Compatible Decompositions and Block Realizations of Finite Metrics

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Abstract

Given a metric D defined on a finite set X, we define a finite collection $\mathcal{D}$ of metrics on X to be a **compatible decomposition** of D if any two distinct metrics in $\mathcal{D}$ are linearly independent (considered as vectors in $\mathbb{R}^{X\times X}$), $D = \sum_{d\in\mathcal{D}} d$ holds, and there exist points $x, x' \in X$ for any two distinct metrics d, d' in $\mathcal{D}$ such that $d(x, y)\, d'(x', y) = 0$ holds for every $y \in X$. In this paper, we show that such decompositions are in one-to-one correspondence with (isomorphism classes of) **block realizations** of D, that is, graph realizations G of D for which G is a **block graph** and for which every vertex in G not labelled by X has degree at least 3 and is a **cut point** of G. This generalizes a fundamental result in phylogenetic combinatorics that states that a metric D defined on X can be realized by a **tree** if and only if there exists a compatible decomposition $\mathcal{D}$ of D such that all metrics $d \in \mathcal{D}$ are split metrics, and lays the foundation for a more general theory of metric decompositions that will be explored in future papers.

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1 Introduction

Given a metric $D : X^2 \to \mathbb{R} : (x, y) \mapsto xy$ defined on a finite set X, i.e., a map D from the set $X^2 := \{(x, y) : x, y \in X\}$ of all (ordered) pairs of elements from X into the real number field $\mathbb{R}$ such that $xx = 0$ and $xy \leq xz+yz$ (and, therefore, also $0 \leq xy = yx$) holds for all $x, y, z \in X$, a **graph realization** of D is a triple (G, ℓ, φ) consisting of a finite connected graph $G = (V, E)$ with vertex set $V = V_G$ and edge set $E = E_G \subseteq \binom{V}{2}$, a **length-assigning map** $\ell : E \to \mathbb{R}_{>0} : \{u, v\} \mapsto \ell(u, v)$ from the edge set E into the set $\mathbb{R}_{>0}$ of positive real numbers that assigns to every edge $e \in E$ its length $\ell(e)$ and satisfies the **triangle inequality**, that is,

$$\ell(u, v) \leq \ell(u, w) + \ell(w, v)$$

holds for all $u, v, w \in V$ with $\{u, v\}, \{u, w\}, \{w, v\} \in E$, and a **labeling map** $\varphi : X \to V$ from X into V such that $xy = D_\ell(\varphi(x), \varphi(y))$ holds for all $x, y \in X$ where D_ℓ denotes the metric induced by ℓ on V, i.e., the (necessarily unique and proper) largest metric defined on V for which $D_\ell(u, v) \leq \ell(u, v)$ holds for every edge $\{u, v\} \in E$.

While a metric can have several (non-equivalent) graph realizations (even if shortest total length $\ell(G) := \sum_{e \in E} \ell(e)$ is required, see for instance [1, 6, 13, 16]), it has been observed occasionally that graph realizations satisfying certain additional, rather specific constraints (mostly structural constraints combined with some shortest-length requirements, but not necessarily implying shortest total length) can sometimes be shown to be uniquely determined — up to canonical isomorphism — by such constraints (see, for example, [6, 7]).

In this note, we will show that there is a canonical one-to-one correspondence between (isomorphism classes of) certain graph realizations and **compatible decompositions** of D.

More specifically,

(i) we define a finite collection $\mathcal{D}$ of metrics on X to be a **compatible decomposition** of D if any two distinct metrics in $\mathcal{D}$ are linearly independent (considered as vectors in $\mathbb{R}^{X \times X}$),

$$D = \sum_{d \in \mathcal{D}} d$$

holds, and there exist points $x', x'' \in X$ for any two distinct metrics d, d' in $\mathcal{D}$ such that either $d(x', y) = 0$ or $d'(x'', y) = 0$ holds for every $y \in X$ and

(ii) we define a graph realization (G, ℓ, φ) of a finite metric D to be a **block realization** of D if the graph $G = (V, E)$ is a **block graph** (i.e., a connected graph whose 2-connected components are cliques, cf. [4, 9, 11]) and every vertex v in $V - \varphi(X)$ has degree at least 3 and is a **cut point** of G, that is, it is a vertex in V such that the graph induced by G on $V - \{v\}$ $\big($i.e., the graph $G^v := (V - \{v\}, E \cap \binom{V-\{v\}}{2})\big)$ is disconnected.

(iii) Further, given any block graph $G = (V, E)$, let $\mathcal{B}(G)$ denote the collection of all **blocks** $B \subseteq V$ of G (i.e., all those subsets B of the vertex set V that make up the vertex set of a 2-connected component of G) and, given a block realization (G, ℓ, φ) of a metric D defined on a finite set X, associate to any **block** $B \in \mathcal{B}(G)$ of G the metric $d_{(\varphi|B|\ell)}$ defined on X by

$$d_{(\varphi|B|\ell)} : X \times X \to \mathbb{R} : (x, y) \mapsto D_\ell(x_{(\varphi|B|\ell)}, y_{(\varphi|B|\ell)})$$

where $x_{(\varphi|B|\ell)}$ denotes, for any $x \in X$, the (necessarily unique!) point in B that minimizes the distance (relative to D_ℓ) to $\varphi(x)$.

Referring to these concepts, the following result will be established in this note:

Theorem 1 *Associating, to any block realization* (G, ℓ, φ) *of a metric* D *defined on a finite set* X*, the collection*

$$\mathcal{D}(\varphi|G|\ell) := \{d_{(\varphi|B|\ell)} : B \in \mathcal{B}(G)\}$$

sets up a canonical one-to-one correspondence between (*isomorphism classes of*) *block realizations and compatible decompositions of* D*.*

In [8], this result will be used to establish that **shortest block realizations** of a finite metric are (essentially) unique.

Remarkably, a very well-known result regarding phylogenetic trees and compatible split systems (cf. [5, 15]) follows immediately from Theorem 1: Note first that a metric D as above can be realized by a tree, i.e., a finite tree $T = (V, E)$ with vertex set V and edge set $E \subseteq \binom{V}{2}$ together with a length-assigning map $\ell : E \to \mathbb{R}_{>0}$ and a labeling map $\varphi : X \to V$ from X into V such that $xy = D_\ell(\varphi(x), \varphi(y))$ holds for all $x, y \in X$ if and only if it has a block realization (G, ℓ, φ) where the underlying graph G is a tree, i.e., it is a block graph such that all blocks of G have cardinality 2. Thus, our results above imply one of the most fundamental results in phylogenetic combinatorics:

Theorem 2 *A metric* D *defined on a finite set* X *can be realized by a tree if and only if there exists a compatible decomposition* $\mathcal{D}$ *of* D *such that all metrics* $d \in \mathcal{D}$ *are split metrics*[1]*.*

This result inspired much further research (cf. [2, 3, 6, 7, 9]) and led, in particular, to thorough investigations of the so-called **tight-span construction** first proposed by John Isbell (cf. [14], see also [6]) and the relationships of this construction to various sorts of decompositions $\mathcal{D}$ of a finite metric D.

[1]i.e., there exist points $x, x' \in X$ such that $d(x, y)\, d(x', y) = 0$ holds for every $y \in X$ in which case this holds for all $x, x' \in X$ with $d(x, x') \neq 0$.

The present paper continues this line of research. By establishing a canonical one-to-one correspondence between block realizations and compatible decompositions $\mathcal{D}$ of a finite metric D, it lays the basis for establishing in future papers that

(i) there is also a one-to-one correspondence between

(a) the subsets of the (necessarily finite) subset $T_0(D)$ of the tight span

$$T(D) = \{f \in \mathbb{R}^X : f(x) = \sup(f(y) - xy : y \in X)\}$$

of D consisting of all those points $f \in T(D)$ that

- are not of the form $f = h_z : X \to \mathbb{R} : x \mapsto xz$ for some $z \in X$,
- do not have a neighbourhood that is homeomorphic to an open interval,
- and for which the space $T(D) - \{f\}$ is disconnected (cf. [12] for terminology)

and

(b) block realizations (G, ℓ, φ) of D for which no pair (B, v) exists that consists of a block B and cut point v of G with $v \in B$ such that the graph $(B, \{\{u, w\} \in \binom{B-\{v\}}{2} : \ell(u, w) < \ell(u, v) + \ell(v, w)\})$ is disconnected,

(ii) and there exists a shortest block realization (G_0, ℓ_0, φ_0) of D that is unique up to canonical isomorphism, shares the above property **(b)**, corresponds to the set $T_0(D)$ considered as a subset of itself, and for which the associated compatible decomposition $\mathcal{D}_0(D) := \mathcal{D}(G_0, \ell_0, \varphi_0)$ is the (also necessarily unique) **finest** compatible decomposition of D defined by the property that all metrics $d \in \mathcal{D}_0(D)$ do not possess any compatible decomposition $\mathcal{D}$ consisting of more than one metric.

