Evolutionary games on complex network

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(Dated: February 22, 2007)

Cooperation is ubiquitous in the real world, ranging from biological systems to economic and social systems. Evolutionary game theory has been considered an important approach to characterizing and understanding the emergence of cooperative behavior in systems consisting of selfish individuals. In this paper, we review some of our works about dynamics of evolutionary games over complex networks, including the effects of network structures on the emergence and persistence of cooperation, resonance type phenomena in evolutionary games, and the memory-based evolution of game dynamics.

PACS numbers: 87.23.Kg, 02.50.Le, 87.23.Ge, 89.65.-s, 89.75.Fb

I. INTRODUCTION

Game theory provides a useful framework for describing the evolution of systems consisting of selfish individuals [1–3]. The prisoner’s dilemma game (PDG) as a metaphor for investigating the evolution of cooperation has drawn considerable attention [4, 5]. In the PDG, two players simultaneously choose whether to cooperate or defect. Mutual cooperation results in payoff \(R\) for both players, whereas mutual defection leads to payoff \(P\) gained both. If one cooperates while the other defects, the defector gains the highest payoff \(T\), while the cooperator bears a cost \(S\). This thus gives a simply rank of four payoff values: \(T > R > P > S\). One can see that in the PDG, it is best to defect regardless of the co-player’s decision to gain the highest payoff \(T\). However, besides the widely observed selfish behavior, many natural species and human being show the altruism that individuals bear cost to benefit others. These observation brings difficulties in evaluating the fitness payoffs for different behavioral patterns, even challenge the rank of payoffs in the PDG. Since it is not suitable to consider the PDG as the sole model to discuss cooperative behavior, the snowdrift game (SG) has been proposed as possible alternative to the PDG, as pointed out in Ref [6]. The main difference between the PDG and the SG is in the order of \(P\) and \(S\), as \(T > R > S > P\) in the SG. This game, equivalent to the hawk-dove game, is also of much biological interest [7, 8]. However, the original PDG and SG cannot satisfyingly reproduce the widely observed cooperative behavior in nature and society. This thus motivates numerous extensions of the original model to better mimic the evolution of cooperation in the real world [9–12].

Since the spatial structure is introduced into the evolutionary games by Nowak and May [13], there has been a continuous effort on exploring effects of spatial structures on the cooperation [6, 14, 15]. It has been found that the spatial structure promotes evolution of cooperation in the PDG [13], while in contrast often inhibits cooperative behavior in the SG [6]. In recent years, extensive studies indicate that many real networks are far different from regular lattices, instead, show small-world and scale-free topological properties. Hence, it is naturally to consider evolutionary games on networks with these kinds of properties [16–21]. An interesting result found by Santos and Pacheco is that “Scale-free networks provide a unifying framework for the emergence of cooperation” [22]. In this paper, we review some of our works in the field of evolutionary games. By means of some simple models, we have studied how an important topological structural feature, the average degree, affect the cooperative behavior [23]. We found there exists the highest cooperation level induced by an optimal value of average degree for different types of networks. Besides, we investigate the randomness effect on the cooperative behavior by introducing both topological and dynamical randomness [24]. We found a resonance type phenomena reflected by the existence of highest level of cooperation in the case of appropriate randomness. Moreover, we propose a memory-based snowdrift game over complex networks [25]. Some very interesting behavior are observed, such as the nonmonotous behavior of frequency of cooperation as a function of payoff parameter, spatial pattern transition and so on.

II. THE PRISONER’S DILEMMA GAME AND SNOWDRIFT GAME

In The individuals can follow only two simple strategies: \(C\) (cooperate) or \(D\) (defect), described by

\[ s = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]  \hspace{1cm} (1)

respectively. Each individual plays the PDG with its “neighbors” defined by their spatial relationships. The total income of the player at the site \(x\) can be expressed as

\[ M_x = \sum_{y \in \Omega_x} s_y^T \cdot P \cdot s_y \]  \hspace{1cm} (2)

where \(s_x\) and \(s_y\) denote the strategy of node \(x\) and \(y\). The sum runs over all the neighboring sites of \(x\) (this set is indicated by \(\Omega_x\)).

