The first part of the theorem follows from the Theorem 2.1. We are going to prove the nonexistence results.

In 2D case, i.e. $d = 2$, let $\varphi(x) \in H^1(\mathbb{R}^2)$ such that $\|\varphi\|_2 = 1$ and $C_b = \|\nabla \varphi\|_2^2 / \|\varphi\|_4^2$ [36]. Consider $\Phi^\varepsilon = (\phi_1^\varepsilon, \phi_2^\varepsilon)^T$, where $\phi_1^\varepsilon(x) = \sqrt{\theta \varepsilon^{-1}} \varphi(x/\varepsilon)$, $\phi_2^\varepsilon(x) = \sqrt{1 - \theta \varepsilon^{-1}} \varphi(x/\varepsilon)$, $\theta \in [0, 1], \varepsilon > 0$. When $\beta_{11} < -C_b$, choose $\theta = 1$, we have

$$E(\Phi^\varepsilon) = \frac{1}{2\varepsilon^2} \|\nabla \varphi\|_2^2 + \frac{\beta_{11}}{2\varepsilon^2} \|\varphi\|_4^2 + O(1) = \frac{1 + \frac{\beta_{11}}{C_b}}{2\varepsilon^2} \|\nabla \varphi\|_2^2 + O(1), \quad \varepsilon \to 0^+,$$

thus $\lim_{\varepsilon \to 0^+} E(\Phi^\varepsilon) = -\infty$. When $\beta_{22} < -C_b$, choose $\theta = 0$, similarly we can draw the same conclusion. When $\beta_{11} \geq -C_b$, $\beta_{22} \geq -C_b$ and $\beta_{12} < -C_b - \sqrt{C_b + \beta_{11}} \sqrt{C_b + \beta_{22}}$, choose
\[ \theta = \frac{\beta_{22} - \beta_{12}}{\beta_{11} + \beta_{22} - 2\beta_{12}}, \]

then

\[ \beta_\theta := \theta^2 \beta_{11} + 2\beta_{12} \theta (1 - \theta) + \beta_{22} (1 - \theta)^2 = \frac{\beta_{11} \beta_{22} - \beta_{12}^2}{\beta_{11} + \beta_{22} - 2\beta_{12}} < -C_b, \]

and

\[ E(\Phi^\varepsilon) = \frac{1 + \frac{\beta_\theta}{C_b}}{2\varepsilon^2} \|\nabla \varphi\|^2_2 + \Theta(1), \quad \varepsilon \to 0^+, \]

then \( \lim_{\varepsilon \to 0^+} E(\Phi^\varepsilon) = -\infty \). Thus there exists no ground state in these cases.

In three dimensions (3D) case, i.e., \( d = 3 \), choose \( \Phi^\varepsilon = (\phi_1^\varepsilon, \phi_2^\varepsilon)^T \), where

\[ \phi_1^\varepsilon(x) = \sqrt{\pi} \exp(-|x|^2/2\varepsilon), \quad \phi_2^\varepsilon(x) = \sqrt{\pi} \exp(-|x|^2/2\varepsilon), \quad \theta \in [0, 1], \quad \varepsilon > 0. \]

When \( \beta_{11} < 0 \), choosing \( \theta = 1 \), we obtain

\[ E(\Phi^\varepsilon) = C_1 \varepsilon^{-1} + \frac{\beta_{11}}{2} (2\pi \varepsilon)^{-3/2} + \Theta(1), \quad \varepsilon \to 0^+, \]

which shows \( \lim_{\varepsilon \to 0^+} E(\Phi^\varepsilon) = -\infty \). When \( \beta_{22} < 0 \), choose \( \theta = 0 \), the same conclusion holds.

When \( \beta_{11} \geq 0, \beta_{22} \geq 0, \beta_{12} < 0 < \beta_{12}^2 > \beta_{11} \beta_{22} \), choose \( \theta = \frac{\beta_{22} - \beta_{12}}{\beta_{11} + \beta_{22} - 2\beta_{12}} \in (0, 1) \), then

\[ \beta_\theta := \theta^2 \beta_{11} + 2\beta_{12} \theta (1 - \theta) + \beta_{22} (1 - \theta)^2 = \frac{\beta_{11} \beta_{22} - \beta_{12}^2}{\beta_{11} + \beta_{22} - 2\beta_{12}} < 0, \]

and

\[ E(\Phi^\varepsilon) = C_1 \varepsilon^{-1} + \frac{\beta_\theta}{2} (2\pi \varepsilon)^{-3/2} + \Theta(1), \quad \varepsilon \to 0^+, \]

thus \( \lim_{\varepsilon \to 0^+} E(\Phi^\varepsilon) = -\infty \). The above results imply that there exists no ground state in such cases.

When \( B \) is nonnegative, we have the following uniqueness results for the ground states of (1.7):

**Theorem 2.3.** Suppose \( V(x) \geq 0 \) satisfying \( \lim_{|x| \to \infty} V(x) = \infty \), the matrix \( B \) is nonnegative satisfying \( \beta_{11} = \beta_{22} \geq 0 \), at least one of the parameters \( \delta, \lambda, \gamma_1 \) and \( \gamma_2 \) are nonzero, and \( \delta \neq 0 \) if \( \beta_{12} - \beta_{11} > 0 \), then the ground state \( \Phi_\varepsilon = (|\phi_1^\varepsilon|, -\text{sign}(\lambda)|\phi_2^\varepsilon|)^T \) of (1.7) is unique.

**Proof.** If \( B \) is nonnegative and \( \beta_{11} = \beta_{22} \geq \beta_{12} \geq 0 \) which immediately implies that \( B \) is positive semi-definite, since at least one of the parameters \( \delta, \lambda, \gamma_1 \) and \( \gamma_2 \) are nonzero, the uniqueness of the ground state \( \Phi_\varepsilon \) follows immediately from Theorem 2.1.

If \( \beta_{12} > \beta_{11} = \beta_{22} \geq 0 \), for any \( \Phi = (\phi_1, \phi_2)^T \in \mathcal{H}_1 \), let

\[ \varphi_1 = \frac{1}{\sqrt{2}}(\phi_1 + \phi_2), \quad \varphi_2 = \frac{1}{\sqrt{2}}(\phi_1 - \phi_2). \]  

(2.36)
Suppose that $\Phi_g = (\phi_1^g, \phi_2^g)^T$ is a nonnegative minimizer of (2.7), then the corresponding $(\varphi_1^g, \varphi_2^g)^T$ is a minimizer of the following energy functional

$$\bar{E}(\varphi_1, \varphi_2) = \int_{\mathbb{R}^d} \left[ \frac{1}{2} (|\nabla \varphi_1|^2 + |\nabla \varphi_2|^2) + V(x) (|\varphi_1|^2 + |\varphi_2|^2) \\ + \delta \text{Re} (\varphi_1 \cdot \bar{\varphi}_2) - 2|\lambda| |\varphi_1|^2 + \frac{\beta_{11} + \beta_{12}}{2} \left[ |\varphi_1|^4 + |\varphi_2|^4 \right] \\ + (3\beta_{11} - \beta_{12}) |\varphi_1|^2 |\varphi_2|^2 \right] dx,$$

(2.37)

under the constraint $\int_{\mathbb{R}^d}(|\varphi_1(x)|^2 + |\varphi_2(x)|^2) dx = 1$.

