On the Euler characteristic of the discrete spectrum

Benedict H. Gross David Pollack

This paper, which is largely expository in nature, seeks to illustrate some of the advances that have been made on the trace formula in the past fifteen years, via a calculation of the Euler characteristic of the $S$-cohomology of the discrete spectrum. As a byproduct of this calculation, we obtain the existence of automorphic representations with certain local behavior at the places in $S$.

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1. The cohomology of the discrete spectrum (cf. [5])

Let $G$ be a simply connected, semi-simple algebraic group defined over $\mathbb{Q}$. Let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$. The group $G(\mathbb{A})$ is locally compact and unimodular; let $dg$ be a fixed Haar measure on $G(\mathbb{A})$. The subgroup $G(\mathbb{Q})$ is discrete in $G(\mathbb{A})$, so $dg$ induces a measure on the quotient $G(\mathbb{Q}) \backslash G(\mathbb{A})$, which has finite volume [4].

The group $G(\mathbb{A})$ acts unitarily, by right translation, on the Hilbert space

$$L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}), dg).$$

Let

$$L = L^2_{\text{disc}} \subset L^2$$

be the discrete spectrum, which decomposes as a Hilbert direct sum of irreducible unitary representations $\pi$ of $G(\mathbb{A})$, with finite multiplicities $m(\pi)$:

$$L = \bigoplus m(\pi)\pi.$$ 

Each irreducible $\pi$ is a restricted tensor product

$$\pi = \otimes \pi_v,$$

with $\pi_v$ an irreducible, unitary representation of $G(\mathbb{Q}_v)$ [10].

The group $G$ is unramified at almost all primes $p$: that is, $G$ is quasi-split over $\mathbb{Q}_p$ and split by an unramified extension of $\mathbb{Q}_p$ [22, 3.9.1]. Let $S$ be a finite set of places of $\mathbb{Q}$ which contains the real place and all finite primes.
$p$ where $G$ is ramified. We may choose an integral model $\mathcal{G}$ for $G$ over the ring $\mathbb{Z}_S$ of $S$-integers, with $\mathcal{G}$ having good reduction at all primes $p$ outside of $S$. For such a good prime $p$, $\mathcal{G}(\mathbb{Z}_p)$ is a hyperspecial maximal compact subgroup of $\mathcal{G}(\mathbb{Q}_p) = G(\mathbb{Q}_p)$. The product

$$G_S(\mathbb{A}) = \prod_{v \in S} G(\mathbb{Q}_v) \times \prod_{p \notin S} G(\mathbb{Z}_p)$$

is locally compact, and open in $G(\mathbb{A})$. Moreover,

$$G(\mathbb{A}) = \lim_{\rightarrow S} G_S(\mathbb{A}).$$

Fix such a finite set $S$ and an integral model $\mathcal{G}$ for $G$, as well as an irreducible, finite-dimensional representation $V$ of the real Lie group $G(\mathbb{R})$, such that $V$ has trivial central character. The tensor product $L \otimes V$ is a continuous, complex representation of the locally compact group $G_S(\mathbb{A})$, and we may define the continuous cohomology groups

$$H^i(G_S(\mathbb{A}), L \otimes V)$$

following [5, Chapter IX]. These complex vector spaces are finite dimensional, and are zero for $i \geq 0$. Indeed, the subgroup

$$K = \mathcal{G}(\hat{\mathbb{Z}}_S) = \prod_{p \notin S} G(\mathbb{Z}_p)$$

of $G_S(\mathbb{A})$ is compact, so only contributes to $H^0$, and we find

$$H^i(G_S(\mathbb{A}), L \otimes V) \simeq H^i \left( \prod_{v \in S} G(\mathbb{Q}_v), L^\mathcal{G}(\hat{\mathbb{Q}}_S) \otimes V \right),$$

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by the Künnett formula. The local continuous cohomology groups are known to be finite dimensional [5, Prop. X.6.3].

We define the Euler characteristic of the discrete spectrum tensored with $V$ by the formula

$$
\chi = \chi(G, S, V) = \sum_{i \geq 0} (-1)^i \dim H^i(G_\mathbb{A}(\mathbb{A}), L \otimes V).
$$

Our goal is to give an explicit formula for $\chi$, under the following two hypotheses:

1. $\text{Card}(S) \geq 2$, so $S$ contains a finite prime,

2. $G(\mathbb{R})$ contains a maximal compact torus.

The first hypothesis is essential, to use results of Arthur to simplify the trace formula, as well as results of Kottwitz to stabilize it. The second hypothesis is not essential, but one finds that $\chi = 0$ for local reasons if it is not met.

When $G(\mathbb{R})$ contains a maximal compact torus $T$, we let $W^c = N(T)/T$ be its Weyl group in $G(\mathbb{R})$ (the compact Weyl group) and $W = N(T_\mathbb{C})/T_\mathbb{C}$ be its Weyl group in $G(\mathbb{C})$. We will see that

$$
\chi = (W : W^c) \cdot \chi^*
$$

with $\chi^*$ equal to the Euler characteristic $\chi(G^*, S, V)$ of any inner form $G^*$ of $G$ which is compact over $\mathbb{R}$ and unramified outside of $S$. Our formula will express the integer $\chi^*$ as a sum of rational numbers. The terms in the sum will be indexed by the rational stable, torsion conjugacy classes in $G$ (or
equivalently, in $G^*$. The global contribution of each torsion class $\gamma$ to the sum is

$$\frac{1}{2^l} L_S(M_\gamma) \cdot \text{Tr}(\gamma|V).$$

Here $\ell = \dim(T)$ is the rank of $G$ over $\mathbb{C}$, and $M_\gamma$ is the Artin-Tate motive of rank $l$ which is associated to the centralizer $G_\gamma$ in [11]. This motive is well-defined by the stable class of $\gamma$, as $G_\gamma$ is determined up to inner twisting over $\mathbb{Q}$. The term $L_S(M_\gamma)$ is the value of the Artin $L$-function of $M_\gamma$, with the Euler factors at $S$ removed, at the point $s = 0$. This special value is known to be a rational number, by results of Siegel [21].
2. The trace formula

We recall the trace formula in an abstract setting. Let $\mathcal{G}$ be a locally compact topological group, and $\Gamma$ a subgroup of $\mathcal{G}$ which is both discrete and co-compact. A Haar measure $dg$ on $\mathcal{G}$ induces a measure on the coset space $\Gamma \backslash \mathcal{G}$, taking counting measure on $\Gamma$.

Let $f$ be compactly supported and measurable on $\mathcal{G}$. We will assume further that $f$ is regular in the sense of Bruhat [6]. If $G$ is a Lie group, this regularity condition is exactly that $f$ be infinitely differentiable. More generally, one can show that $G$ has a subgroup $G_1$ which is the projective limit of Lie groups and for which the kernels of the projections onto these Lie groups are all compact. The regularity condition then asks that $f$ be supported on finitely many cosets of $G_1$, and that on each coset $f$ be the translation of a function coming from an infinitely differentiable function on one of the Lie quotients of $G_1$.

Now let $\varphi$ be the compactly supported measure $\varphi = f \ dg$. Then $\varphi$ gives an endomorphism of the Hilbert space $L^2(\Gamma \backslash \mathcal{G}, dg)$, mapping $F$ to the function

$$\varphi F(x) = \int_{\mathcal{G}} F(xg) \varphi(g).$$

The endomorphism $\varphi$ has a compact kernel

$$K(x, g) = \sum_{\gamma \in \Gamma} f(x^{-1}\gamma g).$$

The sum is finite, as $\Gamma$ is discrete and $f$ has compact support. Note that $K$
is a function on $\Gamma \backslash \mathcal{G} \times \Gamma \backslash \mathcal{G}$. Since the kernel is compact, the endomorphism $\varphi$ has a trace.

