Higher Recursion in Computable Structure Theory.

Antonio Montalbán

University of California, Berkeley

Workshop on Higher Recursion Theory
IMS – NUS – Singapore
May 2019
Summary

1. $\Pi^1_1$-ness and ordinals
2. Hyperarithmeticity
3. When hyperarithmetic is recursive
4. Overspill
5. A structure equivalent to its own jump
Part I

1. $\Pi^1_1$-ness and ordinals
2. Hyperarithmeticy
3. When hyperarithmetical is recursive
4. Overspill
5. A structure equivalent to its own jump
Ordinals

0, 1, 2, ...,
Ordinals

0, 1, 2, ..., ω,
Ordinals

$0, 1, 2, \ldots, \omega, \omega + 1,$
Ordinals

0, 1, 2, ..., \( \omega \), \( \omega + 1 \), \( \omega + 2 \), ...,
Ordinals

0, 1, 2, ..., ω, ω + 1, ω + 2, ..., ω + ω
Ordinals

0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots, \omega + \omega = \omega 2,
Ordinals

0, 1, 2, ..., ω, ω + 1, ω + 2, ..., ω + ω = ω², ω² + 1,
Ordinals

0, 1, 2, ..., ω, ω + 1, ω + 2, ..., ω + ω = ω2, ω2 + 1, ω2 + 2, ..., 
Ordinals

$0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots, \omega + \omega = \omega 2, \omega 2 + 1, \omega 2 + 2, \ldots, \omega 3,$
Ordinals

0, 1, 2, ..., ω, ω + 1, ω + 2, ..., ω + ω = ω², ω² + 1, ω² + 2 ..., ω³, ..., ω⁴, ...,
Ordinals

0, 1, 2, ..., ω, ω + 1, ω + 2, ..., ω + ω = ω2, ω2 + 1, ω2 + 2 ..., ω3, ..., ω4, ...
..., ω · ω =
Ordinals

$0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots, \omega + \omega = \omega 2, \omega 2 + 1, \omega 2 + 2 \ldots, \omega 3, \ldots, \omega 4, \ldots, \omega \cdot \omega = \omega 2,$
Ordinals

$0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots, \omega + \omega = \omega 2, \omega 2 + 1, \omega 2 + 2 \ldots, \omega 3, \ldots, \omega 4, \ldots, \omega \cdot \omega = \omega 2, \ldots \omega 3$
Ordinals

$0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots, \omega + \omega = \omega 2, \omega 2 + 1, \omega 2 + 2 \ldots, \omega 3, \ldots, \omega 4, \ldots, \omega \cdot \omega = \omega 2, \ldots, \omega^3, \ldots, \omega^4, \ldots$
Ordinals

0, 1, 2, ..., \( \omega \), \( \omega + 1 \), \( \omega + 2 \), ..., \( \omega + \omega = \omega^2 \), \( \omega^2 + 1 \), \( \omega^2 + 2 \) ..., \( \omega^3 \), ..., \( \omega^4 \), ..., \( \omega^\omega \), ....
Ordinals

0, 1, 2, ..., ω, ω + 1, ω + 2, ..., ω + ω = ω2, ω2 + 1, ω2 + 2 ..., ω3, ..., ω4, ...
...
ω · ω = ω2, ..., ω3, ..., ω4, ..., ωω, ..., ωωω.
Ordinals

0, 1, 2, ..., ω, ω + 1, ω + 2, ..., ω + ω = ω2, ω2 + 1, ω2 + 2 ..., ω3, ..., ω4, ...
,..., ω · ω = ω2, ..., ω3, ..., ω4, ..., ωω, ..., ωωω, ...
Ordinals

0, 1, 2, ..., ω, ω + 1, ω + 2, ..., ω + ω = ω², ω² + 1, ω² + 2 ..., ω³, ..., ω⁴, ..., ω·ω = ω², ..., ω³, ..., ω⁴, ... ω², ..., ω³, ..., ω⁴, ... ωω, ... ωωω, ...
Ordinals

0, 1, 2, ..., ω, ω + 1, ω + 2, ..., ω + ω = ω2, ω2 + 1, ω2 + 2 ..., ω3, ..., ω4, ...
ω · ω = ω2, ..., ω3, ..., ω4, ..., ωω, ..., ωωω, ...

Definition:
A linear ordering (A; ≤) is well-ordered if every subset has a least element.

An ordinal is an isomorphism type of a well-ordering.

Theorem: Given two well-orderings A and B, one of the following holds:
A is isomorphic to an initial segment of B
B is isomorphic to an initial segment of A

The class of ordinals is itself well-ordered by embeddability.
Ordinals

0, 1, 2, ..., ω, ω + 1, ω + 2, ..., ω + ω = ω², ω² + 1, ω² + 2, ..., ω³, ..., ω⁴, ..., ω · ω = ω², ..., ω³, ..., ω⁴, ..., ωω, ..., ωωω, ...
Ordinals

0, 1, 2, ..., \( \omega \), \( \omega + 1 \), \( \omega + 2 \), ..., \( \omega + \omega = \omega^2 \), \( \omega^2 + 1 \), \( \omega^2 + 2 \), ..., \( \omega^3 \), ..., \( \omega^4 \), ..., \( \omega \cdot \omega = \omega^2 \), ..., \( \omega^3 \), ..., \( \omega^4 \), ..., \( \omega^\omega \), ..., \( \omega^{\omega^\omega} \), .......
Ordinals

0, 1, 2, ..., ω, ω + 1, ω + 2, ..., ω + ω = ω², ω² + 1, ω² + 2 ..., ω³, ..., ω⁴, ...

ω · ω = ω², ..., ω³, ..., ω⁴, ..., ωω, ..., ωωω, ........
Ordinals

0, 1, 2, ..., ω, ω + 1, ω + 2, ..., ω + ω = ω2, ω2 + 1, ω2 + 2 ..., ω3, ..., ω4, ...
ω · ω = ω2, ... ω3, ..., ω4, ... ωω, ... ωωω, ............
Ordinals

0, 1, 2, ..., ω, ω + 1, ω + 2, ..., ω + ω = ω², ω² + 1, ω² + 2 ..., ω³, ..., ω⁴, ...
...
ω · ω = ω², ... ω³, ..., ω⁴, ... ωω, ... ωωω, .............
Ordinals

0, 1, 2, ..., ω, ω + 1, ω + 2, ..., ω + ω = ω2, ω2 + 1, ω2 + 2 ..., ω3, ..., ω4, ...
ω · ω = ω2, ... ω3, ..., ω4, ... ωω, ... ωωω, .................
Ordinals

0, 1, 2, ..., ω, ω + 1, ω + 2, ..., ω + ω = ω2, ω2 + 1, ω2 + 2 ..., ω3, ..., ω4, ...

