Spectral element/spectral vanishing viscosity methods for large eddy simulation of turbulent flows *

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Abstract
A stabilization method for the spectral element computation of incompressible turbulent flow problems is investigated. It is based on a filtering procedure which consists in filtering the velocity field by a spectral vanishing Helmholtz-type operator at each time step. Relationship between this filtering procedure and SVV-stabilization method, introduced recently in [JCP, 2004, 196(2), p680], is established. A number of numerical examples including LES simulation of the cylinder flow, are presented to show the accuracy and stabilization capability of the method.

1 Introduction
Despite the success of the spectral element methods (SEM) in the applications of, among many examples, incompressible flows, severe stability problem has also been encountered in the past, especially when facing problems having weak physical diffusion. This results from the fact that spectral approximations are much less numerically diffusive than low-order ones, even minor errors and under resolution can make the calculation unstable. For example, a well-known difficulty in spectral approximations of the Navier-Stokes equations is the enforcement of the incompressibility constraint on \( u \), and numerical treatment of the non-linear convective term \( u \cdot \nabla u \).

The \( P_N \times P_{N-2} \) SEM introduced in [8] addresses the first problem through the use of compatible velocity and pressure spaces that are free of pressure spurious modes. However the standard \( P_N \times P_{N-2} \) SEM suffers from the coupling of the velocity and pressure field. An alternative method to deal with the incompressibility is of the family of projection methods. All these methods attain exponential convergence in space, and works well in the computation of low Reynolds number flows. However stability problems arise in the applications for moderate to high Reynolds numbers. For a long time, numerous filtering techniques have been proposed to overcome the stability problem. In the frame of spectral element approximations it is however essential to preserve the interelement continuity, as discussed in [1].

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One of the most recent advances in this direction has been proposed in [2], where an interpolation-based stabilization procedure is applied at the end of each time step of the Navier-Stokes discretization. This procedure was then interpreted in [10] as a filter if \( L_N - L_{N-2} \) basis is used to expand the velocity.

Another stabilization technique has been very recently proposed by Xu and Pasquetti in [14], where the spectral vanishing viscosity (SVV) method was integrated into the weak formulation of the Navier-Stokes equations. This technique has been proven to be an efficient stabilization method possessing the properties of the interelement continuity and the spectral accuracy. The SVV method was initially developed for the resolution of hyperbolic equations using the spectral methods [12, 7, 5]. Recently, it has been suggested to use the SVV method, using modal basis, for the large-eddy simulation of turbulent flows [6, 9]. However for the first time we have generalized in [14] the SVV idea to the multi-dimensional case with general deformed elements using nodal basis.

The present work follows the subject of [14], and try to give an interpretation of the SVV-SEM as a filtering technique, the latter takes advantage of being simpler to use, and no need to modifying the existing SEM code. To this end:

First, we recall the SVV stabilized SEM introduced in [14], then try to interpret it as a filtering procedure. We show that the filtering procedure is a reasonable approximation to the SVV-SEM, and that in certain cases, the two approaches are strictly equivalent.

Second, we show how to implement the filtering technique in the frame of a Navier-Stokes spectral element solver.

Third, we consider an analytical solution to study the accuracy property. We will see that the filtering procedure not only stabilizes the calculation, but also improves the accuracy of the overall schema.

Finally, in order to numerically demonstrate the stabilization property of the method, we compute the shear layer roll-up problem at Reynolds numbers \( Re = 10^5 \) and 3D Von Karman flows behind a cylinder at \( Re = 3900 \).