Noticing that the matrix $\begin{pmatrix} \beta_{11} + \beta_{12} & 3\beta_{11} - \beta_{12} \\ 3\beta_{11} - \beta_{12} & \beta_{11} + \beta_{12} \end{pmatrix}$ is positive semi-definite in this case and $\delta$ is nonzero, using the results in the Theorem 2.1, we can obtain the uniqueness of the ground state $(\varphi_1^g, \varphi_2^g)^T$ to the problem (2.37) with $\varphi_1^g \geq 0$. Thus the uniqueness of the ground state $\Phi_g = (|\phi_1^g|, -\text{sign}(\lambda) |\phi_2^g|)^T$ of (1.7) follows immediately.

**Theorem 2.4.** Suppose $V(x) \geq 0$ satisfying $\lim_{|x| \to \infty} V(x) = \infty$ and $\lambda = 0$.

(i) If $\delta \geq 0$, $\beta_{12} \geq \beta_{22}$ and $\beta_{11} > \beta_{22} \geq 0$, then the ground state $\Phi_g = (\phi_1^g, \phi_2^g)^T$ of (1.7) must satisfy $\phi_1^g = 0$ and $|\phi_2^g|$ is unique.

(ii) If $\delta \leq 0$, $\beta_{12} \geq \beta_{11}$ and $\beta_{22} > \beta_{11} \geq 0$, then the ground state $\Phi_g = (\phi_1^g, \phi_2^g)^T$ of (1.7) must satisfy $\phi_2^g = 0$ and $|\phi_1^g|$ is unique.

**Proof:** (i) Suppose $\Phi_g = (\phi_1^g, \phi_2^g)^T$ be a nonnegative minimizer of (1.7). Consider

$$\phi_1(x) = 0, \quad \phi_2(x) = \sqrt{|\phi_1^g(x)|^2 + |\phi_2^g(x)|^2}, \quad x \in \mathbb{R}^d.$$  

(2.38)

Then, $\Phi = (\phi_1, \phi_2)^T \in \mathcal{D}_1$ and satisfies

$$\int_{\mathbb{R}^d} |\nabla \phi_2(x)|^2 dx \leq \int_{\mathbb{R}^d} \left[ |\nabla \phi_1^g(x)|^2 + |\nabla \phi_2^g(x)|^2 \right] dx,$$

$$\int_{\mathbb{R}^d} V(x) (|\phi_1(x)|^2 + |\phi_2(x)|^2) dx = \int_{\mathbb{R}^d} V(x) (|\phi_1^g(x)|^2 + |\phi_2^g(x)|^2) dx,$$

$$\int_{\mathbb{R}^d} \frac{\beta_{22}}{2} |\phi_2(x)|^4 dx \leq \int_{\mathbb{R}^d} \left[ \frac{1}{2} \left( \beta_{11} |\phi_1^g|^4 + \beta_{22} |\phi_2^g|^4 + 2\beta_{12} |\phi_1^g|^2 |\phi_2^g|^2 \right) \right] dx.$$

(2.39)

Thus,

$$E(\Phi) = E(\phi_1, \phi_2) \leq E(\phi_1^g, \phi_2^g) = E(\Phi_g) \leq E(\Phi).$$

(2.40)

So, the above inequalities must be equalities, which leads to our conclusion. The uniqueness of $|\phi_2^g|$ is also easy to see.

(ii) Follow the analogous arguments as in part (i) and the details are omitted.

Lastly, we stress that, if $B$ is not positive semi-definite, the uniqueness of the ground state of (1.7) may not hold. Actually, we have the following result in contrast with Theorem 2.3.
Remark 2.2. is unique under the permutation of subindex.

Equations, i.e.

Thus, for

Since

and

Remark 2.3. (1.7) is not unique.

Remark 2.4. (1.7) is not unique.

Theorem 2.5. Suppose \( V(x) \geq 0 \) satisfying \( \lim_{|x| \to \infty} V(x) = \infty \), \( \delta = 0 \) and \( \beta_{12} > \beta_{11} = \beta_{22} \geq 0 \), then there exists a constant \( \Lambda_0 > 0 \), such that for \( \lambda \in (-\Lambda_0, \Lambda_0) \), the ground state \( \Phi_g = (|\phi_1^g|, \text{sign}(\lambda)|\phi_2^g|)^T \) of (1.7) is not unique.

Proof. Let \( \Phi_1 = (\phi^g, \phi^g)^T \) be the nonnegative minimizer of (2.6) in the set \( \{ \Phi = (\phi_1, \phi_2)^T \in D_1, \phi_1 = \phi_2 \} \) and \( \Phi_2 = (0, \phi^g)^T \) be the nonnegative minimizer of (2.6) in the set \( \{ \Phi = (\phi_1, \phi_2)^T \in D_1, \phi_1 = 0 \} \), then we know

\[
\bar{E}(\Phi_1) = \min_{\Phi = (\phi_1, \phi_2)^T \in D_1} \bar{E}(\Phi)
\]

\[= \min_{\|\phi\|_2 = 1} \int_{\mathbb{R}^d} \left\{ \frac{1}{2} \nabla \phi \cdot \nabla \phi + V(x)|\phi|^2 + \frac{\beta_{11} + \beta_{12}}{4} |\phi|^4 \right\} \, dx - |\lambda|,
\]

and

\[
\bar{E}(\Phi_2) = \min_{\|\phi\|_2 = 1} \int_{\mathbb{R}^d} \left\{ \frac{1}{2} \nabla \phi \cdot \nabla \phi + V(x)|\phi|^2 + \frac{\beta_{11} + \beta_{12}}{2} |\phi|^4 \right\} \, dx.
\]

Since \( \beta_{12} > \beta_{11} \), we have

\[
\Lambda_0 = \min_{\|\phi\|_2 = 1} \int_{\mathbb{R}^d} \left\{ \frac{1}{2} \nabla \phi \cdot \nabla \phi + V(x)|\phi|^2 + \frac{\beta_{11} + \beta_{12}}{4} |\phi|^4 \right\} \, dx
\]

\[> 0.
\]

Thus, for \( \lambda \in (-\Lambda_0, \Lambda_0) \), \( \bar{E}(\Phi_1) > \bar{E}(\Phi_2) \), which implies that for ground state \( \Phi_g = (\phi_1^g, \phi_2^g)^T \) of (1.7), \( |\phi_1^g| \neq |\phi_2^g| \). But under the assumption, we can see that if \( \Phi_g = (\phi_1^g, \phi_2^g)^T \) is a ground state of (1.7), then \( (\phi_2^g, \phi_1^g)^T \) is also a ground state. So, the minimizer \( \Phi_g = (|\phi_1^g|, \text{sign}(\lambda)|\phi_2^g|)^T \) of (1.7) can not be unique.

Remark 2.1. In the above theorem, for ground state \( \Phi_g = (\phi_1^g, \phi_2^g)^T \), we have \( (|\phi_1^g|, |\phi_2^g|) \) is unique under the permutation of subindex.

Remark 2.2. When \( \delta = \lambda = 0 \) and \( \beta_{11} = \beta_{12} = \beta_{22} \geq 0 \), the nonnegative ground state \( \Phi_g \) of (1.7) is not unique.