For $\gamma$ in $\Gamma$, let $\Gamma_\gamma$ be its centralizer in $\Gamma$ and let $\mathcal{G}_\gamma$ be its centralizer in $\mathcal{G}$. Define the orbital integral

$$O_\gamma(\varphi, dg_\gamma) = \int_{\mathcal{G}_\gamma \backslash \mathcal{G}} f(g^{-1} \gamma g) \frac{dg}{dg_\gamma}.$$ 

This depends on the choice of a Haar measure $dg_\gamma$ on $\mathcal{G}_\gamma$. The orbital measure

$$dg_\gamma(\varphi) = O_\gamma(\varphi, dg_\gamma) \cdot dg_\gamma$$ 

on $\mathcal{G}_\gamma$ is invariant and depends only on $\gamma$.

The abstract trace formula is then

$$\text{Tr}(\varphi|L^2(\Gamma \backslash \mathcal{G})) = \sum_\gamma \int_{\Gamma \gamma \backslash \mathcal{G}_\gamma} dg_\gamma(\varphi) = \sum_\gamma \int_{\Gamma \gamma \backslash \mathcal{G}_\gamma} dg_\gamma \cdot O_\gamma(\varphi, dg_\gamma),$$

where $\gamma$ runs through a set of representatives of the conjugacy classes of $\Gamma$. The sum, moreover, is absolutely convergent [17].

When $G(\mathbb{R})$ is compact, we may apply the above formalism to $\mathcal{G} = G(A)$ and $\Gamma = G(\mathbb{Q})$. In this case, the centralizer $G_\gamma$ is a connected, reductive group, and is anisotropic over $\mathbb{Q}$. In particular, every conjugacy class $\gamma$ in $G(\mathbb{Q})$ is semi-simple and elliptic in $G$. (Recall that $\gamma$ is elliptic if it is contained in a maximal anisotropic torus $T$ of $G$ over $\mathbb{Q}$).
When $G(\mathbb{R})$ is not compact, the discrete group $G(\mathbb{Q})$ need not be co-compact in $G(\mathbb{A})$. We need a modification of the trace formula, where we take the trace of $\varphi$ only on the discrete spectrum $L$, and where we only sum over elliptic, semi-simple conjugacy classes in $G(\mathbb{Q})$. This modification will exist for certain measures $\varphi = \Pi \varphi_v$ on $G(\mathbb{A})$, satisfying local conditions.

If $\varphi = \varphi_v$ is a smooth, compactly supported measure on $G(\mathbb{Q}_v)$, and $\pi$ is an irreducible, complex representation of this local group, then the endomorphism

$$\varphi(w) = \int_{G(\mathbb{Q}_v)} g(w) \varphi(g)$$

of $\pi$ has a trace, which we denote $\text{Tr}(\varphi|\pi)$. Similarly, if $\gamma$ is a conjugacy class in $G(\mathbb{Q}_v)$, we define the orbital integral

$$O_\gamma(\varphi, dg_\gamma) = \int_{G_\gamma(\mathbb{Q}_v):G(\mathbb{Q}_v)} f(g^{-1} \gamma g) \frac{dg}{dg_\gamma},$$

which depends on the choice of an invariant measure $dg_\gamma$ on the centralizer $G_\gamma(\mathbb{Q}_v)$. For the convergence of this integral, see [20]. The orbital measure

$$dg_\gamma(\varphi) = O_\gamma(\varphi, dg_\gamma) \cdot dg_\gamma$$

on $G_\gamma$ is again well-defined, independent of choices.

**Proposition** (Arthur). Assume that the smooth, compactly supported measure $\varphi = \Pi \varphi_v$ on $G(\mathbb{A})$ satisfies the following three local conditions:

1. $\text{Tr}(\varphi_\infty|\pi_\infty) = 0$, unless the infinitesimal character of $\pi_\infty$ is regular.
2. $dg_{\gamma_\infty}(\varphi_\infty) = 0$, unless the class $\gamma_\infty$ is both elliptic and semi-simple.

3. $dg_{\gamma_p}(\varphi_p) = 0$, unless the class $\gamma_p$ is both elliptic and semi-simple, for some finite $p$.

Then $\varphi$ is of trace class on the discrete spectrum $L$, and

$$
\text{Tr}(\varphi|L) = \sum_{\gamma} \int_{G_\gamma(\mathcal{Q}) \backslash G_\gamma(\mathbb{A})} dg_\gamma(\varphi)
= \sum_{\gamma} \int_{G_\gamma(\mathcal{Q}) \backslash G_\gamma(\mathbb{A})} dg_\gamma \cdot O_\gamma(\varphi, dg_\gamma)
$$

where the sum is taken over representatives for the elliptic, semi-simple conjugacy classes in $G(\mathcal{Q})$, only finitely many of which have a non-zero orbital integral for $\varphi$.

We now sketch the proof, which follows from Arthur’s general theory. The hypotheses 2) and 3) on vanishing orbital integrals imply that the geometric side of the trace formula is given by the sum of orbital integrals over elliptic, semi-simple conjugacy classes in $G(\mathcal{Q})$:

$$
I(f) = \sum_{\gamma} \tau(G_\gamma)O_\gamma(f).
$$

Here we have used the fact that $G$ is simply connected, so by a result of Steinberg, $G_\gamma$ is connected. This allows us to identify Arthur’s weighting factor $a^G$ with $\tau(G_\gamma)$. 

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The spectral side of trace formula is given by a sum over conjugacy classes of Levi subgroups $M$ of $G$. However, if $M \neq G$, these terms will be a linear combination of representations whose real component has singular infinitesimal character. Since hypothesis 1) implies that these terms vanish for the test measure $\varphi$, one is left with the term for $M = G$, which is just

$$J(f) = \text{Tr}(\varphi|L).$$
3. A test function to compute $\chi(G, S; V)$

To use the trace formula to compute

$$\chi(G, S, V) = \chi(G_S(\mathbb{A}), L \otimes V),$$

we write $L$ as a Hilbert direct sum

$$L = \bigoplus m(\pi)\pi$$

with finite multiplicities. Then

$$\chi(G_S(\mathbb{A}), L \otimes V) = \sum m(\pi)\chi(G_S(\mathbb{A}), \pi \otimes V).$$

The group $G_S(\mathbb{A})$ is a direct product, and the representation $\pi \otimes V$ of $G_S(\mathbb{A})$ is a restricted tensor product: $\pi = \otimes \pi_v$. Since the Euler characteristic is multiplicative, we have

$$\chi(G_S(\mathbb{A}), \pi \otimes V) = \chi(G(\mathbb{R}), \pi_\infty \otimes V) \cdot \prod_{p \in S} \chi(G(\mathbb{Q}_p), \pi_p) \prod_{p \notin S} \chi(G(\mathbb{Z}_p), \pi_p).$$

The term $\chi(G(\mathbb{Z}_p), \pi_p) = \dim \pi_p^{G(\mathbb{Z}_p)}$ is either 0 or 1, so the product of Euler characteristics is either 0 or finite.