## 2 Stabilized spectral element formulation

The spectral element approximation of the incompressible Navier-Stokes equations yields the following semi-discrete problem [8], to be solved at each time-step after the time-discretization: Find \( u_N \in X_N \) and \( p_N \in M_N \) such that

\[
\begin{cases}
(Dp u_N, v_N) + \nu (\nabla u_N, \nabla v_N) - (\nabla \cdot v_N, p_N) = (s_N, v_N), \quad \forall v_N \in X_N \\
(\nabla \cdot u_N, q_N) = 0, \quad \forall q_N \in M_N
\end{cases}
\]  

(1)

where \( u_N, p_N \) and \( s_N \) denote the approximations of the velocity, pressure and source term respectively, \( Dp u_N \) the material derivative of \( u_N \), \( \nu \) the dimensionless viscosity and \((\cdot, \cdot)\) the standard \( L^2(\Omega) \) inner product. The domain \( \Omega \), assumed to be two dimensional for the sake of simplicity, is partitioned into a conforming decomposition

\[
\tilde{\Omega} = \bigcup_{k=1}^{K} \tilde{\Omega}^k, \quad \Omega^k \cap \Omega^{\ell} = \emptyset, \quad \forall k, \ell, k \neq \ell.
\]
The discrete velocity space $X_N$ and pressure space $M_N$ consist of,

$$X_N = P_{N,K}(\Omega)^2 \cap H_0^1(\Omega)^2, \quad M_N = P_{N-2,K}(\Omega) \cap L_0^2(\Omega)$$

with standard notations for the Hilbert spaces $H_0^1(\Omega)$ and $L_0^2(\Omega)$, and with:

$$P_{N,K}(\Omega) = \{ v \in L^2(\Omega); v|_\Omega \circ f^k \in P_N(\Lambda^2), 1 \leq k \leq K \}$$

where $f^k$ is the transformation function from the reference domain $\Lambda^2$, with $\Lambda = (-1, 1)$, to $\Omega^k$ and $P_N$ the space of the polynomials of maximum degree $N$ in each variable. For the reason of simplicity again, homogeneous boundary conditions have been assumed through the use of the $H_0^1(\Omega)$ space.

In [14], a stabilized spectral element approximation of problem (1) was introduced and analyzed. The proposed method consists in adding a stabilization term $V_N$ in (1), yielding the following weak formulation

\[
\begin{align*}
(D_t u_N, v_N) + V_N(\nabla u_N, \nabla v_N) + \nu(\nabla u_N, \nabla v_N) - (\nabla \cdot v_N, p_N) &= (s_N, v_N), \quad \forall v_N \in X_N \\
(\nabla \cdot u_N, q_N) &= 0, \quad \forall q_N \in M_N
\end{align*}
\]

(2)

Where $V_N(\nabla u_N, \nabla v_N)$ is a spectral vanishing viscosity term written in weak form.

Let $f$ be the mapping from $(X, Y)$ in the reference domain $\Lambda^2$, to $(x, y)$ in the spectral element $\Omega^k$ and $g = f^{-1}$, $G$ the transpose of the Jacobian matrix of $g$ and $J$ the Jacobian determinant of $f$. (To simplify the notation we use here $f, g, ..., rather than f^k, g^k, ...$). Then the definition of $V_N$ used in [14] takes the symmetric bilinear form corresponding to the $k$-element:

$$V_N^k(\nabla u_N, \nabla v_N) = \epsilon_N (\tilde{Q}^{1/2}(\tilde{\nabla} \tilde{u}_N), J\tilde{G}Q^{1/2}(\tilde{\nabla} \tilde{v}_N))_{L^2(\Lambda^2)}, \quad k = 1, \cdots, K$$

(3)

where $\epsilon_N = O(1/N)$, $\tilde{\nabla}$ denotes the gradient with respect to the reference domain and with $\tilde{\varphi}_N = \varphi_N \circ f$. The meaning of $\tilde{Q}^{1/2}(\tilde{\nabla} \tilde{u}_N)$ makes use the 1D scalar definition of $Q$:

$$\tilde{Q}^{1/2}(\tilde{\nabla} \tilde{u}_N) = \begin{pmatrix} Q^{1/2}(\partial_X \tilde{u}_N(X, \cdot)) \\ Q^{1/2}(\partial_Y \tilde{u}_N(X, \cdot)) \end{pmatrix}.$$

Finally, global $V_N$ is defined as the summation:

$$V_N(\nabla u_N, \nabla v_N) = \sum_{k=1}^K V_N^k(\nabla u_N, \nabla v_N).$$

(4)

