Evolutionary Dynamics in Complex Networks of Adaptive and Competing Agents

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We use minority game model to investigate evolutionary dynamics in complex networks of adaptive agents competing for limited resources. We show that the dynamics and the phase structures critically depend on the underlying network organizations, and evolution is a key mechanism for the emergence of high-order coordination among the networked agents. A non-growing random directed network admits a clear phase transition from a stable or “critical” state to a “chaotic” state. In contrast, no such phase transition has been detected in a growing directed network. Instead, it permits stable or “critical” dynamics for all the connectivity. The dynamics of a scale-free directed network is found sensitive to its connectivity if a small fraction of link reversal is allowed in the network: The evolutionary dynamics exhibits a gradual phase transition from the stable (or “critical”) state to the “chaotic” regime with increase of the connectivity number $K$; this suggesting that the scale-free directed network is vulnerable to a small fraction of “errors” in its network organization. The underlying sources for the emergence of these dynamics are the organizations of the networks and can be understood with a descendant clusters theory. Evolution dramatically enhances the performance of a system in a stable or “critical” regime, but has no effect on a system in “chaotic” state; such attribute can be empowered to typify the dynamics of a complex network. The impact of evolution on system efficiency lies in its ability in reducing “crowdedness” of agents’ strategies and can be explained by a crowd-anticrowd theory.

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I. INTRODUCTION

Many natural and social systems in the real-world can be characterized by complex networks and study of their organization has attracted intensive research interest [1–3] ever since the seminal works of Strogatz on small-world networks [4] and Barabási and Albert [5] on scale-free networks. The degree distribution $P(k)$, which is defined as the probability of finding a node with exactly $k$ links, has been considered as the most important characterization of complex networks. The power law or scale-free degree distributions have been observed in many real-world complex networks [4, 5].

While organizations and structures of complex network are well understood, the dynamical property of these networks however is obviously less well studied. The dynamics of a complex network can be studied in the context of a system of interactive elements (agents) on the network; it depends on how the network is organized and how the elements interact. The study of network dynamics is pioneered by Kauffman [13, 14] who introduced NK random networks and studied its Boolean dynamics. Recently there are quite a number of studies on different aspects of network dynamics. Aldana and Cluzel demonstrated that the scale-free network favors robust dynamics [15]. Paczuski et al [16] considered the MG model on a random network to study the self-organized process which leads to a stationary but intermittent state. Galstyan [17] studied a network MG, focusing on how the change of the mean connectivity $K$ of a random network affects the global coordination of the system of different capacities. Anghel et al [18] used the MG model to investigate how interagent’s communications across a network lead to a formation of an influence network.

In this paper we investigate the dynamics in the context of an evolutionary minority game of networked agents. The minority game (MG) model was proposed by Challet and Zhang [6], which is a simplification of Arthur’s El Farol bar attendance model [7]. The model serves as an interesting paradigm for studying a system of adaptive agents competing for limited resources. The phase structures of the original MG [8] and the evolutionary version of the game [9–12] have been well understood. Note that in these MG models, the agents are not directly linked to one another but influenced by the global environment created by their collective actions. The network-based MG provides a convenient platform for us to explore dynamics of complex networks.

Our major focuses in the paper is to investigate adaptive and evolutionary dynamics of complex networks, in particular we would like to study: 1) how the dynamics and the associated phase structure of the game depend on the organizations of networks, and 2) how evolution affects the dynamics and phase structure of the game; these issues have not been fully explored in the previous studies. We consider three types of networks: Kauffman’s NK random networks (Kauffman net), growing directed networks (GDNet), and growing directed networks with a fraction of link reversals (GDRNet) Ref. [20]. We apply a simple evolution scheme to investigate the influence of

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evolution on the dynamics.

The paper is organized as follows. In the next section, we describe the network MG model and the various networks we consider in this paper. Section 3 presents our numerical results. Section 4 gives some qualitative analysis and discussion. The last section summarizes.