ω · ω = ω2, ..., ω3, ..., ω4, ..., ωω, ..., ωωω, ....................

Definition: A linear ordering \((A; \leq A)\) is well-ordered if every subset has a least element. An ordinal is an isomorphism type of a well-ordering.

Theorem: Given two well-orderings \(A\) and \(B\), one of the following holds:

- \(A\) is isomorphic to an initial segment of \(B\)
- \(B\) is isomorphic to an initial segment of \(A\)

The class of ordinals is itself well-ordered by embeddability.
Ordinals

0, 1, 2, ..., ω, ω + 1, ω + 2, ..., ω + ω = ω2, ω2 + 1, ω2 + 2 ..., ω3, ..., ω4, ...
ω · ω = ω2, ... ω3, ..., ω4, ... ωω, ... ωωω, .....................
Ordinals

0, 1, 2, ..., \(\omega\), \(\omega + 1\), \(\omega + 2\), ..., \(\omega + \omega = \omega 2\), \(\omega 2 + 1\), \(\omega 2 + 2\) ..., \(\omega 3\), ..., \(\omega 4\), ...

\(\omega \cdot \omega = \omega^2\), ...

\(\omega^3\), ..., \(\omega^4\), ..., \(\omega^\omega\), ..., \(\omega^{\omega^\omega}\), ..........................
Ordinals

$0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots, \omega + \omega = \omega 2, \omega 2 + 1, \omega 2 + 2 \ldots, \omega 3, \ldots, \omega 4, \ldots, \omega \cdot \omega = \omega ^ 2, \ldots, \omega ^ 3, \ldots, \omega ^ 4, \ldots, \omega ^ \omega, \ldots, \omega ^ {\omega ^ \omega}, \ldots$
Ordinals

0, 1, 2, ..., ω, ω + 1, ω + 2, ..., ω + ω = ω², ω² + 1, ω² + 2 ..., ω³, ..., ω⁴, ...
ω · ω = ω², ..., ω³, ..., ω⁴, ..., ωω, ..., ωωω, ........................................

Definition:
A linear ordering \((A; \leq A)\) is well-ordered if every subset has a least element.

An ordinal is an isomorphism type of a well-ordering.

Theorem: Given two well-orderings \(A\) and \(B\), one of the following holds:

- \(A\) is isomorphic to an initial segment of \(B\)
- \(B\) is isomorphic to an initial segment of \(A\)

The class of ordinals is itself well-ordered by embeddability.
Ordinals

0, 1, 2, ..., ω, ω + 1, ω + 2, ..., ω + ω = ω², ω² + 1, ω² + 2 ..., ω³, ..., ω⁴, ...
ω · ω = ω², ... ω³, ..., ω⁴, ... ωω, ..., ωωω, ..............................................
Ordinals

\[ 0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots, \omega + \omega = \omega 2, \omega 2 + 1, \omega 2 + 2 \ldots, \omega 3, \ldots, \omega 4, \ldots, \omega \cdot \omega = \omega^2, \ldots, \omega^3, \ldots, \omega^4, \ldots, \omega^\omega, \ldots, \omega^{\omega^\omega}, \ldots \]
Ordinals

0, 1, 2, ..., ω, ω + 1, ω + 2, ..., ω + ω = ω², ω² + 1, ω² + 2 ..., ω³, ..., ω⁴, ..., ω · ω = ω², ..., ω³, ..., ω⁴, ... ω⁵, ... ωω, ... ωωω, ..................................................
Ordinals

0, 1, 2, ..., ω, ω + 1, ω + 2, ..., ω + ω = ω2, ω2 + 1, ω2 + 2 ..., ω3, ..., ω4, ...
..., ω · ω = ω², ..., ω³, ..., ω⁴, ..., ω⁵, ..., ωω, ..., ωωω, ...........................................
Ordinals

\[ 0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots, \omega + \omega = \omega 2, \omega 2 + 1, \omega 2 + 2 \ldots, \omega 3, \ldots, \omega 4, \ldots, \omega \cdot \omega = \omega 2, \ldots \omega 3, \ldots, \omega 4, \ldots \omega ^{\omega}, \ldots \omega ^{\omega ^{\omega}}, \ldots \]

**Definition:**
A linear ordering \((A; \leq_A)\) is *well-ordered* if every subset has a least element.
Ordinals

0, 1, 2, ..., ω, ω + 1, ω + 2, ..., ω + ω = ω2, ω2 + 1, ω2 + 2 ..., ω3, ..., ω4, ...
ω · ω = ω2, ..., ω3, ..., ω4, ..., ωω, ..., ωωω, ..............................................................

Definition:
A linear ordering \((A; \leq_A)\) is well-ordered if every subset has a least element.
An ordinal is an isomorphism type of a well-ordering.
Definition:
A linear ordering \((A; \leq_A)\) is **well-ordered** if every subset has a least element. 
An **ordinal** is an isomorphism type of a well-ordering.

Theorem: Given two well-orderings \(A\) and \(B\), one of the following holds:
- \(A\) is isomorphic to an initial segment of \(B\)
- \(B\) is isomorphic to an initial segment of \(A\)
Ordinals

0, 1, 2, ..., \( \omega \), \( \omega + 1 \), \( \omega + 2 \), ..., \( \omega + \omega = \omega 2 \), \( \omega 2 + 1 \), \( \omega 2 + 2 \), ..., \( \omega 3 \), ..., \( \omega 4 \), ..., \( \omega \cdot \omega = \omega 2 \), ..., \( \omega 3 \), ..., \( \omega 4 \), ..., \( \omega ^{\omega} \), ..., \( \omega ^{\omega \omega} \), ...

**Definition:**
A linear ordering \((A; \leq_A)\) is *well-ordered* if every subset has a least element. An *ordinal* is an isomorphism type of a well-ordering.

**Theorem:** Given two well-orderings \(A\) and \(B\), one of the following holds:
- \(A\) is isomorphic to an initial segment of \(B\)
- \(B\) is isomorphic to an initial segment of \(A\)

The class of ordinals is itself well-ordered by embeddability.
A $\Pi^1_1$ formula is one of the form $\forall f \in \mathbb{N} \phi(f)$, where $\phi$ is arithmetic.

Theorem: Consider $S \subseteq \mathbb{N}$. $S$ is $\Pi^1_1 \iff$ there is a computable list of linear orders $L_e$ such that $e \in S \leftrightarrow L_e$ is well-ordered.