Remark 2.3. Similar to the results in [9, 10, 16], for any fixed \( \beta_{11} \geq 0 \) and \( \beta_{22} \geq 0 \), the phase of two components of the ground state \( \Phi_g = (\phi_1^g, \phi_2^g)^T \) will be segregated when \( \beta_{12} \to \infty \), i.e. \( \Phi_g \) will converge to a state such that \( \phi_1^g \cdot \phi_2^g = 0 \).

Remark 2.4. If the potential \( V(x) \) in the two equations in (1.1) is chosen different in different equations, i.e. \( V_j(x) \) in the \( j \)th \((j = 1, 2)\) equation, if they satisfy \( V_j(x) \geq 0 \), \( \lim_{|x| \to \infty} V_j(x) = \infty \) \((j = 1, 2)\), then the conclusions in the above Lemmas and Theorem 2.1-2.2 are still valid under the similar conditions.
2.2. For the case without external driving field, i.e. problem (1.13)

If $\alpha = 0$ or 1 in the nonconvex minimization problem (1.13), it reduces to a single component problem and the results were established in [23]. Thus here we assume $\alpha \in (0, 1)$. Denote

$$
\beta'_{11} := \alpha \beta_{11}, \quad \beta'_{22} = (1 - \alpha) \beta_{22}, \quad \beta'_{12} = \sqrt{\alpha(1-\alpha)} \beta_{12}, \quad \alpha' = \alpha(1 - \alpha).
$$

Then the following conclusions can be drawn.

**Theorem 2.6.** (Existence and uniqueness of (1.13)) Suppose $V(x) \geq 0$ satisfying $\lim_{|x| \to \infty} V(x) = \infty$ and at least one of the following condition holds,

(i) $d = 1$;

(ii) $d = 2$ and $\beta'_{11} \geq -C_b$, $\beta'_{22} \geq -C_b$, and $\beta'_{12} \geq -\sqrt{(C_b + \beta'_{11})(C_b + \beta'_{22})}$;

(iii) $d = 3$ and $B$ is either positive semi-definite or nonnegative,

then there exists a ground state $\Phi_g = (\phi^g_1, \phi^g_2)^T$ of (1.13). In addition, $\tilde{\Phi}_g := (e^{i\theta_1} |\phi_1^g|, e^{i\theta_2} |\phi_2^g|)$ is also a ground state of (1.13) with two constants $\theta_1$ and $\theta_2$. Furthermore, if the matrix $B$ is positive semi-definite, the ground state $(|\phi^g_1|, |\phi^g_2|)^T$ of (1.13) is unique. In contrast, if one of the following conditions holds,

(i) $d = 2$ and $\beta'_{11} < -C_b$ or $\beta'_{22} < -C_b$ or $\beta'_{12} < -\frac{1}{2\alpha'} (\alpha \beta_{11} + (1 - \alpha) \beta_{22} + C_b)$;

(ii) $d = 3$ and $\beta_{11} < 0$ or $\beta_{22} < 0$ or $\beta_{12} < -\frac{1}{2\alpha'} (\alpha^2 \beta_{11} + (1 - \alpha)^2 \beta_{22})$.

there exists no ground states of (1.13).

*Proof:* The proof is similar to those of Theorems 2.1 and 2.2 and it is omitted here for brevity.

3. Properties of the ground states

In this section, we will show some properties of the stationary states and find the limiting behavior of the ground states when either $|\lambda| \to \infty$ or $|\delta| \to \infty$.

**Theorem 3.1.** Suppose that $V(x) \geq 0$ and $\beta_{11} = \beta_{12} = \beta_{22} = 0$, for the stationary states of (1.9) under the constraint (1.10), we have

(i) The ground state $\Phi_g = (\phi^g_1, \phi^g_2)^T$ is the global minimizer of $E(\Phi)$ over the unit sphere $S$.

(ii) Any excited state $\Phi_j = (\phi^j_1, \phi^j_2)^T$ ($j = 1, 2, \ldots$) is a saddle point of $E(\Phi)$ over the unit sphere $S$. 


Theorem 3.2. Suppose $V(x) \geq 0$ satisfying $\lim_{|x| \to \infty} V(x) = \infty$ and $B$ is either positive semidefinite or nonnegative. For fixed $V(x)$, $B$ and $\delta$, let $\Phi^\lambda = (\phi^\lambda_1, \phi^\lambda_2)^T$ be a ground state of (1.7) with respect to $\lambda$. Then when $|\lambda| \to \infty$, we have

$$\|\phi^\lambda_j - \phi^\delta_j\|_2 \to 0, \quad j = 1, 2, \quad E(\Phi^\lambda) \approx 2E_1(\phi^\delta) - |\lambda|,$$

where $\phi^\delta$ is the unique positive minimizer [23] of

$$E_1(\phi) = \int_{\mathbb{R}^d} \left[ \frac{1}{2} |\nabla \phi|^2 + V_1(x)|\phi|^2 + \frac{B}{2} |\phi|^4 \right] dx$$

under the constraint

$$\|\phi\|_2^2 = \int_{\mathbb{R}^d} |\phi(x)|^2 dx = \frac{1}{2}.$$
with
\[ V_1(x) = V(x) + \frac{\delta}{2}, \quad \beta = \frac{\beta_{11} + \beta_{22} + 2\beta_{12}}{2}. \] (3.6)

Proof. Without loss of generality, we assume \( \lambda < 0 \) and the ground state \( \Phi^\lambda \) satisfying \( \phi^\lambda_j \geq 0 \) (\( j = 1, 2 \)). Since \( (\phi^g, \phi^g)^T \in \mathcal{D}_1 \), we have
\[ \tilde{E}(|\phi^\lambda_1|, |\phi^\lambda_2|) \leq \tilde{E}(\phi^g, \phi^g). \] (3.7)
Noticing
\[ \tilde{E}(\Phi) = \int_{\mathbb{R}^d} \left[ \frac{1}{2} (|\nabla \phi_1|^2 + |\nabla \phi_2|^2) + V(x) \left( |\phi_1|^2 + |\phi_2|^2 \right) + |\lambda| |\phi_1 - \phi_2|^2 \\
+ \delta |\phi_1|^2 + \left( \frac{1}{2} \beta_{11} |\phi_1|^4 + \frac{1}{2} \beta_{22} |\phi_2|^4 + \beta_{12} |\phi_1|^2 |\phi_2|^2 \right) \right] dx \\
- |\lambda|, \quad \Phi \in \mathcal{D}_1, \] (3.8)
we have
\[ \tilde{E}(\phi^g, \phi^g) = 2E_1(\phi^g) - |\lambda|. \] (3.9)
Plugging (3.9) into (3.7) and noticing (3.8), there exists a constant \( C > 0 \) such that
\[ \|\phi^\lambda_1\|_{L^2} + \|\phi^\lambda_2\|_{L^2} \leq C, \quad \|\phi^\lambda_1 - \phi^\lambda_2\|_2 \leq \frac{C}{|\lambda|}, \quad |\lambda| > 0, \] (3.10)
this immediately implies
\[ \phi^\lambda_1 - \phi^\lambda_2 \longrightarrow 0 \text{ in } L^2, \quad \text{as } |\lambda| \to \infty. \] (3.11)
Using the similar arguments as in the proof of Theorem 2.1, we can see that there exists \( \Phi^\infty = (\phi_1^\infty, \phi_2^\infty)^T \in \mathcal{D}_1 \) such that
\[ \phi^\lambda_1 \to \phi_1^\infty, \quad \phi^\lambda_2 \to \phi_2^\infty, \quad \text{in } L^2 \cap L^4 \cap L^2_V, \\
\nabla \phi^\lambda_1 \to \nabla \phi_1^\infty, \quad \nabla \phi^\lambda_2 \to \nabla \phi_2^\infty, \quad \text{in } L^2, \] (3.12)
and
\[ \tilde{E}(\phi_1^\infty, \phi_2^\infty) \leq \liminf_{|\lambda| \to \infty} \tilde{E}(\phi^\lambda_1, \phi^\lambda_2). \] (3.13)
These together with (3.11) imply that
\[ \phi_1^\infty = \phi_2^\infty := \phi^\infty. \] (3.14)
Plugging (3.14) into (2.6), we obtain
\[ \tilde{E}(\phi^\infty, \phi^\infty) = 2E_1(\phi^\infty) - |\lambda| \leq \liminf_{|\lambda| \to \infty} \tilde{E}(\phi^\lambda_1, \phi^\lambda_2) \leq \limsup_{|\lambda| \to \infty} \tilde{E}(\phi^\lambda_1, \phi^\lambda_2) \leq 2E_1(\phi^g) - |\lambda|, \] (3.15)
and