II. NETWORK MINORITY GAME MODEL

In this section, we introduce network-based minority game model, specify the evolutionary scheme, and describe the three types of the networks we study in the paper.

A. MG of network agents

The network-based MG model consists of $N$ (odd number) agents described by the state variables $s_i = \{0, 1\}, i = 1, 2, ..., N$, each connected to another $K$ agents, $i_1, i_2, ..., i_K$. Each agent has $S$ strategies which are the mapping functions specifying a binary output state (0 or 1) for each possible input vector consisting of the states of her $K$ connected agents. The state or decision of the $i$th agent at the current time step $t$ is determined by the states/decisions of her connected $K$ agents at the previous time step $t-1$, i.e.

$$s_i(t) = F^i_j(s_{i_1}(t-1), s_{i_2}(t-1), ..., s_{i_K}(t-1))$$

where $s_{i_k}(k = 1, 2, ..., K)$ is the state of the $k$th agent that is connected to agent $i$, and $F^i_j, j = 1, ..., S$ are $S$ Boolean functions (strategies) taken from the strategy space consisting of $2^K$ strategies. As in the standard minority game, each agent keeps a record of the cumulative wealth $W_i(t)$ as well as the cumulative (pseudo) scores, $Q^i_s(t), i = 1, 2, ..., N, s = 1, 2, ..., S$, for each of her $S$ strategies.

Before the game starts each agent selects at random one of her $S$ strategies, and the cumulative wealth $W_i(t)$ and strategy scores $Q^i_s(t)$ are initialized to zero. At each time step, each agent decides which of the two groups (0 or 1) to join based on the best-scoring strategy (the strategy that would have made the most winning predictions in the past) among her $S$ strategies. The agent gains (loses) one point in her cumulative wealth for her winning (losing) decision and each strategy gains (loses) one pseudo-point for its winning (losing) prediction. The agents who are among the minority (majority) win (lose). Let $A(t)$ be the number of agents choosing 1 at time step $t$. Then a measure of utilization of the limited resources (system performance) can be defined as the variance of $A(t)$ over a time period $T$:

$$\sigma^2 = \frac{1}{T} \sum_{t=t_0}^{t_0+T} (A(t) - \bar{A})^2$$

where $\bar{A} = \frac{1}{T} \sum_{t=t_0}^{t_0+T} A(t) \sim N/2$ is the mean number of agents choosing 1.

Clearly $\sigma^2$ measures the global coordination among agents. The optimal (smallest) value of variance $\sigma^2$ is 0.25, where the number of winning agents reaches its optimal value, $(N-1)/2$, in every time step. For a random choice game (RCG), where each agent makes decision by coin-tossing, the value of the variance $\sigma^2$ is 0.25N. The game is adaptive as each agent has $S$ strategies to choose from, attempting to increase her chance of winning.

It’s worthwhile to point out the differences between the network MG we discuss here and the original MG defined in Ref. [6]. In the original MG, the input for each agent’s strategies at time $t$ is a vector of the winning decisions of the game in the previous $M$ time steps. In the network (local) MG, however, the input for each agent’s strategies at time $t$ is a vector consisting of the decisions, at the previous time step $t-1$, of the $K$ agents she connects to. Thus, the agents in the original MG use local information while the agents in the network MG use global information. The second key difference is that the progress of the original MG is based on the $M$ time-step history of global information while the network MG employs a one-step forward dynamics.

B. Evolutionary Minority Game

Now we describe the mechanism for network evolutionary minority game (EMG). We first investigate the adaptive dynamics of MG models and used the results as a basis for comparison with the evolutionary MG model, which is our main interest. In an evolutionary dynamics, the quenched strategies can be changed and more generic behaviors will emerge. We use the same evolutionary dynamics as the one described in Refs. [9, 12]. In this scheme, each agent is required to change her $S$ strategies (by choosing $S$ new strategies randomly) whenever her cumulative wealth $W_i(t)$ is below a pre-specified bankruptcy threshold, $-W_c(W_c > 0)$. The bankrupted agents re-set their wealth and strategy pseudo-scores to zero, and the game continues. The network connection, however, does not evolve.