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For the PDG and SG, the payoff matrix has rescaled forms respectively, as follows

\[ P_{PDG} = \begin{pmatrix} 1 & 0 \\ b & 0 \end{pmatrix} \quad P_{SG} = \begin{pmatrix} 1 & 1 - r \\ 1 + r & 0 \end{pmatrix} \] (3)

where \( 1 < b < 2 \) and \( 0 < r < 1 \).

### III. EFFECTS OF AVERAGE DEGREE ON COOPERATION

In this section, we focus on the influence of the average degree of complex networks on the evolution of the networked PDG. Firstly, we construct networks using three typical models - the BA, NW and ER Models. The details of these network models can be seen in Refs. [26], [27] and [28], respectively.

Each site of the network is occupied with an individual. An individual can be either a cooperator or a defector. All pairs of connected individuals play the game simultaneously and gain benefits according to the payoff parameters mentioned in the last section. During the evolutionary process, each player is allowed to learn from one of its neighbors and update its strategy at each round. The probability of a node \( i \) selecting one of its neighbors \( j \) is \( \Pi_{i,j} = k_j / \sum_{k_i} k_i \), where the sum runs over the set of neighbor nodes of \( i \). The assumption of \( \Pi \) takes into account the fact that individuals with more interactions usually cause more attraction in society. In other words, well-known persons will have more influences than the others. Whereafter, the node \( i \) will adopt the selected neighbor’s strategy with a probability determined by the normalized payoff difference between them [29], i.e.,

\[ W = \frac{1}{1 + \exp[(E_i/k_i - E_j/k_j)\beta]}, \] (4)

where \( k_i \) and \( k_j \) respectively represent the degrees of node \( i \) and \( j \), \( E_i \) and \( E_j \) respectively represent the total payoff of node \( i \) and \( j \), and \( \beta \) characterizes the noise introduced to permit irrational choices. Here, \( \beta \) is set to 50. According to the evolutionism, \( W \) reflects the rule of natural selection based on relative fitness.

One of the key quantities for characterizing the cooperative behavior is the density of cooperators \( \rho_c \), which is defined as the fraction of cooperators in the whole population. We study \( \rho_c \) as a function of the average degree \( \langle k \rangle \) for three types of networks. As shown in Fig. 1, one can find that \( \rho_c \) exhibits a nonmonotonous behavior with a peak at some specific values of \( \langle k \rangle \) for the BA network. Simulation results for the NW and ER networks can be seen in Ref. [23]. The common nontrivial behavior shared by the scale-free, small-world networks indicate that the average degree is a crucial feature for the networked evolutionary game. Such nonmonotonous behavior has been well explain in Ref. by means of considering the situation in two limits of \( \langle k \rangle \).

We further concern the relationship between the average degree \( \langle k \rangle \) and the average payoff \( \langle M \rangle \) in the whole population for different types of networks (BA, NW, ER Networks). The average payoff is defined as

\[ \langle M \rangle = \frac{1}{N} \sum_{i=1}^{N} E_i, \] (5)

where \( E_i \) is the total income of individual \( i \). Also we found similar non-monotonous phenomena exhibited in the dependence of \( \rho_c \) on \( \langle k \rangle \). Simulation results of \( \langle M \rangle \) versus \( \langle k \rangle \) for different values of \( b \) with adopting the BA network are reported in Fig. 3. In the following, we provide theoretical predictions for the average payoff of individuals \( \langle M \rangle \) by assuming cooperators are distributed uniformly among the network. \( \langle M \rangle \) can be expressed as

\[ \langle M \rangle = \langle k \rangle \times \rho_c \times ((1 - \rho_c) \times b + \rho_c). \] (6)

Since the density of cooperators \( \rho_c \) cannot be reproduced by the mean-field approach, except for the well-mixed
cases (fully connected networks), $\rho_c$ used in Eq. (6) for calculating $\langle M \rangle$ is obtained by simulations, as shown in Fig. 1. As shown in Fig. 2. In the case of BA networks, analytical results are in very good agreement with numerical ones.

we have clarified the effect of average connectivity on the cooperate behavior. Interestingly, the average connectivity plays a non-trivial role in the cooperation, i.e., there exists the optimum average connectivity resulting in the highest cooperation level.