$$E_1(\phi^\infty) \leq E_1(\phi^\delta).$$

(3.16)

Since $\phi^1_1$ and $\phi^2_2$ are nonnegative and $\phi^1_1$ converges weakly to $\phi^\infty$ in $H^1$, there exists a subsequence such that $\phi^1_n$ converges to $\phi^\infty$ a.e. in any compact subset, which shows $\phi^\infty$ is nonnegative. Recalling that $\|\phi^\infty\|^2 = \|\Phi^\delta\|^2/2 = 1/2$ and $\phi^\delta$ is the unique positive minimizer of (3.4) under the constraint (3.5), we conclude that $\phi^\infty$ must be equal to $\phi^\delta$. Therefore, all inequalities above must hold as equalities. Thus, with (3.11), we can obtain the norm convergence,

$$\|\phi^1_1\|_2 \to \|\phi^\delta\|_2, \quad \|\phi^2_2\|_2 \to \|\phi^\delta\|_2,$$

and

$$\|\nabla \phi^1_1\|_2 \to \|\nabla \phi^\delta\|_2, \quad \|\nabla \phi^2_2\|_2 \to \|\nabla \phi^\delta\|_2.$$  

(3.17)

Now, the weak convergence and the norm convergence would imply the conclusion since $H^1$ is a Hilbert space.

**Theorem 3.3.** Suppose $V(x) \geq 0$ satisfying $\lim_{|x| \to \infty} V(x) = \infty$ and $B$ is either positive semi-definite or nonnegative. For fixed $V(x)$, $B$ and $\lambda$, let $\Phi^\delta = (\phi^\delta_1, \phi^\delta_2)^T$ be a ground state of (1.7) with respect to $\delta$. Then when $\delta \to +\infty$, we have

$$\|\phi^\delta_1\|_2 \to 0, \quad \|\phi^\delta_2\|_2 \to 0,$$

and when $\delta \to -\infty$, we have

$$\|\phi^\delta_1\|_2 \to 0, \quad \|\phi^\delta_2\|_2 \to 0,$$

$$E(\Phi^\delta) \approx E_2(\phi^\delta) + \delta,$$

(3.18)

and

$$E_2(\phi) = \int_{\mathbb{R}^d} \left[ \frac{1}{2} |\nabla \phi|^2 + V(x)|\phi|^2 + \frac{\beta}{2} |\phi|^4 \right] dx$$

(3.20)

under the constraint

$$\|\phi\|_2 = \|\phi\|_2 = \int_{\mathbb{R}^d} |\phi|^2 dx = 1,$$

(3.21)

with $\beta = \beta_{22}$ when $\delta > 0$, and $\beta = \beta_{11}$ when $\delta < 0$.

**Proof.** Using the fact $(0, \phi^\delta)^T \in \mathcal{D}_1$ when $\delta > 0$ and $(\phi^\delta, 0)^T \in \mathcal{D}_1$ when $\delta < 0$, the results can be established by a similar argument as in Theorem 3.2.

### 4. Numerical methods

In this section, we will propose and analyze efficient and accurate numerical methods for computing the ground states of (1.7).
4.1. Continuous normalized gradient flow and its discretization

In order to compute the ground state of two-component BEC with an external driving field (1.7), we construct the following continuous normalized gradient flow (CNGF):

\[
\begin{align*}
\frac{\partial \phi_1(x, t)}{\partial t} &= \left[ \frac{1}{2} \nabla^2 - V(x) - \delta - (\beta_{11} |\phi_1|^2 + \beta_{12} |\phi_2|^2) \right] \phi_1 - \lambda \phi_2 + \mu_\phi(t) \phi_1, \\
\frac{\partial \phi_2(x, t)}{\partial t} &= \left[ \frac{1}{2} \nabla^2 - V(x) - (\beta_{12} |\phi_1|^2 + \beta_{22} |\phi_2|^2) \right] \phi_2 - \lambda \phi_1 + \mu_\phi(t) \phi_2,
\end{align*}
\]

where \( \Phi(x, t) = (\phi_1(x, t), \phi_2(x, t))^T \) and \( \mu_\phi(t) \) is chosen such that the above CNGF is mass or normalization conservative and it is given as

\[
\mu_\phi(t) = \frac{1}{\|\Phi(\cdot, t)\|^2} \int_{\mathbb{R}^d} \left[ \frac{1}{2} (|\nabla \phi_1|^2 + |\nabla \phi_2|^2) + V(x)(|\phi_1|^2 + |\phi_2|^2) + \delta |\phi_1|^2 + \beta_{11} |\phi_1|^4 + \beta_{22} |\phi_2|^4 + 2\beta_{12} |\phi_1|^2 |\phi_2|^2 + 2\lambda \Re(\phi_1 \bar{\phi}_2) \right] dx
\]

\[
= \frac{\mu(\Phi(\cdot, t))}{\|\Phi(\cdot, t)\|^2}, \quad t \geq 0.
\]

For the above CNGF, we have

**Theorem 4.1.** For any given initial data

\[
\Phi(x, 0) = (\phi_1^0(x), \phi_2^0(x))^T := \Phi^{(0)}(x), \quad x \in \mathbb{R}^d,
\]

satisfying \( \|\Phi^{(0)}\|^2 = 1 \), the CNGF (4.1) is mass or normalization conservative and energy diminishing, i.e.

\[
\|\Phi(\cdot, t)\|^2 \equiv \|\Phi^{(0)}\|^2 = 1, \quad E(\Phi(\cdot, t)) \leq E(\Phi(\cdot, s)), \quad 0 \leq s \leq t.
\]


Using an argument similar to that in [33], when \( V(x) \geq 0 \) satisfying \( \lim_{|x| \to \infty} V(x) = \infty \), \( B \) is either positive semi-definite or nonnegative, and \( \|\Phi^{(0)}\| = 1 \), we may get as \( t \to \infty \), \( \Phi(x, t) \) approaches to a steady state solution, which is a critical point of the energy functional \( E(\Phi) \) over the unit sphere \( S \) or an eigenfunction of the nonlinear eigenvalue problem (1.9) under the constraint (1.10). In addition, when the initial data in (4.3) is chosen properly, e.g. its energy is less than that of the first excited state, the ground state \( \Phi_g \) can be obtained from the steady state solution of (4.1), i.e.

\[
\Phi_g(x) = \lim_{t \to \infty} \Phi(x, t), \quad x \in \mathbb{R}^d.
\]

For practical computation, here we also present a second-order in both space and time full discretization for the above CNGF (4.1). For simplicity of notation, we introduce the
The boundary condition (4.6) is discretized as

\[ \Phi(a, t) = \Phi(b, t) = 0, \quad t \geq 0. \]  

(4.6)

Generalizations to higher dimensions are straightforward for tensor product grids.