A different evolution scheme was used in the previous study [16] where the evolution takes place for every epoch which is chosen for “Darwinian” selection, i.e., the network was updated until either the attractor of the dynamics was found, or the length of the attractor was found to be larger than some limiting value which was set at 10 000 time steps. We choose the wealth threshold-based scheme in consideration of its practical significance as the wealth is the central concern in economics. Also considering its practical relevance, we allow evolution at every time step as this imitates real-world situation most closely [25].
C. Description of the networks

The dynamics of the game may crucially depends on the network organization and this is one of the focuses of this paper. Thus, to study the impact of different network organizations on the dynamics, a network configuration (once initialized in each game), remains unchanged throughout the game. We now give a brief description on the three different types of networks we discuss in this paper.

Kauffman NK random network (Kauffman net): There are a few different ways to generate random networks [19, 22]. To study dynamics, it is convenient to use Kauffman’s NK random (directed) networks (Kauffman net). The Kauffman net is generated by specifying $N$ agents first, and then connecting each agent randomly to $K$ (referred to as connectivity, hereafter) other agents. Each agent makes her decision on whether to choose 1 or 0 at time step $t$, is based on the decisions of her $K$ connected agents at time step $t-1$. Thus the degree distribution in Kauffman network is very simple: concentrated at $K$.

Growing directed network (GDNet): A growing directed network is generated according to the description given in Ref. [20]. It starts with an initial cluster of $K+1$ agents, which are mutually connected (two directed links between each pair of agents). At each stage, a new agent is added in and connected to $K$ other agents among those agents already present in the network. The link is directed from the new agent to the existing ones, meaning that the strategies of the new (younger) agent are based on the states of the existing (older) ones. We assume that the probability of connecting a new agent to an existing one with degree $k_i$ is proportional to $k_i^\alpha+1$, where $k_i$ is the number of incoming links to the existing agent. The constant 1 is added to give a nonzero starting weight to the agents that have not been connected to.

For $\alpha = 0$, we have a growing directed random network which we refer to as GDNet-0. For $\alpha > 0$, we have preferential attachment. The special case of $\alpha = 1$ corresponds to a scale-free directed network which we refer to as GDNet-1. In this network, the out-degree is $K$ for all the agents, but the in-degree follows a power law distribution. The undirected version of this model corresponds to the Barabási-Albert model. In the growing networks (random or scale-free), the younger agents are influenced by the older ones, except for the initial $K+1$ agents who are mutually influenced.

Growing directed network with a fraction of link reversals (GDRNet) In order to make our discussion more relevant to many real world networks which typically have a fraction of feedback links, we here modify the above growing network to allow a small fraction of link reversals. Let $p$ be the probability that each agent has a link reversal: when each new agent is connected to other $K$ agents already present in the network, each link has a probability of $q = p/K$ to have its direction reversed. We consider two GDRNet: GDNet-0 with $\alpha = 0$ and GDNet-1 with $\alpha = 1$.

There are two new features for GDRNet that are worth mentioning: 1) Some agents may have more than $K$ strategy inputs, while others may have fewer than $K$ inputs; but the mean number of inputs for an agent remains as $K$. 2) Some younger (added in the network later) agents can influence the older (entered in the network earlier) agents due to link-reversal.

Next we present the numerical results for the Boolean dynamics of these three types networks using a minority game model.

III. NUMERICAL RESULTS

For many highly complex systems in the real world, it’s very often extremely difficult (if possible at all) to derive an analytical solution for their dynamics. As a result, numerical simulation becomes one of the most effective methods. A complex network in which agents are interconnected and their actions are interdependent constitutes one of such complex systems, for which numerical simulation may be taken as the first choice for studying their dynamics. In this section, we apply numerical approach to investigate the evolutionary dynamics of interactive agents connected in the three complex networks we described in the previous section. We use minority game model to study the impact of the organizations of the networks on their adaptive dynamics and apply the evolution scheme (described in the previous section) to investigate how evolution affects the dynamics and the associated phase structure of the systems.