The key notion connecting $\Pi^1_1$-ness and well-orders is well-founded trees.
\( \Pi^1_1 \)-ness and Well-Orders

**Definition**

A \( \Pi^1_1 \) formula is one of the form \( \forall f \in \mathbb{N}^\mathbb{N} \varphi(f) \), where \( \varphi \) is arithmetic.
Definition

A $\Pi^1_1$ formula is one of the form $\forall f \in \mathbb{N}^\mathbb{N} \varphi(f)$, where $\varphi$ is arithmetic.

Theorem: Consider $S \subseteq \mathbb{N}$.

$S$ is $\Pi^1_1$ $\iff$ there is a computable list of linear orders $\mathcal{L}_e$ such that $e \in S \iff \mathcal{L}_e$ is well-ordered.
\( \Pi^1_1 \)-ness and Well-Orders

**Definition**

A \( \Pi^1_1 \) formula is one of the form \( \forall f \in \mathbb{N}^\mathbb{N} \ \varphi(f) \), where \( \varphi \) is arithmetic.

**Theorem:** Consider \( S \subseteq \mathbb{N} \).

\( S \) is \( \Pi^1_1 \) \iff there is a computable list of linear orders \( L_e \) such that \( e \in S \leftrightarrow L_e \) is well-ordered.

The key notion connecting \( \Pi^1_1 \)-ness and well-orders is well-founded trees.
Well-founded trees

**Definition:** A tree $T \subseteq \mathbb{N}^{<\omega}$ is well-founded if it has no infinite paths.
Well-founded trees

**Definition:** A tree $T \subseteq \mathbb{N}^{<\omega}$ is **well-founded** if it has no infinite paths.

**Definition:** Let the **rank** of a tree be defined by transfinite recursion:

$$
\text{rk}(T) = \sup_{n \in \mathbb{N}} (\text{rk}(T_n) + 1),
$$

where $T_n = \{\sigma \in \mathbb{N}^{<\omega} : n \vdash \sigma \in T\}$. 

If $T$ is ill-founded, let $\text{rk}(T) = \infty$.

The rank function is NOT a computable function.

**Lemma:** Given trees $S$ and $T$, 

$$
\text{rk}(S) \leq \text{rk}(T) \iff \text{there is an } \mathrel{\subseteq} -\text{preserving embedding } S \to T.
$$
Well-founded trees

**Definition:** A tree \( T \subseteq \mathbb{N}^{<\omega} \) is well-founded if it has no infinite paths.

**Definition:** Let the rank of a tree be defined by transfinite recursion:

\[
\text{rk}(T) = \sup_{n \in \mathbb{N}} (\text{rk}(T_n) + 1),
\]

where \( T_n = \{ \sigma \in \mathbb{N}^{<\omega} : n \upharpoonright \sigma \in T \} \).

If \( T \) is ill-founded, let \( \text{rk}(T) = \infty \).
Well-founded trees

**Definition:** A tree $T \subseteq \mathbb{N}^{<\omega}$ is **well-founded** if it has no infinite paths.

**Definition:** Let the **rank** of a tree be defined by transfinite recursion:

$$\text{rk}(T) = \sup_{n \in \mathbb{N}} (\text{rk}(T_n) + 1),$$

where $T_n = \{\sigma \in \mathbb{N}^{<\omega} : n \triangleleft \sigma \in T\}$.

If $T$ is ill-founded, let $\text{rk}(T) = \infty$.

The rank function is **NOT** a computable function.

**Lemma:** Given trees $S$ and $T$,

$$\text{rk}(S) \leq \text{rk}(T) \iff \text{there is an } \subseteq\text{-preserving embedding } S \rightarrow T.$$
From linear orderings to trees

Definition: Given a linear ordering $L = (L; \leq_L)$, define the tree of descending sequences:

$$T_L = \{ \langle \ell_0, \ldots, \ell_k \rangle \in L^{<\omega} : \ell_0 >_L \ell_1 >_L \cdots >_L \ell_k \}.$$ 

Obs: $L$ is well-ordered $\iff$ $T_L$ is well-founded.

Furthermore, if $L$ is well-ordered, $\text{rk}(T_L) \sim = L$.

Corollary: Deciding if a linear ordering is WO, is as hard as deciding if a tree is WF.
Definition: Given a linear ordering $\mathcal{L} = (L; \leq_L)$, define the tree of descending sequences:

$$T_L = \{ \langle \ell_0, \ldots, \ell_k \rangle \in L^{<\omega} : \ell_0 >_L \ell_1 >_L \cdots >_L \ell_k \}.$$
From linear orderings to trees

Definition: Given a linear ordering $\mathcal{L} = (L; \leq_L)$, define the tree of descending sequences:

$$T_{\mathcal{L}} = \{\langle \ell_0, \ldots, \ell_k \rangle \in L^{<\omega} : \ell_0 >_L \ell_1 >_L \cdots >_L \ell_k\}.$$

Obs: $\mathcal{L}$ is well-ordered $\iff T_{\mathcal{L}}$ is well-founded.
From linear orderings to trees

**Definition:** Given a linear ordering $\mathcal{L} = (L; \leq_L)$, define the *tree of descending sequences*:

$$T_{\mathcal{L}} = \{ \langle \ell_0, \ldots, \ell_k \rangle \in L^\omega : \ell_0 >_L \ell_1 >_L \cdots >_L \ell_k \}.$$  

**Obs:** $\mathcal{L}$ is well-ordered $\iff$ $T_{\mathcal{L}}$ is well-founded.

Furthermore, if $\mathcal{L}$ is well-ordered, $\text{rk}(T_{\mathcal{L}}) \cong \mathcal{L}$. 

Antonio Montalbán (U.C. Berkeley)  
Higher Recursion and computable structures  
May 2019 7 / 50
From linear orderings to trees

Definition: Given a linear ordering \( \mathcal{L} = (L; \leq_L) \), define the tree of descending sequences:

\[
T_\mathcal{L} = \{ \langle \ell_0, \ldots, \ell_k \rangle \in L^{<\omega} : \ell_0 \succ_L \ell_1 \succ_L \cdots \succ_L \ell_k \}.
\]

Obs: \( \mathcal{L} \) is well-ordered \( \iff \) \( T_\mathcal{L} \) is well-founded.

Furthermore, if \( \mathcal{L} \) is well-ordered, \( \text{rk}(T_\mathcal{L}) \cong \mathcal{L} \).

Corollary: Deciding if a linear ordering is WO, is as hard as deciding if a tree is WF.
The Kleene-Brower ordering on $\mathbb{N}^{<\omega}$ is defined as follows: $\sigma \leq_{\text{KB}} \tau \iff \sigma \supseteq \tau \lor \exists i (\sigma \upharpoonright i = \tau \upharpoonright i \land \sigma(i) < \tau(i))$.

