How Data Dependent is a Nonlinear Subdivision Scheme? 
– A Case Study Based on Convexity Preserving Subdivision

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Abstract:

The regularity of the limit function of a linear subdivision scheme is essentially irrelevant to the initial data. How data dependent, then, is the regularity of the limit of a nonlinear subdivision scheme? The answer is the most obvious it depends. In this paper, we prove that the nonlinear convexity preserving subdivision scheme developed independently by Floater/Micchelli [12] and Kuijt/van Damme [14] exhibits a rather strong nonlinear, data-dependent, behavior: For any $\nu \in (1, 2)$, there exists initial convex data such that the critical Hölder regularity of the limit curve is exactly $\nu$. This result stands in contrast to what are reported in several recent publications on nonlinear subdivision schemes [18, 21, 20, 5], in which various families of nonlinear subdivision schemes are either proved or empirically observed to produce limit curves with smoothness insensitive to initial data.

Keywords. Weak and strong nonlinearity, Subdivision/Refinement scheme, Nonlinear subdivision scheme, Convexity preserving subdivision, Hölder regularity, Homogeneous map, Real projective plane

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Dedication: To the memory of Wong Suk Ling (1935-2004)

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1 Introduction

In recent years, it was either observed empirically or proved that certain nonlinear subdivision schemes arising from signal processing and geometric modeling applications exhibit the following *weakly nonlinear* property: the limit curve produced by the nonlinear subdivision scheme has a critical Hölder regularity the same as that of a related linear subdivision scheme. Notable examples include (i) median- and p-mean-interpolating subdivision schemes [21, 20, 8, 17, 16], (ii) refinement schemes of manifold-valued data [18, 19], and (iii) refinement schemes arising from normal multiresolution analysis [5]. A conceptually related discovery is reported in [3], where it is shown that a irregular grid variant of Dubuc’s 4-point interpolatory subdivision scheme has the exact same critical Hölder regularity as Dubuc’s scheme in the regular grid setting, so long as the irregularity of the successively refined grids is somehow controlled; and it is conjectured [3, 22] that a similar phenomenon holds for a wilder class of irregular grid subdivision schemes.

In this note, we show that the nonlinear convexity preserving subdivision scheme (2.1) by [14, 12] produces limit curves with critical Hölder exponent quite heavily dependent on the initial data, unlike the behavior of a linear or weakly nonlinear subdivision scheme. Note that the convexity preserving subdivision scheme is simply based on the harmonic mean and, similar to the aforementioned weakly nonlinear schemes, may not occur to be data dependent at first glance. (Caveat: This comment is only fair when the convexity preserving scheme (2.1) is applied to strictly convex data. When applied to general data, it is clear from the definition of $H(\cdot, \cdot)$ that the scheme is “explicitly data adaptive”. ) There exist nonlinear refinement schemes that are more data adaptive in appearance, e.g. the edge adapted or ENO refinement schemes in [2, 7].

While this paper is intended to be self-contained, the proof of our main result (Theorem 2.1) uses a key idea from [21], which the latter is described by the commutative diagram in Figure 1.

We reiterate the lame statement that a lot is known about linear subdivision, but little is known about their nonlinear counterparts; there are currently many unsolved open questions in the nonlinear subdivision literature, see the non-exhaustive list: [18, 19, 5, 21, 20, 8, 17, 13, 16, 15, 14, 12, 2]. Subdivision schemes in various geometric and nonlinear settings are of recent practical interests because of their natural connection with multiscale representations of different data types.

2 Convexity Preserving Subdivision

In [14, 12], the following nonlinear subdivision scheme is introduced: $f_{j+1} = Sf_j$ where $S : \ell(Z) \to \ell(Z)$ is defined by

$$f_{j+1,2k} = f_{j,k}, \quad f_{j+1,2k+1} = \frac{f_{j,k} + f_{j,k+1}}{2} - \frac{1}{8} H((\Delta^2 f_j)_{k-1}, (\Delta^2 f_j)_k). \quad (2.1)$$

Here $\Delta$ is forward difference operator, and $H(\cdot, \cdot)$ denotes harmonic mean, i.e. $H(a, b) = 2ab/(a + b)$ if $ab > 0$, and we define $H(a, b) = 0$ if $ab \leq 0$.

