Wellordered Unions of Quotients of Smooth Equivalence Relations.

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This work is joint with William Chan.

We study wellordered (disjoint) unions of quotients of smooth equivalence relations in models of determinacy.

**Definition**
An equivalence relation $E$ on a Polish space $X$ is **smooth** if $X/E$ injects into $\mathbb{R}$.

We say $M$ is a **natural model of determinacy** if

$$M \models ZF + AD^+ + V = L(\mathcal{P}(\mathbb{R})).$$
If $M$ is a natural model of AD$^+$ then any set $A \in (V_\Theta)^M$ is the quotient of an equivalence relation on $\mathbb{R}$.

Thus, a significant part of the theory of “definable sets” is captured by the theory of definable equivalence relations.

An equivalence relation $E$ is said to be countable if every class $[x]_E$ is countable.

**Remark**
As in the Borel context, we have the Feldman-Moore theorem that every countable equivalence relation is induced by the action of a countable group.
The case of countable equivalence relations encompasses all degree notions (e.g., Turing degrees, arithmetical degrees, $L$-degrees, $L[S]$-degrees) and so is of interest in descriptive set theory.

- A reasonable analog of “countable unions” in the Borel context is “wellordered union” in the general definable context.
- Another motivation comes from trying to show preservation of combinatorial properties under various operations, including wellordered unions.
In particular, motivation comes from considering the Jónsson property.

Let $[X]^n$ denote the set of $n$-tuples of distinct elements from $X$.

**Definition**

$X$ has the **Jónsson property** if for every $F: [X]<\omega \to X$ there is a $Y \subseteq X$ with $Y \approx X$ (in bijection) such that $F[[Y]<\omega] \neq X$.

- [J, Ketchersid, Schlutzenberg, Woodin]
  Assuming $AD + V = L(\mathbb{R})$, every $\kappa < \Theta$ is Jónsson (Woodin extended this to models of $AD^+$).

- [Holshauser, J]
  Assuming $AD$, $\mathbb{R} \times \kappa$ is Jónsson for all $\kappa < \Theta$. 
On the other hand, $E_0$ is 2-Jónsson [Holshausr, J], but not 3-Jónsson [Chan, Meehan].

This leads to the following question and conjecture:

**Question**
Assuming AD, which sets are Jónsson?

**Conjecture**
Assuming AD, every wellordered union of Jónsson sets is Jónsson.
Sets which are represented as quotients of smooth equivalence relations are the simplest, so it natural to ask:

**Question**
Does a wellordered union of quotients of smooth equivalence relations on $\mathbb{R}$ have the Jónsson property?

- There is also a connection between wellordered unions of quotients of smooth equivalence relations and work of Woodin on $\omega_1^{<\omega}$ which we mention below.
- We will see that there is a significant difference between the cases where $E$ is countable or a general equivalence relation.
Our first main result is the following.

**Theorem**
Assume ZF + AD$^+$ + V = L($\mathcal{P}(\mathbb{R})$). Let $\kappa \in \text{On}$ and $\{E_\alpha\}_{\alpha < \kappa}$ a sequence of smooth equivalence relations with all classes countable. Then $\bigsqcup_{\alpha < \kappa} R/E_\alpha \approx R \times \kappa$.

**Corollary**
$\bigsqcup_{\alpha < \kappa} R/E_\alpha$ is Jónsson.

**Remark**
There will be two separate arguments to show $R \times \kappa$ injects into $\bigsqcup_{\alpha < \kappa} R/E_\alpha$ and to show $\bigsqcup_{\alpha < \kappa} R/E_\alpha$ injects into $R \times \kappa$. 
Woodin analyzed the (uncountable) cardinals below $\omega_1^\omega$ assuming AD$_\mathbb{R}$. They are $\omega_1$, $\mathbb{R}$, $\omega_1 \sqcup \mathbb{R}$, $\omega_1 \times \mathbb{R}$, and $\omega_1^\omega$.