Obs: A tree $T \subseteq \mathbb{N}^{<\omega}$ is well-founded $\iff (T; \leq_{\text{KB}})$ is well-ordered.

Lemma: $\text{rk}(T) + 1 \leq (T; \leq_{\text{KB}}) \leq \omega \text{rk}(T) + 1$.

Corollary: Deciding if a tree is WF, is as hard as deciding if linear ordering is WO.
From trees to linear orderings

The *Kleene-Brower* ordering on $\mathbb{N}^{<\omega}$ is defined as follows:

$$\sigma \leq_{KB} \tau \iff \sigma \supseteq \tau \lor \exists i \left( \sigma \upharpoonright i = \tau \upharpoonright i \land \sigma(i) < \tau(i) \right).$$
From trees to linear orderings

The *Kleene-Brower* ordering on $\mathbb{N}^{<\omega}$ is defined as follows:

$$\sigma \leq_{KB} \tau \iff \sigma \supseteq \tau \lor \exists i \left( \sigma \upharpoonright i = \tau \upharpoonright i \land \sigma(i) < \tau(i) \right).$$

**Obs:** A tree $T \subseteq \mathbb{N}^{<\omega}$ is well-founded $\iff (T; \leq_{KB})$ is well-ordered.
From trees to linear orderings

The *Kleene-Brower* ordering on $\mathbb{N}^{<\omega}$ is defined as follows:

$$\sigma \leq_{KB} \tau \iff \sigma \supseteq \tau \lor \exists i \left( \sigma \upharpoonright i = \tau \upharpoonright i \land \sigma(i) < \tau(i) \right).$$

**Obs:** A tree $T \subseteq \mathbb{N}^{<\omega}$ is well-founded $\iff (T; \leq_{KB})$ is well-ordered.

**Lemma:** $\text{rk}(T) + 1 \leq (T; \leq_{KB}) \leq \omega^{\text{rk}(T)} + 1$. 
From trees to linear orderings

The *Kleene-Brower* ordering on $\mathbb{N}^{<\omega}$ is defined as follows:

$$\sigma \leq_{KB} \tau \iff \sigma \supseteq \tau \lor \exists i \left( \sigma \upharpoonright i = \tau \upharpoonright i \land \sigma(i) < \tau(i) \right).$$

**Obs:** A tree $T \subseteq \mathbb{N}^{<\omega}$ is well-founded $\iff (T; \leq_{KB})$ is well-ordered.

**Lemma:** $\text{rk}(T) + 1 \leq (T; \leq_{KB}) \leq \omega^{\text{rk}(T)} + 1$.

**Corollary:** Deciding if a tree is WF,

is as hard as deciding if linear ordering is WO.
Kleene’s $\mathcal{O}$

**Definition**

Kleene’s $\mathcal{O}$ is the set of indices $e$ of computable well-orders.
**Kleene’s $\mathcal{O}$**

**Definition**
Kleene’s $\mathcal{O}$ is the set of indices $e$ of computable well-orders.

**Theorem**
Kleene’s $\mathcal{O}$ is $\Pi^1_1$-complete.
Kleene’s $\mathcal{O}$

**Definition**

Kleene’s $\mathcal{O}$ is the set of indices $e$ of computable well-orders.

**Theorem**

*Kleene’s $\mathcal{O}$ is $\Pi^1_1$-complete.*

**Proof:**
Kleene’s $\mathcal{O}$

**Definition**
Kleene’s $\mathcal{O}$ is the set of indices $e$ of computable well-orders.

**Theorem**
*Kleene’s $\mathcal{O}$ is $\Pi^1_1$-complete.*

**Proof:**
- Every $\Sigma^1_1$ formula $\varphi(n)$ is equivalent to
  $$\exists f \in \mathbb{N}^\mathbb{N} \; \theta(f, n)$$
  where $\theta$ is $\Pi^{0}_1$.  

Kleene’s $\mathcal{O}$

**Definition**
Kleene’s $\mathcal{O}$ is the set of indices $e$ of computable well-orders.

**Theorem**
*Kleene’s $\mathcal{O}$ is $\Pi^1_1$-complete.*

**Proof:**
- Every $\Sigma^1_1$ formula $\varphi(n)$ is equivalent to
  \[\exists f \in \mathbb{N}^\mathbb{N} \; \theta(f, n)\] where $\theta$ is $\Pi^0_1$.
- For a $\Pi^0_1$ formula $\theta(f)$, there is a computable tree $T$ with $\theta(f) \iff f \in [T]$. 
Kleene’s $\mathcal{O}$

**Definition**

Kleene’s $\mathcal{O}$ is the set of indices $e$ of computable well-orders.

**Theorem**

*Kleene’s $\mathcal{O}$ is $\Pi^1_1$-complete.*

**Proof:**

- Every $\Sigma^1_1$ formula $\varphi(n)$ is equivalent to 
  $$\exists f \in \mathbb{N}^\mathbb{N} \; \theta(f, n) \text{ where } \theta \text{ is } \Pi^0_1.$$
- For a $\Pi^0_1$ formula $\theta(f)$, there is a computable tree $T$ with $\theta(f) \iff f \in [T]$.
- For a $\Pi^0_1$ formula $\theta(f, n)$, there is computable sequence of trees $T_n$ such that 
  $$\theta(f, n) \iff f \in [T_n].$$
Kleene’s $\mathcal{O}$

**Definition**

Kleene’s $\mathcal{O}$ is the set of indices $e$ of computable well-orders.

**Theorem**

*Kleene’s $\mathcal{O}$ is $\Pi^1_1$-complete.*

**Proof:**

- Every $\Sigma^1_1$ formula $\varphi(n)$ is equivalent to $\exists f \in \mathbb{N}^\mathbb{N} \theta(f, n)$ where $\theta$ is $\Pi^1_0$.
- For a $\Pi^1_0$ formula $\theta(f)$, there is a computable tree $T$ with $\theta(f) \iff f \in [T]$.
- For a $\Pi^1_0$ formula $\theta(f, n)$, there is a computable sequence of trees $T_n$ such that $\theta(f, n) \iff f \in [T_n]$.
- If $S \subseteq \mathbb{N}$ is $\Pi^1_1$ and definable by $\neg \varphi(n)$, then $n \in S \iff T_n$ has no paths.
Kleene’s $\mathcal{O}$

**Definition**
Kleene’s $\mathcal{O}$ is the set of indices $e$ of computable well-orders.

**Theorem**

*Kleene’s $\mathcal{O}$ is $\Pi^1_1$-complete.*

**Proof:**

- Every $\Sigma^1_1$ formula $\varphi(n)$ is equivalent to
  $$\exists f \in \mathbb{N}^\mathbb{N} \ \theta(f, n)$$
  where $\theta$ is $\Pi^0_1$.
- For a $\Pi^0_1$ formula $\theta(f)$, there is a computable tree $T$ with $\theta(f) \iff f \in [T]$.
- For a $\Pi^0_1$ formula $\theta(f, n)$, there is computable sequence of trees $T_n$ such that $\theta(f, n) \iff f \in [T_n]$.
- If $S \subseteq \mathbb{N}$ is $\Pi^1_1$ and definable by $\neg \varphi(n)$, then
  $$n \in S \iff T_n \text{ has no paths} \iff (T_n; \leq_{KB}) \text{ is well-ordered}$$
Kleene’s $\mathcal{O}$

**Definition**

Kleene’s $\mathcal{O}$ is the set of indices $e$ of computable well-orders.

