On the minimal size of a basis for uncountable linear order

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For a class of structures $\mathcal{K}$, a subclass $\mathcal{B}$ is a basis for $\mathcal{K}$ if every structure in $\mathcal{K}$ has a substructure that is in $\mathcal{B}$. We will study the basis for $\mathcal{K}$ when $\mathcal{K}$ is the structure of

- uncountable linear order
or related structures including

- Aronszajn lines
- Aronszajn tree
- $\omega_1$-dense subset of reals
Some examples of uncountable linear orders:

- uncountable subsets of real line
- $\omega_1$
- $\omega_1^*$
- Suslin line

Notation: Those uncountable linear orders which do not contain uncountable separable suborders or copies of $\omega_1$ or $\omega_1^*$ are called Aronszajn lines.
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Suslin line is a Aronszajn line. The existence of Suslin line is independent of ZFC. However, Shelah manages to construct a special type of Aronszajn line under ZFC.

**Theorem (Shelah)**

There is a Countryman line.

Here an uncountable linear order $(X, <)$ is called a Countryman line if $(X^2, <^2)$ can be decomposed into $\omega$ many chain. Here

$$(x_0, y_0) <^2 (x_1, y_1) \iff x_0 < x_1 \land y_0 < y_1.$$
Theorem (Baumgartner)

Assume PFA. Any two $\omega_1$-dense subsets of reals are isomorphic.

Theorem (Moore)

Under PFA, for any Countryman line $C$, all Aronszajn line contains $C$ or $C^*$.

Moore’s theorem can be separated into following substatement:

- any two normal Aronszajn trees are club isomorphic under PFA. (Shelah)
- $MA_{\omega_1}$ implies for any special countryman tree $T$, there is a 2 element basis for suborders of Countryman lines with partition tree $T$. (Todorcevic)
- every Aronszajn line contains a Countryman subline under PFA.

In conclusion, under PFA, there is a 5-element basis of uncountable linear order.
To the contrast, the minimal size of basis could also be huge.

**Theorem (Sierpinski)**

*Under CH, there is no basis for the uncountable separable linear orders of cardinality less than $2^{\omega_1}$.***
The following diagram describes the construction of basis:

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Basis for u.l.o

Basis for suborder of \( \mathbb{R} \)

Basis for Country-man line

Basis for Country-man tree
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Coherent trees

Definition (Todorcevic)

An Aronszajn tree $T \subseteq \omega^{<\omega_1}$ is coherent if for any $s, t \in T$, $D_{s,t} = \{ \alpha < ht(s), ht(t) : s(\alpha) \neq t(\alpha) \}$ is finite.
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Definition (Todorcevic)

For a coherent tree $T$, $\mathcal{I}_T$ is the ideal consists of $\Gamma \subset \omega_1$ disjoint from $\Delta(X)$ for some $X \in [T]^{\omega_1}$ where $\Delta(X) = \{ \min(D_{s,t}) : s \perp t \text{ are in } X \}$. 
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$\mathcal{I}_T$ is an ideal for any coherent tree $T$. 
Simple facts about $\mathcal{I}_T$

**Fact**

If $\Gamma \notin \mathcal{I}_T$ and $\mathcal{P}$ is a forcing with property (K), then $\Vdash \mathcal{P} \Gamma \notin \mathcal{I}_T$.

$FA(K)$ asserts that if $\mathcal{P}$ is a forcing with property (K) and $\mathcal{D}$ is a collection of $\omega_1$ many dense sets of $\mathcal{P}$, then there is a filter meeting all dense sets in $\mathcal{D}$. $FA(K)$ implies that

- any Countryman line can be embedded into any of its uncountable subline,
- for a special coherent tree $T$, the basis for Countryman lines that have partition tree $T$ has size $|\mathcal{P}(\omega_1)/\mathcal{I}_T|$.