- This result need AD$_\mathbb{R}$. The cardinal structures below $\omega_1^{<\omega_1}$ under AD + $V = L(\mathbb{R})$ and AD$_\mathbb{R}$ are different.

Woodin also showed that the cardinal structure below $\omega_1^{<\omega_1}$ is extremely complicated.

Note that if $A = \sqcup_{\alpha<\omega_1} \mathbb{R}/E_\alpha$ where each $E_\alpha$ is smooth, then $\mathbb{R} \sqcup \omega_1$ injects into $A$. 
The next result characterizes those $X \subseteq \omega_1^{<\omega_1}$ which can be represented as wellordered (disjoint) unions of quotients of smooth equivalence relations.

**Theorem**

Assume $ZF + AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. Let $X \subseteq \omega_1^{<\omega_1}$. Then $X \approx \sqcup_{\alpha<\omega_1} \mathbb{R}/E_\alpha$, where each $E_\alpha$ is a smooth equivalence relation on $\mathbb{R}$, iff $\mathbb{R} \sqcup \omega_1$ injects into $X$. 
Along the way we also show the following.

**Theorem**
Assume ZF + AD. For any sequence $\{E_\alpha\}_{\alpha < \kappa}$ of countable equivalence relations on $\mathbb{R}$, $\omega_1^\omega$ does not embed into $\bigcup_{\alpha < \kappa} \mathbb{R}/E_\alpha$. 
We use heavily aspects of Woodin’s AD$^+$ theory.

For $M$ a model of ZF and $S \subseteq \text{On}$, $S \in M$, let $\mathcal{O}^M_S$ be the collection of subsets of $(\mathbb{R}^n)^M$ which are $(\text{OD}_S)^M$.

**Fact (Vopenka’s theorem)**

Let $M$ be a transitive model of ZF and $S \in M$ a set of ordinals.

- Every $x \in \mathbb{R}^M$ induces a $\mathcal{O}^M_S$ generic filter $G_x$ over $\text{HOD}^M_S$. Also $x = \tau[G_x]$ for a canonical name $\tau$.

- Let $K$ be an $\text{OD}_S^M$ set of ordinals and $\varphi$ a formula. Let $p \in \mathcal{O}^M_S$ be $p = \{x \in \mathbb{R}^M : L[K, x] \models \varphi(K, x)\}$. Let $N$ be a transitive inner model with $\text{HOD}^M_S \subseteq N$. Then $N \models p \models_{\mathcal{O}^M_S} L[K, \tau] \models \varphi(K, \tau)$.
If \((g_0, \ldots, g_{n-1})\) is \(n\mathcal{O}_S^M\)-generic over \(N\), then each \(g_i\) is \(\mathcal{O}_S^M\)-generic over \(N\).

We have the following generalization of Silver’s theorem due to Woodin.

**Theorem (Woodin)**

Assume \(ZF + AD^+\). Let \(E\) be an equivalence relation on \(\mathbb{R}\). then either

1. \(\mathbb{R}/E\) is wellorderable
2. \(\mathbb{R}\) injects into \(\mathbb{R}/E\).
A ⊆ R is ∞-Borel if there is an S ⊆ On and a formula ϕ such that for all x ∈ R, x ∈ A ↔ L[S, x] ⊨ ϕ(S, x).

- Part of AD⁺ asserts that every A ⊆ R is ∞-Borel.
- (Woodin) Assume ZF + AD⁺ + V = L(P(R)), and let S ⊆ On. If A ⊆ R is ODₘ, than A has an ODₘ ∞-Borel code.

If A ⊆ κ × R we likewise say (S, ϕ) is an ∞-Borel code for A if (α, x) ∈ A ↔ L(S, x) ⊨ ϕ(S, α, x).

- Assuming ZF + AD⁺ + V = L(P(R)), every A ⊆ κ × R has an ∞-Borel code.
Woodin also showed that a model of $\text{ZF} + \text{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$ is either a model of $\text{AD}_{\mathbb{R}}$ or of the form $L(J, \mathbb{R})$ for some $J \subseteq \text{On}$.