IV. RESONANCE TYPE PHENOMENON IN EVOLUTIONARY GAMES

It is well known that intrinsically noisy and disordered processes can generate surprising phenomena, such as stochastic resonance [30], coherence resonance [31], ordering spatiotemporal chaos by disorder [32], disorder-enhanced synchronization [33], ordering chaos by randomness [34] etc. In this section, we investigate the effect of both the topological randomness in individual relationships and the dynamical randomness in decision makings on the dynamics of evolutionary games. Our work is inspired Perc, who recently studied PDG by introducing the random disorder [35].

In order to explore the topological randomness, we consider a homogeneous small-world network (HAWN) [36]. Starting from a undirected regular graph with fixed connectivity $z$ and size $N$, two-step circular procedure is introduced: (i) choose two different edges randomly, which have not been used yet in step (ii) and (ii) swap the ends of the two edges. Here, duplicate connections and disconnected graphs are avoided. The annealed randomness is characterized by the parameter $p$, which denotes the fraction of swapped edges in the network. (Details can be seen in Ref. [24]). In contrast to the Watts-Strogatz (WS) model [37], this network has small-world effect together with keeping the degree of each individual unchanged, so that the pure topological randomness can be investigated by avoiding any associated heterogeneity of degree distribution [36]. At each time step, each play selects a neighbor at random, and updates its strategy according to Eq. (4).

Figure 3(a) shows the frequencies of cooperators $\rho_c$ on the HAWN as a function of $b$ for different values of the topological randomness $p$ with $T = 0.08$. In the equilibrium state, $\rho_c$ is independent of the initial state and decreases monotonically as $b$ increases. One can find that when $b < 1.04$, cooperators dominate defectors significantly on the regular ring graph ($p = 0$) and the more randomness of the network, the worse the cooperation is. While for $b > 1.04$, the cooperators remain nearly extinct in the case of $p = 0$ and $p = 1$, which correspond to the complete regular network and the complete random network, respectively. However, the cooperators can survive around $p = 0.2$, i.e., intermediate topological randomness, the dependence of $\rho_c$ on the topological randomness $p$ is presented in Fig. 2 (b). It illustrates that there is a clear maximum $\rho_c$ around $p = 0.2$, where cooperation can be revitalized and maintained for substantially large values of $b$. This phenomenon reveals that there exists somewhat resonant behaviors reflected by the optimal cooperation level at intermediate topological randomness, similar to the effects of noise and disorder in nonlinear systems.

To quantify the ability of topological randomness $p$ to facilitate and maintain cooperation for various $b$ more precisely, we study $\rho_c$ depending on $b$ and $p$ together, as shown in Fig.4(a). One can find that when $b < 1.04$, $\rho_c$ is a monotonically decreasing function of $p$. We call it the harmful region (denoted by II in Fig. 4(a)) because the topological randomness $p$ always decreases $\rho_c$. While for $b > 1.04$ (the region is denoted by I in Fig. 4(a)), there exists an optimal level of
Among the previous work, the effects of individuals’ memory have not received much attention in the study of evolutionary games on networks. We argue that individuals usually make decisions based on the knowledge of past records in nature and society, and the historical memory would play a key role in an evolutionary game. Therefore, in the present work, we propose a memory-based snowdrift game (MBSG), in which players update their strategies based on their past experience. Our work is partially inspired by Challet and Zhang [38], who presented a so-called minority game, in which agents make decisions exclusively according to the common information stored in their memories.