It is helpful to bear in mind the linear counterpart of (2.1) based on replacing the nonlinear harmonic mean by the linear arithmetic mean:

$$f_{j+1,2k} = f_{j,k}, \quad f_{j+1,2k+1} = \frac{f_{j,k} + f_{j,k+1}}{2} - \frac{1}{8} \text{Average}((\Delta^2 f_j)_{k-1}, (\Delta^2 f_j)_k)$$

$$= \frac{9}{16} (f_{j,k} + f_{j,k+1}) - \frac{1}{16} (f_{j,k-1} + f_{j,k+2}). \quad (2.2)$$

This is the well-known subdivision scheme by Dubuc [9]; we denote its subdivision operator by $\overline{S} : \ell(Z) \to \ell(Z)$.

For $r = 1, 2$, there exist (nonlinear) subdivision operators $S^{[r]}$ such that

$$S^{[r]} \circ \Delta^r = \Delta^r \circ S. \quad (2.3)$$
In particular, \( S^{[2]} \Delta^2 v = \Delta^2 Sv \); if we write \( w = \Delta^2 v \), we have

\[
(S^{[2]}w)_{2k} = (\Delta^2 Sv)_{2k} = (Sv)_{2k} - 2(Sv)_{2k+1} + (Sv)_{2k+2} \\
= v_k - 2\left[\frac{v_k + v_{k+1}}{2} - \frac{1}{8} H((\Delta^2 v)_{k-1}, (\Delta^2 v)_k) + v_{k+1}\right] \\
= \frac{1}{4} H(w_{k-1}, w_k),
\]

(2.4)

\[
(S^{[2]}w)_{2k+1} = (\Delta^2 Sv)_{2k+1} = (Sv)_{2k+1} - 2(Sv)_{2k+2} + (Sv)_{2k+3} \\
= \frac{v_k + v_{k+1}}{2} - \frac{1}{8} H((\Delta^2 v)_{k-1}, (\Delta^2 v)_k) - 2v_{k+1} + \\
\frac{v_{k+1} + v_{k+2}}{2} - \frac{1}{8} H((\Delta^2 v)_k, (\Delta^2 v)_{k+1}) \\
= \frac{w_k}{2} - \frac{1}{8} [H(w_{k-1}, w_k) + H(w_k, w_{k+1})].
\]

(2.5)

As a comparison, there exist linear subdivision operators \( S^{[r]} \), \( r = 1, 2, 3, 4 \), such that \( S^{[r]} \circ \Delta^r = \Delta^r \circ S \).

It is easy to check, using (2.4)-(2.5), that \( S^{[2]} \) is positivity preserving, consequently \( S^{[1]} \) is monotonicity preserving and \( S \) is convexity preserving. These properties of \( S \) are not shared by its linear counterpart \( S \). More in-depth discussions of the relationships among convexity preservation, nonlinear means and rational interpolation can be found in [14, 12, 11]. We denote by \( \mathbb{R}_+ \) the set of positive real numbers, and let \( \ell_+(\mathbb{Z}) := \{ v \mid v : \mathbb{Z} \to \mathbb{R}_+ \} \), and \( \ell_+^{cc}(\mathbb{Z}) := \ell_+(\mathbb{Z}) \cap \ell^{cc}(\mathbb{Z}) \). By the positivity preserving and locality properties of \( S^{[2]} \), we can view it as operators on either \( \ell_+(\mathbb{Z}) \) or \( \ell_+^{cc}(\mathbb{Z}) \).

![Figure 1: Commutation relations for S.](image)

By (2.4)-(2.5), we have

\[
(S^{[2]}w)_{l=2k,2k+1,2k+2} = D(w_{k-1}, w_k, w_{k+1}),
\]

(2.6)

where the nonlinear map \( D : \mathbb{R}^3 \to \mathbb{R}^3 \) is given by

\[
D(w_1, w_2, w_3) = \left( \frac{H(w_1, w_2)}{4}, \frac{w_2}{2} - \frac{1}{8} (H(w_1, w_2) + H(w_2, w_3)), \frac{H(w_2, w_3)}{4} \right).
\]

(2.7)

Since \( D \) is homogeneous (i.e. \( D(\lambda w) = \lambda D(w) \) for all \( \lambda \in \mathbb{R} \) and \( w \in \mathbb{R}^3 \)), it induces a quotient map \( \pi : P(\mathbb{R}^3) \to P(\mathbb{R}^3) \) via the formula \( \pi([v]_\sim) = [Dv]_\sim \). Here if \( V \) is a real vector space, \( P(V) := V/\sim = \{ [v]_\sim : v \in V \} \) where \( \sim \) is the equivalence relation defined by \( v \sim v' \iff \exists \ c \neq 0 \) such that \( v = cv' \).