**Theorem**

*Kleene’s $\mathcal{O}$ is $\Pi^1_1$-complete.*

**Proof:**

- Every $\Sigma^1_1$ formula $\varphi(n)$ is equivalent to
  \[ \exists f \in \mathbb{N}^\mathbb{N} \; \theta(f, n) \text{ where } \theta \text{ is } \Pi^0_1. \]
- For a $\Pi^1_1$ formula $\theta(f)$, there is a computable tree $T$ with $\theta(f) \iff f \in [T]$.
- For a $\Pi^0_1$ formula $\theta(f, n)$, there is computable sequence of trees $T_n$ such that $\theta(f, n) \iff f \in [T_n]$.
- If $S \subseteq \mathbb{N}$ is $\Pi^1_1$ and definable by $\neg \varphi(n)$, then
  \[ n \in S \iff T_n \text{ has no paths} \iff (T_n; \leq_{KB}) \text{ is well-ordered} \iff \text{index}(T_n; \leq_{KB}) \in \mathcal{O}. \]
An ordinal $\alpha$ is **computable** if 

there is a computable $\leq A \subseteq \omega^2$ with $\alpha \cong (\omega; \leq A)$. 

Obs: The computable ordinals are closed downwards.

Definition: Let $\omega_{CK}$ be the least non-computable ordinal.

Obs: Kleene’s $\Omega$ can compute a copy of $\omega_{CK}$:

$\omega_{CK} \cong = \sum_{e \in \Omega} L_e$ where $L_e$ is the linear ordering with index $e$. 

Antonio Montalbán (U.C. Berkeley) Higher Recursion and computable structures May 2019 10 / 50
An ordinal $\alpha$ is \textit{computable} if there is a computable $\leq_A \subseteq \omega^2$ with $\alpha \approx (\omega; \leq_A)$.

\textbf{Obs: } The computable ordinals are closed downwards.
Omega-one-Church-Kleene

An ordinal $\alpha$ is "computable" if

there is a computable $\leq_\mathcal{A} \subseteq \omega^2$ with $\alpha \simeq (\omega; \leq_\mathcal{A})$.

**Obs:** The computable ordinals are closed downwards.

**Definition:** Let $\omega^1_{CK}$ be the least non-computable ordinal.
An ordinal \( \alpha \) is \textit{computable} if there is a computable \( \leq_A \subseteq \omega^2 \) with \( \alpha \cong (\omega; \leq_A) \).

\textbf{Obs:} The computable ordinals are closed downwards.

\textbf{Definition:} Let \( \omega_1^{CK} \) be the least non-computable ordinal.

\textbf{Obs:} Kleene’s \( O \) can compute a copy of \( \omega_1^{CK} \):
An ordinal $\alpha$ is **computable** if
there is a computable $\leq_A \subseteq \omega^2$ with $\alpha \simeq (\omega; \leq_A)$.

**Obs:** The computable ordinals are closed downwards.

**Definition:** Let $\omega_1^{CK}$ be the least non-computable ordinal.

**Obs:** Kleene’s $O$ can compute a copy of $\omega_1^{CK}$:

$$\omega_1^{CK} \simeq \sum_{e \in O} L_e$$

where $L_e$ is the linear ordering with index $e$. 
Theorem: Let $A \subset \mathcal{O}$ be $\Sigma^1_1$. There is an ordinal $\alpha < \omega_1^{CK}$ such that $L_e < \alpha$ for all $e \in A$. 

$\Sigma^1_1$-bounding
\[ \Sigma^1_1 \text{-bounding} \]

**Theorem:** Let \( A \subset \mathcal{O} \) be \( \Sigma^1_1 \). There is an ordinal \( \alpha < \omega^{CK}_1 \) such that \( \mathcal{L}_e < \alpha \) for all \( e \in A \).

**Proof:**
**Theorem:** Let \( A \subset \mathcal{O} \) be \( \Sigma^1_1 \).
There is an ordinal \( \alpha < \omega_1^{CK} \) such that \( \mathcal{L}_e < \alpha \) for all \( e \in A \).

**Proof:**
- Define \( * \)-operation on trees satisfying \( \text{rk}(T * S) = \min(\text{rk}(T), \text{rk}(S)) \).
Theorem: Let $A \subseteq \mathcal{O}$ be $\Sigma^1_1$. There is an ordinal $\alpha < \omega^{	ext{CK}}_1$ such that $L_e < \alpha$ for all $e \in A$.

Proof:
- Define $\ast$-operation on trees satisfying $\text{rk}(T \ast S) = \min(\text{rk}(T), \text{rk}(S))$.
- Let $\{S_n\}_{n \in \mathbb{N}}$ be a computable sequence of trees s.t. $n \in A \iff \text{rk}(S_n) = \infty$. 
**Theorem:** Let \( A \subset \mathcal{O} \) be \( \Sigma^1_1 \).

There is an ordinal \( \alpha < \omega_1^{CK} \) such that \( \mathcal{L}_e < \alpha \) for all \( e \in A \).

**Proof:**

- Define \( * \)-operation on trees satisfying \( \text{rk}(T \ast S) = \min(\text{rk}(T), \text{rk}(S)) \).
- Let \( \{S_n\}_{n \in \mathbb{N}} \) be a computable sequence of trees s.t. \( n \in A \iff \text{rk}(S_n) = \infty \).
- Define \( \alpha = \sum_{n \in \mathbb{N}} (T_{\mathcal{L}_e} \ast S_n; \leq_{KB}) \).
**Theorem:** Let $A \subset \mathcal{O}$ be $\Sigma^1_1$.
There is an ordinal $\alpha < \omega^\text{CK}_1$ such that $\mathcal{L}_e < \alpha$ for all $e \in A$.

**Proof:**
- Define $\ast$-operation on trees satisfying $\text{rk}(T \ast S) = \text{min}(\text{rk}(T), \text{rk}(S))$.
- Let $\{S_n\}_{n \in \mathbb{N}}$ be a computable sequence of trees s.t. $n \in A \iff \text{rk}(S_n) = \infty$.
- Define $\alpha = \sum_{n \in \mathbb{N}}(T_{\mathcal{L}_e} \ast S_n; \leq_{KB})$.