The rest of the paper is organized as follows: In the next section, we will collect some more basic definitions and notations concerning metric spaces and block graphs. Then, in Section 3, we will show how to go from block realizations to compatible decompositions of metrics (Theorem 3) and in Section 4 how to go back (Theorem 4). In Section 5, we then discuss uniqueness (Theorem 5). Theorem 1 immediately follows from Theorems 3, 4, and 5.

2 Some more Basic Terminology and Facts

In this note, we adopt the following terminology: Given a metric D as above, we denote

- by $\sim_D$ the binary relation defined on X by putting

$$x \sim_D y \iff xy = 0 \ (\iff xz = yz \text{ holds for all } z \in X)$$

  which, in view of the fact that $xy = 0 \iff \forall_{a \in X}\, xa = ya$ holds for all $x, y \in X$, is obviously an equivalence relation,

- by $x/D := \{z \in X : zx = 0\}$ the equivalence class of x relative to this equivalence relation,

- and by X/D the set $\{x/D : x \in X\}$ of all such equivalence classes.

The metric D is called a **proper** metric if $x/D = \{x\}$ holds for all $\in X$. In this case, the pair $M = M_D := (X, D)$ is also called a **metric space**, X is called the **point set** of that space – and every element $x \in X$ a **point** of M.

Note that any metric D induces a (well-defined!) proper metric

$$\overline{D} : X/D \times X/D \to \mathbb{R} : (x/D, y/D) \mapsto xy$$

on the set X/D and, thus, a metric space $M_{\overline{D}} = (X/D, \overline{D})$. So, most concepts defined for proper metrics extend naturally to arbitrary metrics D by just applying them to the induced proper metric $\overline{D}$ and the associated metric space $M_{\overline{D}}$.

Further, given any metric space $M = (X, D)$,

(D1) we denote by $[x, y]$, for any two points $x, y \in X$, the **interval** between x and y, i.e., the set

$$[x, y] = [x, y]_M := \{z \in X : xy = xz + zy\},$$

(D2) we define a subset R of X to be a **gated subset**of M (cf. [10]) if there exists a (necessarily unique) map $\mathbf{gate}_R : X \to X : x \mapsto x_R$, also called the **gate map** of R (relative to M), such that $x_R \in R \cap [x, r]$ holds for all $x \in X$ and $r \in R$ in which case x_R is the unique point r in R that minimizes the distance xr to x (so, one has $x_R = x$ for some $x \in X$ if and only if $x \in R$ holds), (D3) and we define R to be a **retract** of M if it is a gated subset and $x_R, y_R \in [x, y]$ holds for all $x, y \in X$ with $x_R \neq y_R$ implying that also

$$xy = x\,x_R + x_R\,y_R + y_R\,y$$

must hold in this case.

Finally, consider a finite block graph $G = (V, E)$ with vertex set V and edge set $E \subseteq \binom{V}{2}$, and let $\mathcal{B}(G)$ denote the set of blocks of G. The following facts are well known and easily established:

(B1) A vertex $v \in V$ is a cut point of G if and only if v is contained in at least two distinct blocks of G.

(B2) Given any two vertices $u, v \in V$, there exists a unique $k = k(u, v) = k_G(u, v) \in \mathbb{N}$ and a unique finite sequence

$$\mathbf{p}_G(u, v) = (p_0(u, v) := u, p_1(u, v), \ldots, p_k(u, v) := v)$$

of $k + 1$ distinct vertices in V — dubbed the **shortest path** from u to v in G — such that $\{p_{i-1}(u, v), p_i(u, v)\} \in E$ holds for all $i = 1, \ldots, k$ and no block $B \in \mathcal{B}(G)$ contains more than two points from the set $\overline{uv} := \{p_0(u, v), p_1(u, v), \ldots, p_k(u, v)\}$.

(B3) Given, in addition, a length-assigning map $\ell : E \to \mathbb{R}_{>0}$ one has $D_\ell(u, v) = \ell(u, v)$ for all $u, v \in V$ with $\{u, v\} \in E$ as well as, more generally, $D_\ell(u, v) = \sum_{i=1}^{k(u,v)} \ell(p_{i-1}(u, v), p_i(u, v))$ for all $u, v \in V$ for the unique finite sequence $\mathbf{p}_G(u, v) = (p_0(u, v), p_1(u, v), \ldots, p_k(u, v))$ considered above.

(B4) Every block $B \subseteq V$ of G is a retract of V relative to the induced metric D_ℓ.

(B5) The gate map $\mathbf{gate}_B : V \to V : v \mapsto v_B$ that can therefore be associated to any block B of G

   (i) maps a vertex $u \in V$ onto the vertex $v \in B$ if and only if the vertex v is the only vertex in the set $\overline{uv} \cap B$ if and only if there exists a sequence $v'_0 := u, v'_1, \ldots, v'_{k'} := v$ of vertices in V with $\{v'_{i-1}, v'_i\} \in E$ for all $i = 1, \ldots, k'$ such that v is the only element in the intersection $\{v'_0, v'_1, \ldots, v'_{k'}\} \cap B$,

(ii) does, therefore, not depend on the length-assigning map ℓ from E into $\mathbb{R}_{>0}$,

(iii) induces a metric $d_{(B|\ell)}$ on V defined by $d_{(B|\ell)}(u,v) := D_\ell(u_B, v_B)$ for all $u, v \in V$,

(iv) maps any vertex $v \in V - B$ onto a cut point of G,

(v) maps any two vertices $u, v \in V$ with $\{u, v\} \in E$ onto the same point in B unless $u, v \in B$ holds,

(vi) and is, therefore, constant on any other block B' of G,

(vii) and the pre-image $\mathbf{gate}_B^{-1}(v) := \{u \in V : u_B = v\}$ of every point $v \in B$ is a retract of V relative to D_ℓ, and contains always at least one vertex that is not a cut point of G.

(B6) Furthermore, given any two distinct blocks $B, B' \in \mathcal{B}(G)$ and any two elements $v \in B$ and $v' \in B'$, one has $u_B = v'_B$ or $u_{B'} = v_{B'}$ (or, equivalently, $d_{(B|\ell)}(v', u) = 0$ or $d_{(B'|\ell)}(v, u) = 0$) for all $u \in V$ and, therefore, also $u_{B'} = v_{B'}$ for every $u \in V$ with $u_B \in B - \{v'_B\}$.

Indeed, assume that $d_{(B|\ell)}(v', u), d_{(B'|\ell)}(v, u) \neq 0$ holds for some u in V.

Then, denoting

- the image $(u_{B'})_B \in B$ of $u_{B'}$ and, hence, of any element in B' including the element v' relative to the map $\mathbf{gate}_B$ by $u_{B'B}$, and
- the image $(u_B)_{B'} \in B'$ of u_B and, hence, of any element in B including the element v relative to the map $\mathbf{gate}_{B'}$ by $u_{BB'}$,

our assumption $0 \neq d_{(B|\ell)}(v', u), d_{(B'|\ell)}(v, u)$ and, hence,

$$u_{B'B} = v'_B \neq u_B \quad \text{and} \quad u_{BB'} = v_{B'} \neq u_{B'}$$

implies, in view of the fact that B is a retract, that

$$D_\ell(u, u_{B'}) = D_\ell(u, u_B) + D_\ell(u_B, u_{BB'}) + D_\ell(u_{BB'}, u_{B'}) > D_\ell(u, u_B)$$

and, in view of the fact that also B' is a retract, that also

$$D_\ell(u, u_B) = D_\ell(u, u_{B'}) + D_\ell(u_{B'}, u_{B'B}) + D_\ell(u_{B'B}, u_B) > D_\ell(u, u_{B'})$$

must hold which is impossible.