Our task is thus to construct local measures $\varphi_v$, such that for all irreducible representations $\pi_v$ of $G(\mathbb{Q}_v)$:

$$\text{Tr}(\varphi_\infty | \pi_\infty) = \chi(G(\mathbb{R}), \pi_\infty \otimes V)$$

$$\text{Tr}(\varphi_p | \pi_p) = \chi(G(\mathbb{Q}_p), \pi_p) \quad p \in S$$

$$\text{Tr}(\varphi_p | \pi_p) = \chi(G(\mathbb{Z}_p), \pi_p) \quad p \notin S$$
Then we will have

\[ \chi(G_S(A), \pi \otimes V) = \text{Tr}(\varphi|\pi) \quad \text{for all irreducible } \pi, \text{ and hence} \]

\[ \chi(G_S(A), L \otimes V) = \text{Tr}(\varphi|L). \]

Of course, to calculate \( \text{Tr}(\varphi|L) \) using the trace formula, we will have to verify that \( \varphi_\infty \) and \( \varphi_p \) satisfy the local conditions of the Proposition. We will also need to calculate the local orbital measures \( dg_\gamma(\varphi_v) \) of the test measures \( \varphi_v \).

At primes \( p \) which are not in \( S \), the measure

\[ \varphi_p = \frac{ch(G(Z_p))}{\int_{G(Z_p)} dg_p} \cdot dg_p \]

has the desired property, where \( ch \) is the characteristic function of the open compact subset \( G(Z_p) \). Indeed, the endomorphism \( \varphi_p \) of \( \pi_p \) is

\[
\varphi_p(w) = \int_{G(Q_p)} g(w) \varphi(g) = \int_{G(Z_p)} g(w) \cdot \int_{G(Z_p)} dg_p .
\]

This is just the projection of \( w \) to the \( G(Z_p) \) – fixed space in \( \pi_p \), so

\[ \text{Tr}(\varphi_p|\pi_p) = \dim \pi_p^{G(Z_p)}. \]

The calculation of the orbital integrals of the local measure \( \varphi_p \) specified above is a fundamental problem in local harmonic analysis. Clearly this orbital integral is zero unless the conjugacy class \( C(\gamma) \) of \( \gamma \) in \( G(Q_p) \) meets \( G(Z_p) \). In this case, we say \( \gamma \) is integral. There are finitely many \( G(Z_p) \)
orbits on $C(\gamma) \cap G(\mathbb{Z}_p)$, and their stabilizers are open compact subgroups $K_i$ of $G_\gamma(\mathbb{Q}_p)$. The orbital measure is then

$$dg_\gamma(\varphi_p) = \sum_i \frac{1}{ \int_{K_i} dg_\gamma} \cdot dg_\gamma.$$ 

We say an integral, semi-simple class $\gamma$ has good reduction (mod $p$) if, for every root $\alpha$ of $G$, the $p$-integer ($\alpha(\gamma) - 1$) is either 0 or a unit. In other words, the class of $\gamma$ has good reduction if it has no excess intersection (mod $p$) with the discriminant divisor, in the variety of conjugacy classes. In this case, Kottwitz has shown [15, Prop. 7.1] that the group scheme $G_\gamma$ over $\mathbb{Z}_p$ has good reduction (mod $p$), so $G_\gamma(\mathbb{Z}_p)$ is a hyperspecial maximal compact subgroup in $G_\gamma(\mathbb{Q}_p)$. Moreover, if $\gamma$ has good reduction (mod $p$), the group $G(\mathbb{Z}_p)$ has a single orbit on $C(\gamma) \cap G(\mathbb{Z}_p)$, with stabilizer $G_\gamma(\mathbb{Z}_p)$. Hence, in this case, $dg_\gamma(\varphi_p)$ is the unique Haar measure with

$$\int_{G_\gamma(\mathbb{Z}_p)} dg_\gamma(\varphi_p) = 1.$$ 

At finite primes $p$ in $S$, we need a locally constant, compactly supported measure $\varphi_p$ such that

$$\text{Tr}(\varphi_p|\pi_p) = \sum (-1)^i \dim H^i(G(\mathbb{Q}_p), \pi_p).$$ 

Let $\mathcal{F}$ be a facet of maximal dimension in the building of $G(\mathbb{Q}_p)$, and let $\mathcal{F}_j$ be the faces of $\mathcal{F}$. The dimension of $\mathcal{F}$ is the rank $\ell$ of $G$ over $\mathbb{Q}_p$. Let $K_j \subset G(\mathbb{Q}_p)$ be the parahoric subgroup fixing the facet $\mathcal{F}_j$. Then Kottwitz
has shown that the measure
\[ \varphi_p = \sum_j (-1)^{\dim \mathcal{F}_j} \cdot \frac{\text{ch}(K_j)}{\int_{K_j} dg_p} \cdot dg_p \]
has the desired traces. In particular, we have
\[ \sum_i (-1)^i \dim H^i(G(\mathbb{Q}_p), \pi_p) = \sum_j (-1)^{\dim \mathcal{F}_j} \dim(\pi_{K_j}). \]

For example, the Steinberg representation \( St \) of \( G(\mathbb{Q}_p) \) has a line fixed by the Iwahori subgroup \( K \) fixing \( \mathcal{F} \) pointwise, and no fixed vectors under any larger parahoric subgroup. Hence \( \chi(St) = (-1)\ell \); this agrees with the calculation of \( H^i(G(\mathbb{Q}_p), St) \) by Casselman, as the cohomology is zero for \( i \neq \ell \), and one-dimensional for \( i = \ell \).

Kottwitz also calculated the orbital integrals of \( \varphi_p \). For \( \gamma = 1 \), we have
\[ dg_\gamma(\varphi_p) = \sum_j (-1)^{\dim \mathcal{F}_j} \cdot \frac{1}{\int_{K_j} dg_p} \cdot dg_p, \]
which is Serre’s formula for Euler-Poincaré measure on \( G(\mathbb{Q}_p) \). This is the unique invariant measure \( \mu \) such that
\[ \int_{\Gamma \backslash G(\mathbb{Q}_p)} d\mu = \chi(\Gamma) = \sum_i (-1)^i \dim H^i(\Gamma, \mathbb{Q}) \]
for each discrete, co-compact, torsion-free subgroup \( \Gamma \). More generally, Kottwitz has shown that for any \( \gamma \)
\[ dg_\gamma(\varphi_p) = d\mu_\gamma = \text{Euler-Poincaré measure on } G_\gamma(\mathbb{Q}_p). \]

This measure is zero, unless \( \gamma \) is elliptic and semi-simple.
At the real place, we need to construct a smooth compactly supported measure $\varphi_\infty$ on $G(\mathbb{R})$ such that

$$\text{Tr}(\varphi_\infty|_{\pi_\infty}) = \sum (-1)^i \dim H^i(G(\mathbb{R}), \pi_\infty \otimes V).$$

When $G(\mathbb{R})$ is compact, we have $H^i = 0$ for $i \geq 1$ and the Euler characteristic is equal to

$$\dim(\pi_\infty \otimes V)^{G(\mathbb{R})}.$$

In this case, we may take the test measure

$$\varphi_\infty = \frac{\text{Tr}(g_\infty|V)}{\int_{G(\mathbb{R})} dg_\infty} \cdot dg_\infty.$$

Indeed, the endomorphism $\varphi_\infty$ of $\pi_\infty$ is just $\frac{1}{\dim V}$ times the projection onto the $V^\ast$-isotypical space. In the case when $G(\mathbb{R})$ is not compact, a suitable measure $\varphi$ was constructed by Clozel and Delorme [9], who also calculated its orbital integrals. We have

$$dg_\gamma(\varphi_\infty) = \text{Tr}(\gamma|V) \cdot \text{Euler-Poincaré measure on } G_\gamma(\mathbb{R}).$$

This is zero, unless $\gamma$ is semi-simple and elliptic. Also, since any $\pi_\infty$ with cohomology has the same infinitesimal character as $V^\ast$, which is regular, we have $\text{Tr}(\varphi_\infty|\pi_\infty) = 0$ unless $\pi_\infty$ has a regular infinitesimal character.

