Let $\mathcal{H}_\nu$ be weighted Bergman space on a bounded symmetric domain $D = G/K$. It has analytic continuation in the weight $\nu$ and for $\nu$ in the so-called Wallach set $\mathcal{H}_\nu$ still forms unitary irreducible (projective) representations of $G$. We give the irreducible decomposition of the tensor product $\mathcal{H}_{\nu_1} \otimes \mathcal{H}_{\nu_2}$ of the representation for any two unitary weights $\nu$ and we find the highest weight vectors of the irreducible components. We find also certain bilinear differential intertwining operators realizing the decomposition, and they generalize the classical transvectants in invariant theory of $SL(2, \mathbb{C})$. As an application we also find an generalization of the Bol’s lemma.

1. **Introduction**

Let $D$ be a bounded symmetric domain in a complex space $V = \mathbb{C}^d$, and let $G$ be the group of biholomorphic automorphisms of $D$ and $K$ the isotropic subgroup of $0 \in D$. Then $D = G/K$ is a symmetric space. The weighted Bergman spaces $\mathcal{H}_\nu$ of holomorphic functions on $D$ form unitary projective representations of $G$ and they are an important subject of study both in complex analysis and in representation theory. Those spaces have analytic continuation with respect to weight parameter $\nu$ and the whole set of the parameter for which we still have a unitary representation has been determined and is also called the Wallach set (some times Berezin-Wallach set); see §2. In the present paper we study the irreducible decomposition of the tensor product $\mathcal{H}_{\nu_1} \otimes \mathcal{H}_{\nu_2}$ for any $\nu_1$ and $\nu_2$ in the Wallach set. We find the highest weight vector of each irreducible component and we find explicit intertwining operator realizing the component as space of holomorphic functions with values in the symmetric tensor power of the cotangent space. See Theorem 3.3.

The weighted Bergman spaces also form part of the so called holomorphic discrete series for the group $G$. In [19] Repka studied the tensor product decomposition for two holomorphic discrete series, and obtained a recursion formula for the highest weights of sub-representations in the tensor product. Earlier Jakobsen and Vergne, [11], [10], have studied the tensor product decomposition by using the explicit realization and by using expansion along the diagonal. In this paper we will use the analytic realization of the representations and give a complete decomposition of the tensor product. We establish first an abstract decomposition by performing the expansion along the diagonal and by finding the highest weight vectors. The main technical difficulty is for the reducible parameters $\nu_1$ and $\nu_2$. Some special cases of the decomposition have been found earlier in [10] as examples of their method. However the novelty in our paper lies in that we give a fairly concrete approach.

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Our idea is roughly the following: The highest weight vectors are of the form \( \Delta_m(z - w) \) where \( \Delta_m \) are the highest weight vectors in the space of polynomials on \( V \); by using the explicit \( K \)-type structure we can then determine which \( \Delta_m(z - w) \) does actually appear. Next by using some refinements of the ideas of Peetre [16] we find the explicit intertwining operators from the tensor product space to certain spaces of holomorphic functions on \( D \) realizing the decomposition; they are certain bilinear differential operators generalizing the classical transvectants in the case of the unit disk in the complex plane ([12], [25], [17], [15]). Further more by taking some limit case of the bilinear differential operator with respect to the weight parameter we prove that the differential operators \( P_{H \theta^l} \) are intertwining (see Theorem 5.1). This generalizes earlier results of Shimura for the case \( l = 1 \). To our knowledge this result for \( l > 1 \) and \( j < r \) has not been discovered before. The intertwining and vanishing properties of those operators play an important role in the study of realization of the singular holomorphic representations; see e. g [13], [3].

A related problem in the expansion along the diagonal is the module property of the tensor product. The space \( \mathcal{H}_\nu \) for \( \nu \) above the reduction point forms also a module of \( \mathbb{C}[z_1, \ldots, z_d] \) by multiplication by the coordinate functions \( z_1, \ldots, z_d \); see [2]. The tensor product \( \mathcal{H}_{\nu_1} \otimes \mathcal{H}_{\nu_2} \) of holomorphic functions of two variables \( (z, w) \) is then a module of \( \mathbb{C}[z_1, \ldots, z_d] \) by multiplication on the first coordinate functions, so is also the quotient module the subspace of holomorphic functions vanishing of certain degree. In the last section we prove that roughly speaking the restriction to the diagonal gives a realization of the module as a reproducing kernel Hilbert space with a explicitly determined kernel.

As it is pointed to us by the referee, the decomposition of the tensor product can also be understood abstractly through the framework of dual pairs [8] and by using the reciprocity relationship [7] for see-saw dual pairs, at least for a classical group \( G \) and for \( \mathcal{H}_\nu \) with \( \nu \) satisfying certain integral conditions. Indeed one may consider \( G \) as one factor of a reductive dual pair \((G, G')\) in \( Sp_{2n}(\mathbb{R}) \) with compact \( G' \). The metaplectic representation \( \omega_n \) of \( Sp_{2n}(\mathbb{R}) \) decomposes under \((\tilde{G}, G')\) as a sum of representations \( \mathcal{H}_\nu \otimes \sigma_\nu \) with multiplicity free, where \( \tilde{G} \) is a double cover of \( G \). Now \((G \times G, G' \times G')\) and \((G, M')\) form two see-saw dual pairs in \( Sp_{2n}(\mathbb{R}) \). Here \( M' \) is a compact group containing \( G' \times G' \). Again, \( \omega_{2n} \) of \( Sp_{2n}(\mathbb{R}) \) under \((\tilde{G}, M')\) decomposes as above as a sum of \( \mathcal{H}_\nu \otimes \tau_\nu \). The reciprocity relationship gives then:

\[
\text{Multiplicity of } \mathcal{H}_\nu \text{ in } \mathcal{H}_{\nu_1} \otimes \mathcal{H}_{\nu_2} = \text{Multiplicity of } \sigma_{\nu_1} \otimes \sigma_{\nu_2} \text{ in } \tau_\nu.
\]

So we are reduced to a problem of tensor product decomposition of compact groups.

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2. Preliminaries

Let \( D \) be an irreducible bounded symmetric domain in a complex \( d \)-dimensional vector space \( V \). In this section we fix some notation and prove some elementary results on power series expansion of holomorphic functions.
There exists a quadratic form with the Lie product \( [X,Y](z) := X'(z)Y(z) - Y'(z)X(z) \), \( X,Y \in aut(D), z \in D \).

Let \( aut(D) = \mathfrak{g} + \mathfrak{p} \) be the Cartan decomposition of \( aut(D) \) with respect to the involution \( \theta \). The center of \( \mathfrak{g} \) is \( i \mathbb{R} Z \) where \( Z \) acts as the identity transformation. There exists a quadratic form \( Q : V \to End(\overline{V}, V) \) (where \( \overline{V} \) is the complex conjugate of \( V \)), such that \( \mathfrak{p} = \{ \xi_v ; v \in V \} \), where \( \xi_v(z) := v - Q(z)\overline{v} \).

Let \( \{ z, \overline{v}, w \} \) be the polarization of the \( Q(z)v \), i.e.,
\[
\{ z, v, w \} = Q(z + w)\overline{v} - Q(z)\overline{v} - Q(w)\overline{v}.
\]
This defines a \textit{triple product} \( V \times \overline{V} \times V \to V \), with respect to which \( V \) is a \( JB^* \)-\textit{triple}, see [14].

We define \( D(z, \overline{v}) \in \text{End}(V, V) \) by \( D(z, \overline{v})w = \{ z, \overline{v}, w \} \) and let \( B(z, w) \) be the Bergman operator on \( V \),
\[
B(z, w) = 1 - D(z, w) + Q(z)Q(w).
\]
The determinant \( \det B(z, w) \) of \( B(z, w) \) is of the form \( h(z, w)^p \) where \( h(z, w) \) is an irreducible polynomial holomorphic in \( z \) and anti-holomorphic in \( w \) and where \( p \) is the genus of \( D \) (see below).