The rules of the MBSG are described as follows. Consider that \( N \) players are placed on the nodes of a certain network. In every round, all pairs of connected players play the game simultaneously. The total payoff of each player is the sum over all its encounters. After a round is over, each player will have the strategy information (C or D) of its neighbors. Subsequently, each player knows its best strategy in that round by means of self-questioning, i.e., each player adopts its anti-strategy to play a virtual game with all its neighbors, and calculates the virtual total payoff. Comparing the virtual payoff with the actual payoff, each player can get its optimal strategy corresponding to the highest payoff and then record it into its memory. Taking into account the bounded rationality of players, we assume that players are quite limited in their analyzing power and can only retain the last \( M \) bits of the past strategy information. At the start of the next generation, the probability of making a decision (choosing C or D) for each player depends on the ratio of the numbers of C and D stored in its memory, i.e., \( P_C = \frac{N_C}{N_C + N_D} = \frac{N_C}{M} \) and \( P_D = 1 - P_C \), where \( N_C \) and \( N_D \) are the numbers of C and D, respectively. Then, all players update their memories simultaneously. Repeat the above process and the system evolves.

V. MEMORY-BASED SNOWDRIFT GAME

A. MBSG on lattices

First, we investigate the MBSG on two-dimensional square lattices of four and eight neighbors with periodic boundary conditions, as shown in Fig. 5 (a) and (b). In these two figures, four common features should be noted: (i) \( f_C \) has a step structure, and the number of steps corresponds to the number of neighbors on the lattice, i.e., 4 steps for the 4-neighbor lattice and 8 steps for the 8-neighbor lattice; (ii) the two figures have 180°-rotational symmetry about the point (0.5, 0.5); (iii) the memory length \( M \) has no influences on the dividing point \( r_c \) between any two cooperation levels, but has strong effects on the value of \( f_C \) in each level; (iv) for a large payoff parameter \( r \), the system still behaves in a high cooperation level, contrary to the results reported in [6]. It indicates that although selfish individuals make decisions based on the best choices stored in their memories to maximize their own benefits, the cooperative behavior can emerge in the population in spite of the highest payoff of D.

The effects of memory length \( M \) on \( f_C \) in the 4-neighbor...
lattice are shown in the insets of Fig. 6. One can find that \( f_C \) is a monotonous function of \( M \) for both levels and the decreasing velocity of \( f_C \) in the 1st level is faster than that in the 2nd one. In contrast, in the 8-neighbor lattice, \( f_C \) exhibits some non-monotonous behaviors as \( M \) increases. As shown in the bottom inset of Fig. 6(b), \( f_C \) peaks in the 1st level corresponding to \( M = 23 \), and \( f_C \) is an increasing function of \( M \) in the 2nd level. A maximum value of \( f_C \) exists in the 3rd and 4th levels when \( M \) is chosen to be 5, as shown in the top inset of Fig. 1(b). Thus, memory length \( M \) plays a very complex role in \( f_C \) reflected by the remarkably different behaviors in 4 cooperation levels. A typical example with \( M = 1 \) for two types of lattices is shown in the bottom inset of Fig. 1(a). A big oscillation of \( f_C \) is observed. The dividing points of different levels can be obtained via local stability analysis. The details can be seen in Ref. [25].

Next, we reports the spatial pattern for each cooperation level for two types of lattices. As shown in Figs. 7 and 8, there are different patterns for different cooperation levels. The pattern formation can be explained in terms of steady local patterns. In Fig. 7(c), we show the steady local patterns existing in the 1st cooperation level. From the payoff ratio by choosing C and D of individual A, i.e., \( W_C : W_D \), the 3rd local pattern is the most stable one with the highest payoff ratio. In parallel, the 4th local pattern is the counterpart of the 3rd one, so that it is also very stable. Hence, the pattern in Fig. 7(a) has a chessboard-like background together with C lines composed of the 1st and 2nd local patterns. Similarly, the chessboard-like background in Fig. 7(b) is also attributed to the strongest stability of the 4th and 5th local patterns, and the probability of the occurrence of other local patterns is correlated with their payoff ratios.