**Proposition (Todorcevic)**

$MA_{\omega_1}$ implies that $\mathcal{I}_T$ is a maximal ideal, i.e., $|\mathcal{P}(\omega_1)/\mathcal{I}_T| = 2$. 
Lemma (Key Lemma)

The minimal size of a basis for uncountable linear order is \(2^n + 3\) if the following conditions are satisfied:

1. there is a one element basis for uncountable set of reals,
2. \(FA(\mathcal{K})\),
3. any two normal Aronszajn trees are club isomorphic,
4. \(|\mathcal{P}(\omega_1)/I_T| = 2^n\),
5. every Aronszajn line contains a Countryman subline.
Suppose $\Gamma$ is a partition of $\omega_1$ and $T$ is a special coherent tree. 
PFA($\Gamma$) asserts that $\Gamma \cap I_T = \emptyset$, and if $P$ is a proper forcing that does not force $\Gamma \cap I_T \neq \emptyset$ and $D$ is a collection of $\omega_1$ many dense sets of $P$, then there is a filter meeting all dense sets in $D$.

**Lemma**

PFA($\Gamma$) implies all conditions in the Key Lemma are satisfied.

It also can be verified that PFA($\Gamma$) implies MRP.
The consistency

**Theorem**

Suppose that $\Gamma \notin I_T$ and $P$ is a countable support iterated forcing such that each iterant is proper and force $\Gamma \notin I_T$, then $P$ does not force $\Gamma$ to be in $I_T$.

**Corollary**

If there is a supercompact cardinal and $\vec{\Gamma}$ is a partition of $\omega_1$, then there is a model satisfying $PFA(\vec{\Gamma})$.

**Corollary**

If there is a supercompact cardinal, then for each $n < \omega$, there is a model in which the basis of uncountable linear orders has size $2^n + 3$. 
\( \vec{T} \) is a collection of complete special coherent trees. PFA(\( \vec{T} \)) asserts that \( \vec{T} \) are pairwise non-club-isomorphic and if \( \mathcal{P} \) is a proper forcing that does not force any two trees in \( \vec{T} \) to be club-isomorphic and \( \mathcal{D} \) is a collection of \( \omega_1 \) many dense sets of \( \mathcal{P} \), then there is a filter meeting all dense sets in \( \mathcal{D} \).
$\vec{T}$ is a collection of complete special coherent trees.
PFA($\vec{T}$) asserts that $\vec{T}$ are pairwise non-club-isomorphic and if $\mathcal{P}$ is a proper forcing that does not force any two trees in $\vec{T}$ to be club-isomorphic and $\mathcal{D}$ is a collection of $\omega_1$ many dense sets of $\mathcal{P}$, then there is a filter meeting all dense sets in $\mathcal{D}$.

**Theorem**

For any $n < \omega$, Con(a supercompact cardinal) implies Con(PFA($\vec{T}$) for some finite $\vec{T}$) such that $|\vec{T}| = n$. 
Fix some $\vec{T}$ of size $n < \omega$.

**Theorem**

Assume $\text{PFA}(\vec{T})$. Then

- $\text{MA}_{\omega_1}$.
- Every coherent tree contains a subtree club-isomorphic to one of $\vec{T}$.
- The basis for Countryman lines has size $2n$.

It is unknown whether $\text{PFA}(\vec{T})$ implies that every Aronszajn tree has a subtree club-isomorphic to a coherent tree.
Theorem (Abraham, Rubin, Shelah)

For any $n < \omega$, $\text{Con}(\text{MA} + \text{the minimal size of a basis for uncountable subset of reals is } n)$.

The forcing in the proof is a finite support iterated c.c.c forcing. No good forcing theory for the class of $n$ element basis preserving proper forcing is known.
When $n = 2$, the following stronger property is easy to use.