**Theorem:** Let $\mathcal{A} \subset 2^\mathbb{N}$ be a $\Sigma^1_1$ set of well-orderings of $\mathbb{N}$.
There is an ordinal $\alpha < \omega^\text{CK}_1$ such that all $\mathcal{L} < \alpha$ for all $\mathcal{L} \in \mathcal{A}$.
\[ \Sigma^1_1 \text{-bounding} \]

**Theorem:** Let \( A \subset \mathcal{O} \) be \( \Sigma^1_1 \). There is an ordinal \( \alpha < \omega^1_{CK} \) such that \( L_e < \alpha \) for all \( e \in A \).

**Proof:**

- Define \( * \)-operation on trees satisfying \( \text{rk}(T * S) = \min(\text{rk}(T), \text{rk}(S)) \).
- Let \( \{S_n\}_{n \in \mathbb{N}} \) be a computable sequence of trees s.t. \( n \in A \iff \text{rk}(S_n) = \infty \).
- Define \( \alpha = \sum_{n \in \mathbb{N}} (T_{L_e} * S_n; \leq_{KB}) \).

**Theorem:** Let \( \mathcal{A} \subset 2^\mathbb{N} \) be a \( \Sigma^1_1 \) set of well-orderings of \( \mathbb{N} \). There is an ordinal \( \alpha < \omega^1_{CK} \) such that all \( L < \alpha \) for all \( L \in \mathcal{A} \).

**Proof:**
**Theorem:** Let $A \subset \mathcal{O}$ be $\Sigma^1_1$. There is an ordinal $\alpha < \omega_{1}^{CK}$ such that $\mathcal{L}_e < \alpha$ for all $e \in A$.

**Proof:**
- Define $\ast$-operation on trees satisfying $\text{rk}(T \ast S) = \min(\text{rk}(T), \text{rk}(S))$.
- Let $\{S_n\}_{n \in \mathbb{N}}$ be a computable sequence of trees s.t. $n \in A \iff \text{rk}(S_n) = \infty$.
- Define $\alpha = \sum_{n \in \mathbb{N}}(T_{\mathcal{L}_e} \ast S_n; \leq_{KB})$.

**Theorem:** Let $\mathcal{A} \subset 2^{\mathbb{N}}$ be a $\Sigma^1_1$ set of well-orderings of $\mathbb{N}$. There is an ordinal $\alpha < \omega_{1}^{CK}$ such that all $\mathcal{L} < \alpha$ for all $\mathcal{L} \in \mathcal{A}$.

**Proof:** Let $A = \{e \in \mathbb{N} : (\exists \mathcal{L} \in \mathcal{A}) \mathcal{L}_e \preceq \mathcal{L}\}$. 
Theorem: Let \( A \subset \mathcal{O} \) be \( \Sigma^1_1 \). There is an ordinal \( \alpha < \omega_1^{CK} \) such that \( \mathcal{L}_e < \alpha \) for all \( e \in A \).

Proof:
- Define \( \ast \)-operation on trees satisfying \( \text{rk}(T \ast S) = \min(\text{rk}(T), \text{rk}(S)) \).
- Let \( \{S_n\}_{n \in \mathbb{N}} \) be a computable sequence of trees s.t. \( n \in A \iff \text{rk}(S_n) = \infty \).
- Define \( \alpha = \sum_{n \in \mathbb{N}} (T_{\mathcal{L}_e} \ast S_n; \leq_{KB}) \).

Theorem: Let \( \mathcal{A} \subset 2^\mathbb{N} \) be a \( \Sigma^1_1 \) set of well-orderings of \( \mathbb{N} \). There is an ordinal \( \alpha < \omega_1^{CK} \) such that all \( \mathcal{L} < \alpha \) for all \( \mathcal{L} \in \mathcal{A} \).

Proof: Let \( A = \{e \in \mathbb{N} : (\exists \mathcal{L} \in \mathcal{A}) \mathcal{L}_e \preceq \mathcal{L}\} \).
\( A \) is \( \Sigma^1_1 \) and \( A \subseteq \mathcal{O} \).
**Theorem:** Let \( A \subset \mathcal{O} \) be \( \Sigma^1_1 \). There is an ordinal \( \alpha < \omega^{	ext{CK}}_1 \) such that \( L_e < \alpha \) for all \( e \in A \).

**Proof:**
- Define \( \ast \)-operation on trees satisfying \( \text{rk}(T \ast S) = \min(\text{rk}(T), \text{rk}(S)) \).
- Let \( \{S_n\}_{n \in \mathbb{N}} \) be a computable sequence of trees s.t. \( n \in A \iff \text{rk}(S_n) = \infty \).
- Define \( \alpha = \sum_{n \in \mathbb{N}} (T_{L_e} \ast S_n; \leq_{KB}) \).

**Theorem:** Let \( \mathcal{A} \subset 2^\mathbb{N} \) be a \( \Sigma^1_1 \) set of well-orderings of \( \mathbb{N} \). There is an ordinal \( \alpha < \omega^{	ext{CK}}_1 \) such that all \( L < \alpha \) for all \( L \in \mathcal{A} \).

**Proof:** Let \( A = \{e \in \mathbb{N} : (\exists L \in \mathcal{A}) L_e \preceq L\} \).
\( A \) is \( \Sigma^1_1 \) and \( A \subseteq \mathcal{O} \). Let \( \alpha < \omega^{	ext{CK}}_1 \) be a bound for \( A \).
**Σ₁¹-bounding**

**Theorem:** Let \( A \subset O \) be \( \Sigma_1^{1} \). There is an ordinal \( \alpha < \omega_1^{CK} \) such that \( L_e < \alpha \) for all \( e \in A \).

**Proof:**
- Define \(*\)-operation on trees satisfying \( \text{rk}(T * S) = \min(\text{rk}(T), \text{rk}(S)) \).
- Let \( \{S_n\}_{n \in \mathbb{N}} \) be a computable sequence of trees s.t. \( n \in A \iff \text{rk}(S_n) = \infty \).
- Define \( \alpha = \sum_{n \in \mathbb{N}} (T_{L_e} * S_n; \leq_{KB}) \).

**Theorem:** Let \( \mathcal{A} \subset 2^{\mathbb{N}} \) be a \( \Sigma_1^{1} \) set of well-orderings of \( \mathbb{N} \). There is an ordinal \( \alpha < \omega_1^{CK} \) such that all \( L \in \mathcal{A} \) with \( L < \alpha \).