Since $\#S \geq 2$, the test measure $\varphi = \prod \varphi_\gamma = f \cdot dg$ satisfies all the conditions of the Proposition. Hence

$$\chi(G, S, V) = \text{Tr}(\varphi|L) = \sum_{\gamma} \int_{G_\gamma(\mathbb{Q})\backslash G_\gamma(\mathbb{A})} dg_\gamma \cdot \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} f(g^{-1}g) \frac{dg}{dg_\gamma}.$$
4. The stable trace formula

We henceforth take Tamagawa measure $dg_{\gamma}$ on the adèlic group $G_\gamma(\mathbb{A})$, so

$$
\int_{G_\gamma(\mathbb{Q}) \backslash G_\gamma(\mathbb{A})} dg_{\gamma} = \tau(G_\gamma)
$$

is, by definition, the Tamagawa number. For a discussion of Tamagawa measure see [8]. The trace formula then reads

$$
\chi(G, S, V) = \text{Tr}(\varphi|L) = \sum_\gamma \tau(G_\gamma)O(\varphi, dg_\gamma).
$$

The sum is over semi-simple classes $\gamma$ of $G(\mathbb{Q})$ which are elliptic in $G(\mathbb{Q}_v)$ for all $v \in S$.

Since the support of $\varphi_p$ is the union of compact open subgroups for all $p$, the class $\gamma$ must be integral to contribute a non-vanishing orbital integral. Since $\gamma$ is also elliptic over $\mathbb{R}$, it is contained in a compact subgroup $K$ of $G(\mathbb{A})$. But $K \cap G(\mathbb{Q})$ is finite, so $\gamma$ is a torsion conjugacy class. Hence the above sum is over torsion classes which are also elliptic at the finite primes in $S$.

The problem in using this trace formula to calculate $\chi(G, S, W)$ is that semi-simple conjugacy classes $\gamma$ in $G(\mathbb{Q})$ are difficult to describe. For example, when $G = \text{SL}_2$, there are infinitely many conjugacy classes of order 4, all conjugate over $\mathbb{Q}$. Using the Euler-Poincaré test measure $\varphi_p$, Kottwitz was able to convert the above expression into a sum over stable conjugacy classes in the quasi-split inner form $G'$ of $G$ over $\mathbb{Q}$. We describe his formula below, and use it to compute $\chi$ in the next section.
We say two semi-simple elements of $G'\left( \mathbb{Q} \right)$ are stably conjugate if they are conjugate in $G'\left( \overline{\mathbb{Q}} \right).$ (We recall that the group $G'$ is simply-connected, which simplifies the definition of stable conjugacy.) Let $T$ denote a (finite) set of representatives for the stable, torsion conjugacy classes in $G'\left( \mathbb{Q} \right).$ Fix an inner twisting $\psi : G' \to G$ over $\overline{\mathbb{Q}}$ and, for $t \in T,$ consider the set of $\gamma$ in $G(\mathbb{A})$ that are conjugate to $\psi(t)$ in $G(\overline{\mathbb{A}}).$ The group $G(\mathbb{A})$ acts on this set by conjugation, and for each $\gamma$ we have the adèlic centralizer $G_{\gamma}(\mathbb{A}).$ If $\gamma$ is conjugate to an element in $G(\mathbb{Q}),$ then $G_{\gamma}(\mathbb{A})$ contains the discrete subgroup $G_{\gamma}(\mathbb{Q}).$

We can always define Tamagawa measure $dg_{\gamma}$ on $G_{\gamma}(\mathbb{A}),$ using the inner twisting. Indeed, let $dg'_{\gamma}$ be Tamagawa measure on $G'_{\gamma}(\mathbb{A}),$ and fix a product decomposition: $dg'_{\gamma} = \otimes (dg'_{\gamma})_v.$ For each place $v,$ $G_{\gamma_v}$ is an inner twist of $G_{\gamma_v}$ over $\mathbb{Q}_v,$ so we may transfer the measure $(dg'_{\gamma})_v$ to a measure $(dg)_{\gamma_v}$ on $G_{\gamma_v}(\mathbb{Q}_v).$ We then define

$$dg_{\gamma} = \otimes (dg)_{\gamma_v}.$$  

If $\gamma$ is in $G(\mathbb{Q}),$ this agrees with usual Tamagawa measure, and we can define $\tau(G_{\gamma}).$ In general, there is no Tamagawa number, but we can still define the adèlic orbital integral

$$O_{\gamma}(\varphi, dg_{\gamma}) = \int_{G_{\gamma}(\mathbb{A}) \setminus G(\mathbb{A})} f(g^{-1}\gamma g) \frac{dg}{dg_{\gamma}}.$$  

We may also attach a sign $e(\gamma) = \pm 1$ to the adèlic class $\gamma,$ by the formula

$$e(\gamma) = \prod_v e(G_{\gamma_v}).$$
The local invariants \(e(G_{\gamma})\) are defined in [14]. If \(\gamma\) is in \(G(\mathbb{Q})\), \(e(\gamma) = +1\).

**Proposition** (Kottwitz [15]).

\[
\chi(G, S, V) = \sum_T \sum_{\gamma} e(\gamma) O_{\gamma}(\varphi, dg_{\gamma}),
\]

where the first sum is over the stable torsion classes in \(G'\), and the second is over the \(G(\mathbb{A})\)-conjugacy classes of \(\gamma\) in \(G(\mathbb{A})\), which are conjugate to \(\psi(t)\) in \(G(\mathbb{A})\).

We sketch the proof. As usual, there are an infinite number of \(\gamma\) in the inner sum, but only finitely many have a non-zero orbital integral. Kottwitz writes the geometric side of the trace formula as a triple sum

\[
\sum_T \sum_{\gamma} \sum_\kappa \langle \text{obs}(\gamma), \kappa \rangle e(\gamma) O_{\gamma}(\varphi, dg_{\gamma}),
\]

where \(\kappa\) runs over elements of a finite abelian group, which depends only on \(t\) and is used to detect whether or not \(\gamma\) comes from a class in \(G(\mathbb{Q})\). We switch the inner sums, and exploit the fact that \(\varphi_p\) is the Euler-Poincaré function at a finite prime in \(S\).

Then Kottwitz shows that for \(\kappa \neq 1\):

\[
\sum_\gamma \langle \text{obs}(\gamma), \kappa \rangle e(\gamma) O_{\gamma}(\varphi, dg_{\gamma}) = 0.
\]

Hence we obtain the simple stable formula in the proposition.

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5. A comparison of measures

The stable formula for $\chi(G,S,V)$ is still not readily computable, as we have only evaluated the local orbital integrals for $\varphi$. To convert $O_\gamma(\varphi,dg_\gamma)$ into a product of local integrals, we need to express Tamagawa measure $dg_\gamma$ on $G_\gamma(\mathbb{A})$ as a product of local measures.

To do this, we use the results of [11]. For $v \in S$, we let $d\mu_v$ be Euler-Poincaré measure on $G_\gamma(\mathbb{Q}_v)$. For $p$ not in $S$, we let $G'_\gamma$ be the quasi-split inner form of $G_\gamma$, and we let $d\mu_p$ be the measure on $G_\gamma(\mathbb{Q}_p)$ transferred from the Haar measure on $G'_\gamma(\mathbb{Q}_p)$ which gives the connected component of a certain special compact subgroup volume 1. This measure on $G_\gamma(\mathbb{Q}_p)$ is denoted $L(M_{G'_\gamma}(1)) \cdot |\omega_{G_\gamma}|$ in [11, Sect. 4]. When $G_\gamma$ is unramified at $p$ and corresponds to the model $G_\gamma$ over $\mathbb{Z}_p$ with good reduction, we have $\int_{G_\gamma(\mathbb{Q}_p)} d\mu_p = 1$. Hence we can form the product measure $d\mu_\gamma = \otimes_{v \in S} d\mu_v$ on $G_\gamma(\mathbb{A})$.