- In a model of $\text{ZF} + \text{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$ every set is OD from a set $S \subseteq \text{On}$. 
Fact
Assume ZF + AD$^+$ + $V = L(P(R))$. Let $\kappa \in \text{On}$ and $\{E_\alpha\}_{\alpha < \kappa}$ a sequence of smooth equivalence relations on $R$ with all classes countable. Then $R \times \kappa$ injects into $\sqcup_{\alpha < \kappa} R/E_\alpha$.

Remark
This lower-bound does not hold in general without the assumption that each $E_\alpha$ is a countable equivalence relation. A counterexample is given by

$$xE_\alpha y \leftrightarrow (x, y \notin \text{WO}_\alpha) \vee (x = y).$$
Sketch of proof:
Let \((S, \varphi)\) be an \(\infty\)-Borel code for \(\{E_\alpha\}_{\alpha<\kappa}\). This uniformly gives \(\infty\)-Borel codes for the \(E_\alpha\).

For an eq. rel. \(E\) on \(\mathbb{R}\) (with code \(S\)) there are 2 cases in the Woodin/Silver dichotomy:

Case 1. \(\forall^* d \in \mathcal{D} \ \forall a \in L(S, d)\) there is some \(E\)-component \(A \in \bigcirc_{S}^{L(S, d)}\) which contains \(a\).

In this case we get a wellordering of \(\mathbb{R}/E\) using the wellfoundedness of \(\prod_{\mathcal{D}}\text{On}/\nu\) (Martin measure on \(\mathcal{D}\)).
**Case 2.** There is an $x \in \mathbb{R}$ and an $a \in L(S, x)$ which does belong to any $E$-component in $\mathcal{O}_S^{L(S,d)}$.

Fix such an $x$. In $L(S, x)$, let $u$ be the set of all $a$ which do not belong any $E$-component in $\mathcal{O}_S^{L(S,d)}$. So, $u \in \mathcal{O}_S^{L(S,x)}$.

As in Harrington’s proof of the Silver dichotomy,

$$(u, u) \Vdash \mathcal{O}_S^{L(S,d)} \times \mathcal{O}_S^{L(S,d)} \models (\tau_L E \tau_R).$$

[The proof makes an appeal to $2\mathcal{O}_S^{L(S,x)}$]

This gives an injection of $\mathbb{R}$ into $\mathbb{R}/E$. 
This injection is obtained uniformly from $S$, $x$, and an enumeration of the dense sets of $\bigcirc_S^{L(S,x)}$ in $L(S,x)$.

Returning to the case $\{E_\alpha\}_{\alpha<\kappa}$, it suffices to find a single $x \in \mathbb{R}$ which works for all $\alpha < \kappa$. (For each $\alpha < \kappa$, almost all $x \in \mathcal{D}$ work).

- Pick a real $a \notin \text{OD}_S$. For each $\alpha < \kappa$, $a$ must fall under Case 2.
- There are only countably many distinct values for $[a]_{E_\alpha}$.

The last fact follows from: each $[a]_{E_\alpha}$ is a countable $\text{OD}_{S,a}$ sets and hence $[a]_{E_\alpha} \subseteq \text{HOD}_{S,a}$. But $(2^\omega)^{\text{HOD}_{S,a}}$ is countable, and a wellordered sequence of reals is countable.
For these countably many $\alpha$ giving distinct $[a]_{E_\alpha}$, intersect the corresponding cones.

Note that if $[a]_{E_\alpha} = [a]_{E_\beta}$, then $x$ works for $E_\alpha$ iff $x$ works for $E_\beta$.

This produces an $x \in \mathbb{R}$ which works (satisfies Case 2) for all $\alpha < \kappa$. 
Fact
Assume ZF + AD⁺ + V = L(\mathcal{P}(\mathbb{R})). Let \kappa \in \text{On} and \{E_\alpha\}_{\alpha < \kappa} a sequence of smooth equivalence relations on \mathbb{R} with all classes countable. Then \bigcup_{\alpha < \kappa} \mathbb{R}/E_\alpha \text{ injects into } \mathbb{R} \times \kappa.