(B7) In particular, given two distinct blocks $B, B' \in \mathcal{B}(G)$ as above, two elements $w, w' \in X$, two distinct elements $u, v \in B$, and two distinct elements $u', v' \in B'$ with $w_B = u, v_{B'} = v', v'_B = v$, and $w'_{B'} = u'$, one has $w_{B'} = v'$ and $w'_B = v$ and, therefore, also

$$D_\ell(w, w') = D_\ell(w, u) + D_\ell(u, v) + D_\ell(v, v') + D_\ell(v', u') + D_\ell(u', w').$$

Indeed, one has $u_{B'} = v_{B'} = v'$ and $u'_B = v'_B = v$ (as the gate map is constant on blocks) and, therefore

$$\begin{aligned} D_\ell(w, u') &= D_\ell(w, u) + D_\ell(u, v) + D_\ell(v, u') && (\Leftarrow u = w_B \neq v = v'_B) \\ &= D_\ell(w, u) + D_\ell(u, v) + D_\ell(v, v') + D_\ell(v', u') && (\Leftarrow v' = v_{B'}) \\ &= D_\ell(w, v') + D_\ell(v', u') && (\Leftarrow u = w_B \neq v = u'_B) \end{aligned}$$

implying that $w_{B'} = v'$ must hold. So, by symmetry, we also have $w'_B = v$. Altogether, this implies that also

$$\begin{aligned} D_\ell(w, w') &= D_\ell(w, u) + D_\ell(u, v) + D_\ell(v, w') && (\Leftarrow u = w_B \neq v = w'_B) \\ &= D_\ell(w, u) + D_\ell(u, v) + D_\ell(v, v') + D_\ell(v', u') + D_\ell(u', w') \end{aligned}$$

must hold — the latter because $D_\ell(v, w') = D_\ell(v, v') + D_\ell(v', u') + D_\ell(u', w')$ must hold in view of $v' = v_{B'} \neq u' = w'_{B'}$.

(B8) And finally, $D_\ell(u, v) = \sum_{B \in \mathcal{B}(G)} d_{(B|\ell)}(u, v)$ must hold for all $u, v \in V$ because denoting, for each $i = 1, \ldots, k(u, v)$, the unique block B in $\mathcal{B}(G)$ with $\{p_{i-1}(u, v), p_i(u, v)\} \subseteq B$ by $B^i = B(i : u|v)$, we have

$$d_{(B|\ell)}(u, v) = \begin{cases} \ell\big(p_{i-1}(u, v), p_i(u, v)\big) & \text{in case } B = B^i, \\ 0 & \text{in case } B \notin \{B^1, \ldots, B^{k(u,v)}\} \end{cases}$$

and, therefore,

$$\begin{aligned} D_\ell(u, v) &= \sum_{i=1,\ldots,k} \ell\big(p_{i-1}(u, v), p_i(u, v)\big) \\ &= \sum_{i=1,\ldots,k} d_{(B^i,\ell)}(u, v) \\ &= \sum_{B \in \mathcal{B}(G)} d_{(B|\ell)}(u, v), \end{aligned}$$

as claimed.

3 From Block Realizations to Compatible Collections of Metrics

Now, assume that $(G,\ell,\varphi) = \big((V,E),\ell,\varphi\big)$ is a block realization of a proper metric

$$D : X^2 \to \mathbb{R} : (x,y) \mapsto xy$$

defined on a finite set X.

Noting that the labeling map φ must be injective for any block realization of a proper metric D, we can assume without loss of generality that X is a subset of V and that φ coincides with the identity $\mathbf{Id}_X$ on X in which case the vertex $x_{(\varphi|B|\ell)} = x_{(\mathbf{Id}_X|B|\ell)}$ associated above to any $x \in X$ and any block $B \in \mathcal{B}(G)$ is simply the gate x_B of x in B.

Now, consider a fixed block B of $G = (V,E)$. First, we claim

Lemma 3.1 *With $D, G = (V,E), B, \ell$, and $\varphi = \mathbf{Id}_X$ as above, the restriction*

$$\mathbf{gate}_B|_X : X \to B : x \mapsto x_B$$

of the gate map $\mathbf{gate}_B$ associated with B to X is always surjective.

Proof: Indeed, (B7,vii) implies that there exists some point $w \in \mathbf{gate}_B^{-1}(v)$ for any $v \in B$ that is not a cut point and hence, by assumption, a point in $X \subseteq V$ implying that $v \in \{x_B : x \in X\}$ must hold. ∎

Next note that there are three metrics canonically associated to B,
— the metric ℓ^B defined on B that maps any pair (u,v) of vertices $u,v \in B$ onto the positive real number $\ell(u,v)$ in case $u \neq v$ (and, hence, $\{u,v\} \in E$) and onto 0 in case $u = v$, giving rise to the metric space $M_\ell(B) := (B, \ell^B)$,
— the metric $d_{(B|\ell)}$ defined on V that maps any pair $(u,v) \in V^2$ onto $\ell^B(u_B, v_B)$, giving rise to the induced metric space $M_{\overline{d_{(B|\ell)}}} = (V/d_{(B|\ell)}, \overline{d_{(B|\ell)}})$, and
— and the metric $d_{(X|B|\ell)} := d_{(B|\ell)}|_{X\times X}$ defined on X by restricting the metric $d_{(B|\ell)}$ defined on V to X which apparently coincides with the metric $d_{(\varphi|B|\ell)}$ introduced above and gives rise to the induced metric space $M_{\overline{d_{(X|B|\ell)}}} = (X/d_{(X|B|\ell)}, \overline{d_{(X|B|\ell)}})$.

We claim

Lemma 3.2 *With $D, G = (V, E), \ell, \varphi$, and B as above, the maps*

$$\mathbf{gate}_B : V \to B : v \mapsto v_B \quad \text{and} \quad \mathbf{gate}_B|_X : X \to B : x \mapsto x_B$$

induce bijective isometries from the metric spaces $M_{\overline{d_{(B|\ell)}}}$ and $M_{\overline{d_{(X|B|\ell)}}}$, respectively, onto the metric space $M_\ell(B) = (B, \ell^B)$.

Proof: It follows immediately from the definitions that the map $\mathbf{gate}_B$ induces a bijective isometry from $M_{\overline{d_{(B|\ell)}}}$ onto $M_\ell(B)$, and that the map $\mathbf{gate}_B|_X$ induces an injective isometry from $M_{\overline{d_{(X|B|\ell)}}}$ into $M_\ell(B)$. Moreover, Lemma 3.1 confirms that the map $\mathbf{gate}_B|_X$ must also be surjective. ∎

Now, we show

Theorem 3 *With $D, G = (V, E), \ell$, and φ as above, the associated collection $\mathcal{D}(\varphi|G|\ell)$ is a compatible decomposition of D.*

Proof: Note first that

$$\begin{aligned} xy &= D_\ell(x, y) \\ &= \sum_{B \in \mathcal{B}(G)} d_{(B|\ell)}(x, y) \\ &= \sum_{B \in \mathcal{B}(G)} d_{(X|B|\ell)}(x, y) \end{aligned}$$

holds for all $x, y \in X$. So, we have indeed $D = \sum_{d \in \mathcal{D}(\varphi|G|\ell)} d$, as required.

Further, as we have seen already in (B6), $d_{(B|\ell)}(v', u) = 0$ or $d_{(B'|\ell)}(v, u) = 0$ holds for every $u \in V$ for any two distinct blocks $B, B' \in \mathcal{B}(G)$ and any two elements $v \in B$ and $v' \in B'$. So, we must also have

$$d_{(X|B|\ell)}(x, y) = d_{(B|\ell)}(x, y) = 0 \quad \text{or} \quad d_{(X|B'|\ell)}(x', y) = d_{(B'|\ell)}(x', y) = 0$$

for all $y \in X$ for any two elements $x, x' \in X$ with $x_B = v'_B$ and $x'_{B'} = v_{B'}$ — it follows from Lemma 3.1 that such elements must exist.

It is also obvious in view of Lemma 3.2 that none of the metrics of the form $d_{(X|B|\ell)}$ can vanish.