**B. MBSG on scale-free networks**

Going beyond two-dimensional lattices, we also investigate the MBSG on scale-free (SF) networks, since such structural property is ubiquitous in natural and social systems. Figure 5 shows the simulation results on the Barabási-Albert networks [26], which are constructed by the preferential attachment mechanism. Each data point is obtained by averaging over 30 different network realizations with 20 different initial states of each realization. Figures 9 (a1) and (a2) display \( f_C \) depending on \( r \) on BA networks in the cases of average degree \( \langle k \rangle = 4 \) and \( \langle k \rangle = 8 \) for different memory lengths \( M \). There are some common features in these two figures: (i) in sharp contrast...
to the cases on lattices, $f_c$ is a non-monotonous function of $r$ with a peak at a specific value of $r$. This interesting phenomenon indicates that properly encouraging selfish behaviors can optimally enhance the cooperation on SF networks; (ii) it is the same as the cases on lattices that the continuity of $f_c$ is broken by some sudden decreases. The number of continuous sections corresponds to the average degree $\langle k \rangle$; (iii) two figures have a $180^{\circ}$-rotational symmetry about the point $(0.5, 0.5)$; (iv) the memory length $M$ does not influence the values of $r$, at which sudden decreases occur, as well as the trend of $f_c$, but affects the values of $f_c$ in each continuous section. Then, we investigate the effect of $M$ on $f_c$ in detail. Due to the inverse symmetry of $f_c$ about point $(0.5, 0.5)$, our study focus on the range of $0 < r < 0.5$. We found that in both SF networks, there exists a unique continuous section, in which $M$ plays different roles in $f_c$. For the case of $\langle k \rangle = 4$, the special range is from $r = 0.34$ to $0.49$, as shown in Fig. 9(a1). In this region $f_c$ is a function of $M$ is displayed in Fig. 9(b1). One can find that for $r = 0.42$, $f_c$ is independent of $M$. For $0.34 < r < 0.42$, $f_c$ is a decreasing function of $M$; while for $0.42 < r < 0.49$, $f_c$ becomes an increasing function of $M$. Similar phenomena are observed in the SF network with $\langle k \rangle = 8$, as exhibited in Fig. 5(b2). $r = 0.45$ is the dividing point, and for $r < 0.45$ and $r > 0.45$, $f_c$ shows decreasing and increasing behaviors respectively as $M$ increases. In the case of $M = 1$, the system has big oscillations as shown in the inset of Fig. 5(a2). Similar to the cases on lattices, the behavior of large proportion of individuals’ strategy switches that induces the big oscillation of $f_c$ in the SF network.

In order to give an explanation for the non-monotonous behaviors reported in Figs. 9(a1) and (a2), we study the average degree $\langle k_s \rangle$ of cooperators and defectors depending on $r$. In Figs. 9(c1) and (c2), $\langle k_s \rangle$ of $C$ vs $r$ shows almost the same trend as that of $f_c$ in Figs. 9(a1) and (a2), also the same sudden decreasing points at specific values of $r$. When $r$ is augmented from 0, large-degree nodes are gradually occupied by D, reflected by the enhancement of D’s $\langle k_s \rangle$. The detailed description of the occupation of nodes with given degree can be seen in Fig. 10. One can clearly find that on the 4-neighbor lattice, in the case of low value of $f_c$ (Fig. 10(a)), all high degree nodes are occupied by cooperators and most low degree nodes are occupied by defectors; while at the peak value of $f_c$ (Fig. 10(b)), cooperators on most high degree nodes are replaced by defectors and on low degree nodes cooperators take the majority. Similarly, as $f_c$ increases in the 8-neighbor lattices, defectors gradually occupy those high degree nodes, together with most very low degree nodes taken by cooperators (Fig. 10(c) and (d)). Moreover, note that in SF networks, large-degree nodes take the minority and most