The efficient implementation of the stabilized spectral element approximation (2) is quite technique, we refer to [14] for a reasoning for the definition (3) and its detailed implementation.
3 Filter-based stabilizing procedure

In this section, we will first introduce a filtering procedure, then try to give an interpretation of the SVV stabilization method presented in the previous section. We first go back to the standard $P_N \times P_{N-2}$ spectral element semi-discrete problem (1) without stabilization. We introduce the following second-order rotational projection schema [3, 4] plus a filtering step to discretize in time the problem (1):

- Diffusion step (computation of the intermediate velocity):
  Find $u_N^s \in X_N$, such that for all $v_N \in X_N$,

  $\frac{3u_N^s - 4u_{N-1}^s + u_{N-2}^s}{2\Delta t} + \nu(\nabla u_N^s, \nabla v_N) = (\nabla \cdot v_N, p_N) + (s_{N+1}^s, v_N), \quad (5)$

  where $\Delta t$ is the time step. $u_{N-1}^s, u_{N-2}^s$ are the transport at different instant of the previous solution on the characteristics. A detailed description on their computation was given in [13].

- Filtering step:
  Once $u_N^s$ is obtained, we filter it by the following procedure: Find $\bar{u}_N^s$, such that

  $w_N = \bar{u}_N^s - u_N^s \in X_N$, satisfying

  $(w_N, v_N) + \frac{2\Delta t}{3} V_N(\nabla w_N, \nabla v_N) = \frac{2\Delta t}{3} V_N(\nabla u_N^s, \nabla v_N), \forall v_N \in X_N \quad (6)$

  where the bilinear form $V_N$ is defined in (4). It is readily seen that the filtering procedure preserves the interelement continuity and the boundary conditions, i.e.

  $\bar{u}_N^s|_{\partial \Omega} = u_N^s|_{\partial \Omega}$.

  In fact, we see that (6) is equivalent to the following problem: Find $\bar{u}_N^s$, such that

  $\bar{u}_N^s - u_N^s \in X_N$, and

  $(\bar{u}_N^s, v_N) + \frac{2\Delta t}{3} V_N(\nabla \bar{u}_N^s, \nabla v_N) = (u_N^s, v_N), \quad \forall v_N \in X_N \quad (7)$

  It is worth noting that, according to the definition (4) of the operator $V_N$, problem (7) is indeed a spectral vanishing Helmholtz problem. Since the SVV term $V_N$ is only active on the last highest spatial frequencies (i.e. on modes $L_i$ for $i > m_N$ in 1D case), the corresponding filtering procedure has effect of damping the component of the high spatial frequencies of $u_N^s$.

  Hereafter we denote the stabilization procedure (7) by

  $\bar{u}_N^s = F_H u_N^s \quad (8)$

  with $F_H$ representing the spectral vanishing Helmholtz operator in strong form.

- Projection step:
Find \( u_{N+1}^n \in P_{N,K}(\Omega)^2, p_{N+1}^n \in M_N \) such that

\[
\begin{cases}
\left( \frac{3u_{N+1}^n - 3\tilde{a}_N^*}{2\Delta t}, v_N \right) + \left( \nabla(p_{N+1}^n - p_N^n + \nu \nabla \cdot \tilde{a}_N^*), v_N \right) = 0, \\
(u_{N+1}^n, \nabla q_N) = 0, \quad \forall q_N \in M_N.
\end{cases}
\]

(9)

**Remark 1:** The step (9) is a realization of projecting \( \tilde{a}_N^* \) into a space of weak divergence free. In the implementation, we take \( v_N = \nabla q_N \) in the first equation of (9), so that we obtain a discrete Poisson equation for the pressure increment \( p_{N+1}^n - p_N^n + \nu \nabla \cdot \tilde{a}_N^* \).

**Remark 2:** It is the intermediate velocity \( u_N^* \) that we choose to filter, so that the weak divergence-free property of the final velocity \( u_{N+1}^n \) is preserved. This property is desirable in the calculation of the velocity transport \( u_{N+1}^n \).