Let \( a, b \) be the Peirce multiplicities as in [14]. The \textit{genus} \( p = p(D) \) is defined by
\[
p := (r - 1)a + b + 2.
\]

Let further \( \otimes^j V \) and \( \otimes^j (V) \) be the \( j \)-fold tensor product and its subspace of symmetric \( j \)-fold tensors, respectively. We recall also that symmetric tensor product of two symmetric tensors \( \otimes^n v \) and \( \otimes^m w \) is defined by
\[
(\otimes^n v) \otimes (\otimes^m w) = \frac{1}{\binom{n+m}{n}} \sum_{s \in S_{n+m}/S_n \times S_m} s(\otimes^n v \otimes \otimes^m w)
\]
where \( S_k \) is the symmetric group of \( k \)-elements and \( S_{n+m}/S_n \times S_m \) is the left coset space.

Let \( V' \) be the dual of \( V \). We will use the following notational convention for the pairing and inner product: If \( W \) is a Hilbert space the inner product of \( u \) and \( v \) will be denoted by \( \langle u, v \rangle \) while the pairing between \( W \) and the dual space \( W' \) will be denoted by \( (u, v') \). We fix an Hermitian inner product on \( V \), or equivalently Hermitian bilinear form on \( V \times \overline{V} \),
\[
\langle v, w \rangle = \langle v, \overline{w} \rangle = \frac{1}{p} \text{tr} D(v, \overline{w}).
\]
We equip \( \otimes^j V \) with the natural inner product induced from that of \( V \). Now the natural pairing between \( \otimes^j V' \) and \( \otimes^j V \) is defined by \( (x_1' \otimes \cdots \otimes x_j', x_1 \otimes \cdots \otimes x_j) = (x_1', x_1) \cdots (x_j', x_j) \).

Let \( f(w) \) be a holomorphic function on \( D \) with values in vector space \( W \). We let \( \partial \) be the holomorphic differentiation, and the differential
\[
\partial f = \frac{\partial f}{\partial w_1} dw_1 + \cdots + \frac{\partial f}{\partial w_d} dw_d
\]
takes then value in \( W \otimes V' \) where \( (w_1, \ldots, w_d) \) are the coordinates of \( w \) under an orthonormal basis of \( W \); the higher order derivative \( \partial^n f \),
\[
\partial^n f = \sum_{|\alpha|=n} \partial^n f \otimes^{a_1} dw_1 \otimes \cdots \otimes \otimes^{a_d} dw_d,
\]
with \( \alpha = (\alpha_1, \ldots, \alpha_d) \) being tuples of nonnegative integers and \(|\alpha| = \alpha_1 + \cdots + \alpha_d\) and 
\( \partial^{\alpha} f = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} f \), takes then values in \( W \otimes (\mathcal{O}^n V') \).

We recall the following result (see [18], Corollary 6.26, p. 269), which holds for any Stein domain \( D \), in particular for a convex domain.

**Lemma 2.1.** Let \( f(z, w) \) be a holomorphic function in \( D \times D \), then there exist holomorphic functions \( f_j(z, w), j = 1, \ldots, d \), on \( D \times D \) such that

\[
f(z, w) - f(z, z) = \sum_{j=1}^d (w_j - z_j) f_j(z, w)
\]

The above lemma can be rephrased as follows: There exist a holomorphic function \( F(z, w) \) with value in \( V' \) such that

\[
f(z, w) - f(z, z) = (F(z, w), w - z).
\]

Repeatedly using Lemma 2.1 we get the following Taylor expansion formula.

**Lemma 2.2.** Let \( f(z, w) \) be a holomorphic function in \( D \times D \) and \( m \geq 0 \) an integer. Then there exist a holomorphic function \( f^{[m+1]}(z, w) \) on \( D \times D \) with value in \( \mathcal{O}^{m+1} V' \), and unique holomorphic functions \( f^{(j)}(z) \) on \( D \) with value in \( \mathcal{O}^j V' \), \( j = 0, \ldots, m \) such that

\[
f(z, w) = \sum_{j=0}^m \left( f^{(j)}(z), \mathcal{O}^j (w - z) \right) + \left( f^{[m+1]}(z, w), \mathcal{O}^j (w - z) \right).
\]

For \( \nu > p - 1 \) we consider the weighted measure

\[
d\mu_\nu(z) = c_\nu h(z, z)^{\nu-p} dm(z)
\]

and the weighted Bergman space \( \mathcal{H}_\nu \) of holomorphic functions \( f \) on \( D \) so that

\[
\|f\|_\nu^2 = \int_D |f(z)|^2 d\mu_\nu(z) < \infty.
\]

The group \( G \) acts unitarily on \( H^\nu \) via the following

\[
\pi_\nu(g) f(z) = (J_{g^{-1}}(z))^\# f(g^{-1} z), \quad g \in G,
\]

and it gives an irreducible unitary (projective) representation of \( G \). One may also consider more generally the actions of \( G \) on vector-valued \( C^\infty \)-functions on \( D \); see (3.1) below.

We recall now the Hua-Schmid decomposition ([9], [20]) and Faraut-Koranyi expansion [6] of the reproducing kernel \( h(z, w)^{-\nu} \).

Let \( \gamma_1 > \cdots > \gamma_r \) be the Harish-Chandra strongly orthogonal roots.

**Lemma 2.3.** The symmetric tensor product \( \mathcal{O}^m V' \) is decomposed under \( K \) into irreducible as follows, with multiplicity one.

\[
\mathcal{O}^m V' = \bigoplus_{|m| = m} \mathcal{O}^{m'} V'
\]

where each \( \mathcal{O}^{m'} V' \) has lowest weight \( -m = -(m_1 \gamma_1 + \cdots + m_r \gamma_r) \) with \( m_1 \geq \cdots \geq m_r \geq 0 \) and \( m_1 + m_2 + \cdots + m_r = m \).
There is a natural identification between the polynomial space $P^m(V)$ and $\odot^m V'$. The highest weight vector in the space $P^m(V)$ has been constructed earlier, see [6] and references therein; we follow the notation there and denote by $\Delta_m \in P^m(V)$ the lowest weight vector.

Let $F(V)$ be the Fock space of entire functions on $V$ defined by

$$(f, g)_F = f(\partial)g^*(0)$$

for polynomials $f$ and $g$, where $g^*$ is obtained from $g$ by taking formal complex conjugates of the coefficients in the expansion of $g$ in terms of monomials. Let $K_m$ be as in [6] the reproducing kernel of the subspace $P_m$ with the Fock norm. Then

$$(2.3) \quad h^{-\nu}(z, w) = \sum_m (\nu)_m K_m(z, w),$$

where

$$(\nu)_m = \prod_{j=1}^r (\nu - \frac{a}{2}(j - 1))_{m_j} = \prod_{j=1}^r \prod_{k=1}^{m_j} (\nu - \frac{a}{2}(j - 1) + k - 1).$$

In particular

$$(2.4) \quad (f, f)_F = (\nu)_m(f, f)_{\nu}$$

for $f \in P_m$.

It follows from this expansion that the kernel $h^{-\nu}(z, w)$ is positive definite and defines a Hilbert space if and only if $\nu$ is in

$$(2.5) \quad W(D) = \{0, \frac{a}{2}, \ldots, \frac{a}{2}(r - 1)\} \cup \left(\frac{a}{2}(r - 1), \infty\right),$$

also called the Wallach set.

Let $\mathcal{H}^K_{\nu}$ be the algebraic sum of polynomials in the space $\mathcal{H}_\nu$. If $\nu = \frac{a}{2}(j - 1)$ is in the discrete Wallach set, only certain subspaces $P_m$ are in the Hilbert space $H^{\nu} = H^{\frac{a}{2}(j - 1)}$; more precisely

$$(2.6) \quad H^{\frac{a}{2}(j - 1)} = \sum_{m, m_j=0} P_m.$$

Moreover it forms an irreducible representation of $G$ with the action $\pi_{\nu}$. In particular, $\mathcal{H}^K_{\nu}$ forms an irreducible representation of the Lie algebra $g^C$.