**Definition**

Let $A \subseteq \mathbb{R}$ be uncountable, $A$ is called an increasing set, if in every uncountable set of pairwise disjoint finite sequences from $A$, there are two sequences $(a_0, \ldots, a_{n-1}), (b_0, \ldots, b_{n-1})$ having the same length such that $a_i < b_i$ for any $i \leq n - 1$

**Proposition (Abraham, Shelah)**

Assume $MA_{\omega_1}$ and there is an increasing set $A$, then the minimal size of a basis for uncountable subset of reals is 2.
Forcing preserves increasing set

PFA(inc(A)) asserts that for A is increasing and if \( \mathcal{P} \) is a proper forcing that forces A is increasing and \( \mathcal{D} \) is a collection of \( \omega_1 \) many dense sets of \( \mathcal{P} \), then there is a filter meeting all dense sets in \( \mathcal{D} \).

**Theorem**

Assume PFA(inc(A)). Then

- \( MA_{\omega_1} \).
- any two normal Aronszajn trees are club isomorphic.
- every Aronszajn line contains a Countryman subline.
- the minimal size of a basis for uncountable subset of reals is 2.

However, no good iteration theory for inc(A) is known. In particular, it is unknown how to force PFA(inc(A)) using iterated forcing. We use the \( \mathbb{P}_{\text{max}} \) machinery instead.
Following Woodin’s generalized version of $\mathbb{P}_{\text{max}}$ forcing, we define the variant $\mathbb{P}^{\text{inc}}_{\text{max}}$ as follows: The partial order $\mathbb{P}^{\text{inc}}_{\text{max}}$ consists of all pairs $\langle (M, I), a, K \rangle$ such that:

1. $M$ is a countable transitive model of $ZFC^-$,
2. $I \in M$ and in $M$, $I$ is a normal ideal on $\omega_1$,
3. $(M, I)$ is iterable,
4. $M$ think $a$ is increasing subset of reals.
5. $K \in M$ and $K$ is a set of pairs $\langle (N, J), b, E \rangle, j \rangle$ such that
   - $\langle (N, J), b, y, E \rangle \in \mathbb{P}^{\Gamma}_{\text{max}} \cap H(\omega_1)^M$,
   - $j$ is an iteration of $(N, J)$ of length $\omega_1^M$ such that $j(J) = I \cap j(N)$ and $j(b) = a$,
   - $j(E) \subset K$,

   with the property that for each $p \in \mathbb{P}^{\Gamma}_{\text{max}}$ there is at most one $j$ such that $(p, j) \in X$.

Say $\langle (M', I'), a', K' \rangle \prec \langle (M, I), a, K \rangle$ if there exists a $j$ such that $\langle (M, I), a, K \rangle, j \rangle \in K'$. 

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basis for uncountable linear order

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By verifying the iteration lemma, we have:

**Proposition**

\[ P_{\Gamma_{\text{max}}} \text{ forces the following:} \]

- \( A_G \) is increasing.
- \( \text{NS}_{\omega_1} \) is saturated, or \( \text{Sat} (\text{NS}_{\omega_1}) \) holds.
- \( \text{MA}_{\omega_1} \).
- There is a 2 element basis for countryman line.
- If ground model satisfies \( V = L(P(\mathbb{R})) + AD_{\mathbb{R}} \), then \( \text{BPFA}(\text{inc}(A)) \) holds.

Konig–Larson-Moore-Velickovic shows that \( \text{Sat}(\text{NS}_{\omega_1}) + \text{BPFA} \) implies every Aronzajn line contains a Countryman line. By analysing the forcing poset used in their proof, \( \text{Sat}(\text{NS}_{\omega_1}) + \text{BPFA}(\text{inc}(A)) \) also works for the purpose. Putting these together, we have

**Theorem**

Assuming \( V = L(P(\mathbb{R})) + AD_{\mathbb{R}} \), \( P_{\Gamma_{\text{max}}} \) forces the minimal size of basis for uncountable linear order is 6.
As a natural question, we are curious about basis of arbitrary size.

For any fixed \( n \geq 5 \), is it consistent that the minimal size of basis for uncountable linear order is \( n \)?

Thank you for your attention!