### 3.1 Relationship with the SVV formulation

It can be shown that the filtering procedure (5)-(7)-(9) is a reasonable approximation to the SVV formulation (2). To see that, we need to establish the equations satisfied by the filtered velocity.

Substituting \( u_N^* \) in (5) by \( F_H^{-1} \tilde{u}_N^* \) gives

\[
\left( \frac{3F_H^{-1} \tilde{u}_N^* - 4 \hat{a}_N^* + \hat{a}_N^{n-1}}{2\Delta t}, v_N \right) + \nu \left( \nabla \cdot F_H^{-1} \tilde{u}_N^*, \nabla v_N \right) = \left( \nabla \cdot v_N, p_N^1 \right) + \left( s_N^{n+1}, v_N \right).
\]

From the definition of \( F_H \), it is readily seen that

\[
\begin{align*}
\left( \frac{3\tilde{a}_N^* - 4\hat{a}_N^* + \hat{a}_N^{n-1}}{2\Delta t}, v_N \right) + V_N(\nabla \tilde{a}_N^*, \nabla v_N) + \nu(\nabla \tilde{a}_N^*, \nabla v_N) \\
- \frac{2\Delta t}{3} \nu \epsilon_N(\nabla \cdot Q \nabla \tilde{a}_N^*, \nabla v_N) = \left( \nabla \cdot v_N, p_N^1 \right) + \left( s_N^{n+1}, v_N \right),
\end{align*}
\]

(10)

If we ignore the high order term \( \frac{2\Delta t}{3} \nu \epsilon_N(\nabla \cdot Q \nabla \tilde{a}_N^*, \nabla v_N) \), then combining (10) and (9) gives

\[
\begin{align*}
\left( \frac{3\tilde{a}_N^* - 4\hat{a}_N^* + \hat{a}_N^{n-1}}{2\Delta t}, v_N \right) + V_N(\nabla \tilde{a}_N^*, \nabla v_N) + \nu(\nabla \tilde{a}_N^*, \nabla v_N) \\
= \left( \nabla \cdot v_N, p_N^1 \right) + \left( s_N^{n+1}, v_N \right),
\end{align*}
\]

\[
\begin{cases}
\left( \frac{3u_{N+1}^n - 3\tilde{a}_N^*}{2\Delta t}, v_N \right) + \left( \nabla(p_{N+1}^n - p_N^n + \nu \nabla \cdot \tilde{a}_N^*), v_N \right) = 0, \\
(u_{N+1}^n, \nabla q_N) = 0, \quad \forall q_N \in M_N.
\end{cases}
\]

which is nothing else than the second order rotational projection schema of the stabilized spectral element formulation (2).

It can be easily shown that in the Fourier case \(-\langle \nabla \cdot Q \nabla \tilde{a}_N^*, \nabla v_N \rangle \) is purely diffusive, hence helps to stabilizing the schema. In the general case (not periodic), we
are unable to prove the diffusibility, however we note that this additional term has a small magnitude tending to zero when $\Delta t\nu \epsilon_N$ tends to zero. Hence we can expect that the difference on the stabilization capability between the proposed filtering procedure and SVV-SEM is negligible. This point will be confirmed through the computation of the shear layer roll-up problem and 2D high Reynolds number wake of a cylinder.

We should note that in the case where the physical diffusion is absent (i.e. $\nu = 0$), the filtered-SEM and SVV-SEM are strictly equivalent.

### 3.2 Implementation technique

The filtering procedure (7) can be handled in standard way by spectral element methods. However, in view of numerical efficiency, it would be of interest to employ the existing Helmholtz solver.