3. **DECOMPOSITION OF THE TENSOR PRODUCT $\mathcal{H}_{\nu_1} \otimes \mathcal{H}_{\nu_2}$**

We first introduce certain group actions on the space $H(D, \odot^m V')$ of $\odot^m V'$-valued holomorphic functions $f$. For any $\nu > 0$ and $f$ in the space we let

$$(3.1) \quad \pi_{\nu, m}(g) f(z) = (J_{g^{-1}})(z) \odot^m (dg^{-1})^t f(g^{-1} z), \quad g \in G,$$

where $\odot^m (dg^{-1})^t$ on $\odot^m V'$ is the induced action of $(dg^{-1})^t$ on $V'$. We will first consider the action $G$ on the whole space $H(D, \odot^m V')$, and eventually we shall specify some spaces for which we have a unitary irreducible action.

Given a holomorphic function $f(z, w)$ on $D \times D$, written as in Lemma 2.2 we let

$$R_m f(z, w) = f^{[m]}(z)$$
where \( f^{(m)}(z) \) is the \( \otimes^m V' \)-component of \( f^{(m)}(z) \in \otimes^m V' \). Lemma 2.2 asserts that \( R_m \) is well-defined. Denote
\[
\mathcal{K}_m = \{ f \in \mathcal{H}_{\nu_1} \otimes \mathcal{H}_{\nu_2}; f(z, w) = (\otimes^m (z - w), g(z, w)), g \in H(D \times D, \otimes^m V') \}
\]
and its subspace \( \mathcal{K}_m \) with \( g(z, w) \) above being a \( \otimes^m V' \)-valued holomorphic function. Clearly,
\[
\mathcal{K}_0 \supset \mathcal{K}_1 \supset \cdots \supset \mathcal{K}_m \supset \mathcal{K}_{m+1} \supset \cdots
\]
Furthermore it is to prove that \( \mathcal{K}_m \) are closed \( G \)-invariant subspaces of \( \mathcal{H}_{\nu_1} \otimes \mathcal{H}_{\nu_2} \).

**Lemma 3.1.** The operator \( R_m \), \( |m| = m \), is a formal non-zero \( G \)-intertwining operator from \( (K_m, \pi_{\nu_1} \otimes \pi_{\nu_2}) \) into \( H(D, \otimes^m V') \) with the action \( \pi_{\nu_1+\nu_2, m} \).

**Proof.** If \( f \in \mathcal{K}_m \), then
\[
f(z, w) = (\otimes^m (z - w), f^{(m)}(z)) + (\otimes^{m+1} (z - w), f^{(m+1)}(z, w)),
\]
We note that \( R_m f(z) \) is determined only by the coefficient of \( \otimes^m (z - w) \) in \( f(z, w) \) when \( w \to z \). For any \( g \in G \), we have, by definition,
\[
g z - gw = dg(z)(z - w) + o(\|z - w\|), \quad z \to w
\]
where \( o(\|z - w\|) \) involves only higher tensors. Thus
\[
f(gz, gw) J_g(z)^{\nu_1 \nu_2} J_g(w)^{\nu_1 \nu_2} = (\otimes^m dg(z) \otimes^m (z - w), f^{(m)}(g z)) J_g(z)^{\nu_1 \nu_2} + o(\|z - w\|^m)
\]
\[
= ((z - w), \otimes^m dg(z)^t f^{(m)}(g z)) J_g(z)^{\nu_1 \nu_2} + o(\|z - w\|^m).
\]
This proves that
\[
R_m(\pi_{\nu_1} \otimes \pi_{\nu_2})(g^{-1}) = \pi_{\nu_1+\nu_2, m}(g^{-1}) R_m.
\]
We see that \( R_m \) is nonzero we take a vector \( x \in \otimes^m V' \) and let \( f(z, w) = (x, \otimes^j (z - w)). \) Then clearly
\[
R_m f(z) = x
\]
the constant vector-valued function. Thus \( R_m \) is not zero. \( \square \)

**Lemma 3.2.** Any \( g^C \)-lowest weight vector \( F \) in \( \mathcal{H}_{\nu_1}^K \otimes \mathcal{H}_{\nu_2}^K \) is a sum of the vectors of the form
\[
F_m = \Delta_m (z - w).
\]

**Proof.** Let \( v \in p^+ \). Its action on \( F \) is given by [24, Section 4]
\[
\pi_{\nu_1} \otimes \pi_{\nu_2}(v) F = \partial_1^v F(z, w) + \partial_2^v F(z, w)
\]
where \( \partial_1^v \) and \( \partial_2^v \) denote the differentiation with respect to the first variable \( z \) and the second variable \( w \), respectively. Thus \( \partial_1^v F(z, w) + \partial_2^v F(z, w) = 0 \) and its implies that \( F \) is of the form \( F(z, w) = f(z - w) \) for some polynomial \( f \). Let \( X \in \mathfrak{g}^C \) be a compact positive root vector, then we have
\[
\pi_{\nu_1} \otimes \pi_{\nu_2}(X) F(z, w) = \pi_{\nu_1+\nu_2}(X) f(z - w) = 0
\]
which implies that \( f \) is a \( \mathfrak{g}^C \)-highest weight vector; any such vector \( f \) is a sum of \( \Delta_m \) by Lemma 2.3. \( \square \)
It follows then that there is a lowest weight representation of $g^C$ with $(\nu_1 + \nu_2)Z^* - (m_1 \gamma_1 + \cdots + m_r \gamma_r)$, where $Z^*$ is the dual of the center $Z$ of $t$, $Z^*(Z) = 1$, and moreover it is unitarizable since it is a sub-representation of the tensor product. It is well-known (see e.g. [3]) that any such representation can be realized as a subspace of holomorphic functions on $D$ with values in the space $\otimes^m V'$. The reproducing kernel of this space is then
\[
h^{-\nu_1}(z, u) h^{-\nu_2}(z, u) \otimes^m B^t(z, u)^{-1}
\]
by the transformation property of $B^t(z, u)$. We denote $H_{\nu_1 + \nu_2}(\otimes^m V')$ be the corresponding Hilbert space.

**Theorem 3.3.** The tensor product $H_{\nu_1} \otimes H_{\nu_2}$ is decomposed under $G$ into irreducible representations as follows

\[
H_{\nu_1} \otimes H_{\nu_2} = \sum_{m \geq 0} H_{\nu_1 + \nu_2}(\otimes^m V')
\]

if $\nu_1 \geq \nu_2 > \frac{\alpha}{2}(r - 1)$. If $\nu_2 = \frac{\alpha}{2}(j - 1)$ is a reducible point and $\nu_1 \geq \nu_2$ then

\[
H_{\nu_1} \otimes H_{\nu_2} = \sum_{m = [m_1, \ldots, m_j, 0, \ldots, 0] \geq 0} H_{\nu_1 + \nu_2}(\otimes^m V')
\]