A. Adaptive and evolutionary dynamics of Kauffman net

First, Let us use minority game (MG) model to explore the adaptive dynamics of Kauffman random NK network (Kauffman net). The result is shown in Fig. 1. We see clearly from the figure that when $K = 2$, the normalized variance $\sigma^2/N$ has very large fluctuations (four orders of magnitude for $N = 401$ and five orders of magnitude for $N = 901$). This suggests that the system dynamics is at a critical state for $K = 2$. For $K \geq 3$ the system is at "chaotic" regime and performs like a random choice game. The observation we obtained here is consistent with the well-known result for the Boolean dynamics on Kauffman net: when $K = 2$ the system is at the “edge of chaos” and for $K \geq 3$ the system is chaotic [13, 21].

The dynamics in the original MG depends on two variables (parameters): $N$, the system size and $M$, the memory size of the agents. There are two different phases for
FIG. 1: The normalized variances (of the number of agents who choose 1), $\sigma^2/N$, their mean and standard deviation (S.D.), for the network MG as a function of $K$ on Kauffman NK random network. The mean and the S.D. are computed from 100 simulations. The dash-line in the middle is for $\sigma^2/N = 0.25$ (RCG), and the dash-dot line at bottom is for $\sigma^2/N = 0.25/N$ (the theoretical lower limit). The system parameters are: $S = 2$, $N = 401$ (left plot) and 901 (right plot).

FIG. 2: The normalized variances, $\sigma^2/N$, their mean and standard deviation (S.D.) of the network EMG on the Kauffman net as a function of $K$. The mean and the S.D. are computed from 100 simulations. The dash-line in the middle is for $\sigma^2/N = 0.25$ (RCG), and the dash-dot line at bottom is for $\sigma^2/N = 0.25/N$. The system parameters are: $S = 2$, $N = 401$ (left plot) and 901 (right plot), and $W_c = 1024$.

different memory value $M$, described by a Savit curve [8]. The critical value for an optimal global coordination is $M_c \sim \ln(N)$, which depends on $N$. For the network MG on the Kauffman net, however, the two phases (except the trivial case of $K = 1$) of the dynamics are: critical for $K = 2$, and chaotic for $K \geq 3$. So the dynamics of MG for Kauffman network depends on the connectivity $K$ only, and the chaotic regime dominates.

Now we study how the dynamics of the Kauffman net will be affected if a simple evolution scheme is applied in the game. Here we use the evolution scheme described in the previous section to examine the impact of evolution on the dynamics of Kauffman net. The results are shown in Fig. 2. from which, we have some observations as follows:

- The evolution shifts the critical point from $K_c = 2$ (for a non-evolutionary game) to $K_c = 3$;
- The evolution helps dramatically improve the system performance for the game with small connectivity ($K \leq 3$), implying that the systems with small connectivity numbers ($K \leq 3$) are at either stable or “critical” state as evolution is able to “bring out” some order for a complex system when it is in a stable or critical regime.
- It has no effect on the system performance for the game with large connectivity number ($K \geq 4$), suggesting that the network with large connectivity number is too chaotic to allow evolution to have any impact on its chaotic dynamics.

Now a natural question arises: what is the underlying evolutionary mechanism that shifts the critical value of connectivity number $K_c$ in Kauffman network? To find out an answer will surely help us obtain a better understanding of the evolutionary dynamics of Kauffman network, therefore we will leave it for our future study. Next we discuss the other two types of networks.

B. Dynamics of Growing Directed Networks

Here we investigate dynamical property of growing directed networks for which two limiting cases are examined: 1) GDNet-0, the growing random directed network ($\alpha = 0$); 2) GDNet-1, the scale-free directed network (growing directed network with preferential attachment, $\alpha = 1$). First, Let us look at the adaptive dynamics of these two growing directed networks. Shown in Fig. 3 are the results of the MG on these growing directed networks.