We start by approximating problem (7) by using Gauss quadrature integral in each element $\Omega^k, k = 1, \cdots K$: Given $u^*_N, \tilde{u}^*_N - u^*_N \in X_N$, such that

$$
(\tilde{u}^*_N, v_N) + \frac{2\Delta t}{3} \mathcal{V}_N(\nabla \tilde{u}^*_N, \nabla v_N) = (u^*_N, v_N)_N, \quad \forall v_N \in X_N
$$

(11)

where $(\cdot, \cdot)_N$ stands for the summation of the Gauss numerical integral over $\Omega^k$ by Gauss-Lobatto quadrature formulas. $\mathcal{V}_N$ is defined by

$$
\mathcal{V}_N(\nabla \tilde{u}^*_N, \nabla v_N) = \sum_{k=1}^{K} \mathcal{V}^k_N(\nabla \tilde{u}^*_N, \nabla v_N),
$$

with $\mathcal{V}^k_N(\nabla \tilde{u}^*_N, \nabla v_N)$ is an approximation of $V^k_N(\nabla \tilde{u}^*_N, \nabla v_N)$, defined in (3), by using Gauss-Lobatto quadrature.

In general case of deformed domain decomposition, a simple calculation shows that $\mathcal{V}^k_N$ takes form

$$
\mathcal{V}^k_N(\nabla \tilde{u}^*_N, \nabla v_N) = \epsilon_N \sum_{i,j=0}^{N} [F^k_1 Q^{1/2} \partial_X \tilde{u}^*_N Q^{1/2} \partial_X v^k_N + F^k_2 Q^{1/2} \partial_Y \tilde{u}^*_N Q^{1/2} \partial_Y v^k_N + F^k_3 (Q^{1/2} \partial_X \tilde{u}^*_N Q^{1/2} \partial_Y v^k_N + Q^{1/2} \partial_Y \tilde{u}^*_N Q^{1/2} \partial_X v^k_N)](\xi_{ij}) \frac{\rho_{ij}}{J^k(\xi_{ij})}
$$

where $\tilde{u}^*_N = u_N \circ f^k$, $\xi_{ij} = (\xi_i, \xi_j)$, $\rho_{ij} = \rho_i \rho_j$ with $\xi_i, \rho_i, i = 0, \cdots, N$ denoting the Gauss-Lobatto points and corresponding weights in $[-1, 1]$. $F^k_1, F^k_2, F^k_3$ are three geometric factors, defined as

$$
F^k_1 := (\partial_Y f^k_1)^2 + (\partial_Y f^k_2)^2 = (J^k)^2[(\partial_Y g^k_1)^2 + (\partial_X g^k_1)^2]
$$
$$
F^k_2 := (\partial_X f^k_1)^2 + (\partial_X f^k_2)^2 = (J^k)^2[(\partial_Y g^k_2)^2 + (\partial_X g^k_2)^2]
$$
$$
F^k_3 := -(\partial_X f^k_1 \partial_Y f^k_2 + \partial_X f^k_2 \partial_Y f^k_1) = (J^k)^2[\partial_Y g^k_2 \partial_Y g^k_1 + \partial_X g^k_2 \partial_X g^k_1]
$$

where $f^k_1$ and $f^k_2$ are two components of the mapping $f^k$, notations $g^k_1, g^k_2$ are similar.
By choosing each test function \( v_N \) to be the Lagrangian basis \( h_{mn} \) for \( X_N \) and expressing \( \tilde{u}_N \) in terms of these basis, we arrive at a matrix statement of the elemental term \( V_N^k(\nabla \tilde{u}_N, \nabla v_N^k) \) corresponding to the basis at interior mesh points, with \( \tilde{u}_N^k \) for \( \tilde{u}_N^k(\xi_{ij}) \):