Choose time step \( k = \Delta t > 0 \) and let time steps be \( t_n = n k = n \Delta t \) for \( n = 0, 1, 2, \ldots; \) and choose spatial mesh size \( h = \Delta x > 0 \) with \( h = (b - a)/M \) a positive integer and let the grid points be \( x_j = a + j h, j = 0, 1, 2, \ldots, M. \) Let \( \Phi^n = (\Phi_{1,j}^n, \Phi_{2,j}^n)^T \) be the numerical approximation of \( \Phi(x_j, t_n) \) and \( \Phi^n \) be the solution vector with component \( \Phi_{1,j}^n. \) In addition, denote \( \Phi_{1,j}^{n+1/2} = (\Phi_{1,j}^{n+1/2}, \Phi_{2,j}^{n+1/2})^T \) with

\[ \Phi_{1,j}^{n+1/2} = \frac{1}{2} \left( \Phi_{1,j}^{n+1} + \Phi_{1,j}^n \right), \quad j = 0, 1, 2, \ldots, M, \quad l = 1, 2. \]  

(4.7)

Then a second-order full discretization for the CNGF (4.1) is given, for \( j = 1, 2, \ldots, M - 1 \) and \( n \geq 0, \) as

\[
\frac{\Phi_{1,j}^{n+1} - \Phi_{1,j}^n}{k} = \frac{\Phi_{1,j+1}^{n+1/2} - 2 \Phi_{1,j}^{n+1/2} + \Phi_{1,j-1}^{n+1/2}}{h^2} - \left[ V(x_j) + \delta - \mu_{\Phi,h} \right] \Phi_{1,j}^{n+1/2} - \lambda \Phi_{2,j}^{n+1/2} - \frac{1}{2} \left[ \beta_{11} \left( |\Phi_{1,j}^{n+1/2}|^2 + |\Phi_{1,j}^n|^2 \right) + \beta_{12} \left( |\Phi_{2,j}^{n+1/2}|^2 + |\Phi_{2,j}^n|^2 \right) \right] \Phi_{1,j}^{n+1/2},
\]

(4.8)

\[
\frac{\Phi_{2,j}^{n+1} - \Phi_{2,j}^n}{k} = \frac{\Phi_{2,j+1}^{n+1/2} - 2 \Phi_{2,j}^{n+1/2} + \Phi_{2,j-1}^{n+1/2}}{h^2} - \left[ V(x_j) - \mu_{\Phi,h} \right] \Phi_{2,j}^{n+1/2} - \lambda \Phi_{1,j}^{n+1/2} - \frac{1}{2} \left[ \beta_{12} \left( |\Phi_{1,j}^{n+1/2}|^2 + |\Phi_{1,j}^n|^2 \right) + \beta_{22} \left( |\Phi_{2,j}^{n+1/2}|^2 + |\Phi_{2,j}^n|^2 \right) \right] \Phi_{2,j}^{n+1/2},
\]

(4.9)

where

\[
\mu_{\Phi,h} = \frac{D_{\Phi,h}^{n+1/2}}{h \sum_{j=0}^{M-1} \left( |\Phi_{1,j}^{n+1/2}|^2 + |\Phi_{2,j}^{n+1/2}|^2 \right)}, \quad n \geq 0,
\]

(4.10)

with

\[
D_{\Phi,h}^{n+1/2} = h \sum_{j=0}^{M-1} \sum_{l=1}^{2} \left( \frac{1}{2h^2} |\Phi_{1,j+l}^{n+1/2} - \Phi_{1,j}^{n+1/2}|^2 + V(x_j)|\Phi_{1,j}^{n+1/2}|^2 \right) + \delta |\Phi_{1,j}^{n+1/2}|^2 \\
+ \frac{1}{2} \beta_{11} \left( |\Phi_{1,j}^{n+1/2}|^2 + |\Phi_{1,j}^n|^2 \right) |\Phi_{1,j}^{n+1/2}|^2 + \frac{1}{2} \beta_{22} \left( |\Phi_{2,j}^{n+1/2}|^2 + |\Phi_{2,j}^n|^2 \right) |\Phi_{2,j}^{n+1/2}|^2 \\
+ \frac{1}{2} \beta_{12} \left( \left( |\Phi_{1,j}^{n+1/2}|^2 + |\Phi_{2,j}^n|^2 \right) |\Phi_{1,j}^{n+1/2}|^2 + \left( |\Phi_{1,j}^{n+1/2}|^2 + |\Phi_{2,j}^n|^2 \right) |\Phi_{2,j}^{n+1/2}|^2 \right) \\
+ 2\lambda \operatorname{Re} \left( \Phi_{1,j}^{n+1/2} \Phi_{2,j}^{n+1/2} \right).
\]

(4.11)

The boundary condition (4.6) is discretized as

\[
\Phi_{1,0}^{n+1} = \Phi_{1,M}^{n+1} = \Phi_{2,0}^{n+1} = \Phi_{2,M}^{n+1} = 0, \quad n = 0, 1, 2, \ldots.
\]

(4.12)
The initial data (4.3) is discretized as
\[ \phi_{1,j}^0 = \phi_{1}^0(x_j), \quad \phi_{2,j}^0 = \phi_{2}^0(x_j), \quad j = 0, 1, \ldots, M. \] (4.13)

Similarly, for the above full discretization for the CNGF, we have

**Theorem 4.2.** For any given time step \( k > 0 \) and mesh size \( h > 0 \) as well as initial data \( \Phi(0) \) in (4.3) satisfying \( \| \Phi(0) \| = 1 \), the full discretization (4.8)-(4.13) for CNGF (4.1) is mass or normalization conservative and energy diminishing, i.e.

\[ N_{\Phi,h}^n := h \sum_{j=0}^{M-1} \sum_{l=1}^{2} |\phi_{l,j}^n|^2 \equiv N_{\Phi,h}^0 := h \sum_{j=0}^{M-1} \sum_{l=1}^{2} |\phi_{l}^0(x_j)|^2, \quad n \geq 0, \] (4.14)

\[ E_{\Phi,h}^n \leq E_{\Phi,h}^{n-1} \leq \cdots \leq E_{\Phi,h}^0, \quad n \geq 0, \] (4.15)

where the discretized energy \( E_{\Phi,h}^n \) is defined as

\[ E_{\Phi,h}^n = h \sum_{j=0}^{M-1} \left( \sum_{l=1}^{2} \left( \frac{1}{2h^2} |\phi_{l,j+1}^n - \phi_{l,j}^n|^2 + V(x_j) |\phi_{l,j}^n|^2 \right) + \delta |\phi_{l,j}^n|^2 \right) + \frac{1}{2} \beta_{11} |\phi_{1,j}^n|^4 + \frac{1}{2} \beta_{22} |\phi_{2,j}^n|^4 + \beta_{12} |\phi_{1,j}^n|^2 |\phi_{2,j}^n|^2 + 2\lambda Re \left( \phi_{1,j}^n \phi_{2,j}^n \right) \] (4.16)

**Proof:** Follow the analogous arguments in [7] for spin-1 BEC and we omitted the details here.

In the above full discretization, at every time step, we need to solve a fully nonlinear system which is very tedious in practical computation. Below we present a more efficient discretization for the CNGF (4.1) for computing the ground states.