**Proof.** It follows from general considerations (see [19]) that the irreducible sub-representations in $H_{\nu_1} \otimes H_{\nu_2}$ are all lowest weight modules. Any lowest weight vector in $H_{\nu_1} \otimes H_{\nu_2}$ is of the form $\Delta_m(z - w)$. If $\nu_1 \geq \nu_2 > \frac{\alpha}{2}(r - 1)$ then clearly $\Delta_m(z - w)$ is in the space $H_{\nu_1} \otimes H_{\nu_2}$, so we get the first decomposition. If $\nu_1 \geq \nu_2 = \frac{\alpha}{2}(j - 1)$ we claim that $\Delta_m(z - w)$ is in the tensor product if and only if $m_j = 0$. Suppose $\Delta_m(z - w)$ is in the tensor product. We consider the circle action on the variable $w$, $e^{i\theta} : \Delta_m(w - z) \mapsto \Delta_m(w - e^{i\theta} z)$. $\Delta_m(w - e^{i\theta} z)$ is also in the tensor product since $H_{\nu_1}$ is invariant under the circle action. Therefore

\[
1 \otimes \Delta_m(w) = \frac{1}{2\pi} \int_0^{2\pi} \Delta_m(w - e^{i\theta} z) d\theta
\]
is also in $H_{\nu_1} \otimes H_{\nu_2}$ and consequently $\Delta_m(z)$ is in $H_{\nu_2}$. This implies that $m_j = 0$ by (2.6). Conversely suppose $m_j = 0$. By taking the pairing of the binomial expansion

\[
\otimes^m(z - w) = \sum_{n + n' = m} (-1)^n \binom{m}{n} \otimes^n z \otimes^{n'} w
\]

with the highest weight vector $\delta_m$ in $\otimes^m V'$ dual to the lowest weight vector $\Delta_m$ we see that $\Delta_m(z - w) = (\otimes^m(z - w), \delta_m) = \sum_{n + n' = m} (-1)^n \binom{m}{n} \sum_{n' \in W} (P_m(P_n(\otimes^n z) \otimes P_{n'}(\otimes^{n'} w)), \delta_m)

But clearly those terms $P_m(P_n(\otimes^n z) \otimes P_{n'}(\otimes^{n'} w))$ for which $n_j > 0$ respectively $n'_j > 0$ are vanishing by simple tensor product argument and we have then

\[
\Delta_m(z - w) = (\otimes^m(z - w), \delta_m) = \sum_{n + n' = m} (-1)^n \binom{m}{n} \sum_{n_j = 0, n'_j = 0} (P_m(P_n(\otimes^n z) \otimes P_{n'}(\otimes^{n'} w)), \delta_m)
\]

and each term $(P_m(P_n(\otimes^n z) \otimes P_{n'}(\otimes^{n'} w)), \delta_m)$ is now in the tensor product $H_{\nu_1} \otimes H_{\nu_2}$, so is $\Delta_m(z - w)$. This proves the second decomposition formula. \(\square\)

This theorem, together with Lemma 3.1, implies immediately that
Corollary 3.4. The map $R_m$ is up to constant a unitary operator from

$$\mathcal{K}_m \oplus \mathcal{K}_{m+1}$$

onto the space $\mathcal{H}_{\nu_1 + \nu_2} \left( \mathcal{O}^{m} V' \right)$.

In the next section we will find an explicit intertwining from the whole space $\mathcal{H}_{\nu_1} \otimes \mathcal{H}_{\nu_2}$ onto $\mathcal{H}_{\nu_1 + \nu_2} \left( \mathcal{O}^{m} V' \right)$.

4. BILINEAR DIFFERENTIAL INTERTWINING OPERATORS

The Hilbert space $\mathcal{H}_{\nu_1 + \nu_2} \left( \mathcal{O}^{m} V' \right)$ has a canonical representation as certain Hilbert space of holomorphic functions on the domain $D$. In [16] Peetre introduced a certain $\text{End}(V)$-valued symmetric differential forms and use it to construct intertwining operator for tensor product of weighted Bergman space, namely for $\nu_1, \nu_2 > p - 1$. We prove that Peetre’s construction is valued for the general tensor product for $\nu_1, \nu_2 > \frac{p}{2}(r - 1)$ and we give a refined version when one of the parameter is in the discrete Wallach set.

Following Peetre [16] we define

$$\Omega(z; w_1, w_2) = d_2 B(z, w_1) B(z, w_1)^{-1} - d_2 B(z, w_2) B(z, w_2)^{-1}$$

and

$$\omega(z; w_1, w_2) = -\frac{1}{p} \text{tr} \Omega(z; w_1, w_2).$$

(4.1)

Then $\Omega$, for fixed $w_1$ and $w_2$, is a $\text{End}(V)$-valued holomorphic differential form on $D$, namely it takes values in $\text{End}(V) \otimes V'$. It transforms under $G$ as follows

$$dg(z)^4 \Omega(gz, gw_1, gw_2) = dg(z) \Omega(z; w_1, w_2) dg(z)^{-1}$$

where $dg(z)^4 : V' \rightarrow V'$ in the left hand side is as before the adjoint of $dg(z) : V \rightarrow V$ and that $dg(z)$ in the right hand side is an element of $\text{End}(V)$; see [16]. Therefore, taking trace one finds

$$dg(z)^4 \omega(gz, gw_1, gw_2) = \omega(z; w_1, w_2)$$

(4.2)

as anti-holomorphic functions of $w_1, w_2$ taking values in $V'$.

We can express $\Omega$ in terms of quasi-inverses in the sense of Jordan triples [14]. Recall that for $\tilde{w} \in \bar{V}$ and $z \in V$ the quasi-inverse $\tilde{w}^z \in \bar{V}$ of $\tilde{w}$ with respect to $z$ is defined by

$$q_z(w) = \tilde{w}^z = B(\tilde{w}, z)^{-1}(\tilde{w} - Q(\tilde{w}) z)$$

whenever it is defined. We will identify $q_z$ with a holomorphic differential form $v \in V \mapsto (v, q_z(v))$ defined by the bilinear form on $V \times \bar{V}$.

The following formula is essential in [16], which can be proved simply by using the formula (JB31) in [10].

Lemma 4.1. Identifying the space $\tilde{w} \in V$ with the space $V'$ via the Hermitian form $(u, v) = \frac{1}{p} \text{tr} D(u, \bar{v})$, we have

$$\omega(z; w_1, w_2) = -\frac{1}{p} \text{tr} \Omega(z; w_1, w_2) = q_z(w_1) - q_z(w_2).$$
A direct computation using \( h(z, w)^p = \det B(z, w) \) shows also that

\[
\partial h^{-\nu}(z, w) = \nu h^{-\nu}(z, w)\bar{w}^z \tag{4.3}
\]

which we will also need.

We define an operator from \( \mathcal{H}_{\nu_1} \otimes \mathcal{H}_{\nu_2} \) into the space of \( \otimes_m \mathcal{V}' \)-valued holomorphic functions by

\[
J_{\mathbf{m}}(f_1, f_2)(z) = (h(z, \cdot)^{-\nu_1} h(z, \cdot)^{-\nu_2} \mathbf{P}_m \otimes_m \omega(z; \cdot, \cdot), f_1(\cdot) \otimes f_2(\cdot))
\]

Here \( h(z, w_1)^{-\nu_1} h(z, w_2)^{-\nu_2} \mathbf{P}_m \otimes_m \omega(z; w_1, w_2) \) is viewed as an element in the dual space \( \mathcal{H}_{\nu_1}^* \otimes \mathcal{H}_{\nu_2}^* \) of \( \mathcal{H}_{\nu_1} \otimes \mathcal{H}_{\nu_2} \).

**Lemma 4.2.** Suppose \( \nu > \frac{a}{2}(r - 1) \). Then the reproducing kernel \( K(z, w) = h^{-\nu}(z, w) \) on the symmetric domain \( D \) satisfies the following condition: The functions

\[
\partial_{u_1} \cdots \partial_{u_n} \partial_{\bar{u}_1} \cdots \partial_{\bar{u}_n} K(z, w)
\]

for any fixed \( w \in D \) and for all \( u_1, \ldots, u_m \in V, \bar{u}_1, \ldots, \bar{u}_n \in V \), are in the space \( \mathcal{H}_{\nu} \).