From these plots, we have the following observations:

- There are very large fluctuations in the values of the normalized variance $\sigma^2/N$ for all connectivity numbers (except the trivial case of $K = 1$), and they are independent of the connectivity number $K$, radically different from that of the Kauffman network;
- There seems to be no significant difference between the MG dynamics from the two growing network models, the random growing network (GDNet-0), and the scale-free network(GDNet-1).

These observations suggest that for a simple adaptive (non-evolutionary) dynamical process, all growing directed networks (irrespective of its value of preferential attachment exponent $\alpha \in [0, 1]$) have similar dynamics and the scale-free network is not special. This is due to the construction process of growing networks, which leads to a maximum state cycle length of $2^{K+1}$, irrespective of
the value of \( \alpha \) as was pointed out in Ref. [20]. In contrast, the Kauffman network is not generated through a “growing” process and therefore it lacks an ordered structure; each agent appears in the network at the same time and is randomly connected with other \( K \) agents. Due to the absence of a state cycle, the Kauffman network is in chaotic state for most connectivity numbers \( (K > 2) \).

Now we study the impact of evolution on their dynamics for these two growing directed networks, GDNet-0 and GDNet-1. To see how evolution affects the dynamics, we plot, in Fig. 4, the results of system performance for the EMG on the GDNet. By comparing Fig. 3 and Fig. 4, we observe that:

- Evolution helps reduce the normalized variance \( \sigma^2/N \) so dramatically (by more than two orders of magnitude) so that the mean of \( \sigma^2/N \) is far below the value of an RCG.

- The normalized variance \( \sigma^2/N \) still exhibits substantial fluctuation, implying that the dynamics is still in a critical regime although the variance is substantially reduced.

- The EMG on GDNet-0 and on GDNet-1 produce similar results, suggesting that the growing directed network is insensitive to its preferential attachment parameters and scale-free network \( (\alpha = 1) \) is no special;

- By comparing the results for growing networks (shown in Figs. 3-4) with the results for the Kauffman net (shown in Figs. 1-2), we see that these two types of networks have substantial different impacts on the dynamics. The dynamics of Kauffman net depends on connectivity and exhibits a clear phase transition. However, in growing directed network the dynamics is insensitive to their connectivity number and, as a result, there is no phase transition.

C. Dynamics of Growing Directed Networks with Link-Reversals

Now we consider the growing directed network with link reversal.

To see how some “mistakes” in a growing directed network affects the dynamics, we now examine the performance of the MG and the EMG on the growing networks with link reversals, GDRNet-0 \((\alpha = 0)\) and GDRNet-1 \((\alpha = 1)\). The results for the MG and the EMG are
phase structure. The key results we obtained here can be summarized as follows:

- **GDNet-0** and **GDRNet-0** produce similar results (see the top panel of Fig. 3 and Fig. 5): The systems are in stable or “critical” state, irrespective of their connectivity, meaning that a small fraction of link-reversal has no significant impact on the dynamics for growing random directed network;

- A small fraction of “mistakes” in link-reversal causes the dynamics of a scale-free network dependent on its connectivity: the larger the connectivity, the more pronounced the impact of the link-reversal. This may be explained by the fact that the larger the connectivity, the more link-reversals each agent may have.

   In this section we have investigated the impact of different networks on the dynamics and the associated phase structure. The key results we obtained here can be summarized as follows:

   - The dynamics and the associated phase structure critically depend on how the underlying complex network is organized and how the networked agents interact and evolve.

   - For a non-growing random directed network, the dynamics and the associated phase transition are determined by its connectivity variable: the system stays in stable or critical regime for the low connectivity ($K < 3$ for adaptive dynamics, $K < 4$ for evolutionary dynamics); but it remains at chaotic for large connectivity numbers.

   - For a growing directed network, the dynamical property for both adaptive and evolutionary dynamics is insensitive to its connectivity and the systems remain in stable or “critical” regime for all their connectivity.

   - When there exist a small fraction of “mistakes” in link-direction reversal in growing directed network, the dynamics of random network remains unaffected, but the evolutionary dynamics of the scale-free network becomes dependent on its connectivity: the larger the connectivity, the closer the dynamics to chaotic state.