**Proof:** Let \( A = \{ e \in \mathbb{N} : (\exists L \in \mathcal{A}) L_e \preceq L \} \).
\( A \) is \( \Sigma_1^{1} \) and \( A \subset O \). Let \( \alpha < \omega_1^{CK} \) be a bound for \( A \). Then \( \alpha \) is a bound for \( \mathcal{A} \) too.
A set is $\Delta^1_1$ if it is both $\Pi^1_1$ and $\Sigma^1_1$. 
$\Delta^1_1$ sets

A set is $\Delta^1_1$ if it is both $\Pi^1_1$ and $\Sigma^1_1$.

For $\alpha < \omega^*_1$, let $O(\leq \alpha)$ be the set of indices of computable ordinals $\leq \alpha$. 

\[ \]
\( \Delta^1_1 \) sets

A set is \( \Delta^1_1 \) if it is both \( \Pi^1_1 \) and \( \Sigma^1_1 \).

For \( \alpha < \omega^\text{CK}_1 \), let \( O(\leq \alpha) \) be the set of indices of computable ordinals \( \leq \alpha \).

Observation: \( O(\leq \alpha) \) is \( \Delta^1_1 \).
$\Delta^1_1$ sets

A set is $\Delta^1_1$ if it is both $\Pi^1_1$ and $\Sigma^1_1$.

For $\alpha < \omega_1^{CK}$, let $O_{(\leq \alpha)}$ be the set of indices of computable ordinals $\leq \alpha$.

Observation: $O_{(\leq \alpha)}$ is $\Delta^1_1$.

$\Sigma^1_1$-bounding: If $A \subseteq O$ is $\Sigma^1_1$, then $A \subseteq O_{(\leq \alpha)}$ for some $\alpha < \omega_1^{CK}$. 
$\Delta_1^1$ sets

A set is $\Delta_1^1$ if it is both $\Pi_1^1$ and $\Sigma_1^1$.

For $\alpha < \omega_1^{CK}$, let $O(\leq \alpha)$ be the set of indices of computable ordinals $\leq \alpha$.

Observation: $O(\leq \alpha)$ is $\Delta_1^1$.

$\Sigma_1^1$-bounding: If $A \subseteq O$ is $\Sigma_1^1$, then $A \subseteq O(\leq \alpha)$ for some $\alpha < \omega_1^{CK}$.

Theorem: $A \subseteq \omega$ is $\Delta_1^1$ $\iff$ $A \leq_m O(\leq \alpha)$ for some $\alpha < \omega_1^{CK}$. 
$\Delta^1_1$ sets

A set is $\Delta^1_1$ if it is both $\Pi^1_1$ and $\Sigma^1_1$.

For $\alpha < \omega^\text{CK}_1$, let $O_{(\leq \alpha)}$ be the set of indices of computable ordinals $\leq \alpha$.

Observation: $O_{(\leq \alpha)}$ is $\Delta^1_1$.

$\Sigma^1_1$-bounding: If $A \subseteq O$ is $\Sigma^1_1$, then $A \subseteq O_{(\leq \alpha)}$ for some $\alpha < \omega^\text{CK}_1$.

Theorem: $A \subseteq \omega$ is $\Delta^1_1$ $\iff$ $A \leq_m O_{(\leq \alpha)}$ for some $\alpha < \omega^\text{CK}_1$.

Proof: ($\leq$)
Δ₁¹ sets

A set is Δ₁¹ if it is both Π₁¹ and Σ₁¹.

For \( \alpha < \omega₁^{CK} \), let \( \mathcal{O}(\leq \alpha) \) be the set of indices of computable ordinals \( \leq \alpha \).

Observation: \( \mathcal{O}(\leq \alpha) \) is Δ₁¹.

Σ₁¹-bounding: If \( A \subseteq \mathcal{O} \) is Σ₁¹, then \( A \subseteq \mathcal{O}(\leq \alpha) \) for some \( \alpha < \omega₁^{CK} \).

Theorem: \( A \subseteq \omega \) is Δ₁¹ \( \iff \) \( A \leq_m \mathcal{O}(\leq \alpha) \) for some \( \alpha < \omega₁^{CK} \).

Proof: (≤) Both Π₁¹ sets and Σ₁¹ sets are closed downward under \( \leq_m \).
$\Delta^1_1$ sets

A set is $\Delta^1_1$ if it is both $\Pi^1_1$ and $\Sigma^1_1$.

For $\alpha < \omega^\text{CK}_1$, let $O(\leq \alpha)$ be the set of indices of computable ordinals $\leq \alpha$.

Observation: $O(\leq \alpha)$ is $\Delta^1_1$.

$\Sigma^1_1$-bounding: If $A \subseteq O$ is $\Sigma^1_1$, then $A \subseteq O(\leq \alpha)$ for some $\alpha < \omega^\text{CK}_1$.

Theorem: $A \subseteq \omega$ is $\Delta^1_1$ if and only if $A \leq_m O(\leq \alpha)$ for some $\alpha < \omega^\text{CK}_1$.

Proof: ($\Leftarrow$) Both $\Pi^1_1$ sets and $\Sigma^1_1$ sets are closed downward under $\leq_m$.

($\Rightarrow$)
A set is $\Delta^1_1$ if it is both $\Pi^1_1$ and $\Sigma^1_1$.

For $\alpha < \omega_1^{CK}$, let $O_{(\leq \alpha)}$ be the set of indices of computable ordinals $\leq \alpha$.

**Observation:** $O_{(\leq \alpha)}$ is $\Delta^1_1$.

**$\Sigma^1_1$-bounding:** If $A \subseteq O$ is $\Sigma^1_1$, then $A \subseteq O_{(\leq \alpha)}$ for some $\alpha < \omega_1^{CK}$.

**Theorem:** $A \subseteq \omega$ is $\Delta^1_1$ $\iff$ $A \leq_m O_{(\leq \alpha)}$ for some $\alpha < \omega_1^{CK}$.

**Proof:** ($\leq$) Both $\Pi^1_1$ sets and $\Sigma^1_1$ sets are closed downward under $\leq_m$.

($\Rightarrow$) Let $f : A \leq_m O$. 
\(\Delta^1_1\) sets

A set is \(\Delta^1_1\) if it is both \(\Pi^1_1\) and \(\Sigma^1_1\).

For \(\alpha < \omega^\text{CK}_1\), let \(O(\leq \alpha)\) be the set of indices of computable ordinals \(\leq \alpha\).

**Observation:** \(O(\leq \alpha)\) is \(\Delta^1_1\).

**\(\Sigma^1_1\)-bounding:** If \(A \subseteq O\) is \(\Sigma^1_1\), then \(A \subseteq O(\leq \alpha)\) for some \(\alpha < \omega^\text{CK}_1\).