Since \( D \) leaves \( \mathbb{R}^3_+ \) invariant, so does \( \pi \) to \( \{ [v]_\sim : v \in \mathbb{R}^3_+ \} \). Clearly \( \{ [v]_\sim : v \in \mathbb{R}^2_+ \} \) can be identified with \( \mathbb{R}^2_+ \) by pairing \( [x, 1, y]^T \) with \( (x, y) \) (here \( x, y > 0 \)). Viewing \( \pi \) as a map on \( \mathbb{R}^2_+ \) under this identification, we have, by (2.7),

\[
\pi(x, y) = \left( \frac{2x(1+y)}{2+x+y}, \frac{2y(1+x)}{2+x+y} \right).
\]

(2.8)

We have the following facts pertaining to this map:

[P1] \( (x, x) \) is a fixed point of \( \pi \) for any \( x > 0 \).
[P2] For any \(x, y > 0\), there is a unique \(r > 0\) such that \(\lim_{n \to \infty} \pi^n(x, y) = (\pi, \pi).

Proof: [P1] takes care of the case of \(x = y\). Assume \(x > y\), the other case is symmetrical. The observation that
\[
1 < \frac{xy + x}{xy + y} = \frac{\pi(x, y) \pi_1}{\pi(x, y) \pi_2} = \frac{x}{y} + \frac{1}{y} < \frac{x}{y}
\]
implies that as \(n\) increases the first component of \(\pi^n(x, y)\) decreases monotonically to a limit value \(\pi\), whereas the second component of \(\pi^n(x, y)\) increases monotonically to the same value. ■

[P3] For any \(0 < a < b\), \(\pi\) leaves the square \(R := [a, b] \times [a, b]\) invariant, i.e. \(\pi(R) \subseteq R\).

Proof: Let \((x, y) \in R\). Without loss of generality, assume \(x \geq y\). We seek to show that \((x', y') := \pi(x, y)\) continues to belong to \(R\). By (2.9), we have (i) \(x' \leq x\), (ii) \(y' \geq y\) and (iii) \(x' \geq y'\). Assume the contrary that \((x', y') \notin R\), then either \(x' < a\) or \(y' > b\). If \(x' < a\), then \(y' < a\) by (iii), so \(y < a\) by (ii), contradicting the assumption that \((x, y) \in R\). A similar contradiction can be generated if \(y' > b\). ■

The following is a more quantitative version of [P2], and is due to Sinan Güntürk. Not used in the proof of our main result, it may be useful for proving more refined properties of \(S\). See Section 3.

[P4] \(L(x, y) := xy/(2 + x + y)\) is invariant under \(\pi\), i.e. \(L(\pi(x, y)) = L(x, y)\). Therefore, for any \((x', y') \in \mathbb{R}_+^2\), the orbit \(\{\pi^n(x, y) : n = 0, 1, 2, \ldots\}\) lies on the level curve \(C = \{(x, y) : L(x, y) = x'y'/(2 + x' + y')\}\) and converges (in a monotonic fashion as described in [P2]) to the limit point \((\overline{\pi}, \overline{\pi})\) where \(\overline{\pi}\) is the positive root of the quadratic equation \(L(\pi, \pi) = x'y'/(2 + x' + y')\).

Next we consider the shrinking factor: for \((x, y) \neq (1, 1)\), define
\[
s(x, y) := \cfrac{\|\Delta D|[x, 1, y]|\|_{\infty}}{\|\Delta D|[x, 1, y]|\|_{\infty}} = \max \left\{ \cfrac{1}{2} - \cfrac{1}{2} H(1, x) - \cfrac{1}{2} H(1, y), \cfrac{1}{2} - \cfrac{1}{2} H(1, x) - \cfrac{1}{2} H(1, y) \right\}.
\]
We define \(s(1, 1) = 1/4\) in order to make \(s(x, x) = [2(1 + x)]^{-1}\) continuous on \(x > 0\). It is elementary to verify that

[S1] \(s(x, y) \in (0, 1/2)\) for \((x, y) \in \mathbb{R}_+^2\).