Finally, no two metrics of the form $d_{(X|B|\ell)}$ and $d_{(X|B'|\ell)}$ can be linearly dependent in case $B \neq B'$ because, if $d_{(X|B'|\ell)} = \alpha\, d_{(X|B|\ell)}$ would hold for some $\alpha \in \mathbb{R}$, we must have $\alpha > 0$, and the existence of elements $x, x' \in X$ for which $d_{(X|B|\ell)}(x, y) = 0$ or $d_{(X|B'|\ell)}(x', y) = \alpha\, d_{(X|B|\ell)}(x', y) = 0$ holds for all $y \in X$ would imply that both, $d_{(X|B|\ell)}$ and $d_{(X|B'|\ell)}$, would be scalar multiples of the split metric

$$\delta_S : X^2 \to \mathbb{R} : (z, z') \mapsto \begin{cases} 0 & \text{if } z, z' \in A, \\ 0 & \text{if } z, z' \in A', \\ 1 & \text{else} \end{cases}$$

associated with the bipartition — or **split** — $S = \{A, A'\}$ of X into the two non-empty disjoint subsets

$$\begin{aligned} A &:= \{y \in X : d_{(X|B|\ell)}(x, y) = 0\} \\ &= \{y \in X : y_B = x_B\} \\ &= \{y \in X : d_{(X|B'|\ell)}(x, y) = 0\} \\ &= \{y \in X : y_{B'} = x_{B'}\} \end{aligned}$$

and

$$\begin{aligned} A' &:= \{y \in X : d_{(X|B'|\ell)}(x', y) = 0\} \\ &= \{y \in X : y_{B'} = x'_{B'}\} \\ &= \{y \in X : d_{(X|B|\ell)}(x', y) = 0\} \\ &= \{y \in X : y_B = x'_B\}. \end{aligned}$$

In view of Lemma 3.2, we would therefore have $B = \{x_B, x'_B\}$ and $B' = \{x_{B'}, x'_{B'}\}$. Furthermore, switching notation if necessary and writing, as in Assertion (B6), $w_{BB'}$ for $(w_B)_{B'}$ and $w_{B'B}$ for $(w_{B'})_B$ for every vertex $w \in V$, we can assume without loss of generality that

$$x_{BB'} = x'_{BB'} = x_{B'}$$

holds. Then, however, we must also have

$$x_{B'B} = x'_{B'B} = x'_B.$$

Indeed, putting

$$v := x_B, \qquad u := x'_B,$$
$$v' := x_{B'}, \qquad u' := x'_{B'}\ ,$$

so that, by assumption, $v_{B'} = u_{B'} = v'$ holds, we see that A coincides with the set

$$\{y \in X : y_B = v\} = \{y \in X : y_{B'} = v'\}$$

and, therefore, contains x, and A' with

$$\{y \in X : y_{B'} = u'\} = \{y \in X : y_B = u\}$$

and, therefore, contains x'.

Hence, if $x_{B'B} = x'_{B'B} = x'_B$ or, equivalently, $v'_B = u'_B = u$ would not hold, we would have $v'_B = u'_B = v$ and, hence,

$$V = \{w \in V : w_{B'} = v_{B'} = v'\} \cup \{w \in V : w_B = v'_B = v\}$$

in view of (B6) and, therefore,

$$\{w \in V : w_{B'} = v_{B'} = u'\} \cap \{w \in V : w_B = u\} = \emptyset$$

in contradiction to the fact that A' is not empty and $a'_{B'} = u'$ and $a'_B = u$ holds for every element $a' \in A'$. So, we may assume that, with B, B', x, x' as chosen and

$$v := x_B,\ u := x'_B,\ v' := x_{B'},\ u' := x'_{B'},$$
$$A := \{y \in X : d_{(X|B|\ell)}(x, y) = 0\},\ A' := \{y \in X : d_{(X|B|\ell)}(x', y) = 0\}$$

as defined above, the following assertions all hold:

$$B = \{u, v\}, \quad B' = \{u', v'\},$$

$$v = x_B, \quad u = x'_B = v'_B = u'_B,$$
$$v' = x_{B'} = v_{B'} = u_{B'}, \quad u' = x'_{B'},$$
$$x \in A = \{y \in X : y_B = v\} = \{y \in X : y_{B'} = v'\},$$
$$x' \in A' = \{y \in X : y_{B'} = u'\} = \{y \in X : y_B = u\},$$
$$A \cup A' = X, \quad A \cap A' = \emptyset,$$
$$V = \{w \in V : w_{B'} = v'\} \cup \{w \in V : w_B = u\},$$

$$\{w \in V : w_{B'} = u'\} \cap \{w \in V : w_B = v\} = \emptyset.$$

Furthermore, $A \cap A' = \emptyset$ implies that there can be no $y \in X$ with $y_B = u$ and $y_{B'} = v'$. In particular, there can be no $y \in X$ with $y = u$ as this would imply that $y_B = u_B = u$ and $y_{B'} = u_{B'} = v'$.

Consequently, there must be at least three edges e_1, e_2, e_3 in E that all contain u. So, one of these, say e_1 must be distinct from $\{u, v\}$ and the edge $\{p_0(u, u'), p_1(u, u')\}$, the first edge on the shortest path

$$\mathbf{p}_G(u, u') = (p_0(u, u') = u, p_1(u, u'), \ldots, p_k(u, u') = u')$$

from u to u' (which cannot have length 0 as u must be distinct from u' in view of $v' = u_{B'}$).

Now, let u'' denote the unique vertex in e_1 with $u'' \neq u$, and let B'' denote the unique block with $e_1 \subseteq B''$. Clearly, u is the gate in B for every element in B'' and the gate in B'' for every element in B. So, $\mathbf{gate}_{B''}^{-1}(u'') \subseteq \mathbf{gate}_B^{-1}(u)$ must hold. Further, $v' = u_{B'}$ is the gate in B' for every element in B'', and we must also have $\mathbf{gate}_{B''}^{-1}(u'') \subseteq \mathbf{gate}_{B'}^{-1}(v')$ because u'' cannot be the gate in B'' for the vertices in B' as, being connected by an edge to u and distinct from the first vertex $p_1(u, u')$ in the shortest path from u to u', it cannot be a member of the shortest path $\mathbf{p}_G(u, u')$ in G from u to u'. However, choosing any $y \in X$ with $y_{B''} = u''$ (which must exist in view of Lemma 3.2), this would imply $y_B = u$ and $y_{B'} = v'$ which however, as observed already above, is impossible in view of $A \cap A' = \emptyset$.

So, given any two distinct blocks B and B' in $\mathcal{B}(G)$, the two associated metrics $d_{(X|B|\ell)}$ and $d_{(X|B'|\ell)}$ cannot be linearly dependent.

Altogether, this establishes Theorem 3. ∎

4 From Compatible Collections of Metrics to Block Realizations

Next, assume that $\mathcal{D}$ is a compatible decomposition of a proper metric D defined on a finite X.

Noting that the collection $\{\sim_d : d \in \mathcal{D}\}$ is a **compatible** collection of equivalence relations defined on X — i.e. (cf. [9]), a collection of equivalence relations defined on X such that, for any two of its members, say $\sim_d$ and $\sim_{d'}$, there exist equivalence classes $C \in X/\sim_d$ and $C' \in X/\sim_{d'}$ with $C \cup C' = X$ — it follows from [9] (cf. also [15, Section 4.3]) that one can always find a finite tree $T = (V_T, E_T)$ with vertex set V_T and edge set $E_T \subseteq \binom{V_T}{2}$ and two maps

$$\varphi : X \to V_T \text{ and } \kappa : \mathcal{D} \to V_T$$

such that

- $\#\big(e \cap \kappa(\mathcal{D})\big) \leq 1$ holds for all $e \in E_T$,
- and $d(x, y) > 0$ holds for two points $x, y \in X$ and some $d \in \mathcal{D}$ if and only if the (unique!) shortest path $\mathbf{p}_T(x, y)$ from $\varphi(x)$ to $\varphi(y)$ meets the point $\kappa(d)$.

We begin by collecting some simple consequences of this observation that have not been included explicitly in [9]:

(i) It is obvious that the images $\varphi(X) \subseteq V_T$ and $\kappa(\mathcal{D}) \subseteq V_T$ of φ and κ must be disjoint because, if $x = \kappa(d)$ would hold for some $x \in X$ and some $d \in \mathcal{D}$, the path $\mathbf{p}_T(x, y)$ would meet the point $\kappa(d)$ even in case $y := x$ in contradiction to the fact that $d(x, y) = d(x, x) = 0$ holds in this case.