FIG. 9: (Color online). $f_c$ as a function of $r$ in BA networks with (a1) average degree $\langle k \rangle = 4$ and (a2) $\langle k \rangle = 8$ for different $M$. A time series of $f_c$ for $M = 1$ is shown in the inset of (a2). (b1) and (b2) are $f_c$ as a function of $M$ in the case of $\langle k \rangle = 4$ and $\langle k \rangle = 8$ for a special range of $r$. (c1) and (c2) are average degrees $\langle k_s \rangle$ of C and D players depending on $r$ in the case of $M = 7$ for $\langle k \rangle = 4$ and $\langle k \rangle = 8$, respectively. The network size is 10000. Each data point is obtained by averaging over 30 different network realizations with 20 different initial state of each realization. $f_c$ for each simulation is obtained by averaging from MC time step $t = 5000$ to $t = 10000$, where the system has reached a steady state.

FIG. 10: (Color online). Distributions of strategies in BA networks. Cooperators and defectors are denoted by gray bars and black bars, respectively. Each bar adds up to a total fraction of 1, by cooperators (Fig. 10(c) and (d)). Moreover, note that in SF networks, large-degree nodes take the minority and most
neighbors of small-degree nodes are those large-degree ones, so that when more and more large-degree nodes are taken by D, more and more small-degree nodes have to choose C to gain payoff $1 - r$ from each D neighbor. Thus, it is the passive decision making of small-degree nodes which take the majority in the whole populations that leads to the increase of $f_C$. However, for very large $r$, poor benefit of C results in the reduction of $f_C$. Therefore, $f_C$ peaks at a specific value of $r$ on SF networks. In addition, it is worthwhile to note that in the case of high $f_C$, the occupation of large degree nodes in the MBSG on SF networks is different from recently reported results in Ref. [39]. The authors found that all (few) high degree nodes are occupied by cooperators, whereas defectors only manage to survive on nodes of moderate degree. While in our work, defectors take over almost all high degree nodes, which induces a high level of cooperation.

We have studied the memory-based snowdrift game on networks, including lattices and scale-free networks. Transitions of spatial patterns are observed on lattices, together with the step structure of the frequency of cooperation versus the pay-off parameter. The memory length of individuals plays different roles at each cooperation level. In particular, non-monotonous behavior are found on SF networks, which can be explained by the study of the occupation of nodes with give degree. Interestingly, in contrast to previously reported results, in the memory-based snowdrift game, the fact of high degree nodes taken over by defectors leads to a high cooperation level on SF networks. Furthermore, similar to the cases on lattices, the average degrees of SF networks is still a significant structural property for determining cooperative behavior. The memory effect on cooperative behavior investigated in our work may draw some attention from scientific communities in the study of evolutionary games.

VI. CONCLUSION

We have reviewed three of our work in the field of evolutionary games on complex networks. We have studied the network effect, in particular the average degree, on the cooperative behavior. It is observed that the average degree plays a universal role on the prisoner’s dilemma game over scale-free, small-world and random networks, that is the density of cooperator peaks at some specific values of the average degree. We as well have studied both the topological and dynamical randomness effects on the cooperation level. We found a resonance type phenomenon in the prisoner’s dilemma game, which is reflected by the existence of the optimal amount of randomness, leading to the highest level of cooperation. At last, we have investigate a new kind of evolutionary games, i.e., memory-based snowdrift game, with respect to the influence of historical experience of individuals on the decision making. Interestingly, we found in the framework of this new game with adopting scale-free networks, properly encouraging selfish behaviors can optimally enhance cooperative behavior. We as well found the phase transition of spatial patterns over regular lattices. Our work may shed some new light on the study of evolutionary games over complex networks.

VII. ACKNOWLEDGEMENT

This work was supported by the National Basic Research Program of China (973 Project No.2006CB705500), by the National Natural Science Foundation of China (Grant Nos.10635040, 10532060, and 10472116), by the President Funding of Chinese Academy of Science, and by the Specialized Research Fund for the Doctoral Program of Higher Education of China.