Sketch of proof:
Hjorth prooved an extension of the Harrington-Kechris-Louveau dichotomy:

Theorem (Hjorth)
Assume ZF + AD⁺. Let E be an equivalence relation on \mathbb{R}. Then either:
1.) There is a wellordered separeaing family for E.
2.) There is a \phi: \mathbb{R} \to \mathbb{R} such that \text{xE}_0\text{y iff } \phi(x)E\phi(y).
Let \((S, \varphi)\) be a code for \(E\). There are again two cases.

**Case 1.** For all \(x \in \mathbb{R}\) and all \(a, b \in L(S, x)\) there is a \(C \in O^L_{S, x}\) which is \(E\)-invariant with \(a \in c, b \notin C\).

Using the wellfoundedness of \(\prod_{\mathcal{D}} \omega_1/\nu\) we get a separating family for \(E\).

**Case 2.** There is an \(x \in \mathbb{R}\) and \(a, b \in L(S, x)\) with \(\neg(a E b)\) such that there is no \(E\)-invariant \(C \in O^L_{S, x}\) with \(a \in C, b \notin C\).

In this case one produces an embedding of \(\mathbb{R}/E_0\) into \(\mathbb{R}/E\).
Let now $\{E_\alpha\}_{\alpha<\kappa}$ be a sequence of smooth equivalence relations on $\mathbb{R}$ with countable classes.

- Let $S$ be an $\infty$-Borel code for $\{E_\alpha\}_{\alpha<\kappa}$, which uniformly gives codes for the $E_\alpha$.
- For each $\alpha<\kappa$, Case 2 cannot hold. The argument of Case gives uniformly in $\alpha$ an injection $\Phi_\alpha : \mathbb{R}/E_\alpha \rightarrow \mathcal{P}(\delta)$ where $\delta = \prod_{D} \omega_1/\nu$.
- We get an injection to $\kappa \times \mathbb{R}$ by using the following uniformization result.
Fact
Assume $ZF + AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. Let $\delta \in On$, $X$ a set. Suppose $R \subseteq \mathcal{P}(\delta) \times X$ with all sections countable. Then $R$ has a uniformization.

Proof.
$R$ is OD from a set $S$ of ordinals. Each section $R_T$ is thus OD from a set of ordinals. Then (case $X = \mathbb{R}$) each section has an $OD_{S,T} \infty$-Borel code. This shows $R_T \subseteq HOD_{S,T}$.

\[\square\]
Now consider $X \subseteq \omega_1^{<\omega_1}$ and assume $\mathbb{R} \sqcup \omega_1$ injects into $X$.

Write $X = X_0 \sqcup \omega_1$, and let $X_0^\alpha = \{ f \in X_0 : \sup(f) = \alpha \}$.

Let $E_\alpha$ be the equivalence relation on $\text{WO}^\alpha = \{ x \in (\text{WO})^\omega : \sup |x(n)| = \alpha \}$ given by $xE_\alpha y$ iff $x, y \in \text{WO}^\alpha$ and code the same set $A \in X_0^\alpha$ or $x, y \notin \text{WO}^\alpha$ or don’t code a set in $X_0^\alpha$. 

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Wellordered Unions of Quotients of Smooth Equivalence Relations.
Each $\mathbb{R}/E_\alpha$ is in bijection with $\mathbb{R}$ or is countable.

Let $A = \{ \alpha : |\mathbb{R}/E_\alpha| = \omega \}$, $B = \omega_1 - A$.

For $\alpha \in A$, the $\mathbb{R}/E_\alpha$ are uniformly wellorderable using an $\infty$-Borel code for the sequence $\{E_\alpha\}_{\alpha < \omega_1}$.

If $B$ is countable, then $X \approx \omega_1 \sqcup \mathbb{R}$, which is a union of smooth quotients.

If $B$ is uncountable, this gives a bijection between $X$ and $\sqcup_{\alpha < \omega_1} \mathbb{R}/E_\alpha$. 