**Proof.** Fix a point \( w \in D \). We have

\[
\partial_z h^{-\nu}(z, w) = (-\nu) (\partial_z h(z, w)) h^{-\nu-1}(z, w)
\]

where \( \partial_z h(z, w) \) is a polynomial of \( z \) of degree \( r - 1 \), namely \( \partial_z h^{-\nu}(z, w) \) is obtained by the multiplication by a polynomial on the function \( h^{-\nu-1}(z, w) \). We prove that \( h^{-\nu-1}(z, w) \) is in \( H^\nu \), the result follows then since the multiplications by the coordinates functions are bounded on \( H^\nu \) \([2]\). Indeed by the expansion (2.3) we have

\[
\|h^{-\nu-1}(\cdot, w)\|_\nu^2 = \sum_m \frac{(\nu + 1)^2}{(\nu)_m} K_m(w, w).
\]

We let \( \nu_0 > \frac{a}{2}(r - 1) \) be a large positive real number so that

\[
(\nu_0 - \frac{a}{2}(j - 1))(\nu - \frac{a}{2}(j - 1)) > (\nu + 1 - \frac{a}{2}(j - 1))^2, \quad \nu_0 + \nu - a(j - 1) > 2(\nu + 1 - \frac{a}{2}(j - 1)),
\]

for all \( j = 1, \ldots, r \). With this choice we have

\[
\frac{(\nu + 1)^2}{(\nu)_m} \leq (\nu_0)_m,
\]

and

\[
\|h^{-\nu-1}(\cdot, w)\|^2 \leq \sum_m (\nu_0)_m K_m(w, w)
\]

which converges to \( h^{-\nu_0}(w, w) \) and is finite. Thus \( h^{-\nu-1}(\cdot, w) \) is in \( \mathcal{H}_{\nu} \). This completes the proof for the case when \( m = 1, n = 0 \). The higher order derivatives can be done similarly. \( \square \)

**Proposition 4.3.** Suppose \( \nu_1, \nu_2 > \frac{a}{2}(r - 1) \). The operator \( J_{\mathbf{m}} \) is an nonzero intertwining operator from the tensor product \( \mathcal{H}_{\nu_1} \otimes \mathcal{H}_{\nu_2} \) onto the Hilbert space \( \mathcal{H}_{\nu_1 + \nu_2} \otimes \mathcal{V}' \) of holomorphic functions on \( D \) with values in \( \otimes_m \mathcal{V}' \).

**Proof.** The formula (4.3) implies that \( h(z, w_1)^{-\nu_1} h(z, w_2)^{-\nu_2} \mathbf{P}_m \otimes_m \omega(z; w_1, w_2) \), as function of \( (w_1, w_2) \), are the differentiation of the function \( h(z, w_1)^{-\nu_1} h(z, w_2)^{-\nu_2} \) and thus are in the space \( \mathcal{H}_{\nu_1} \otimes \mathcal{H}_{\nu_2} \), by the previous lemma, so that \( J_{\mathbf{m}} \) makes sense. Now the intertwining relation follows by (4.2) and by the transformation property of the reproducing kernels. To
see that $J_{\mathbf{m}}$ is nonzero let it act on the function $f(w_1, w_2) = \Delta_{\mathbf{m}}(w_1 - w_2)$ in the tensor product and evaluate at $z = 0$. In that case $\tilde{w}^0 = \tilde{w}$, $\omega(0; w_1, w_2) = \tilde{w}_1 - \tilde{w}_2$ and
\[ J_{\mathbf{m}}(f)(0) = \|f\|_{\mathcal{H}_{\nu_1} \otimes \mathcal{H}_{\nu_2}}^2 \neq 0. \]
That $J_{\mathbf{m}}$ is onto follows now by Theorem 3.3.

For the case of weighted Bergman spaces the operator $J_{\mathbf{m}}(f_1, f_2)(z)$ is computed in [16] with a sketch of a proof. We provide here a refined version of that formula for all $\nu_1$ and $\nu_2$.

**Theorem 4.4.** Suppose $\nu_1, \nu_2 > \frac{r}{2}(r - 1)$. Then the operator $J_{\mathbf{m}}$ is given by
\[ J_{\mathbf{m}}(f_1 \otimes f_2)(z) = \sum_{|\mathbf{m}| + |\mathbf{m}'| = m} (-1)^{|\mathbf{m}|} \binom{|\mathbf{m}|}{|\mathbf{m}'|} \frac{1}{(\nu_1)^{|\mathbf{m}'|} (\nu_2)^{|\mathbf{m}'|}} P_{\mathbf{m}}(P_{\mathbf{m}} \partial^{\mathbf{m}} f_1(z) \otimes P_{\mathbf{m}'} \partial^{\mathbf{m}'} f_2(z)). \]
If $\nu_1 \geq \nu_2 = \frac{r}{2}(j - 1)$ then for $\mathbf{m}$ with $m_j = 0$, the following operator
\[ J_{\mathbf{m}}(f_1 \otimes f_2)(z) = \sum_{|\mathbf{m}| + |\mathbf{m}'| = m, n_j = 0, n'_j = 0} (-1)^{|\mathbf{m}|} \binom{|\mathbf{m}|}{|\mathbf{m}'|} \frac{1}{(\nu_1)^{|\mathbf{m}'|} (\nu_2)^{|\mathbf{m}'|}} P_{\mathbf{m}}(P_{\mathbf{m}} \partial^{\mathbf{m}} f_1(z) \otimes P_{\mathbf{m}'} \partial^{\mathbf{m}'} f_2(z)) \]
defines a nonzero intertwining operator from $\mathcal{H}_{\nu_1} \otimes \mathcal{H}_{\nu_2}$ onto the Hilbert space $\mathcal{H}_{\nu_1 + \nu_2}(\otimes \mathbf{m} \mathbf{V}')$.

We need some technical lemmas. The next result will also be used in the next sections.

**Lemma 4.5.** Suppose $\nu > \frac{r}{2}(r - 1)$. Then we have, for any $\mathbf{m}$ and $f \in \mathcal{H}_{\nu}$,
\[ \frac{1}{(\nu)^{|\mathbf{m}|}} P_{\mathbf{m}} \partial^{|\mathbf{m}|} f(z) = (h(z, \cdot)^{-\nu} P_{\mathbf{m}}(\otimes^{|\mathbf{m}|} q_{\nu}(\cdot)), f(\cdot)|_{\mathcal{H}_{\nu}}. \]

**Proof.** The reproducing kernel formula gives, denoting for simplicity the paring as $(\cdot, \cdot)$,
\[ f(z) = (h_{\nu}^{-\nu}, f) \]
where we have temporally written the function $h(z, \cdot)$ as $h_z$. Let $v \in V$ and perform the differentiation $\partial_{\nu}$. Lemma 4.2 guarantees that we can carry the differentiation with respect to $h_z$
\[ \partial_{\nu}^{|\mathbf{m}|} f(z) = \nu(\partial_{\nu}^{|\mathbf{m}|} h_z^{-\nu}, f). \]
Let $m = 1$. We have
\[ \partial_{\nu} h(z, w)^{-\nu} = \nu h(z, w)^{-\nu}(v, q_z(w)) \]
by (4.3). This proves our lemma for $m = 1$. We perform now a second differentiation using Leibnitz rule,
\[ \partial_{\nu}^2 h(z, w)^{-\nu} = \nu \partial_{\nu} h(z, w)^{-\nu}(v, q_z(w)) + \nu h(z, w)^{-\nu}(v, \partial_{\nu} q_z(w)). \]
The first term is
\[ \nu(\partial_{\nu} h_z^{-\nu}) q_z(v) = \nu^2 h_z^{-\nu} q_z(v) q_z(v) = \nu^2 h_z^{-\nu}(v \otimes v, q_z \otimes q_z) \]
\[ = \nu^2 h_z^{-\nu}(v \otimes v, P_{|2,0|} q_z \otimes q_z + (v \otimes v, P_{|1,1|} q_z \otimes q_z)) \]
since $v \otimes v \in V \otimes V = \otimes^{(2,0)} V \oplus \otimes^{(1,1)} V$. Here $P_{|2,0|}$ and $P_{|1,1|}$ stands for the corresponding projections. To handle the second term in (4.4) we recall that $\tilde{w}^{z+v} = (\tilde{w}^z)^v$ (see [14]. Appendix), thus
\[ \partial_{\nu} q_z(w) = Q(\tilde{w}^z)v = Q(q_z(w))v, \]
and
\[ (v, \partial_{\nu} q_z(w)) = (v, Q(q_z(w))v). \]
The form \((v, Q(\bar{u})v) = (Q(v)\bar{u}, \bar{u})\) is quadratic in \(v\) and in \(\bar{u}\) and is \(K\) invariant, thus it is a linear combination of the \(K\)-invariant polynomials \(K_{(2,0)}(v, u)\) and \(K_{(1,1)}(v, u)\). In other words, \((v, Q(\bar{u})v)\) is a linear combination of the paring between \(P_{(2,0)}(v \otimes v)\) and \(P_{(2,0)}(u \otimes u)\) and respectively the paring between \(P_{(1,1)}(v \otimes v)\) and \(P_{(1,1)}(u \otimes u)\); namely \((v, \partial_v q_z)\) is also of the form