(ii) In consequence, κ and φ must both be injective:

Indeed, $d, d' \in \mathcal{D}$, $d \neq d'$, and $\kappa(d) = \kappa(d')$ would imply that $d(x, y) = 0 \iff d'(x, y) = 0$ and, therefore, $z/d = z/d'$ would hold for all $x, y, z \in X$. So, the fact that we have assumed that some $x_0, y_0 \in X$ with $X = x_0/d \cup y_0/d'$ exist implies that also $X = x_0/d \cup y_0/d = y_0/d \cup y_0/d'$ must hold, implying in turn that the two subsets $x_0/d = x_0/d'$ and $y_0/d = y_0/d'$ must form a split S of X and both, d and d', are scalar multiples of the associated split metric δ_S in contradiction to our assumption that no two of the metrics in $\mathcal{D}$ are linear multiples of each other.

And $x, y \in X$ and $\varphi(x) = \varphi(y)$ implies that $d(x, y) = 0$ must hold for all $d \in \mathcal{D}$ and, hence, $x = y$ in view of $xy = \sum_{d \in \mathcal{D}} d(x, y) = 0$. More generally, it follows from this argument that any path from x to y must meet at least one point in $\kappa(\mathcal{D})$ in case $x \neq y$.

So, from now on, we will assume that, as above, the set X is a subset of V_T and that $\varphi = \mathbf{Id}_X$ holds.

(iii) We then must also have $\kappa(\mathcal{D}) \subseteq V_{int} := \{v \in V_T : \deg_T(v) > 1\}$ because no shortest path between two vertices distinct from a leaf would ever meet that leaf while none of the equivalence relation $\sim_d$ $(d \in \mathcal{D})$ can be the trivial equivalence relation.

(iv) So, without loss of generality, we can also assume that all leaves of T are contained in X because we can always eliminate any leaf not contained in X and its pending edge without (seriously) interfering with our assumptions. Similarly, we can assume that no edge $e = \{u, v\} \in E_T$ with $e \cap \kappa(\mathcal{D}) = \emptyset$ exists as we can always contract any such edge without (seriously) interfering with our assumptions. And we can always insert a vertex in-between any two distinct vertices from $\kappa(\mathcal{D})$ forming an edge.

(v) So, altogether, we see that, given a compatible decomposition $\mathcal{D}$ of a proper metric D defined on a finite X, we can always find a finite tree

$$T = T(\mathcal{D}) = \big(V_T(\mathcal{D}), E_T(\mathcal{D})\big)$$

and an injective map κ from $\mathcal{D}$ into the vertex set $V_T(\mathcal{D})$ of $T(\mathcal{D})$ such that

- X and $\kappa(\mathcal{D})$ are disjoint subsets of $V_T(\mathcal{D})$ implying that

  $$V_{\mathcal{D}} := V_T(\mathcal{D}) - \kappa(\mathcal{D})$$

  contains X,
- X contains all leaves of T,
- $\#\big(e \cap \kappa(\mathcal{D})\big) = 1$ holds for all $e \in E_T(\mathcal{D})$, i.e., the bipartition of $V_T(\mathcal{D})$ into the subset $V_{\mathcal{D}}$ and its complement $\kappa(\mathcal{D})$ is a bipartition of the vertex set $V_T(\mathcal{D})$ of the tree $T(\mathcal{D})$ into two disjoint subsets such that all leaves of T are contained in $V_{\mathcal{D}}$ and $\#(e \cap V_{\mathcal{D}}) = 1$ holds for all $e \in E_T(\mathcal{D})$, and
- $d(x, y) \neq 0$ holds for two points $x, y \in X$ and some $d \in \mathcal{D}$ if and only if the shortest path $\mathbf{p}_T(x, y)$ from x to y meets the point $\kappa(d)$.

Next, recall that, given a tree $T = (V_T, E_T)$, there is exactly one bipartition $\Pi = \Pi_T$ of its vertex set V_T into two disjoint subsets such that $\#(e \cap V) = 1$ holds for all edges $e \in E_T$ and each subset $V \in \Pi_T$, and that associating, to any subset $V \in \Pi_T$, the graph $G_V := \big(V, E_V\big)$ where E_V denotes the subset of $\binom{V}{2}$ consisting of all 2-subsets $\{u, v\}$ of V for which the shortest path $\mathbf{p}_T(u, v)$ from u to v in T has length 2 (i.e., for which there exists some (necessarily unique) vertex $w \in V_T - V$ with $\{u, w\}, \{v, w\} \in E_T$), one obtains a block graph whose blocks correspond, in a one-to-one fashion, to the vertices in $V_T - V$ that are not a leaf of T — a block of cardinality N corresponding to a vertex of degree N.

In particular, one can recover T up to canonical isomorphism from G_V in case V contains all leaves of T, implying that this construction gives rise to a canonical one-to-one correspondence between (isomorphism classes of) finite block graphs on the one, and finite trees T for which one of the two subsets in Π_T contains all leaves of T on the other (see also [4, Section 2]).

Moreover, a vertex $v \in V$ is a cut point of G_V if and only if v is not a leaf of T and, given any two vertices $u, v \in V$, the shortest path $\mathbf{p}_{G_V}(u, v)$ between u and v in G_V can be obtained from the shortest path $\mathbf{p}_T(u, v) = (p_0 := u, w_1, p_1, w_2, p_2, \ldots, w_k, p_k := v)$ between u and v in T by just eliminating every second vertex in that path, i.e., the vertices $w_1, w_2, \ldots, w_k$ that are contained in $V_T - V$.

It follows that, with $\mathcal{D}$, $T = T(\mathcal{D})$, $V_T(\mathcal{D})$, $E_T(\mathcal{D})$, $V_\mathcal{D}$, and κ as above, assigning to $\mathcal{D}$ the block graph

$$G_\mathcal{D} := G_{V_\mathcal{D}} = (V_\mathcal{D}, E_\mathcal{D} := E_{V_\mathcal{D}})$$

(as defined above), and to any edge $\{u, v\} \in E_\mathcal{D}$ the (well-defined!) length $\ell_\mathcal{D}(u, v) := d(x, y)$ where d is the unique metric in $\mathcal{D}$ for which the two pairs $\{u, \kappa(d)\}$ and $\{v, \kappa(d)\}$ are edges in $E_T(\mathcal{D})$, and x and y are chosen in X so that u is the first vertex traversed by the shortest path $\mathbf{p}_T\big(\kappa(d), x\big)$ from $\kappa(d)$ to x in T, and v is the first vertex traversed by $\mathbf{p}_T\big(\kappa(d), y\big)$, we obtain a block graph $G_\mathcal{D}$ whose vertex set contains X, together with a length-assigning map $\ell_\mathcal{D} : E_\mathcal{D} \to \mathbb{R}_{>0} : \{u, v\} \mapsto \ell_\mathcal{D}(u, v)$.

We claim

Theorem 4 *Given a compatible decomposition $\mathcal{D}$ of a proper metric D defined on a finite X, the block graph $G_\mathcal{D} := (V_\mathcal{D}, E_\mathcal{D})$ together with the embedding $\varphi := \mathbf{Id}_X$ of X into $V_\mathcal{D}$ and the length assignment map $\ell_\mathcal{D} : E_\mathcal{D} \to \mathbb{R}_{>0}$ is a block realization of D.*

Proof: We have to show that $xy = D_{\ell_\mathcal{D}}(x, y)$ holds for all $x, y \in X$ and that every vertex in $V_\mathcal{D} - X$ has degree at least 3 and is a cut point of G.

To establish the first claim, consider the shortest path

$$\mathbf{p}_T(x, y) = (p_0 := x, w_1, p_1, w_2, p_2, \ldots, w_k, p_k := y)$$

between x and y in $T = T(\mathcal{D})$ and the associated shortest path

$$\mathbf{p}_{G_\mathcal{D}}(x,y) = (p_0, p_1, \ldots, p_k)$$

between x and y in $G_\mathcal{D}$. Next, choose metrics $d_1, \ldots, d_k \in \mathcal{D}$ with $\kappa(d_i) = w_i$ for all $i = 1, 2, \ldots, k$, and note that $d_i(x,y) = \ell_\mathcal{D}(p_{i-1}, p_i) > 0$ will then hold for all $i = 1, 2, \ldots, k$, while $d(x,y) = 0$ will hold for all d in the complement $\mathcal{D} - \{d_1, d_2, \ldots, d_k\}$ of $\{d_1, d_2, \ldots, d_k\}$. Thus, the length $k = k(x,y)$ of the path coincides with

$$\begin{aligned} D_{\ell_\mathcal{D}}(x,y) &= \sum_{i=1}^{k} \ell_\mathcal{D}(p_{i-1}, p_i) \\ &= \sum_{i=1}^{k} d_i(x,y) \\ &= \sum_{d \in \mathcal{D}, d(x,y) \neq 0} d(x,y) \\ &= \sum_{d \in \mathcal{D}} d(x,y) \\ &= xy \end{aligned}$$

for all $x, y \in X$, as required.