   - Evolution is able to enhance the performance dramatically when the system is in stable or critical regime, but fails to affect a chaotic dynamics; this contribute may be empowered to classify complex...
networks according to their adaptive and evolutionary dynamics.

Now a challenging issue is how to understand the underlying mechanism for the emergence of these key findings. We try to explore some explanations for it next.

IV. SOME ANALYSIS AND DISCUSSION

In this section, we try to give some analysis and explanation for the underlying mechanism on how different network organizations affect the dynamics and why evolution is able to enhance the system performance of adaptive and competing agents. As stated in the introduction, the dynamics of the network MG depends on how the network is organized, how the agents interact through the network, and how evolution changes the under-performers’ strategies. Due to its complexity of interdependency of the agents’ strategies, it’s extremely difficult (if possible at all) to deduct a simple cause-effect relationship for the dynamics. As pointed out in [24], in many cases such complexity precludes any deductive attempt and leaves inductive reasoning as the only choice. Therefore, here our analysis and discussion will be qualitative rather than quantitative.

A. Key Factors of Dynamics

It’s well known that herding behavior causes market inefficiency and anti-crowd improves the market efficiency. Thus a key variable in analyzing the dynamics of competing game is the “crowdedness” which can be represented by the imbalance of the two groups of the agents who take opposite actions. The crowdedness can be influenced by both how the network is organized and how the system evolves; consequently it’s natural to investigate the issue via these two perspectives.

The impact of network organization on the crowd effect can be understood by examining how the interdependence of the strategies of networked agents forms crowds. Although degree distribution is regarded as one of the most important characterizations of a complex network, one of our early works suggests that a descendant cluster of agents has more direct impact on the system dynamics [20]. This is so because a crowd is formed through an influence network of strategy interdependence of a descendant cluster of agents. The size of a descendant cluster controls its dynamical state-cycle of the connected agents in the cluster, so that the distribution of the sizes of the descendant clusters in a network determine the state-cycle of the system. Thus here we use such descendant clusters as an alternative approach to analyze the impact of network organization on the dynamics.

Evolution is another important mechanism for influencing the crowd behavior. In an evolutionary minority game (EMG) where agents are not networked and their strategies are based upon the public market information, a crowd-anticrowd theory [12, 23] gives a solid explanation on the underlying mechanism. Although the dynamics of the strategies in the network MG is radically different from that of the non-networked MG, in the end, the performance (efficiency) of each game is determined by the balance of the two numbers of the agents who have chosen 1 or 0. Thus the crow-anticrowd theory can be employed to analyze herding behavior and system efficiency of EMG of networked agents as well.

B. Descendant Cluster Theory

A descendant cluster of an agent is defined as the set of all agents whose strategies are linked and influenced by this agent through single or multiple steps tracing along a path of directed links [20]. Now we use descendant cluster theory to explain some of the key findings summarized in the previous section. Here we use the descendant clusters theory to analyze the impact of complex network organizations on their dynamics and the phase structure and apply crow-anticrowd theory to discuss the system efficiency in the next subsection separately.

Dynamics of Non-growing random directed network

For the non-growing random directed network, a key observation for the dynamics and phase structure is as follows:

- The dynamics and the associated phase transition are determined by its connectivity variable: the system stays in stable or critical regime for the low connectivity ($K < 3$ for adaptive dynamics, $K < 4$ for evolutionary dynamics); but it remains chaotic for large connectivity numbers.

For a Boolean dynamics in this network, some detailed analysis is available and the dynamics and the phase structure is well understood [13, 14].