\[
(v, \partial_v q_z) = c_1(v \otimes v, P_{(2,0)}(q_z \otimes q_z)) + c_2(v \otimes v, P_{(1,1)}(q_z(w) \otimes q_z)).
\]

Hence, (4.4) is of the form

\[
\partial_v^2 h_z^{-\nu} = c_{(2,0)} h_z^{-\nu}(v \otimes v, P_{(2,0)}(q_z \otimes q_z)) + c_{(1,0)} h_z^{-\nu}(v \otimes v, P_{(1,1)}(q_z \otimes q_z))
\]

for some constants \(c_{(2,0)}\) and \(c_{(1,0)}\). So each differentiation produces an extra tensor factor of \(q_z\) and the whole tensor is symmetric in \(q_z\). Thus, generally,

\[
\partial_v^m h_z^{-\nu} = \sum_m c_m h_z^{-\nu}(\otimes^m v, P_m(\otimes^m q_z))
\]

for some constants \(c_m\). Consequently

\[
P_m \partial_v^m f(z) = c_m h_z^{-\nu} P_m(\otimes^m q_z, f)
\]

To find the constant \(c_m\) we take a polynomial \(f\) and \(z = 0\); the formula now reads

\[
P_m \partial_v^m f(0) = c_m P_m(\otimes^m q_0, f).
\]

However \(P_m \otimes^m q_0(w) = P_m \otimes^m w\) and \((P_m \otimes^m q_0, f) = (v)_m P_m \partial_v^m f(0)\) by (2.4) and by the definition of the Fock space norm. This completes the proof. 

We prove now Theorem 4.4.

**Proof.** Let \(\nu_1, \nu_2 > \frac{q}{2}(r - 1)\). We expand \(\otimes^m \omega(z; w_1, w_2) = \otimes^m (q_z(w_1) - q_z(w_2))\) using the binomial formula,

\[
\otimes^m \omega(z; w_1, w_2) = \sum_{m+n'=m} (-1)^n \binom{m}{n} \otimes^n q_z(w_1) \otimes \otimes^{n'} q_z(w_2)
\]

\[
= \sum_{\|n\|+\|n'\|=m} (-1)^n \binom{m}{\|n\|} (P_{m,n} \otimes^n q_z(w_1)) \otimes (P_{m,n'} \otimes^{n'} q_z(w_2)).
\]

Consequently

\[
J_{m}(f_1, f_2)(z) = \sum_{\|n\|+\|n'\|=m} (-1)^n \binom{m}{\|n\|} P_{m}(h_z^{-\nu_1} P_{n} \otimes^n q_z, f_1) \otimes (h_z^{-\nu_2} P_{n'} \otimes^{n'} q_z, f_1)f_2)
\]

and our result follows then from Lemma 4.5.

We prove now the second part using analytic continuation. First observe that, for \(\nu_1, \nu_2 > \frac{q}{2}(r - 1)\), if \(m\) is such that \(m_j = 0\) then the formula for \(P_m \otimes^m \omega(z; w_1, w_2)\) becomes

\[
P_m \otimes^m \omega(z; w_1, w_2) = \sum_{\|n\|+\|n'\|=m, n_j=0, n'_j=0} (-1)^n \binom{m}{n} P_{m}(P_{n} \otimes^n q_z(w_1) \otimes P_{n'} \otimes^{n'} q_z(w_2))
\]
since those terms $P_n (P_n \otimes^n q_z (w_1) \otimes P_n' \otimes^{n'} q_z (w_2))$ with $n_j > 0$ or $n_j' > 0$ vanish by a simple tensor product argument. Thus $J_m$ for $m_j = 0$ is given as in the Theorem. The intertwining relation

$$J_m (\pi_{\nu_1} (g) \otimes \pi_{\nu_2} (g) (f_1 \otimes f_2)) = \pi_{\nu_1 + \nu_2, m} (g) J_m (f_1 \otimes f_2)$$

is meromorphic in $\nu_1$ and $\nu_2$ and holds for $\nu_1, \nu_2 > \frac{a}{2} (r - 1)$. Moreover it is finite for $\nu_1 \geq \nu_2 = \frac{a}{2} (j - 1)$ since $(\nu_1)_n \neq 0$, $(\nu_2)_n' \neq 0$ in the summation. Thus it still holds for our given $\nu_1$ and $\nu_2$.

\[ \Box \]

5. LIMIT CASES OF THE BILINEAR DIFFERENTIAL OPERATORS

When $D$ is a tube domain the holomorphic differential operator $\Delta (\partial)^l$ for any positive integer $l$ intertwines the group action $\pi_{\frac{a}{2} (r - 1) + 1 - l}$ with $\pi_{\frac{a}{2} (r - 1) + 1 + l}$ acting on all holomorphic functions; see [21] and [1]; in the case of the unit disk this is classical known as the Bol’s lemma. Shimura proved further that the operators $P_{l, j} \partial^j$ with $\frac{a}{2} (j - 1) \leq \nu_1, \nu_2 \leq \frac{a}{2} (r - 1)$ are also intertwining operator; see also [3]. By taking certain limit procedure we will derive the intertwining properties for all operators $P_{l, j} \partial^j$ on general domains and thus generalize the above results.

We recall again that the intertwining relation (4.5) is meromorphic in $\nu_1$ and $\nu_2$, and it holds for all polynomials $f_1$ and $f_2$; as it is a local relation so by taking the Taylor expansion for any holomorphic functions $f_1$ and $f_2$ we see it holds true for all such $f_1$ and $f_2$. Moreover the actions $\pi_{\nu_1} (g), \pi_{\nu_2} (g)$ and $\pi_{\nu_1 + \nu_2, m} (g)$ are all holomorphic functions of $\nu_1$ and $\nu_2$ for fixed $g \in G$.

**Theorem 5.1.** Let $1 \leq j \leq r$ and $l$ be a positive integer. The operator $P_{l, j} \partial^j$ has the following intertwining property

$$P_{l, j} \partial^j (\pi_{\frac{a}{2} (j - 1) + 1 - l} (g) f) = \pi_{\frac{a}{2} (j - 1) + 1 - l, l} (g) P_{l, j} \partial^j (f), \quad g \in G$$

for any holomorphic function $f$ on $D$.