The main global result of [11] then gives the ratio of measures on $G_\gamma(\mathbb{A})$:

$$d\mu_\gamma/dg_\gamma = L_S(M_t)/\prod_{v \in S} e(\gamma_v)c(\gamma_v).$$

Here $L_S(M_t)$ is the value of the Artin $L$-series of the motive of $G_\gamma$ at $s = 0$, which only depends on the stable class $t$ of $\gamma$, and the sign $e(\gamma_v) = e(G_{\gamma_v}) = \pm 1$ is the local invariant defined by Kottwitz [14]. The invariant $c(\gamma_v)$ is defined as follows.

For primes $p$ in $S$,

$$c(\gamma_p) = \#H^1(\mathbb{Q}_p, G_\gamma).$$

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This depends only on the stable class of \( t \) over \( \mathbb{Q}_p \), and gives the number of classes \( \gamma_p \) in the stable class (as \( H^1(\mathbb{Q}_p, G) = 1 \)).

At the real place, we have

\[
c(\gamma_{\infty}) = \frac{\#H^1(\mathbb{R}, T)}{\#\ker(H^1(\mathbb{R}, T) \rightarrow H^1(\mathbb{R}, G, \gamma))}.
\]

Here \( T \subset G_{\gamma} \subset G \) is a maximal anisotropic torus, so \( \#H^1(\mathbb{R}, T) = 2^\ell \), with \( \ell = \dim T \).

We now replace the measure \( dg/dg_\gamma \) on \( G_{\gamma}(\mathbb{A}) \backslash G(\mathbb{A}) \) by the equivalent term

\[
dg/d\mu_\gamma \cdot L_S(M_t)/\prod_{v \in S} e(\gamma_v)c(\gamma_v).
\]

This allows us to write the adèlic orbital integral as a product of local integrals

\[
e(\gamma)O_\gamma(\varphi, dg_\gamma) = L_S(M_t) \cdot \prod_{v \in S} O_{\gamma_v}(\varphi_v, d\mu_v)/c(\gamma_v) \cdot \prod_{p \notin S} O_{\gamma_p}(\varphi_p, d\mu_p) e(\gamma_p).
\]

Since each adèlic class \( \gamma \) in the stable class of \( t \) is the product of local classes \( \gamma_v \) in the stable classes of the \( t_v \), we have

\[
\chi(G, S, V) = \sum_T L_S(M_t) \cdot \prod_v SO^*_v(\varphi_v, d\mu_v).
\]

Each term in the product is now a stable orbital integral, slightly modified at primes \( v \) in \( S \):

\[
SO^*_v = SO_t(\varphi_p, d\mu_p) = \sum_{\gamma_p} e(\gamma_p)O_{\gamma_p}(\varphi_p, d\mu_p) \quad p \notin S,
\]

\[
SO^*_v = \sum_{\gamma_v} c(\gamma_v)^{-1}O_{\gamma_v}(\varphi_v, d\mu_v) \quad v \in S.
\]
The sums are taken over the finitely many classes \( \gamma_v \) in \( G(\mathbb{Q}_v) \), which are in the stable class of \( \psi(t) \) in \( G(\mathbb{Q}_v) \).

We now turn to the evaluation of the stable local terms \( SO_t^* \). When \( v = p \) is a finite prime in \( S \), we have \( O_{\gamma_v}(\varphi_v, d\mu_v) = 1 \). This is assuming that \( \gamma_v \) is elliptic; if not, \( L_S(M_t) = 0 \). The constant \( c(\gamma_v) = c(t) \) is the number of local classes in the stable class of \( t \). Hence \( SO_t^*(\varphi_v, d\mu_v) = 1 \).

When \( v = \infty \) and \( \gamma_v \) is elliptic, we have \( O_{\gamma_v}(\varphi_v, d\mu_v) = \text{Tr}(\gamma_v|V) \). This depends only on the stable class \( t \) of \( \gamma_v \). The constant \( c(\gamma_v) \) was described above, and this gives

\[
SO_t^*(\varphi_v, d\mu_v) = \frac{\text{Tr}(t|V)}{2^t} \cdot \sum_{\gamma_v} \# \ker(H^1(\mathbb{R}, T) \to H^1(\mathbb{R}, G_{\gamma_v}))
\]

\[
= \frac{\text{Tr}(t|V)}{2^t} \cdot \# \ker(H^1(\mathbb{R}, T) \to H^1(\mathbb{R}, G)).
\]

The latter kernel has cardinality \((W : W^c)\). Hence we have shown

\[
\chi(G, S, V) = (W : W^c) \sum_T \frac{1}{2^t} L_S(M_t) \text{Tr}(t|V) \cdot \prod_{p \notin S} SO_t(\varphi_p, d\mu_p).
\]

Finally, we consider the stable orbital integrals at the primes \( p \) not in \( S \).

For each class \( t \), almost all of these terms are equal to 1. For example, if \( p \) does not divide the order of \( t \), then there is a single class \( \gamma \) in the stable class over \( \mathbb{Q}_p \) which meets \( G(\mathbb{Z}_p) \), and for this class we have seen that \( O_{\gamma}(\varphi_p, d\mu_p) = 1 \).

Since \( G_{\gamma} \) is unramified in this case, \( e(\gamma) = 1 \) and hence \( SO_t(\varphi_p, d\mu_p) = 1 \). We are left with the formula

\[
\chi(G, S, V) = (W : W^c) \sum_T \frac{1}{2^t} L_S(M_t) \text{Tr}(t|V) \cdot \prod_{\substack{p \mid \text{order}(t) \atop p \notin S}} SO_t(\varphi_p, d\mu_p). \quad (1)
\]

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If, for example, the torsion primes for $G$ are all contained in $S$, we have a complete formula (as the product is empty). In all cases, the primary contribution of the stable torsion class $t$ to $\chi$ is

$$ (W : W^e) \cdot \frac{1}{2t} L_S(M_t) \text{Tr}(t|V), $$

as claimed earlier.

The remaining calculation of $SO_t(\varphi_p, d\mu_p)$ is a central local problem. For each $\gamma_p$ in $G(\mathbb{Q}_p)$ which is stably conjugate to $t$, we must write

$$ Cl(\gamma_p) \cap G(\mathbb{Z}_p) = \prod_i K_i \setminus G(\mathbb{Z}_p). $$

Then

$$ SO_t(\varphi_p, d\mu_p) = \sum_{\gamma_p} e(\gamma_p) \cdot \sum_i \frac{1}{f_{K_i}} d\mu_p. \tag{2} $$

Unfortunately, even the first step of decomposing the integral elements of $Cl(\gamma_p)$ into integral conjugacy classes is not readily computable. Our approach to computing the stable orbital integrals $SO_t(\varphi_p, d\mu_p)$ in the next section of this paper is rather round-about. We will see in the next section that the Euler characteristic $\chi(G, S, V)$ can be computed directly for certain $G$ and small $S, V$. We may use these values in equation (1) to get a system of equations in the unknowns $SO_t(\varphi_p, d\mu_p)$. We are able to compute enough values of $\chi(G, S, V)$ to solve for all of the remaining $SO_t(\varphi_p, d\mu_p)$ when $G$ is $\text{SL}_2$, $\text{Sp}_4$, or $G_2$. We give these values in section 7 and use them to compute more values of $\chi(G, S, V)$ via (1).