It remains to show that every vertex in $V_\mathcal{D} - X$ has degree at least 3 and is a cut point of $G_\mathcal{D}$. However, as none of the vertices in $V_\mathcal{D} - X$ is a leaf in $T(\mathcal{D})$, all these vertices must be cut points in $G_\mathcal{D}$. And if there would be a vertex $v \in V_\mathcal{D} - X$ that has degree 2, let v_1 and v_2 denote the two vertices in $V_\mathcal{D}$ with $\{v, v_1\}, \{v, v_2\} \in E_\mathcal{D}$, and let $D_1, D_2 \in \mathcal{D}$ denote the two metrics in $\mathcal{D}$ with $\{v, \kappa(D_1)\}, \{v, \kappa(D_2)\}, \{v_1, \kappa(D_1)\}, \{v_2, \kappa(D_2)\} \in E_T(\mathcal{D})$. Clearly, $B_1 := \{v, v_1\}$ and $B_2 := \{v, v_2\}$ must be two blocks of cardinality 2 implying that the two partitions of X/D_1 and X/D_2 must be splits of X and that the two metrics D_1 and D_2 must be scalar multiples of the associated split metrics δ_{X/D_1} and δ_{X/D_2}. However, X/D_1 consists of the two subsets $\mathbf{gate}_{B_1}^{-1}(v)$ and $\mathbf{gate}_{B_1}^{-1}(v_1)$, and X/D_2 consists of the two subsets $\mathbf{gate}_{B_2}^{-1}(v)$ and $\mathbf{gate}_{B_2}^{-1}(v_2)$ and, in view of $v_{B_1} = (v_2)_{B_1} = v$ and $v_{B_2} = (v_1)_{B_2} = v$, we must have

$$\mathbf{gate}_{B_1}^{-1}(v_1) \subseteq \mathbf{gate}_{B_2}^{-1}(v) = X - \mathbf{gate}_{B_2}^{-1}(v_2)$$

and

$$\mathbf{gate}_{B_2}^{-1}(v_2) \subseteq \mathbf{gate}_{B_1}^{-1}(v) = X - \mathbf{gate}_{B_1}^{-1}(v_1)$$

while, in view of $v \notin X$ and $\deg_{G_{\mathcal{D}}}(v) = 2$, there can be no $x \in X$ with $x_{B_1} = x_{B_2} = v$. So, we must also have $\mathbf{gate}_{B_1}^{-1}(v) \cap \mathbf{gate}_{B_2}^{-1}(v) = \emptyset$ and, therefore, $\mathbf{gate}_{B_1}^{-1}(v_1) = \mathbf{gate}_{B_2}^{-1}(v)$ and $\mathbf{gate}_{B_2}^{-1}(v_2) = \mathbf{gate}_{B_1}^{-1}(v)$ implying that $X/D_1 = X/D_2$ must hold. Thus, D_1 and D_2 would both be scalar multiples of the same split metric $\delta_{X/D_1} = \delta_{X/D_2}$ in contradiction to our assumption that D_1 and D_2 are linearly independent. This concludes the proof of Theorem 4. ∎

5 Back and Forth: The Problem of Uniqueness

In this section, we want to establish the following result:

Theorem 5 *Given a proper finite metric space $M = (X, D)$, a compatible decomposition $\mathcal{D}$ of D, and a block realization $(G, \ell, \mathbf{Id}_X)$ of D, one has $\mathcal{D} = \mathcal{D}(\mathbf{Id}_X|G|\ell)$ if and only if $(G, \ell, \mathbf{Id}_X)$ is isomorphic to $(G_{\mathcal{D}}, \ell_{\mathcal{D}}, \mathbf{Id}_X)$, i.e., there exists a (necessarily unique) bijection α from the vertex set $V_{\mathcal{D}}$ of $G_{\mathcal{D}}$ onto the vertex set V_G of G such that $\alpha(x) = x$ holds for all $x \in X$, a 2-subset $\{u, v\}$ of $V_{\mathcal{D}}$ is an element of the edge set $E_{\mathcal{D}} = E_{V_{\mathcal{D}}}$ of $G_{\mathcal{D}}$ if and only if $\{\alpha(u), \alpha(v)\}$ is an element of the edge set E_G of G, and $\ell_{\mathcal{D}}(u, v)$ coincides with $\ell\big(\alpha(u), \alpha(v)\big)$ for all $u, v \in V_{\mathcal{D}}$ with $\{u, v\} \in E_{\mathcal{D}}$.*

Proof: It is an immediate consequence of the description of the construction of $G_{\mathcal{D}}$ and $\ell_{\mathcal{D}}$ provided in the previous section that $\mathcal{D} = \mathcal{D}(\mathbf{Id}_X|G|\ell)$must hold in case $(G, \ell, \mathbf{Id}_X) = (G_{\mathcal{D}}, \ell_{\mathcal{D}}, \mathbf{Id}_X)$.

So, it remains to establish the following proposition:

Proposition 5.1 *Given two block realizations*

$$(G_1, \ell_1, \mathbf{Id}_X) \text{ and } (G_2, \ell_2, \mathbf{Id}_X)$$

and a compatible decomposition $\mathcal{D}$ of D with

$$\mathcal{D} = \mathcal{D}(G_1|\ell_1|\mathbf{Id}_X) = \mathcal{D}(G_2|\ell_2|\mathbf{Id}_X),$$

there exists

(i) *a (necessarily unique) bijection α from the vertex set V_{G_1} of G_1 onto the vertex set V_{G_2} of G_2 such that $\alpha(x) = x$ holds for all $x \in X$,*

(ii) *a 2-subset $\{u, v\}$ of V_{G_1} is contained in the edge set E_{G_1} of G_1 if and only if $\{\alpha(u), \alpha(v)\}$ is is contained in the edge set E_{G_2} of G_2 ,*

(iii) *and $\ell_1(u, v) = \ell_2\big(\alpha(u), \alpha(v)\big)$ holds for all $u, v \in V$ with $\{u, v\} \in E_{G_1}$.*

To establish this proposition, note first that, there exists, for every metric $d \in \mathcal{D}$, unique blocks $B_1 = B_1(d) \in \mathcal{B}(G_1)$ and $B_2 = B_2(d) \in \mathcal{B}(G_2)$ with $d = d_{(\mathbf{Id}_X|B_1|\ell_1)} = d_{(\mathbf{Id}_X|B_2|\ell_2)}$ implying that there exists a unique bijection $\alpha_{\mathcal{B}}$ from $\mathcal{B}(G_1)$ onto $\mathcal{B}(G_2)$ such that

(i) $\alpha_{\mathcal{B}}(B_1) = B_2$ holds for some $B_1 \in \mathcal{B}(G_1)$ and some $B_2 \in \mathcal{B}(G_2)$ if and only if there exists some $d \in \mathcal{D}$ with $B_1 = B_1(d)$ and $B_2 = B_2(d)$,

(ii) the maps

$$\mathbf{gate}_{B_1}|_X : X \to B_1 : x \mapsto x_{B_1} \quad \text{and} \quad \mathbf{gate}_{B_2}|_X : X \to B_2 : x \mapsto x_{B_2}$$

induce canonical bijective isometries from $M_{\overline{d}} = (X/d, \overline{d})$ onto the metric spaces $M_{\ell_1}(B_1) = (B_1, \ell_1^{B_1})$ and $M_{\ell_2}(B_2) = (B_2, \ell_2^{B_2})$,

(iii) implying also that there exists a bijective isometry $\alpha_{B_1} : B_1 \to B_2$ from $M_{\ell_1}(B_1)$ onto $M_{\ell_2}(B_2)$ defined by mapping any $v_1 \in B_1$ onto the unique element $v_2 \in B_2$ for which $v_2 = x_{B_2}$ holds for some — or all — $x \in X$ with $v_1 = x_{B_1}$.

Further, given an element $v_1 \in V_{G_1}$ and two distinct blocks B_1, B_1' in $\mathcal{B}(G_1)$ with $v_1 \in B_1 \cap B_1'$, we claim that $v_2 := \alpha_{B_1}(v_1)$ coincides with $v_2' := \alpha_{B_1'}(v_1)$.