**Proof.** We write $\nu_0 = \frac{a}{2} (j - 1) + 1 - l$ for simplicity. Fix $\nu_2 > \frac{a}{2} (r - 1)$ and let $\nu = \nu_1$. The operator $(\nu)_{l, j} J_{l, j}$ satisfies the same intertwining relation as $J_{l, j}$ and

$$(\nu)_{l, j} J_{l, j} (f \otimes F) = P_{l, j} \partial^j f +$$

$$+ (\nu)_{l, j} \sum \frac{1}{n + |n| + |n'|} P_{l, j} (P_n \partial^n f \otimes P_{n'} \partial^{n'} F (z)),$$

with the leading term explicitly written, where $n \leq m$ means that $n_i \leq m_i, 1 \leq i \leq r$; the appearance of the conditions in the summation is due to the fact that only those terms are possibly nonzero, by a simple tensor product argument. Now the operator $(\nu)_{l, j} J_{l, j} (f \otimes F)$ and consequently its intertwining relation are holomorphic at $\nu = \nu_0 = \frac{a}{2} (j - 1) + 1 - l$. We take now the limit for $\nu \rightarrow \nu_0$,

$$\lim_{\nu \rightarrow \nu_0} (\nu)_{l, j} J_{l, j} (\pi_{\nu} (g) f \otimes \pi_{\nu_2} (g) F) = \lim_{\nu \rightarrow \nu_0} (\nu)_{l, j} \pi_{\nu + \nu_2, l} (g) J_{l, j} (f \otimes F)$$

$$= \pi_{\nu_0 + \nu_2, l} (g) \lim_{\nu \rightarrow \nu_0} (\nu)_{l, j} J_{l, j} (f \otimes F).$$
However
\[
\lim_{\nu \to \nu_0} (\nu)_{\mathbf{l} \mathbf{l}^j} J_{\mathbf{l} \mathbf{l}^j} = P_{\mathbf{l} \mathbf{l}^j} \partial^{l^j}
\]
\[
+ \sum_{|l^j|+|l| = l^j} \left( -1 \right)^{|l^j|} \left( \begin{array}{c} l^j \\ l \end{array} \right) \lim_{\nu \to \nu_0} \frac{\nu}{\nu_0} \frac{\nu}{\nu_2} (\nu)_{\mathbf{n}} P_{\mathbf{n}} (\partial_{\mathbf{u}} \nu \mathbf{u} \otimes \partial_{\mathbf{w}} \nu \mathbf{w}),
\]
the summation is vanishing since
\[
\frac{\nu}{\nu_0} \frac{\nu}{\nu_2} (\nu)_{\mathbf{n}} = \left( \prod_{i=1}^{j-1} \prod_{k=0}^{l-n_k} (\nu - a_2(i - 1) + k) \right) \left( \prod_{k=0}^{l-n_j} (\nu - a_2(j - 1) + k) \right),
\]
and the second product \( \prod_{k=0}^{l-n_j} (\nu - \frac{a_2}{2}(j - 1) + k) \) is zero when \( \nu = \nu_0 \). Namely
\[
\lim_{\nu \to \nu_0} (\nu)_{\mathbf{l} \mathbf{l}^j} J_{\mathbf{l} \mathbf{l}^j} = P_{\mathbf{l} \mathbf{l}^j} \partial^{l^j}
\]
and the resulting intertwining relation is
\[
(\partial_{\mathbf{u}} \nu \mathbf{u} \otimes \nu \mathbf{u} \mathbf{v} (g) f) \otimes \nu \mathbf{u} \mathbf{v} (g) F = \nu \mathbf{u} \mathbf{v} (g) (\partial_{\mathbf{u}} \nu \mathbf{u} \mathbf{v} (g) (\partial_{\mathbf{u}} \nu \mathbf{u} \mathbf{v} (g) f) \otimes F)
\]
\[
= \nu \mathbf{u} \mathbf{v} (g) P_{\mathbf{l} \mathbf{l}^j} \partial^{l^j} (g) \otimes \nu \mathbf{u} \mathbf{v} (g) F.
\]
Taking \( F = 1 \) proves our relation. \( \square \)

When \( D \) is a tube domain and when \( j = r \) we have \( \mathbf{l} \mathbf{r} \mathbf{r} \) is one-dimensional and
\[
\pi_{\nu_0+\mathbf{l} \mathbf{r}} = \pi_{\nu_0+\mathbf{u}}, \quad \partial_{\mathbf{r}} \partial_{\mathbf{r}} = c \Delta (\partial)^r
\]
for some non-zero constant \( c \) [26]. The intertwining relation is now
\[
\Delta (\partial)^r \pi_{\nu_0+\mathbf{l} \mathbf{r}} = \pi_{\nu_0+\mathbf{l} \mathbf{r}} \Delta (\partial)^r
\]
which was proved earlier by Shimura [21]. He proved further [22] the intertwining relation of \( P_{\mathbf{l} \mathbf{r}} \) for \( 1 \leq j \leq r \) for classical domain.

This result has some other applications and we hope to pursue them in the future.

6. APPLICATION TO HOMOGENEOUS OPERATOR TUPLES

In [27] it is proved by an elementary method that the diagonal map \( f(z, w) \mapsto f(z, z) \) is a unitary operator from the subspace \( (\mathcal{H}_{\nu_1} \otimes \mathcal{H}_{\nu_2}) \otimes K_1 \) to the space with the reproducing kernel \( h(z, w)^{-\nu_1-\nu_2} \), namely the Schur product of two reproducing kernels, and thus giving the first component in the decomposition. In the case when both spaces are weighted Bergman spaces this then proves that the function module tensor product [5] of the two weighted Bergman space is isometric to another weighted Bergman space. For smaller parameter with \( \nu > \frac{a_2}{2}(r - 1) \), the multiplication operators by the coordinate functions on \( \mathcal{H}_{\nu} \) are still bounded and form a homogeneous tuple of operators. They act then on the tensor product \( \mathcal{H}_{\nu_1} \otimes \mathcal{H}_{\nu_2} \) for \( \nu_1, \nu_2 > \frac{a_2}{2}(r - 1) \) as a multiplication on the first factor. We will apply our previous considerations and consider certain quotients of the tensor product and provide a larger class of homogeneous tuples of operators in the Cowen-Douglas class. In the case of the unit disk \( D = SU(1,1)/U(1) \) it is proved in [23] that all \( G \)-homogeneous tuples of the Cowen-Douglas class are obtained this way. It might be interesting to understand if some similar results hold in the general case.

We use the same notation for the operators defined by
\[
M_j f(z, w) = z_j f(z, w).
\]
Since $M_j$ on $\mathcal{H}_{\nu_1}$ is bounded we have that $M_j$ on $\mathcal{H}_{\nu_1} \otimes \mathcal{H}_{\nu_2}$ is also bounded.

Recall the subspace $K_j$ defined in \S 3, so if $f \in K_m$, then
\begin{equation}
    f(z, w) = (g_m(z, w), \otimes^m(w - z)),
\end{equation}
with $g_m(z, w)$ a $\otimes^m V'$-valued function on $D \times D$. Let
\begin{equation}
    \mathcal{H}_{\nu_1} \otimes \mathcal{H}_{\nu_2} / K_{m+1} = \mathcal{H}_{\nu_1} \otimes \mathcal{H}_{\nu_2} \otimes K_{m+1}
\end{equation}
be the quotient Hilbert space. We consider the tuple $M = (M_1, \ldots, M_n)$ of the multiplication operators on quotient defined by
\begin{equation}
    M_j(f + K_{m+1}) = M_j f + K_{m+1},
\end{equation}
where $M_j f(z, w) = z_j f(z, w)$. This is indeed well-defined, for if $f \in K_{m+1}$, then $M_j \in K_{m+1}$. Clearly the operator tuple $M$ is homogeneous with respect to $G$, namely for any $\psi \in G$ we have $(\pi_{\nu_1} \otimes \pi_{\nu_2})(\psi)^{-1} M(\pi_{\nu_1} \otimes \pi_{\nu_2})(\psi) = \psi(M)$ where $\psi(M)$ is the operator tuple of multiplication by the coordinate functions $(\psi(z), \ldots, \psi(z)_d)$ of $\psi(z)$.