[S2] \(s(x, y)\) is discontinuous at \((1, 1)\).

[S3] \(s(x, y) \leq s(\pi(x, y))\) with equality holds iff \(x = y\).

See also Figure 2. Combining [S3] with [P3] and that \(s(x, x)\) is decreasing in \(x\), we have
\[
\sup_{\theta \in [a, b] \times [a, b]} s(\theta) = s(a, a) = [2(1 + a)]^{-1}. \tag{2.11}
\]

Let \(M, N\) be integers, \(M < N\). It is well-known in approximation theory, see e.g. [1, 6], that for \(f \in C([M, N])\), for \(\alpha > 0\), \(r \in \mathbb{Z}_+^+, r > \alpha\), we have
\[
f \in \operatorname{Lip} \alpha \iff \max_{2^r M \leq k \leq 2^r N - r} |(\Delta^r f_j)_k| = O(2^{-j\alpha}), \tag{2.12}
\]
where \(f_j\) is the sequence \((f_j)_k = f(2^{-j} k)\). This equivalence implies that the critical Hölder regularity exponent of \(f\) can be determined from the exact asymptotic decay rate of \(\max_{2^r M \leq k \leq 2^r N - r} |(\Delta^r f_j)_k|\) for a large enough differencing order \(r\), i.e.
\[
\sup\{\alpha : f \in \operatorname{Lip} \alpha\} = \sup \left\{ \alpha : \max_{2^r M \leq k \leq 2^r N - r} |(\Delta^r f_j)_k| = O(2^{-j\alpha}) \right\}. \tag{2.13}
\]

For a possibly unbounded continuous function \(f : \mathbb{R} \to \mathbb{R}\), we say \(f \in \operatorname{Lip} \alpha\) if \(f|_{[M, N]} \in \operatorname{Lip} \alpha\) for any \(M < N\).

Remark. Since \(S\) is (point-)interpolatory, the subdivision data \(S^j f_0\) is exactly the limit function \(f\) sampled on the grid \(2^{-j} \mathbb{Z}\), so the above result is directly applicable to analyzing the smoothness of \(f\). For other
subdivision schemes, linear or nonlinear, more subtle arguments related to stability are needed. See [21, Section 3] and the references therein.

As suggested by Figure 1, we shall use \( r = 3 \) to analyze the limit functions generated by the nonlinear convexity preserving scheme \( S \).

A essential fact based on the locality property of \( S \) is that if we specify the initial data \( f_{0,k} \) at the integers \( k = M - 2, \ldots, N + 2 \), then the limit function restricted to the interval \([M, N] \) is uniquely determined. For \( v \in \ell(\mathbb{Z}) \), we denote by \( S^\infty v \) or \( f_v : \mathbb{R} \to \mathbb{R} \) the limit function; it is shown in [14] that \( f_v \) is \( C^1 \) smooth for arbitrary strictly convex initial data \( v \), i.e. \( \Delta^2 v \in \ell_+^\infty(\mathbb{Z}) \). (Note: Property \([S1]\) already says that \( \|\Delta^2 S^j v\|_\infty = O((1/2)^j) \) if \( \Delta^2 v \in \ell_+^\infty(\mathbb{Z}) \), which implies that \( f_v \) is almost Lipschitz.)

Our main result is:

**Theorem 2.1** Let \( v \in \ell(\mathbb{Z}) \) be such that \( (\Delta^2 v)_{i-1} = (\Delta^2 v)_{i+1} \) and \( (\Delta^2 v)_i > 0 \) for all \( i \). Assume

\[
\mu := \frac{(\Delta^2 v)_{2i}}{(\Delta^2 v)_{2i-1}} \in (0, 1).
\]

(In particular, \( \Delta^2 v \in \ell_+^\infty(\mathbb{Z}) \) and is a 2-periodic sequence.) Then

\[
\sup\{\alpha : f_v \in \text{Lip} \alpha\} = \log_2 (1 + \mu).
\]

Therefore, by adjusting the value \( \mu \in (0, 1) \), one can construct strictly convex initial data \( v \) such that the limit function \( f_v \) has a critical Hölder exponent equals to any value in \((1, 2)\).