To prove this assertion, we first establish the following

Lemma 5.2 *Given a compatible decomposition $\mathcal{D}$ of D, the following holds:*

(a) *Given any two distinct metrics d, d' in $\mathcal{D}$, there exists exactly one pair (C, C') of equivalence classes $C = C(d|d') \in X/d$ and $C' = C(d'|d) \in X/d'$ whose union coincides with X.*

(b) *Given, in addition, any two points $x, y \in X$ with $d(x, y), d'(x, y) \neq 0$, one has either*

$$x/d = C(d|d') \ \textit{and} \ \ y/d' = C(d'|d)$$

or

$$y/d = C(d|d') \quad \textit{and} \quad x/d' = C(d'|d),$$

but never $x/d = x/d'$ *and* $y/d = y/d'$.

Furthermore, the first assertion $x/d = C(d|d')$ *and* $y/d' = C(d'|d)$ *is equivalent with each of the following seven assertions*

(i) $x/d \cup y/d' = X$,

(ii) $x \in C(d|d')$,

(iii) $y \in C(d'|d)$,

(iv) $x/d' \subseteq x/d$ *and* $y/d \subseteq y/d'$,

(v) $x/d' \subsetneq x/d$, *or* $x/d' = x/d$ *and* $y/d \subsetneq y/d'$,

(vi) $x/d' \subsetneq x/d$, *or* $x/d' = x/d$ *and* d' *is a split metric,*

(vii) $x/d' \subsetneq x/d$, *or* $x/d' = x/d$ *and* $y/d \subseteq y/d'$.

(c) *In particular, the binary relation "$\preceq_x^y$" defined on the set*

$$\mathcal{D}(x|y) := \{d' \preceq_x^y d \iff x/d' \subseteq x/d \quad \textit{and} \quad y/d \subseteq y/d'$$

for any two metrics $d, d' \in \mathcal{D}(x|y)$ *is a linear order on* $\mathcal{D}(x|y)$.

Proof: **(a)** Assume that $C_1 \cup C_1' = C_2 \cup C_2' = X$ would hold for some C_1, C_2 in X/d and C_1', C_2' in X/d' with, say, $C_1 \neq C_2$. Then, also

$$X = C_2 \cup C_2' \subseteq (X - C_1) \cup C_2' \subseteq C_1' \cup C_2'$$

would hold, implying that also C_1' and C_2' cannot coincide. Thus, we would get

$$C_2 \subseteq X - C_1 \subseteq C_1' \subseteq X - C_2' \subseteq C_2$$

and, therefore, $C_2 = C_1', C_1 = C_2'$ and $X/d = X/d' = \{C_1, C_2\}$ in which case $S := \{C_1, C_2\}$ would be a split of X and both, d and d', would be scalar multiples of the associated split metric δ_S in contradiction to the fact that d and d' are supposed to be linearly independent.

(b) Note first that $X = C(d|d') \cup C(d'|d)$ implies that $z \in C(d|d')$ and, hence, $z/d = C(d|d')$, or $z \in C(d'|d)$ and, hence, $z/d' = C(d'|d)$ must hold for all $z \in X$.

So, as $d(x,y), d'(x,y) \neq 0$ implies $x/d \neq y/d$ and $x/d' \neq y/d'$, either $x/d = C(d|d')$ and $y/d' = C(d'|d)$ or $x/d' = C(d'|d)$ and $y/d = C(d|d')$ must hold for all $x, y \in X$ with $d(x,y), d'(x,y) \neq 0$ while $x/d = x/d'$ and $y/d = y/d'$ can never hold in view of the fact that, according to (a), there can be only one pair of subsets $C \in X/d$ and $C' \in X/d'$ with $C \cup C' = X$.

Thus, assuming that $x/d = C(d|d')$ and $y/d' = C(d'|d)$ holds, it is obvious that also the assertions (i) to (iii) must hold while none of these assertions can hold in case $y/d = C(d|d')$ and $x/d' = C(d'|d)$. So, all of these assertions must indeed be equivalent with the assertion that $x/d = C(d|d')$ and $y/d' = C(d'|d)$ holds.

Further, (i) implies $x/d' \subseteq X - y/d' \subseteq x/d$ and $y/d \subseteq X - x/d \subseteq y/d'$, i.e., (i) implies (iv) while, conversely, if (iv) holds and (i) would not hold, we would have $X = x/d' \cup y/d$ and, hence, $x/d \subseteq x/d'$ and $y/d' \subseteq y/d$ which together with (iv) would imply $x/d = x/d'$ and $y/d' = y/d$ which, as observed above, is impossible. So, (i) is indeed also equivalent to (iv).

It follows from the same observation that (iv) implies (v), and it is obvious that $x/d' = x/d$ together with $x/d \cup y/d' = X$ implies $X/d' = \{x/d', y/d'\}$ and, hence, that d' is a split metric. So, (iv) implies (v) and (vi).

Next, it is trivial that (v) implies (vii) while (vi) implies that either $x/d' \subsetneq x/d$ holds or $x/d' = x/d$ and, therefore, $y/d \subseteq X - x/d \subseteq X - x/d' = y/d'$. So, also (vi) implies (vii).

And finally, if (vii) holds and (iv) would not hold, we would have $x/d \subseteq x/d'$ and $y/d' \subseteq y/d$ and, therefore, $x/d = x/d'$ and $y/d' = y/d$ which, as we know, is impossible. So, (vii) also implies (iv), as claimed

(c) It follows immediately that the binary relation "$\preceq_x^y$" defines a linear order on $\mathcal{D}(x|y)$: "$\preceq_x^y$" is transitive as $d, d', d'' \in \mathcal{D}(x|y)$, $d' \prec_x^y d$ and $d'' \prec_x^y d'$ implies $x/d'' \subseteq x/d' \subseteq x/d$ and $y/d \subseteq y/d' \subseteq y/d''$ and, hence, $d'' \prec_x^y d$ (in view of (**(b)**,iv)). "$\preceq_x^y$" is a partial order as $d' \prec_x^y d$ and $d \prec_x^y d'$ implies $x/d' = x/d$ and $y/d = y/d'$ and, hence, $d = d'$ for all $d, d' \in \mathcal{D}(x|y)$. And it is a linear order as either $d = d'$ or $d' \prec_x^y d$ or $d \prec_x^y d'$ holds for all $d, d' \in \mathcal{D}(x|y)$ in view of the first assertion in **(b)**. ∎

Now,

(i) assume that $(G, \ell, \mathbf{Id}_X)$ of D is a block realization for which $\mathcal{D}$ coincides with $\mathcal{D}(\mathbf{Id}_X|G|\ell)$,

(ii) consider the shortest path

$$\mathbf{p}_G(x,y) = (p_0 := p_0(x,y), p_1 := p_1(x,y), \ldots, p_k := p_k(x,y))$$

from x to y in G of length $k = k_G(x,y) = \#\mathcal{D}(x|y)$,

(iii) let

$$B^i := B(i : x|y)$$

denote the unique block in $\mathcal{B}(G)$ with $\{p_{i-1}, p_i\} \subseteq B^i$, and

(iv) let $d_i := d_{(B^i|\ell_1)}$ denote the metric in $\mathcal{D}$ corresponding to B^i,

implying that $\mathcal{D}(x|y) = \{d_1, d_2, \ldots, d_k\}$ must hold.