Before going on, we note that from the expression (2), it follows that $SO_t$ is a rational number, which is positive whenever $t$ is regular. In the regular
case, \( e(\gamma_p) = 1 \) and \( d\mu_p \) has volume 1 on the connected component \( T^0(\mathbb{Z}_p) \) of the Néron model of \( T = G_{\gamma_p} \). Hence

\[
SO_i(\varphi_p, d\mu_p) = \sum_{\gamma_p} \sum_i (T^0(\mathbb{Z}_p) : K_i).
\]

These “indices” can have denominators \( (T(\mathbb{Z}_p) : T^0(\mathbb{Z}_p)) \). However, in all cases where we have been able to determine \( SO_i \), it turns out to be an integer (which can be negative for non-regular \( t \)).
6. Algebraic Modular Forms

For this section we drop the requirement that $G$ be simply connected, but insist that $G(\mathbb{R})$ be compact. This guarantees that $G(\mathbb{Q})$ is discrete and co-compact in $G(\mathbb{A})$. For a given representation $V$ of $G$ over $\mathbb{Q}$ and an open compact subgroup $K$ of $G(\hat{\mathbb{Q}})$ (where $\hat{\mathbb{Q}} = \hat{\mathbb{Z}} \otimes \mathbb{Q}$ is the ring of finite adéles) we define the space of modular forms on $G$ of weight $V$ and level $K$ to be the rational vector space [12]:

$$M_G(V, K) = \{ F: G(\mathbb{A})/(G(\mathbb{R})_+ \times K) \to V : F(\gamma g) = \gamma F(g), \text{ for all } \gamma \in G(\mathbb{Q}) \},$$

where $G(\mathbb{R})_+$ is the connected component of the identity in $G(\mathbb{R})$.

If $K$ is a product $K = \prod_p K_p$, with each $K_p$ open and compact in $G(\mathbb{Q}_p)$, then the Hecke algebras $\mathcal{H}(G(\mathbb{Q}_p), K_p)$ each act on $M(V, K)$, and commute with each other in $\text{End}(M(V, K))$. We will fix a finite set $S$ of places of $\mathbb{Q}$ containing those for which $G$ is ramified, and an integral model $\underline{G}$ for $G$ over the ring $\mathbb{Z}_S$ with good reduction at all $p$ not in $S$. For $p$ not in $S$, we let $K_p = G(\mathbb{Z}_p)$. For primes $p$ in $S$, we let $K_p$ be an Iwahori subgroup of $G(\mathbb{Q}_p)$, which fixes a maximal facet in the Bruhat-Tits building pointwise.

The Steinberg representation of $G(\mathbb{Q}_p)$ has a vector fixed by the Iwahori subgroup, so gives rise to a 1-dimensional representation of the Hecke algebra $\mathcal{H}(G(\mathbb{Q}_p), K_p)$. We call a character of this algebra special if it is the twist of the Steinberg character by a character of the fundamental group $\Omega$ of $G$. We may twist by such characters as $\Omega \cong G(\mathbb{Q}_p)/G(\mathbb{Q}_p)_s$, where $G(\mathbb{Q}_p)_s \supset K_p$ is the normal subgroup of elements of $G(\mathbb{Q}_p)$ that preserve the types of vertices

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in the building. Thus, special representations are those representations of $G(\mathbb{Q}_p)$ with an Iwahori-fixed vector, where the standard generators of the simply-connected Hecke algebra act by $-1$. We denote by $M_G(V, K)^{\text{St}}$ the subspace of $M_G(V, K)$ on which the Hecke algebras $\mathcal{H}(G(\mathbb{Q}_p), K_p)$ act by special characters for all $p$ in $S$.

**Proposition** (Padowitz [19]). Assume that $G$ is absolutely simple and simply-connected, and let $r_s = \sum_{p \in S} \text{rank } G(\mathbb{Q}_p)$. Let $V$ be an absolutely irreducible representation of $G$ over $\mathbb{Q}$ with trivial central character, and define $K = \prod K_p$ as above.

Then
\[
\chi(G, S, V) = (-1)^{r_s} \dim M_G(V^*, K)^{\text{St}},
\]
expect in the case when $V$ is the trivial representation and $r_s > 0$. In the exceptional case,

\[
\chi(G, S, V) = 1 + (-1)^{r_s} \dim M_G(V^*, K)^{\text{St}}.
\]

**Proof.** The dimension of $M_G(V^*, K)^{\text{St}}$ is the number of irreducible automorphic representations $\pi$ (counted with their multiplicities in the discrete spectrum) which satisfy:

- $\pi_{\infty} \cong V^*$

- $\pi_p$ is the Steinberg representation for $p \in S$

- $\pi_p$ has a vector fixed by $G(\mathbb{Z}_p)$ for $p \notin S$. 
Each such representation contributes a space of dimension $m(\pi)$ in $H^r_s(G_S(\mathbb{A}), L \otimes V)$. Moreover, by results of Casselman [7], these are the only unitary representations contributing to cohomology (except when $V$ is trivial and $r_S > 0$, in which case $\pi = \mathbb{C}$ contribute a line to $H^0(G_S(\mathbb{A}), L)$). This completes the proof.

Since we will actually compute the spaces $M_G(V, K)^{St}$ for groups $G$ of adjoint type, we need a lemma to compare spaces for isogenous groups. Let $G$ be a reductive group (such as $GL_n$ or $GSp_{2n}$) with the following property: the derived subgroup $G_0$ is simply-connected, and the center $C$ of $G$ is a split torus. Put $\bar{G} = G/C$, which is a group of adjoint type, and let $f : G_0 \to \bar{G}$ be the corresponding isogeny.

Let $V$ be an irreducible representation of $\bar{G}$, which we may also view as a representation of $G_0$ with trivial central character. Let $K_0$ be an open compact subgroup of $G_0(\hat{\mathbb{Q}})$, defined as above, and let $\bar{K}$ be such a subgroup of $\bar{G}(\hat{\mathbb{Q}})$ which contains $f(K_0)$.

The map $f : G_0 \to \bar{G}$ then induces a linear map of $\mathbb{Q}$-vector spaces $M_G(V, \bar{K}) \to M_{G_0}(V, K_0)$ which is equivariant for the action of the Hecke algebras. The comparison lemma we need is the following easily proved fact.

**Lemma** The induced map

$$M_G(V, \bar{K})^{St} \to M_{G_0}(V, K_0)^{St}$$

is an isomorphism.

The proposition and the lemma together allow us to use the calculations
of $M_G(V, K)^{\text{St}}$ in [18] to get the values of

$$\chi^* = \frac{1}{(W : W^c)} \chi(G, S, V).$$
7. Examples

We now give some examples. By interpreting \( G(\mathbb{A})/K \) geometrically, and making heavy use of a computer, the spaces \( M_G(V, K) \) and \( M_G(V, K)^{St} \) are worked out for certain \( G, V, K \) in [18]. In particular the calculations there work with the (unique) form of \( G_2 \) which is compact over \( \mathbb{R} \) and with the forms of PGSp\(_4\) which are ramified at \( \{2, \infty\} \) and at \( \{3, \infty\} \).

The calculation of the \( M(V, K) \) is computationally intensive and so has only been carried out for small weights and levels. We now tabulate the values of \( \chi^* \) we derive from these direct calculations. The corresponding values when \( G \) is the split form of SL\(_2\) are well known.