**Proof:** Write \( f_j := S^j v \in \ell(\mathbb{Z}) \). As \( r := 3 > 2 > \log_2 (1 + \mu) \), it suffices to use \( r = 3 \) in (2.13) and prove

\[
\|\Delta^3 f_j\|_\infty \approx [2(1 + \mu)]^{-j}, \quad j \to \infty.
\]

(2.16)

Note that \( s(x, x) = [2(1 + x)]^{-1} \) according to (2.10). The key point here is to relate the decay rate of \( \|\Delta^3 f_j\|_\infty \) to the maps \( \pi : \mathbb{R}_+^2 \to \mathbb{R}_+^2 \) and \( s : \mathbb{R}_+^2 \to (0, 1/2) \) introduced in (2.8) and (2.10).

Recall that \( \Delta^2 f_j = (S^{[2]})^j w \), where \( w := \Delta^2 v \). Recall also that the map \( D : \mathbb{R}^3 \to \mathbb{R}^3 \) in (2.7) describes the operator \( S^{[2]} \) via (2.6). Now, if we write

\[
\mathbb{R}_+^3 \ni w_{j,k} := (\{S^{[2]}\}^j w)_{i=k-2, k-1, k}, \quad k = 0, \ldots, 2^j,
\]

then, in virtue of (2.6) we have:

\[
w_{j+1,2k} = D(w_{j,k}), \quad k = 0, \ldots, 2^j, \quad w_{j+1,2k+1} = \left[w_{j+1,2k}^2, (w_{j+1,2k})_3 = (w_{j+1,2k+2})_2, (w_{j+1,2k+2})_3\right]^T, \quad k = 0, \ldots, 2^j - 1.
\]

(2.18)
Define $\theta_{j,k} = ((w_{j,k})_1, (w_{j,k})_3)/(w_{j,k})_2 \in \mathbb{R}^2_+$. Then (2.18) gives
\[
\begin{align*}
\theta_{j+1,2k} &= \pi(\theta_{j,k}), \quad k = 0, \ldots, 2^j, \\
\theta_{j+1,2k+1} &= \Xi(\theta_{j+1,2k}, \theta_{j+1,2k+2}), \quad k = 0, \ldots, 2^j - 1,
\end{align*}
\]
(2.19)
where $\Xi : \mathbb{R}^2_+ \times \mathbb{R}^2_+ \rightarrow \mathbb{R}^2_+$, $\Xi((x,y),(x',y')) = (1/y, 1/x')$. (See Figure 3.)

Under these notations, the assumption (2.14) of the theorem is equivalent to saying
\[
\theta_{0,0} = (\mu, \mu), \text{ and } \theta_{0,1} = (1/\mu, 1/\mu),
\]
and (2.16) is, by symmetry of the data and the subdivision scheme, equivalent to
\[
\max_{k=0,\ldots,2^j} \|\Delta w_{j,k}\|_\infty \asymp [2(1+\mu)]^{-j}, \quad j \to \infty.
\]
By the overlapping properties of $w_{j,k}$’s, recall (2.17), we do not need to use all the spatial indices $k$ for a given scale $j$, and (2.21) is equivalent to
\[
\max_{k=0,\ldots,2^j} \|\Delta w_{j+1,2k}\|_\infty \asymp [2(1+\mu)]^{-(j+1)}, \quad j \to \infty.
\]
(2.22)

Let $R_\mu := [\mu, 1/\mu] \times [\mu, 1/\mu]$. It is clear that $\Xi(\theta, \theta') \in R_\mu$ if $\theta, \theta' \in R_\mu$, together with property [P3] ($\pi(R_\mu) \subset R_\mu$), we conclude that
\[
\theta_{j,k} \in R_\mu, \quad \forall \, j, k.
\]
(2.23)
So by (2.11), we upper bound all the shrinking factors as:
\[
s(\theta_{j,k}) \leq s(\mu, \mu) = [2(1+\mu)]^{-1}, \quad \forall \, j, k.
\]
(2.24)
Notice also that
\[
\theta_{j,0} = (\mu, \mu), \quad \forall \, j \geq 0.
\]
(2.25)
Since
\[
\|\Delta w_{j+1,2k}\|_\infty = s(\theta_{j,k}) \|\Delta w_{j,k}\|_\infty,
\]
together with (2.24) we have
\[
\max_{k=0,\ldots,2^j} \|\Delta w_{j+1,2k}\|_\infty = O([2(1+\mu)]^{-(j+1)}).
\]
(2.26)
On the other hand $\max_{k=0,\ldots,2^j} \|\Delta w_{j+1,2k}\|_\infty = \Omega([2(1+\mu)]^{-(j+1)})$:
\[
\max_{k} \|\Delta w_{j,k}\|_\infty \geq \|\Delta w_{j,0}\|_\infty = \|\Delta w_{0,0}\|_\infty \sum_{l=1}^j s(\theta_{l,0}) \geq \|\Delta w_{0,0}\|_\infty s(\mu, \mu)^j.
\]
So we have proved (2.16).