\[
\begin{align*}
\epsilon_N & \sum_{i=0}^{N} \left[ \frac{\rho_{in}}{J^k(\xi_{in})} \right] F_{1,in}^k(QD)_{im} \left( \sum_{p=0}^{N} (QD)_{ip} \tilde{u}_N^k \right) \\
+ & \epsilon_N \sum_{j=0}^{N} \left[ \frac{\rho_{mj}}{J^k(\xi_{mj})} \right] F_{2,mj}^k(QD)_{jn} \left( \sum_{q=0}^{N} (QD)_{jq} \tilde{u}_N^k \right) \\
+ & \epsilon_N \sum_{j=0}^{N} \left[ \frac{\rho_{mj}}{J^k(\xi_{mj})} \right] F_{3,mj}^k(QD)_{jn} \left( \sum_{p=0}^{N} (QD)_{mp} \tilde{u}_N^k \right) \\
+ & \epsilon_N \sum_{i=0}^{N} \left[ \frac{\rho_{in}}{J^k(\xi_{in})} \right] F_{3,in}^k(QD)_{im} \left( \sum_{q=0}^{N} (QD)_{nq} \tilde{u}_N^k \right)
\end{align*}
\]

where \( D \) is the Legendre derivative matrix, \( Q \) stands for the spectral viscosity operator matrix, which can be expressed in form

\[
Q = M^{-1} \text{diag}(Q_i)^{1/2} M
\]

with \( M \) for passage matrix from physical space to spectral space.

The expression corresponding to \( m,n = 0 \) or \( N \) should take into account the elemental interface stiffness summation.

In practice we just need replace the classical derivative matrix \( D \) by its modified matrix \( QD \). Use of this combination technique allows us to maintain the same cost as the standard Helmholtz.

Note that, since operator \( Q \) is just active on the last highest modes and the “artificial viscosity” \( \frac{\rho_{in}}{J^k(\xi_{in})} \) is generally small, the filtering problem (11) results in a well-conditioned symmetric positive definite system, hence can be inverted by standard elliptic solvers like conjugate gradient algorithm.

4 Numerical results

The purpose of this section is to investigate the convergence and stability property of the filtered-SEM. To this end we first consider the Navier-Stokes equations in \( \Lambda^2 \), with a stiff analytical solution, to demonstrate that the filtered-SEM not only keeps exponentially accurate, but also improves the accuracy as compared to the unfiltered case. Then the shear layer roll-up problem and Von-Karman flow past a cylinder is computed to show the stabilization capability of the method.

4.1 An analytical solution

In this first test, \( \nu \) is fixed to be \( 10^{-2} \). In order to check numerically the effect of the filtering procedure (7) on the spectral element solution, we choose the analytical
exact solution:
\[
\begin{align*}
    u_1(x, y, t) &= \sin(2\pi x) \cos(2\pi y) \sin(t) \\
    u_2(x, y, t) &= -\cos(2\pi x) \sin(2\pi y) \sin(t)
\end{align*}
\]
and the computational domain $\Lambda^2$ is partitioned into $10 \times 10$ square elements.

Fig. 1 shows some velocity errors at $t = 1$ in $H^1, L^\infty$ and $L^2$ norms obtained with the filtered-SEM and the non-filtered case respectively. Here, as motivated by the numerical experiences performed in [14], the filtering is applied with parameters $m_N = N - 2$ and $\epsilon_N = 1/N$. Surprisingly, contrast to our previous results [14] obtained by solving the steady Helmholtz equations, the stabilized-SEM not only preserves the exponential convergence rate (the errors show an exponential decay when the polynomial degree is increased), but also is more accurate than the standard (non-filtered) SEM.

In order to know whether the SVV-SEM introduced in [14] possesses similar property, this time-dependent solution is also calculated by using SVV-SEM. The $L^2$-errors on $u$ as a function of time $t$ is plotted in Fig. 2, showing that more accurate solution is obtained by both SVV-SEM and filtered-SEM, and that the difference between the SVV-SEM and filtered-SEM is insignificant. This confirms our analysis given in section 3.1.

The non-linear convective term may be responsible for different behaviors between the steady Helmholtz equations and unsteady Navier-Stokes equations: in the latter case the spurious "high-frequency modes" resulting from aliasing effects may be damped when the SVV or filtering is activated, resulting in better results.