<table>
<thead>
<tr>
<th>( S )</th>
<th>( \dim V = )</th>
<th>( \lambda = (0,0) )</th>
<th>( \lambda = (0,1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( {\infty, 2} )</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( {\infty, 2, 3} )</td>
<td>1</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>( {\infty, 2, 5} )</td>
<td>-1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( {\infty, 2, 7} )</td>
<td>-4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( {\infty, 2, 11} )</td>
<td>-33</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( {\infty, 3, 5} )</td>
<td>-8</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table of \( \chi^*(G, S, V) \) for \( G = Sp_4 \)
Table of $\chi^*(G,S,V)$ for $G = G_2$

<table>
<thead>
<tr>
<th>$S$</th>
<th>$\dim V$</th>
<th>(0,0)</th>
<th>(1,0)</th>
<th>(0,1)</th>
<th>(2,0)</th>
<th>(1,1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>${\infty,2}$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>${\infty,3}$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>${\infty,5}$</td>
<td>2</td>
<td>7</td>
<td>11</td>
<td>31</td>
<td></td>
<td></td>
</tr>
<tr>
<td>${\infty,7}$</td>
<td>13</td>
<td>54</td>
<td>120</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>${\infty,11}$</td>
<td>135</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>${\infty,13}$</td>
<td>386</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>${\infty,2,3}$</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>${\infty,2,7}$</td>
<td>253</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For the three split, simply-connected groups $SL_2$, $Sp_4$, and $G_2$ over $\mathbb{Q}$, we will now tabulate the rational stable torsion classes. Since our groups are simply-connected, these are just the stable torsion classes that meet the group of rational points. We group the classes $t$ and $zt$, for $z$ in the center, as these have the same contribution to the stable trace formula for $\chi$. There are 3 groups for $SL_2$, 12 groups for $Sp_4$, and 14 rational stable torsion classes for $G_2$. Similarly, one can show there are 102 rational stable torsion classes for $F_4$, and 785 rational stable torsion classes for $E_8$.

The stable class of an element $t$ in $SL_2$, $Sp_4$, or $G_2$ is determined by its characteristic polynomial on the fundamental representation of dimension 2, 4, or 7 respectively. Since $t$ is torsion, this is a product of cyclotomic polynomials $\phi_m$. We tabulate this polynomial, as well as the value $L(M_t)$.  

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Using equation 2, the data in the two preceding tables, and a separate calculation of $\chi(\text{Sp}_4, \{p\}, V)$ for $p$ prime and $V$ trivial, we are able to solve for the values of $SO_t(\varphi_p, d\mu_p)$. Recall that we know that all but finitely many of these values are equal to 1. We include in our tables only those values of $SO_t(\varphi_p, d\mu_p)$ which are not equal to 1. With these values computed, we are then able to tabulate the integers

$$\chi^* = \frac{1}{(W : W^c)} \chi(G, S, V)$$

for many pairs $(S, V)$ beyond those values obtained directly from looking at modular forms. The value of $\chi^*$ depends only on the inner class of $G$ over $\mathbb{Q}$.

**Torsion Classes in $\text{SL}_2$**

<table>
<thead>
<tr>
<th>order $t$</th>
<th>char poly</th>
<th>$L(M_t)$</th>
<th>$SO_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 2</td>
<td>$\phi_1^2, \phi_2^2$</td>
<td>$\frac{1}{12}$</td>
<td></td>
</tr>
<tr>
<td>3, 6</td>
<td>$\phi_3, \phi_6$</td>
<td>$\frac{1}{3}$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$\phi_4$</td>
<td>$\frac{1}{2}$</td>
<td></td>
</tr>
</tbody>
</table>
### Torsion Classes in $\text{Sp}_4$

<table>
<thead>
<tr>
<th>order $t$</th>
<th>char poly $t$</th>
<th>$L(M_t)$</th>
<th>$SO_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1, 2$</td>
<td>$\phi_1^4, \phi_2^4$</td>
<td>$-\frac{1}{1440}$</td>
<td></td>
</tr>
<tr>
<td>$2$</td>
<td>$\phi_1^2 \phi_2^2$</td>
<td>$\frac{1}{144}$</td>
<td>$SO_t(\varphi_2) = 7$</td>
</tr>
<tr>
<td>$3, 6$</td>
<td>$\phi_1^2 \phi_3, \phi_1^2 \phi_6$</td>
<td>$\frac{1}{36}$</td>
<td></td>
</tr>
<tr>
<td>$3, 6$</td>
<td>$\phi_3^2, \phi_6^2$</td>
<td>$\frac{1}{36}$</td>
<td></td>
</tr>
<tr>
<td>$4$</td>
<td>$\phi_4^2$</td>
<td>$\frac{1}{24}$</td>
<td></td>
</tr>
<tr>
<td>$4, 4$</td>
<td>$\phi_1^2 \phi_4, \phi_2^2 \phi_4^2$</td>
<td>$-\frac{1}{24}$</td>
<td></td>
</tr>
<tr>
<td>$6, 6$</td>
<td>$\phi_1^2 \phi_6, \phi_2^2 \phi_3$</td>
<td>$\frac{1}{36}$</td>
<td></td>
</tr>
<tr>
<td>$5, 10$</td>
<td>$\phi_5, \phi_{10}$</td>
<td>$\frac{2}{6}$</td>
<td></td>
</tr>
<tr>
<td>$6$</td>
<td>$\phi_3 \phi_6$</td>
<td>$\frac{1}{9}$</td>
<td>$SO_t(\varphi_2) = 4$</td>
</tr>
<tr>
<td>$8$</td>
<td>$\phi_8$</td>
<td>$\frac{1}{2}$</td>
<td></td>
</tr>
<tr>
<td>$12$</td>
<td>$\phi_{12}$</td>
<td>$\frac{1}{6}$</td>
<td></td>
</tr>
<tr>
<td>$12, 12$</td>
<td>$\phi_3 \phi_4, \phi_6 \phi_4$</td>
<td>$\frac{1}{6}$</td>
<td></td>
</tr>
</tbody>
</table>
## Torsion Classes in $G_2$

<table>
<thead>
<tr>
<th>order $t$</th>
<th>char poly $t$</th>
<th>$L(M_t)$</th>
<th>$SO_t$</th>
<th>$SO_t(\varphi_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\phi_1^7$</td>
<td>$\frac{1}{3024}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$\phi_1^3\phi_2^4$</td>
<td>$\frac{1}{144}$</td>
<td>$SO_1(\varphi_2) = 31$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$\phi_1\phi_3^3$</td>
<td>$\frac{1}{54}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$\phi_1^3\phi_3$</td>
<td>$-\frac{1}{36}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$\phi_1\phi_2^2\phi_4^2$</td>
<td>$-\frac{1}{24}$</td>
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<td></td>
</tr>
<tr>
<td>4</td>
<td>$\phi_1^3\phi_4^2$</td>
<td>$-\frac{1}{24}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$\phi_1\phi_3^2\phi_6^2$</td>
<td>$-\frac{1}{36}$</td>
<td>$SO_1(\varphi_2) = -2$</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$\phi_1^3\phi_6^2$</td>
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<td></td>
</tr>
<tr>
<td>6</td>
<td>$\phi_1\phi_2^2\phi_3\phi_6$</td>
<td>$\frac{1}{5}$</td>
<td>$SO_1(\varphi_2) = 4$</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>$\phi_1\phi_7$</td>
<td>$\frac{4}{7}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>$\phi_1\phi_2\phi_8$</td>
<td>$\frac{1}{2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>$\phi_1\phi_4\phi_8$</td>
<td>$\frac{1}{2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>$\phi_1\phi_2^2\phi_{12}$</td>
<td>$\frac{1}{6}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>$\phi_1\phi_3\phi_{12}$</td>
<td>$\frac{1}{6}$</td>
<td>$SO_1(\varphi_2) = 4$</td>
<td></td>
</tr>
</tbody>
</table>
Table of $\chi^*(G, S, V)$ for $G = SL_2$