Here we try to use descendant clusters theory to give an alternative explanation for the dynamics we observed. Based on the theory, the key to understand the statistical property of a dynamics is to find out the distribution of the sizes of the descendant clusters in the network. In non-growing random directed network (e.g., the Kauffman net), each agent enters into the network simultaneously and is randomly connected to other $K$ agents upon whom his strategies depend. Any given agent has the same potential to influence $K$ other agents. In such a network, the chance of embracing large clusters is an increasing function of the connectivity: the larger the connectivity, the more clusters of big sizes. Consequently, many large clusters may exist in the network when the connectivity is large (enough); thus a chaotic dynamics emerges as a consequence. However, when the connectivity number is small, large clusters are absent in the
network, the state-cycle from each agent remains small as a result. Consequently, the dynamics of the system stays in a stable or critical regime, and it is impossible to admit. Fig.7 plots the distribution of the descendant cluster sizes for Kauffman net.

We see from the plot that the larger the connectivity $K$, the more clusters of big sizes. The distribution of the descendant cluster sizes displayed in the figure supports the above explanation [26].

**Dynamics of growing directed network**

The key feature of the dynamics of the growing network we observed is:

- The dynamical property for both adaptive and evolutionary dynamics is insensitive to its connectivity and the systems remain in stable or “critical” regime for all their connectivity;

This can be explained by growing process of the network, which leads to a power law distribution for the sizes of the descendant clusters, irrespective of the value of preferential attachment $\alpha$ [20]. Showing in Fig.8 are the distributions of the degree and the sizes of the descendant clusters for growing network.

From this figure, we can see that descendant cluster size distribution is in power law irrespective of its attachment parameter value ($\alpha \in [0, 1]$).

We have shown, in our early work, that such power-law distribution of descendant cluster size can be analytically derived and its exponent is obtained to be equal to $-\frac{1+K}{K}$ [20]. Fig.9 shows the cluster size distribution for different connectivity number $K$. Such scale-free distribution of cluster sizes is responsible for a dynamics to be in a stable or critical regime, disregarding their connectivity.

**Dynamics of growing directed network with link-reversal**

For the growing directed network with link-reversal, a key observation is as follows.

- The adaptive dynamics remains unaffected by the link-reversal, but the evolutionary dynamics of the scale-free network becomes dependent on its connectivity: the larger the connectivity, the closer the dynamics to chaotic state.

The impact of link-reversal on the sizes of “descendant clusters” may depends on the connectivity number. On the one hand, it may split large “descendant clusters” into smaller clusters; on the other hand, it may link up small clusters to form larger clusters. According to the generating process of the network, the chance for each agent to have link-reversals increases with the increase of its connectivity. For a very large connectivity, this may lead to form a few very large clusters (resulting in a few very large state-cycles), driving the system into a chaotic state.

Next we use crow-anticrowd theory to analyze the impact of evolution on system performance.

**C. Crowd-Anticrowd Theory**

The performance of an MG in each simulation is determined by the number of agents whose strategies are anti-crowdedly paired among the $N$ agents in the game; when $N - 1$ agents out of the $N$ agents in the game are anti-crowdedly paired at each time step. Thus the key idea in the crowd-anticrowd theory [23] is to find out the number of the anti-crowdedly paired agents. Such a task has been done in the context of EMG where the agents’ strategies are dependent on the historical record.
of cluster sizes is described by a power-law [20]. As a
reducing the crowdedness (due to herding behavior). For
system performance for all types networks is similar:
underlying mechanism why evolution helps enhance
anti-crowdedly paired agents. This in turn leads
to the reduction of the number of the most unpaired agents
(due to herding behavior). For the growing random directed network, the distribution
of cluster sizes is described by a power-law [20]. As a
result, the dynamical property for both adaptive and
evolutionary dynamics remain in stable or "critical"
regime for all their connectivity. Evolution does not
change the characteristics of the dynamics and the phase
structure as it cannot change the cluster size distributional-
property of the underlying network. However, it
does help enhance the system performance significantly
as a result of its facility to reduce the crowdedness of
herding agents in their strategies.

Growing random directed network with link-
reversal
The key to understand the dynamics of scale-free di-
rected network with link-reversal is to examine how such
link-reversal change the descendant clusters distribution
in the network. However for an adaptive process, the
dynamics accommodate a lot of "noises" (short-term fluctuations) so that the essential dynamical property of
the underlying networks may be concealed. Evolution
plays an important role here: It can effectively reduce
such noises so that it not only helps to enhance system
performance but also help to reveal the underlying
principal property of the dynamics of the network.
Evolution helps to make it clear of the dependence of the
dynamics and the associated structure on their underly-
ing network interdependence property characterized by
the distribution of the sizes of descendant clusters in the
network.