### 4.2. Gradient flow with discrete normalization and its discretization

Another more efficient way to discretize the CNGF (4.1) is through the construction of the following gradient flow with discrete normalization (GFDN):

\[ \frac{\partial \phi_1}{\partial t} = \left[ \frac{1}{2} \nabla^2 - V(x) - \delta - (\beta_{11} |\phi_1|^2 + \beta_{12} |\phi_2|^2) \right] \phi_1 - \lambda \phi_2, \]
\[ \frac{\partial \phi_2}{\partial t} = \left[ \frac{1}{2} \nabla^2 - V(x) - (\beta_{12} |\phi_1|^2 + \beta_{22} |\phi_2|^2) \right] \phi_2 - \lambda \phi_1, \quad t_n \leq t < t_{n+1}, \] (4.17)

followed by a projection step as

\[ \phi_l(x, t_{n+1}) := \phi_l(x, t_{n+1}^+) = \sigma_{l+1}^{n+1} \phi_l(x, t_{n+1}^-), \quad l = 1, 2, \quad n \geq 0, \] (4.18)

where \( \phi_l(x, t_{n+1}^+) = \lim_{t \to t_{n+1}^+} \phi_l(x, t) \) \( (l = 1, 2) \) and \( \sigma_{l+1}^{n+1} \) \( (l = 1, 2) \) are chosen such that

\[ \|\Phi(x, t_{n+1})\|^2 = \|\phi_1(x, t_{n+1})\|^2 + \|\phi_2(x, t_{n+1})\|^2 = 1, \quad n \geq 0. \] (4.19)
The above GFDN (4.17)-(4.18) can be viewed as applying the first-order splitting method to the CNGF (4.1) and the projection step (4.18) is equivalent to solving the following ordinary differential equations (ODEs)

\[
\frac{\partial \phi_1(x, t)}{\partial t} = \mu_\phi(t) \phi_1, \quad \frac{\partial \phi_2(x, t)}{\partial t} = \mu_\phi(t) \phi_2, \quad t_n \leq t \leq t_{n+1},
\]

which immediately suggests that the projection constants in (4.18) are chosen as

\[
\sigma_1^{n+1} = \sigma_2^{n+1}, \quad n \geq 0.
\]

Plugging (4.21) and (4.18) into (4.19), we obtain

\[
\sigma_1^{n+1} = \sigma_2^{n+1} = \frac{1}{\| \Phi(\cdot, t_{n+1}^-) \|} = \frac{1}{\sqrt{\| \phi_1(\cdot, t_{n+1}^-) \|^2 + \| \phi_2(\cdot, t_{n+1}^-) \|^2}}, \quad n \geq 0.
\]

In fact, the gradient flow (4.17) can be viewed as applying the steepest decent method to the energy functional \( E(\Phi) \) in (1.7) without constraints, and (4.18) project the solution back to the unit sphere \( S \). In addition, (4.17) can also be obtained from the CGPES (1.1) by the change of variable \( t \rightarrow -t \), that is why this kind of algorithm is usually called as the imaginary time method in the physics literature [2,4,14,32]. From numerical point of view, the GFDN is much easier to discretize since the gradient flow (4.17) can be solved via traditional techniques and the normalization (4.19) is simply achieved by a projection (4.18) at the end of each time step.

For the above DNGF, we have

**Theorem 4.3.** Suppose \( V(x) \geq 0 \) and \( \beta_{11} = \beta_{12} = \beta_{22} = 0 \), then for any time step \( k > 0 \) and initial data \( \Phi^{(0)} \) in (4.3) satisfying \( \| \Phi^{(0)} \| = 1 \), the GFDN (4.17)-(4.18) is energy diminishing, i.e.

\[
E(\Phi(\cdot, t_{n+1}^-)) \leq E(\Phi(\cdot, t_n^-)) \leq \cdots \leq E(\Phi(\cdot, 0)) = E(\Phi^{(0)}), \quad n = 0, 1, 2, \ldots.
\]

**Proof.** Follow the analogous arguments in [4] for single-component BEC and we omitted the details here.

Again, for practical computation, here we also present a modified backward Euler finite difference (MBEFD) discretization for the above GFDN (4.17)-(4.18) in a bounded domain \( \Omega = (a, b) \) with homogeneous Dirichlet boundary condition (4.6):

\[
\Phi_{1,j}^* - \Phi_{1,j}^n = \frac{1}{2h^2} \left[ \Phi_{1,j+1}^* - 2 \Phi_{1,j}^* + \Phi_{1,j-1}^* \right] - \left[ (V(x_j) + \delta + \alpha) \Phi_{1,j}^* - \lambda \Phi_{2,j}^* \right] - \left( \beta_{11} |\Phi_{1,j}^n|^2 + \beta_{12} |\Phi_{2,j}^n|^2 \right) \Phi_{1,j}^n + \alpha \Phi_{1,j}^n, \quad 1 \leq j \leq M - 1,
\]

\[
\Phi_{2,j}^* - \Phi_{2,j}^n = \frac{1}{2h^2} \left[ \Phi_{2,j+1}^* - 2 \Phi_{2,j}^* + \Phi_{2,j-1}^* \right] - \left[ (V(x_j) + \alpha) \Phi_{1,j}^* - \lambda \Phi_{2,j}^* \right] - \left( \beta_{12} |\Phi_{1,j}^n|^2 + \beta_{22} |\Phi_{2,j}^n|^2 \right) \Phi_{2,j}^n + \alpha \Phi_{2,j}^n, \quad 1 \leq j \leq M - 1,
\]

\[
\Phi_{l,j}^{n+1} = \frac{\Phi_{l,j}^*}{\| \Phi^* \|_h}, \quad j = 0, 1, \ldots, M, \quad n \geq 0, \quad l = 1, 2;
\]
where \( \alpha \geq 0 \) is a stabilization parameter chosen such that the time step \( k \) is independent of the external driving field \( \lambda \) and

\[
\| \Phi^* \|_h := \sqrt{h \sum_{j=1}^{M-1} \left| \phi_{1,j}^* \right|^2 + \left| \phi_{2,j}^* \right|^2}. \tag{4.25}
\]

The initial and boundary conditions are discretized similarly as those for CNGF.

For the above full discretization for the GFDN, we have

**Theorem 4.4.** Suppose \( V(x) \geq 0 \) and \( \beta_{11} = \beta_{12} = \beta_{22} = 0 \), if \( \alpha \geq |\lambda| + \max(0, -\delta) \), then the MBEFD discretization (4.24) is energy diminishing for any time step \( k > 0 \) and initial data \( \Phi^{(0)} \) satisfying \( \| \Phi^{(0)} \|_h = 1 \), i.e.

\[
E_{\Phi,h}^{n+1} \leq E_{\Phi,h}^n \leq \cdots \leq E_{\Phi,h}^0 = E_{\Phi^{(0)},h}, \quad n \geq 0, \tag{4.26}
\]

where the discretized energy \( E_{\Phi,h}^n \) is defined in (4.16) with \( \beta_{11} = \beta_{12} = \beta_{22} = 0 \).