<table>
<thead>
<tr>
<th>$V = V_\lambda$</th>
<th>$\lambda$</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
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<td>3</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>11</td>
</tr>
<tr>
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Table of $\chi^*(G, S, V)$ for $G = Sp_4$

<table>
<thead>
<tr>
<th>$V = V_\lambda$</th>
<th>$\lambda$</th>
<th>(0,0)</th>
<th>(0,1)</th>
<th>(2,0)</th>
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<th>(2,1)</th>
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<tbody>
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<td>$S$ dim V =</td>
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<tr>
<td>${\infty, 3}$</td>
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<td>0</td>
<td>0</td>
<td>-1</td>
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<tr>
<td>${\infty, 5}$</td>
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<td>-1</td>
<td>-1</td>
<td>-6</td>
</tr>
<tr>
<td>${\infty, 7}$</td>
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<td>-8</td>
<td>-27</td>
</tr>
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<td>-2</td>
<td>-2</td>
<td>-5</td>
</tr>
<tr>
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<td>-1</td>
<td>-7</td>
<td>-14</td>
<td>-18</td>
<td>-43</td>
</tr>
</tbody>
</table>

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Table of $\chi^*(G, S, V)$ for $G = G_2$

\[
\begin{array}{ccccccc}
V = V_\lambda & \lambda = & (0,0) & (1,0) & (0,1) & (2,0) & (1,1) & (3,0) \\
S & \text{dim } V = & 1 & 7 & 14 & 27 & 64 & 77 \\
{\infty}, 2} & & 1 & 0 & 0 & 0 & 1 & 0 \\
{\infty}, 3} & & 1 & 0 & 0 & 2 & 3 & 3 \\
{\infty}, 5} & & 2 & 7 & 11 & 31 & 71 & 76 \\
{\infty}, 7} & & 13 & 54 & 120 & 231 & 523 & 642 \\
{\infty}, 2, 3} & & 2 & 8 & 17 & 33 & 79 & 95 \\
{\infty}, 2, 5} & & 35 & 218 & 460 & 863 & 2029 & 2476 \\
\end{array}
\]

For groups of higher rank, one can enumerate the classes $t$ and determine the motives $M_t$ of their centralizers. The local stable orbital integrals $SO_t(\varphi_p, d\mu_p)$ at primes $p$ dividing the order of $t$ are difficult to calculate. However, a good estimate for $\chi^*$ comes from the central terms in the trace formula, which together contribute the rational number

\[
\#Z \cdot \frac{1}{2^t} L_s(M_G) \cdot \text{dim } V.
\]

For $G = F_4$, this estimate suggests that $\chi^* > 10^3$ whenever $S \neq \{\infty, 2\}$, and for $G = E_8$, this estimate suggests that $\chi^* > 10^{30}$ for all pairs $(S, V)$. 

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8. Discrete series and a conjecture

How can one account for the term \((W : W^c)\), which is the only non-stable factor in the formula for \(\chi(G, S, V)\):

\[
\chi(G, S, V) = (W : W^c) \cdot \chi^*(G, S, V)?
\]

On one hand, \((W : W^c)\) is the Euler characteristic of the trivial representation \(\mathbb{C}\) of \(G_S(\mathbb{A})\), arising from the cohomology of the trivial representation of \(G(\mathbb{R})\). Indeed, if \(K\) is a maximal compact subgroup of \(G(\mathbb{R})\) and \(p = \text{Lie}(G) / \text{Lie}(K)\), then:

\[
H^\bullet(G(\mathbb{R}), \mathbb{C}) = (\hat{\Lambda}p)^K.
\]

On the other hand, \((W : W^c)\) is the number of discrete series representations \(\pi_\infty\) of \(G(\mathbb{R})\) with a fixed central and infinitesimal character. This leads us to make the following optimistic prediction.

**Conjecture.** Let \(\pi\) be an irreducible representation of \(G(\mathbb{A})\) which occurs in \(L = L^2_{\text{disc}}\) and has non-zero \(G_S(\mathbb{A})\)-cohomology \(H^\bullet(G_S(\mathbb{A}), \pi \otimes V)\) when tensored with the finite-dimensional representation \(V\) of \(G(\mathbb{R})\).

Then either:

1. \(\pi\) is the trivial representation of \(G(\mathbb{A})\) and \(V = \mathbb{C}\), or

2. \(\pi_\infty\) is a discrete series representation of \(G(\mathbb{R})\) with trivial central character and the same infinitesimal character as \(V^*\), and for all finite places \(v \in S\), \(\pi_v\) is the Steinberg representation.
Even more should be true. Let $G'$ be any inner form of $G$, with good reduction outside of $S$. Let $\pi = \pi_\infty \otimes \bigotimes_{v \in S} St_v \otimes \pi^S$ be the local factorization of a representation of type $2$) in $L$, with $\pi^S$ unramified. If $\pi'_\infty$ is any discrete series for $G' (\mathbb{R})$ with the same infinitesimal and central character as $\pi_\infty$, then we would expect that:

$$\dim \text{Hom}_{G' (\mathbb{A})} \left( \pi'_\infty \otimes \bigotimes_{v \in S} St'_v \otimes \pi^S, L' \right) = 1.$$ 

If this is true, we can use the fact that discrete series representations of $G(\mathbb{R})$ and the Steinberg representation of $G(\mathbb{Q}_p)$ contribute cohomology of dimension $1$ in a single degree, to count the number of distinct automorphic representations of a fixed local type.

**Conjecture.** Let $d_\infty$ be a fixed discrete series for $G(\mathbb{R})$, with infinitesimal character equal to the infinitesimal character of $V^*$. Then the number of distinct irreducible representations $\pi = \bigotimes'_v \pi_v$ of $G(\mathbb{A})$ with local components

$$\begin{cases} 
\pi_\infty \simeq d_\infty \\
\pi_v \simeq St_v, \quad \text{for all } v \in S \\
\pi_{p}^{G(\mathbb{Q}_p)} \neq 0, \quad \text{for all } p \notin S
\end{cases}$$

which appear in the discrete spectrum $L$ of $G$ is equal to the absolute value of the integer $\chi^*(G, S, V)$ (except in the case when $V = \mathbb{C}$ and the group $G_S (\mathbb{A})$ is noncompact, when this number is the absolute value of the integer $\chi^*(G, S, V) - 1$).

For example, when $G = G_2$, $S = \{\infty, 5\}$, and $V = \mathbb{C}$, we saw that $\chi^*(G, S, V) = 2$. Hence, for any discrete series $d_\infty$ of $G_2 (\mathbb{R})$ with infinitesimal
character $\rho$, there should be a unique automorphic irreducible representation $\pi$ of the form
\[
\pi = d_\infty \otimes St_5 \otimes \bigotimes_{p \neq 5} \pi_p
\]
with $\pi_p$ unramified for all $p \neq 5$. For the anisotropic form $G'$ of $G_2$, this is true by calculations of Lansky and Pollack (who also determined $\pi_2$ and $\pi_3$). The representation $\pi'$ of $G'(A)$ lifts to $\mathrm{PGSp}_6(A)$ via an exceptional theta correspondence, and yields a holomorphic Siegel modular form $F$ of weight 4, whose level is the Iwahori subgroup at 5 in $\mathrm{PGSp}_6(\mathbb{Z})$ [13, Prop. 5.8].
References