We claim:

Lemma 5.3 *One has $d_i \preceq_x^y d_j$ for some $i, j \in \{1, 2, \ldots, k\}$ if and only if $i \leq j$ holds.*

Proof: To simplify notation, denote the gate v_B of any $v \in V_G$ relative to any block $B \in \mathcal{B}(G)$ also by $B(v)$. Clearly, we have

$$B^i(x) = B^i(p_0) = B^i(p_1) = \cdots = B^i(p_{i-1}) = p_{i-1}$$

and

$$p_i = B^i(p_i) = B^i(p_{i+1}) = \cdots = B^i(p_k) = B^i(y).$$

Thus, according to (B6), we must also have

$$X = \{u \in X : d_i(x,u) = 0\} \cup \{u \in X : d_{i-1}(y,u) = 0\} = x/d_i \cup y/d_{i-1}$$

and therefore also $x/d_{i-1} \subseteq x/d_i$ and $y/d_i \subseteq y/d_{i-1}$, i.e., $d_{i-1} \prec_x^y d_i$ for all $i = 2, 3, \ldots, k$ as claimed. ∎

In consequence, referring to the terminology introduced in Proposition 5.1, also the following must hold:

Corollary 5.4 *Given a compatible decomposition $\mathcal{D}$ of D and two block realizations*

$$(G_1, \ell_1, \mathbf{Id}_X) \text{ and } (G_2, \ell_2, \mathbf{Id}_X)$$

with

$$\mathcal{D} = \mathcal{D}(G_1|\ell_1|\mathbf{Id}_X) = \mathcal{D}(G_2|\ell_2|\mathbf{Id}_X),$$

the order in which the shortest path from x to y in either G_1 or G_2 traverses the blocks of G_1 and G_2 is respected by the bijection $\alpha_{\mathcal{B}} : \mathcal{B}(G_1) \to \mathcal{B}(G_2)$, if this path traverses the blocks in $\mathcal{B}(G_1)$ corresponding to metrics in $\mathcal{D}(x|y)$ in the order $B^1, B^2, \ldots, B^k$, it traverses the blocks in $\mathcal{B}(G_2)$ corresponding to those metrics in the order $\alpha_{\mathcal{B}}(B^1), \alpha_{\mathcal{B}}(B^2), \ldots, \alpha_{\mathcal{B}}(B^k)$.

Continuing with the terminology introduced in Proposition 5.1, let us now return to the proof of our claim that, given an element $v_1 \in V_{G_1}$ and two distinct blocks B_1, B_1' in $\mathcal{B}(G_1)$ with $v_1 \in B_1 \cap B_1'$, the vertex $v_2 := \alpha_{B_1}(v_1)$ coincides with $v_2' := \alpha_{B_1'}(v_1)$. To simplify notation, put $B_2 := \alpha_{\mathcal{B}}(B_1)$ and $B_2' := \alpha_{\mathcal{B}}(B_1')$, and recall that $\alpha_{B_1}(v_1)$ and $\alpha_{B_1'}(v_1)$ are determined by first choosing some $x, y \in X$ with $B_1(y) = B_1'(x) = v_1$ and then putting $\alpha_{B_1}(v_1) := B_2(y)$ and $\alpha_{B_1'}(v_1) := B_2'(x)$.

Next, note that we may choose x and y as follows: First, we choose some $u \in B_1 - \{v_1\}$ and some $u' \in B_1' - \{v_1\}$, and then we choose $x, y \in X$ so that $B_1(x) = u$ and $B_1'(y) = u'$ holds.

According to (B7) (replacing w by x and w' by y, and putting $v = v' := v_1$), this implies $B_1'(x) = B_1(y) = v_1$, as required. Furthermore, it follows from the fact that the terms

$$D_\ell(u, v_1) = D_\ell\big(B_1(x), B_1(y)\big) = d_{(B_1|\ell_1)}(x, y)$$

and

$$D_\ell(v_1, y) = D_\ell\big(B_1'(x), B_1'(y)\big) = d_{(B_1'|\ell_1)}(x, y)$$

in the sum

$$D_\ell(x, y) = D_\ell(x, u) + D_\ell(u, v_1) + D_\ell(v_1, u') + D_\ell(u', y)$$

do not vanish that the shortest path

$$\mathbf{p}_{G_1}(x, y) = (p_0^1(x, y) := x, p_1^1(x, y), \ldots, p_k^1(x, y) := y)$$

from x to y in G_1 of length $k = \#\mathcal{D}(x|y)$ must pass through the vertices u, v_1, u' (in that order), that is, there must exist some index $i_0 \in \{1, 2, \ldots, k-1\}$ with $p_{i_0-1}^1(x, y) = u$, $p_{i_0}^1(x, y) = v_1$, and $p_{i_0+1}^1(x, y) = u'$.

Thus, denoting, for all $i = 1, \ldots, k$, the block $B \in \mathcal{B}(G_1)$ that contains $p^1_{i-1}(x,y)$ and $p^1_i(x,y)$ by $B_1(i : x|y)$, we must have $B_1 = B_1(i_0 : x|y)$ and $B'_1 = B_1(i_0+1 : x|y)$ while v_1 is the unique vertex in V_{G_1} that is contained in the intersection of $B_1(i_0 : x|y)$ and $B_1(i_0+1 : x|y)$. Consequently, we have $v_1 = p^1_{i_0}$ and

$$B'_1(x) = B'_1\big(p^1_0(x,y)\big) = B'_1\big(p^1_1(x,y)\big) = \cdots = B'_1\big(p^1_{i_0}(x,y)\big) = p^1_{i_0} = v_1$$

as well as

$$v_1 = p^1_{i_0} = B_1\big(p^1_{i_0}(x,y)\big) = B_1\big(p^1_{i_0+1}(x,y)\big) = \cdots = B_1\big(p^1_k(x,y)\big) = B_1(y).$$

Furthermore, the corresponding sequence

$$B_2(1 : x|y) := \alpha_{\mathcal{B}}\big(B_1(1 : x|y)\big), \ldots, B_2(k : x|y) := \alpha_{\mathcal{B}}\big(B_1(k : x|y)\big)$$

of blocks in G_2 must exactly coincide with the sequence of blocks in G_2 that are traversed by the shortest path

$$\mathbf{p}_{G_2}(x,y) = (p^2_0(x,y) := x, p^2_1(x,y), \ldots, p^2_k(x,y) := y)$$

from x to y in G_2. In particular,

$$B_2(i_0 : x|y) = \alpha_{\mathcal{B}}\big(B_1(i_0 : x|y)\big) = \alpha_{\mathcal{B}}(B_1) = B_2$$

and

$$B_2(i_0+1 : x|y) = \alpha_{\mathcal{B}}\big(B_1(i_0+1 : x|y)\big) = \alpha_{\mathcal{B}}\big(B'_1\big) = B'_2$$

must hold which in turn implies that also

$$\alpha_{B'_1}(v_1) = B'_2(x) = B'_2\big(p^2_0(x,y)\big) = B'_2\big(p^2_1(x,y)\big) = \cdots = B'_2\big(p^2_{i_0}(x,y)\big) = p^2_{i_0}$$

as well as

$$p^2_{i_0} = B_2\big(p^1_{i_0}(x,y)\big) = B_2\big(p^2_{i_0+1}(x,y)\big) = \cdots = B_2\big(p^2_k(x,y)\big) = B_2(y) = \alpha_{B_1}(v_1)$$

must hold and, hence, $v'_2 = \alpha_{B'_1}(v_1) = p^2_{i_0} = \alpha_{B'_1}(v_1) = v'_1$ as claimed.

It is now easy to finish the proof of Proposition 5.1: For every block B_1 in $\mathcal{B}(G_1)$, we have a length-preserving bijection α_{B_1} from B_1 onto $\alpha_{\mathcal{B}}(B_1)$ for which $\alpha_{B_1}(x_{B_1}) = x_{\alpha_{\mathcal{B}}(B_1)}$ holds for all $x \in X$. And we have just seen that any two of these bijections coincide on all those vertices in V_{G_1} on which both of them are defined.

Thus, together, they give rise to a unique bijection from V_{G_1} onto V_{G_2} that is easily checked to satisfy all the conditions stated in Proposition 5.1. ∎

The following consequence of our results seems worth mentioning:

Corollary 5.5 *Given a block realization $(G, \ell, \mathbf{Id}_X)$ of a proper finite metric D, a point $x \in X$, and a block $B \in \mathcal{B}(G)$, the following assertions are equivalent:*

(i) *one has $x \in B$,*

(ii) *one has $x/d_{(\mathbf{Id}_X|B|\ell)} \subseteq x/d_{(\mathbf{Id}_X|B'|\ell)}$ for all blocks $B' \in \mathcal{B}(G) - \{B\}$ with $x/d_{(\mathbf{Id}_X|B|\ell)} \cup x/d_{(\mathbf{Id}_X|B'|\ell)} \neq X$ and, in addition, either $x/d_{(\mathbf{Id}_X|B|\ell)} \neq x/d_{(\mathbf{Id}_X|B'|\ell)}$ for all such B' or $\#B = 2$,*

(iii) *one has $d_{(\mathbf{Id}_X|B|\ell)} \preceq_x^y d'$ for all $y \in X - \{x\}$ and all $d' \in \mathcal{D}(x|y)$,*

(iv) *there exists some $y \in X - \{x\}$ such that $d_{(\mathbf{Id}_X|B|\ell)} \preceq_x^y d'$ holds for all $d' \in \mathcal{D}(x|y)$.*

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