![Figure 1: Errors on u in the $H^1$, $L^\infty$ and $L^2$ norms as a function of the polynomial degree $N$, obtained without filtering and with filtering.](image)

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In Fig. 2, the $L^2$-errors on $u$ as a function of time $t$ are plotted for different polynomial degrees $N$. The figure compares the errors obtained with and without filtering for each norm $H^1$, $L^\infty$, and $L^2$. The error is plotted on a log scale for better visualization. The results show a significant reduction in error for higher polynomial degrees, especially for the $L^2$ norm. The filtering appears to improve the convergence rate, as the errors decrease exponentially with increasing degree $N$.
Figure 2: $L^2$ Errors on $u$ as a function of the time, obtained by SEM, SVV-SEM and filtered SEM.

4.2 Applications

In this subsection the efficiency of the proposed filtered SEM is demonstrated through the computation of the shear layer roll-up problem and 2D high Reynolds number Vortex-Karman flow behind a cylinder.

4.2.1 Shear layer roll-up problem

As a well-known test problem [2], shear layer roll-up flow is computed in the domain $[0, 1]^2$, with doubly-periodic boundary conditions and following initial conditions

$$u_1(x, y, t) = \begin{cases} 
\tanh(30(y - 0.25)) & \text{for } y \leq 0.5 \\
\tanh(30(0.75 - y)) & \text{for } y > 0.5 
\end{cases}, \quad u_2(x, y, t) = 0.05 \sin(2\pi x).$$

In all case, the domain is broken into 16 \times 16 equal square elements. The Reynolds number is defined by $Re = \frac{L u_{\text{max}}}{\nu}$, where $L$ and $u_{\text{max}}$ are respectively the dimensioned side length and maximum training velocity. Fig.3(a) shows the vorticity for Reynolds number $Re = 10^3$ at $t = 1.055$ for $N = 8$, $t = 1.035$ for $N = 16$, just prior to blow up for the standard SEM. Increasing $N$ and $K$ and decreasing $\Delta t$ up to reasonable values do not help to stabilize the simulation, as already indicated in [2]. Filtering the intermediate velocity dramatically improves the stability, as shown in Fig.3(c), where vorticity contours at $t = 1.5$ are plotted for filtered SEM with $N = 8$ and $N = 16$ respectively. In order to avoid the insignificant zero contour, an even number of contour levels has been used. Also shown in Fig.3(b) is the result by SVV-SEM: no significant difference is observed as compared with the filtered SEM.
Figure 3: Vorticity contours (-70 < ω < 70, 14 equidistant levels) for different methods: (a) Standard SEM; (b) SVV-SEM; (c) Filtered-SEM.
4.2.2 Turbulent wake flow past a cylinder at Re=3900

Several calculations have been carried out for the flow around an impulsively started circular cylinder, for various Reynolds numbers. Our goal here is to demonstrate the capabilities of the SVV filtered-SEM.

The configuration of the computational domain is: $]-3.6, 12[ \times ]-3.6, 3.6[0,4[$, with the cylinder axis located at $x = 0, y = 0$. The method employed in the simulation is the Legendre Spectral Element for $(x,y)$ plan, and Fourier method in $z$-direction. In our calculations, the number of the elements in $(x,y)$ has been fixed to $K = 310$, the polynomail degree is $N = 7$, and the Fourier grid number is 60.

The Reynolds number is defined as $Re = U_\infty D/\nu$, where $U_\infty$ is the free stream velocity and $D$ is the cylinder diameter. As already pointed out in [14], in this domain decomposition, using the standard SEM we were unable to compute flows at $Re \geq 500$ at any reasonable resolution.

The stabilization effect of the filtering is checked by the long time simulation of the turbulent wake at $Re = 3900$. Fig.4 shows the isosurfaces of the streamwise component of the vorticity for 20 equi-distributed values at $t = 120$. Visualization of the vorticity shows that the flow is fully 3D and quickly oscillating. Fig.5 shows the isosurfaces of the Q-criterion, which points out the turbulent feature of the flow. Detailed comparison with the existing numerical and experiential results is on going.

References


Figure 4: Isosurfaces of the streamwise component of $\omega$ for 20 equi-distributed values at $t = 120$. 
Figure 5: Isosurfaces of the Q-criterion at $t = 120$. 