In this section, we have explored some explanation
for the key observations for the dynamics, the phase
structure, and the system performance. The analysis
here is qualitative, focused on the underlying causes,
instead of the exact mechanism for the emergence of
these important observations. This is partly due to the
fact that the dynamical processes here we investigated
are quite complicated (to be consistent with the real-
world systems), involving both adaptive and evolution
processes; a simple analytical solution does not exist.
Therefore, our results and analysis should be regarded
as a first-step effort in understanding dynamics of com-
plex networks.

V. SUMMARY
We have presented an intensive numerical study on
dynamics of various directed complex networks. We
compared the dynamics for three types of complex
networks and we have demonstrated that the dynamics
and the associated phase structure critically depend
on the organizations of the underlying networks. The
underlying mechanism for such dependency is originated
from the distributional property of the descendant
clusters of the networked agents and such property can
be analyzed with a descendant clusters theory. For the
dynamics of interconnected agents, a key gauge for a

Non-growing random directed network
The complexity of non-growing random directed network
(Kauffman net) is a function of its connectivity number
$K$: the larger the connectivity, the more complex the
network. The numerical results show that evolution can
only have its effect on those low complex random net-
works, but not on the high complex ones. This implies
that the effectivity of evolution lies in its capability
to find "order" out of "randomness". If there exists no
order at all as of the case of the Kauffman network with
large connectivity, evolution cannot have any impact on
the dynamics at all. On the other hand, if the network
complexity is not so high as of the case of Kauffman
network with low connectivity, the evolution may be
able to bring out some order out of the randomness. The
direct impact of evolution on crowdedness of agents is
the reduction of the number of the most unpaired agents
(under-performer agents) and the increase of the number
of the anti-crowdedly paired agents. This in turn leads
to a performance enhancement of the system. To find
out an exact solution for the evolutionary dynamics is a
challenging and significant task and thus we reserve it
as our future efforts.

Growing random directed network
The underlying mechanism why evolution helps enhance
system performance for all types networks is similar:
reducing the crowdedness (due to herding behavior). For
the growing random directed network, the distribution
of cluster sizes is described by a power-law [20]. As a

![FIG. 9: The descendant cluster size distributions for $\alpha=0$, $K=1,2,3,5, 8$, and $N=200000$. The slope of the lines drawn are given by $-\frac{1+K}{K}$.]()}
complex network is its distributional property of the sizes of the descendant clusters: a scale-free distribution leads to critical dynamics, a distribution concentrated at small cluster sizes produces a stable dynamics and a distribution skewed towards to large cluster sizes may result in a chaotic dynamics.

The system performance (efficiency) is found mainly due to the impact of evolution in reducing the crowdedness of the strategies of the interconnected and interdependent agents in the network. This is explained through a crowd-anticrowd theory and the underlying mechanism responsible for such influence of evolution on system dynamics can be understood through “Darwinian” selection powered in the evolution process.

The dynamics we studied here is built upon extremely complex network systems in which several dynamical processes entangle: 1) agents are adaptive as they build multiple strategies so as to optimize their own performance adaptively; 2) agents’ strategies are interdependent and large influential networks may emerge; 3) evolution process keeps changing on the strategies of those underperformance agents. As a result, it’s extremely difficult (if possible at all) to develop an analytical solution; therefore we hope our numerical results and the related analysis presented in this paper provide an important starting point in understanding the dynamics, the associated phase structure and the system efficiency of various complex networks.

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[25] Since the impact of evolution on the dynamics may depend on the mechanics specified, a careful justification is necessary to reach a pertinent result.
[26] Note that the descendant cluster sizes impose an upper limit for the dynamical state cycle. There are chances that agents’ strategies dependence functions piece some large clusters.