The following result is elementary, we include a proof for the sake of completeness. Let temporarily $D$ be any bounded complex domain in $V = \mathbb{C}^d$. Suppose that $K(z, w)$ is the reproducing kernel of a Hilbert space of holomorphic functions on $D$ (so that the point-evaluation is continuous).

**Lemma 6.1.** Suppose the kernel $K(z, w)$ is a reproducing kernel. Then for nonnegative integer $n$ in the matrix-values kernel
\[ (\partial^\alpha \overline{\partial}^\beta K(z, w))_{0 \leq |\alpha|, |\beta| \leq N} \]
on $D \times D$ is also positive definite.

**Proof.** We first observe that for any fixed $w \in D$ the functions $\overline{\partial}^\beta K(z, w)$ are also in the Hilbert space defined by $K(z, w)$. This is equivalent to that the linear functional $f \rightarrow \partial_j f (w)$ is a bounded linear functional, for any $j = 1, \ldots, d$, which can be easily proved by using Cauchy formula and the reproducing kernel property (which we omit). Performing differentiation $\partial^\alpha \overline{\partial}^\beta$ to the reproducing property $K(z, w) = (K_w, K_z)$ we have
\[ \partial^\alpha \overline{\partial}^\beta K(z, w) = (\overline{\partial}^\beta K_w, \overline{\partial}^\alpha K_z). \]
Therefore for any finitely many points $\{z_i\}$ and finite sets $\{c_i\}$ and $\{b_\alpha\}_{|\alpha| \leq m}$ of complex numbers we have
\[ \sum_{i, j} \sum_{\alpha, \beta} c_i \overline{c_j} b_\alpha \overline{b_\beta} \partial^\alpha \overline{\partial}^\beta K(z_i, z_j) = \sum_{i, j} \sum_{\alpha, \beta} c_i \overline{c_j} b_\alpha \overline{b_\beta} (\partial^\beta K_{z_j}, \overline{\partial}^\alpha K_{z_i}) \]
\[ = \| \sum_{i, \alpha} b_\alpha c_j \overline{\partial}^\alpha K_{z_j} \|^2 \geq 0. \]
This completes the proof. \[ \square \]

The kernel
\begin{equation}
    h(z, w)^{-\nu_1} (\partial^\beta \overline{\partial}^\alpha h(z, w)^{-\nu_2})_{0 \leq |\alpha|, |\beta| \leq m}
\end{equation}
with $\nu_1, \nu_2 > \frac{d}{2}(r - 1)$ is therefore positive definite, for it is the Schur product of two positive definite kernels. We denote $Q_m$ the corresponding Hilbert space of holomorphic functions.
with values in $\sum_{j=0}^{m} \otimes^j V'$, more concretely it consists all linear combinations

$$f(z) = \left( \sum_{i, 0 \leq |\alpha| \leq m} c_{i\beta} h(z, z_i)^{-\nu_1} \partial^\alpha \bar{\partial}^\beta h(z, z_i)^{-\nu_2} \right)_{0 \leq |\beta| \leq m},$$

where $\{z_j\}$ is a finite subset of $D$, as a dense subspace with inner product

$$\langle f, f \rangle = \sum_{i, j, 0 \leq |\alpha| \leq |\beta| \leq m} c_{i\bar{j}} c_{\alpha\beta} h(z_i, z_j)^{-\nu_1} \partial^\alpha \bar{\partial}^\beta h(z_i, z_j)^{-\nu_2}.$$

With some abuse of the notation, we denote the tuple of multiplication operators on $Q_m$ also by $M$.

**Proposition 6.2.** Suppose $\nu_1, \nu_2 > \frac{D}{2}(r - 1)$. The map $f(z, w) \mapsto \sum_{n=0}^{m} (\partial^n_{w} f)(z, w)$ induces a unitary operator $I_m$ from $H_{\nu_1} \otimes H_{\nu_2}/\mathcal{K}_{m+1}$ onto the Hilbert space $Q_m$ of $\sum_{n=0}^{m} \otimes^n V'$-valued holomorphic functions with the matrix-valued reproducing kernel (6.3) and it intertwines the two the multiplication operator tuples $M$ on the corresponding spaces.

**Proof.** We claim first that the elements

$$h(\cdot, z)^{-\nu_1} \otimes \bar{\partial}_{1}^{\alpha_1} \cdots \bar{\partial}_{d}^{\alpha_d} h(\cdot, z)^{-\nu_2}$$

for $\alpha_1 + \cdots + \alpha_d \leq m$ and $w \in D$, span a dense subspace of of $H_{\nu_1} \otimes H_{\nu_2} \otimes \mathcal{K}_{m+1}$, where $\bar{\partial}_{j}^{\alpha_j} = \bar{\partial}_{z_j}^{\alpha_j}$ and $w_j$ are the coordinates of $w$ under an orthonormal basis. Indeed, a function $f \in H_{\nu_1} \otimes H_{\nu_2}$ is in the subspace $\mathcal{K}_{m+1}$ if and only if

$$\partial_{w_1}^{\alpha_1} \cdots \partial_{w_d}^{\alpha_d} f(z, w) \big|_{w = z} = 0$$

by Lemma 2.2. In terms of the reproducing kernel,

$$\partial_{w_1}^{\alpha_1} \cdots \partial_{w_d}^{\alpha_d} f \big|_{w = z} = \langle f(\cdot, \cdot), h(\cdot, z)^{-\nu_1} \otimes \bar{\partial}_{1}^{\alpha_1} \cdots \bar{\partial}_{d}^{\alpha_d} h(\cdot, z)^{-\nu_2} \rangle = 0$$

where $\bar{\partial}_{i}$ are with respect to the anti-holomorphic variables. This proves our claim. For any $f = \sum_{\alpha, j} c_{\alpha j} h(\cdot, z_j)^{-\nu_1} \otimes \bar{\partial}_{z_j}^{\alpha_1} \cdots \bar{\partial}_{d}^{\alpha_d} h(\cdot, z_j)^{-\nu_2}$ in the span, it follows from the proof of the previous lemma that its norm square in $H_{\nu_1} \otimes H_{\nu_2}$ is given by

$$\|f\|^2 = \sum_{\alpha, j} \sum_{\beta, \bar{\beta}} c_{\alpha j} c_{\beta \bar{\alpha}} h(z_i, z_j)^{-\nu_1} \partial_{1}^{\beta_1} \cdots \partial_{d}^{\beta_d} \bar{\partial}_{1}^{\alpha_1} \cdots \bar{\partial}_{d}^{\alpha_d} h(z_i, z_j)^{-\nu_2};$$

its image under the our map $I_m$ is

$$\left( \sum_{\alpha, j} c_{\alpha j} h(\cdot, z_j)^{-\nu_1} \partial_{1}^{\beta_1} \cdots \partial_{z_d}^{\beta_d} \bar{\partial}_{1}^{\alpha_1} \cdots \bar{\partial}_{d}^{\alpha_d} h(\cdot, z_j)^{-\nu_2} \right),$$

whose norm square in the reproducing kernel space $\widehat{Q}_m$ is precisely the above $\|f\|^2$, by the definition. So that $I_m$ establishes an isometry between a dense subspace of $Q_m$ and that of $\widehat{Q}_m$. This proves the unitarity. The rest of the claim is trivial. \qed

We note that in [4] some general results are obtained on equivalent realization of quotient function modules on complex domains, namely when the multiplication operators by holomorphic polynomials are bounded by their maximum norm, our $H_\nu$ for $\nu > \frac{D}{2}(r - 1)$ are however not function modules.
It follows further from Corollary 3.4 that the space $Q_m$ is then unitary $G$-equivalent to the sum

$$
\sum_{|\mu| \leq m} H_{\nu_\mu + \nu_1}(\mathbb{H}^2/V);
$$

however the homogeneous tuple of multiplication operators on $Q_m$ is not unitary $G$-equivalent to a direct sum of the later. It would be interesting to study further the spectral properties of those operator tuples; see e.g. [4] for related questions.

REFERENCES


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