**Proof.** Denote

\[
\Phi^n = \left( \Phi^n_{1,1}, \Phi^n_{1,2}, \ldots, \Phi^n_{1,M-1}, \Phi^n_{2,1}, \Phi^n_{2,2}, \ldots, \Phi^n_{2,M-1} \right)^T,
\]

\[
F = \text{diag}(V(x_1), V(x_2), \ldots, V(x_{M-1}), V(x_1), V(x_2), \ldots, V(x_{M-1})),
\]

\[
D = \begin{pmatrix} G & 0 \\ 0 & G \end{pmatrix}, \quad D_1 = \begin{pmatrix} \delta I_{M-1} & \lambda I_{M-1} \\ \lambda I_{M-1} & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} (\alpha + \delta) I_{M-1} & \lambda I_{M-1} \\ \lambda I_{M-1} & \alpha I_{M-1} \end{pmatrix},
\]

where \( I_{M-1} \) is the \( (M-1) \times (M-1) \) identity matrix and \( G \) is an \( (M-1) \times (M-1) \) tridiagonal matrix with \( 1/h^2 \) at the diagonal entries and \( -1/2h^2 \) at the off-diagonal entries. Let

\[
T = D + F + D_2 = D + F + D_1 + \alpha I_{2M-2}. \tag{4.27}
\]

When \( \beta_{11} = \beta_{12} = \beta_{22} = 0 \), the MBEFD discretization (4.24) reads

\[
\frac{\Phi^* - \Phi^n}{k} = -(D + F + D_2)\Phi^* + \alpha \Phi^n = -T \Phi^* + \alpha \Phi^n,
\]

\[
\Phi^{n+1} = \frac{\Phi^*}{\| \Phi^* \|_h}, \quad n \geq 0, \tag{4.28}
\]

and the discretized energy \( E_{\Phi,h}^n \) in (4.16) with \( \beta_{11} = \beta_{12} = \beta_{22} = 0 \) can be written as

\[
E_{\Phi,h}^n = h(\Phi^n)^T (D + F + D_1) \Phi^n = h(\Phi^n, T \Phi^n) - \alpha \| \Phi^n \|_h^2, \tag{4.29}
\]

where \((\cdot, \cdot)\) is the standard inner product. From (4.28), we have

\[
(I + k T)\Phi^* = (1 + \alpha k)\Phi^n, \quad n \geq 0. \tag{4.30}
\]
If $\alpha \geq |\lambda| + \max(0, -\delta)$, then $T$ is positive semi-definite, notice (4.29) and (4.30), using Lemma 2.8 in [4], we get

$$E_{\Phi,h}^{n+1} - \alpha\|\Phi^{n+1}\|_{h}^{2} = h(\Phi^{n+1}, T\Phi^{n+1}) = \frac{(\Phi^{n}, T\Phi^{n})}{((1 + k\alpha)\Phi^{n}, (1 + k\alpha)T\Phi^{n})} \leq \frac{(1 + k\alpha)\Phi^{n}, (1 + k\alpha)T\Phi^{n}}{(1 + k\alpha)\Phi^{n}, (1 + k\alpha)\Phi^{n}} \leq 1.$$  

Thus the conclusion follows immediately from the above inequality and $\|\Phi^{n}\|_{h} = \|\Phi^{n+1}\|_{h} = 1$.

In fact, when $\alpha = 0$, the MBEFD discretization (4.24) collapses to the standard backward Euler finite difference scheme [4]. In addition, from the proof in the above Theorem, in practical computation, we can choose $\alpha = |\lambda| + \max(0, -\delta)$.

5. Numerical results

In this section, we will report the ground states of (1.7) in 1D computed by our numerical method MBEFD (4.24). In our computation, the ground state is reached when $\|\Phi^{n+1} - \Phi^{n}\| \leq \varepsilon := 10^{-7}$. In addition, for ground state of two-component BEC with an
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Figure 2: Ground states $\Phi_g = (\phi_1, \phi_2)^T$ in Example 1 when $\delta = 0$ and $\beta = 100$ for different $\lambda$.

external driving field (1.7), we have $\lambda \leftrightarrow -\lambda \iff \phi_2^g \leftrightarrow -\phi_2^g$, and thus we only present results for $\lambda \leq 0$.

**Example 1.** Ground states of two-component BEC with an external driving field when $B$ is positive definite, i.e. we take $d = 1, V(x) = \frac{1}{2}x^2$ and $\beta_{11} : \beta_{12} : \beta_{22} = (1 : 0.94 : 0.97)\beta$ in (1.7) [2, 21, 22]. In this case, since $\lambda \leq 0$ and $B$ is positive definite when $\beta > 0$, thus we know that the positive ground state $\Phi_g = (\phi_1, \phi_2)^T$ is unique. In our computations, we take the computational domain $\Omega = [-16, 16]$ with mesh size $h = \frac{1}{32}$ and time step $k = 0.1$. The initial data in (4.3) is chosen as

$$\phi_0^g(x) = \phi_2^0(x) = \frac{1}{\pi^{1/4} \sqrt{2}} e^{-x^2/2}, \quad x \in \mathbb{R}. \quad (5.1)$$

In fact, we have checked with other types of initial data in (4.3) and the computed ground state is the same.

Figure 1 plots the ground states $\Phi_g$ when $\delta = 0$ and $\lambda = -1$ for different $\beta$, and Figure 2 depicts similar results when $\delta = 0$ and $\beta = 100$ for different $\lambda \leq 0$. Figure 3 shows mass of each component $N(\phi_j) = ||\phi_j||^2$ ($j = 1, 2$), energy $E := E(\Phi_g)$ and chemical potential $\mu := \mu(\Phi_g)$ of the ground states when $\delta = 0$ for different $\lambda$ and $\beta$. Figure 4 shows similar results when $\beta = 100$ and $\delta = 0, 1$ for different $\lambda$, and Figure 5 for results when $\beta = 100$ and $\lambda = 0, -5$ for different $\delta$. 
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Figure 3: Mass of each component \( N(\phi_j) = \|\phi_j\|^2 \) \( (j = 1, 2) \), energy \( E := E(\Phi_g) \) and chemical potential \( \mu := \mu(\Phi_g) \) of the ground states in Example 1 when \( \delta = 0 \) for different \( \lambda \) and \( \beta \).