In contrast, it is well-known that for Dubuc’s scheme $\mathfrak{S}$,
\[
\sup \{ \nu : \mathfrak{S}^\nu v \in \text{Lip } \nu \} = \begin{cases} 2, & \text{if } v \in \ell(\mathbb{Z}) \backslash \Pi_3|_\mathbb{Z} \\ \infty, & \text{if } v \in \Pi_3|_\mathbb{Z} \end{cases}
\]
(2.27)
In other words, except for low order polynomial initial data, the critical Hölder regularity of the limit curve is 2. \footnote{More precisely: $f_0$ satisfies $|f_0'(x+t) - f_0'(x)| = O(t \log(1/t))$ and this bound cannot be improved unless for data sampled from a cubic polynomial.} This is characteristic of linear subdivision schemes and of the weakly nonlinear schemes mentioned in Section 1.
On the other hand, a classical result of A. D. Alexsandrov (see, e.g., [10]) asserts that regularity is strictly less than 2, meaning that at those dyadic rationals property [P2] and that except that it can be more easily explained than those linear examples in [4, Section 4]: On the one hand,

But a proof or disproof is yet to be found.

Figure 3: Recursive definition of $\theta_{j,k}$

![Recursive definition of $\theta_{j,k}$](image)

Figure 4: For each $\theta_{j,k} \in \mathbb{R}_+^2$ defined by (2.19), a circle with center $\theta_{j,k}$ and radius which decreases linearly with $j$ is drawn. Notice that $\theta_{j,k}$ tends to cluster at $(1,1)$.

### 3 Observations

Not pursued here, it is possible to determine the critical Hölder regularity of more general initial data based on the machinery in the proof of Theorem 2.1. This will require understanding the “path” in Figure 3 that gives the worst asymptotic shrinking rate for a general initial configuration $\theta_{0,0} = (x,y)$, $\theta_{0,1} = (1/y,z)$, $x, y, z > 0$. In the special case of $x = y = 1/z \in (0,1)$, the “worst shrinking path” is $k_j = 0$ for all $j$, i.e. $\max_k \| \Delta w_{j,k} \|_\infty \asymp \| \Delta w_{j,0} \|_\infty$. Computational experiments suggest that for general initial data,

$$\max_k \| \Delta w_{j,k} \|_\infty \asymp \| \Delta w_{j,0} \|_\infty, \quad k_j = 0, \text{ or } 2^{j-1}, \text{ or } 2^j.$$  

But a proof or disproof is yet to be found.

For linear subdivision schemes, it is typical that the local Hölder exponents at dyadic irrationals are higher than the global critical Hölder regularity, see [4, Section 4]. (Here we assume that the subdivision scheme is binary.) A similar property holds for the nonlinear convexity preserving scheme in this paper, except that it can be more easily explained than those linear examples in [4, Section 4]: On the one hand, property [P2] and that $s(\pi, \pi) < 1/4$ for $\pi < 1$ imply that at least many dyadic rationals, the local Hölder regularity is strictly less than 2, meaning that at those dyadic rationals $f_\nu$ cannot be twice differentiable. On the other hand, a classical result of A. D. Alexsandrov (see, e.g., [10]) asserts that $f_\nu$, being a convex
function in $\mathbb{R}$, must be twice differentiable almost everywhere.

(Alexsandrov’s theorem partly accounts for the observed clustering of $\theta_{j,k}$ about $(1,1)$ for arbitrary initial configuration $x, y, z > 0$ (see e.g. Figure 4) – a fact that seems difficult to explain by elementary means.)

References