From Figures 1-5 and additional numerical results not shown here for brevity, we can draw the following conclusions for the ground states in this case: (i) the positive ground state is unique when at least one of the parameters \( \beta, \lambda \) and \( \delta \) are nonzero which confirm the results in Theorem 2.1 (cf. Figs. 1 & 2); (ii) when \( \beta = 0 \) and \( \delta = 0 \), \( \phi_1 = \phi_2 \) when \( \lambda < 0 \), and \( \phi_1 = -\phi_2 \) when \( \lambda > 0 \) (cf. Fig. 1); (iii) for fixed \( \beta \) and \( \delta \), when \( \lambda \to -\infty \), \( \phi_1 - \phi_2 \to 0 \) and when \( \lambda \to +\infty \), \( \phi_1 + \phi_2 \to 0 \) (cf. Fig. 2) which confirm the analytical results in Theorem 3.2; (iv) when \( \delta = 0 \), \( N(\phi_1) \) decreases and \( N(\phi_2) \) increases when \( \lambda \neq 0 \) (cf. Fig. 3) which is due to \( \beta_{11} > \beta_{22} \); (v) for fixed \( \delta \) and \( \lambda \), when \( \beta \gg 1 \), \( E = O(\beta^{1/3}) \).
and $\mu = O(\beta^{1/3})$ which can be confirmed by a re-scaling $x \to e^{1/2} x$ and $\Phi \to e^{-d/4} \Phi$ with $\varepsilon = \beta^{-d/(d+2)}$ in the energy functional $E(\Phi)$ in (1.5) and the chemical potential $\mu(\Phi)$ in (1.11) [3, 38]; (vi) for fixed $\beta > 0$ and $\delta$, when $|\lambda| \to \infty$, then $N(\phi_1) - N(\phi_2) \to 0$, $E \approx -|\lambda| + C_1$ and $\mu \approx -|\lambda| + C_2$ with $C_1$ and $C_2$ two constants independent of $\lambda$ (cf. Fig. 4) which confirm the analytical results in Theorem 3.2; (vii) for fixed $\beta > 0$ and $\lambda$, when $\delta \to +\infty$, $N(\phi_1) \to 0$, $N(\phi_2) \to 1$, $E \approx C_3$ and $\mu \approx C_4$ with $C_3$ and $C_4$ two constants independent of $\delta$; and when $\delta \to -\infty$, $N(\phi_1) \to 1$, $N(\phi_2) \to 0$, $E \approx \delta + C_5$ and $\mu \approx \delta + C_6$ with $C_5$ and $C_6$ two constants independent of $\delta$ (cf. Fig. 5) which confirm the results in Theorem 3.3. In addition, when $\delta = 0$ and $\lambda = 0$, $N(\phi_1) = 1/3$ and $N(\phi_2) = 2/3$ which are independent of $\beta$ (cf. Fig. 3). In fact, in this case, the energy functional can be written as

$$E(\Phi) = \int_{\Omega} \left[ \frac{1}{2} (|\nabla \phi_1|^2 + |\nabla \phi_2|^2) + V(x) (|\phi_1|^2 + |\phi_2|^2) + \frac{\beta}{2} \left( |\phi_1|^4 + 0.97|\phi_2|^4 + 2 \times 0.94|\phi_1|^2|\phi_2|^2 \right) \right] dx. \quad (5.2)$$
Denote $\rho(x) = \sqrt{\phi_1(x)^2 + \phi_2(x)^2}$, using the Cauchy inequality, we have

$$E(\Phi) \geq \int_\Omega \left[ \frac{1}{2} |\nabla \rho|^2 + V(x) |\rho|^2 + \frac{0.94\beta}{2} |\rho|^4 + \frac{\beta}{2} (0.06|\phi_1|^4 + 0.03|\phi_2|^4) \right] dx$$

and the above equality holds only if $2|\phi_1|^2 = |\phi_2|^2$. Notice that the functional $E_2(\rho) = \int_\Omega \left( \frac{1}{2} |\nabla \rho|^2 + V(x) |\rho|^2 + \frac{0.96\beta}{2} |\rho|^4 \right) dx$ admits a unique positive minimizer $\rho_g$ under constraint $\|\rho\|_2 = 1$ [23], then $\Phi_g = (\sqrt{1/3}\rho_g, \sqrt{2/3}\rho_g)^T$ is a ground state of the original problem, which justifies our numerical observation in Fig. 3.

**Example 2.** Ground states of two-component BEC with an external driving field when $B$ is nonnegative, i.e. we take $d = 1$, $V(x) = \frac{1}{2} x^2 + 24 \cos^2(x)$ and $\beta_{11} : \beta_{12} : \beta_{22} = (1.03 : 1 : 0.97)\beta$ in (1.7) [2, 17, 18]. In our computations, we take the computational domain $\Omega = [-16, 16]$ with mesh size $h = \frac{1}{32}$ and time step $k = 0.1$.

Figure 6 plots the ground states $\Phi_g$ when $\delta = 0$ and $\lambda = -1$ for different $\beta$, and Figure 7 depicts similar results when $\delta = 0$ and $\beta = 100$ for different $\lambda$. Figure 8 shows mass...
Ground states of two-component BEC with external driving field

Figure 6: Ground states \( \Phi_g = (\phi_1, \phi_2)^T \) in Example 2 when \( \delta = 0 \) and \( \lambda = -1 \) for different \( \beta \).

of each component \( N(\phi_j) = \|\phi_j\|^2 \) (\( j = 1, 2 \)), energy \( E := E(\Phi_g) \) and chemical potential \( \mu := \mu(\Phi_g) \) of the ground states when \( \delta = 0 \) for different \( \lambda \) and \( \beta \).

From Figures 6-8 and additional numerical results not shown here for brevity, same conclusions as those in (ii)-(vii) in Example 1 can be drawn. Moreover, the numerical results show that the positive ground state is unique in this case. Due to the appearance of the optical lattice potential \( 24 \cos^2(x) \) in the trapping potential \( V(x) \), there are several peaks in the ground state and the distance between two nearby peaks is roughly as \( \pi \) which is the period of the optical lattice potential (cf. Fig. 6-7). In addition, when \( \delta = 0 \), \( \lambda = 0 \), \( N(\phi_1) = 0 \) and \( N(\phi_2) = 1 \) are independent of \( \beta \) (cf. Fig. 8), which can be explained by Theorem 2.4.

6. Conclusions

We have studied the ground states of coupled Gross-Pitaevskii equations for two-component Bose-Einstein condensates with an external driving field both analytically and numerically. Along the analytic front, we proved the existence and uniqueness results for the ground states of the problem when the interaction matrix \( B \) is either positive semi-definite or non-negative. Limiting behavior of the ground states were also found when either \( |\delta| \to \infty \) or \( |\lambda| \to \infty \). In addition, we also showed that the ground state is a global minimizer and
all excited states are saddle points of the energy functional over the unit sphere $S$ when $B = 0$. Along the numerical front, we presented two efficient and accurate numerical methods for computing the ground states. One was based on the continuous normalized gradient flow which is mass conservative and energy diminishing for any time step $k > 0$ and initial data. The other one was based on gradient flow with discrete normalization which was discretized by the modified backward Euler finite difference (MBEFD) with a proper stabilization term and a proper choice of the projection constants. The former numerical method was well-understood mathematically, but it is more tedious and expensive in computation; where the latter numerical method was well-understood mathematically in the linear case and it is more efficient in practical computation. In practise, we suggest to use the MBEFD to compute the ground state of two-component BEC with an external driving filed. Finally, the ground states and their energy and chemical potential diagrams were reported for different parameters to confirm our analytical results and to demonstrate the efficiency and accuracy of our numerical methods.

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Figure 8: Mass of each component $N(\phi_j) = \|\phi_j\|^2$ ($j = 1, 2$), energy $E := E(\Phi_g)$ and chemical potential $\mu := \mu(\Phi_g)$ of the ground states in Example 2 when $\delta = 0$ for different $\lambda$ and $\beta$.

References


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