**Theorem:** \(A \subseteq \omega\) is \(\Delta^1_1\) \iff \(A \leq_m O(\leq \alpha)\) for some \(\alpha < \omega^\text{CK}_1\).

**Proof:**  
(\(\leq\)) Both \(\Pi^1_1\) sets and \(\Sigma^1_1\) sets are closed downward under \(\leq_m\).
(\(\Rightarrow\)) Let \(f : A \leq_m O\). Since \(A\) is \(\Sigma^1_1\), so is \(f[A] \subseteq O\).
A set is $\Delta^1_1$ if it is both $\Pi^1_1$ and $\Sigma^1_1$.

For $\alpha < \omega^1_{CK}$, let $O(\leq \alpha)$ be the set of indices of computable ordinals $\leq \alpha$.

Observation: $O(\leq \alpha)$ is $\Delta^1_1$.

$\Sigma^1_1$-bounding: If $A \subseteq O$ is $\Sigma^1_1$, then $A \subseteq O(\leq \alpha)$ for some $\alpha < \omega^1_{CK}$.

Theorem: $A \subseteq \omega$ is $\Delta^1_1$ $\iff$ $A \leq_m O(\leq \alpha)$ for some $\alpha < \omega^1_{CK}$.

Proof: ($\leq$) Both $\Pi^1_1$ sets and $\Sigma^1_1$ sets are closed downward under $\leq_m$.

(=>) Let $f : A \leq_m O$. Since $A$ is $\Sigma^1_1$, so is $f[A] \subseteq O$.

Let $\alpha < \omega^1_{CK}$ be a bound for $f[A]$. 
A set is $\Delta^1_1$ if it is both $\Pi^1_1$ and $\Sigma^1_1$.

For $\alpha < \omega^{CK}_1$, let $O(\leq \alpha)$ be the set of indices of computable ordinals $\leq \alpha$.

**Observation:** $O(\leq \alpha)$ is $\Delta^1_1$.

**$\Sigma^1_1$-bounding:** If $A \subseteq O$ is $\Sigma^1_1$, then $A \subseteq O(\leq \alpha)$ for some $\alpha < \omega^{CK}_1$.

**Theorem:** $A \subseteq \omega$ is $\Delta^1_1 \iff A \leq_m O(\leq \alpha)$ for some $\alpha < \omega^{CK}_1$.

**Proof:** ($\leq$) Both $\Pi^1_1$ sets and $\Sigma^1_1$ sets are closed downward under $\leq_m$.

($\Rightarrow$) Let $f : A \leq_m O$. Since $A$ is $\Sigma^1_1$, so is $f[A] \subseteq O$.

Let $\alpha < \omega^{CK}_1$ be a bound for $f[A]$. Then $f : A \leq_m O(\leq \alpha)$. 
Finding paths through trees

Observation: $\mathcal{O}$ can compute paths through any computable tree.
Finding paths through trees

Observation: $\mathcal{O}$ can compute paths through any computable tree.

Lemma: Every non-empty $\Sigma^1_1$ class of reals has a member $\leq_T \mathcal{O}$.
Observation: \( \mathcal{O} \) can compute paths through any computable tree.

Lemma: Every non-empty \( \Sigma^1_1 \) class of reals has a member \( \leq_T \mathcal{O} \).

Theorem (Spector-Gandy)

Every non-empty \( \Sigma^1_1 \) class of reals has a member \( \leq_T \mathcal{O} \) and low for \( \omega_1 \)

..., where a real \( X \) is low for \( \omega_1 \) if \( \omega_1^X = \omega_1^{CK} \).
Part II

1. $\Pi^1_1$-ness and ordinals
2. Hyperarithmeticy
3. When hyperarithmetical is recursive
4. Overspill
5. A structure equivalent to its own jump
Arithmetic sets

Vocabulary of arithmetic: 0, 1, +, ×, ≤.

Definition: A set \( A \subseteq \mathbb{N} \) is arithmetic if it is definable in \( \mathbb{N} \) by a first-order formula of arithmetic.

\[
A = \{ n \in \mathbb{N} : (\mathbb{N}; 0, 1, +, \times, \leq) = \varphi(n) \}.
\]

The following are equivalent:

- \( A \) is arithmetic
- \( A \) is computable in \( 0^n \) for some \( n \)
- \( A \) is \( \leq^m O(\leq^\omega n) \) for some \( n \)
Arithmetic sets

Vocabulary of arithmetic: 0, 1, +, ×, ≤.
Arithmetic sets

Vocabulary of arithmetic: 0, 1, +, ×, ≤.

**Definition:** A set $A \subseteq \mathbb{N}$ is *arithmetic* if it is definable in $\mathbb{N}$ by a first-order formula of arithmetic.

$$A = \{ n \in \mathbb{N} : (\mathbb{N}; 0, 1, +, \times, \leq) \models \varphi(n) \}.$$
Arithmetic sets

Vocabulary of arithmetic: 0, 1, +, ×, ≤.

**Definition:** A set \( A \subseteq \mathbb{N} \) is *arithmetic* if it is definable in \( \mathbb{N} \) by a first-order formula of arithmetic.

\[
A = \{ n \in \mathbb{N} : (\mathbb{N}; 0, 1, +, \times, \leq) \models \varphi(n) \}.
\]

The following are equivalent:

- \( A \) is arithmetic
- \( A \) is computable in \( 0^{(n)} \) for some \( n \),
Arithmetic sets

Vocabulary of arithmetic: 0, 1, +, ×, ≤.

**Definition:** A set $A \subseteq \mathbb{N}$ is *arithmetic* if it is definable in $\mathbb{N}$ by a first-order formula of arithmetic.

$$A = \{ n \in \mathbb{N} : (\mathbb{N}; 0, 1, +, \times, \leq) \models \varphi(n) \}.$$ 

The following are equivalent:

- $A$ is arithmetic
- $A$ is computable in $0^{(n)}$ for some $n$,
- $A$ is $\leq_m O(\leq \omega^n)$ for some $n$
Infinitary 1st-order formulas

In 1st-order languages, $\forall$ and $\exists$ range over the elements of the structure. In infinitary languages, conjunctions and disjunctions can be infinitary.
Infinitary 1st-order formulas

In 1st-order languages, $\forall$ and $\exists$ range over the elements of the structure. In infinitary languages, conjunctions and disjunctions can be infinitary.

Example:

In a group $G = (G; e, \ast)$:
Infinitary 1st-order formulas

In 1st-order languages, \( \forall \) and \( \exists \) range over the elements of the structure. In infinitary languages, conjunctions and disjunctions can be infinitary.

Example: \( \text{torsion}(x) \equiv \)

In a group \( \mathcal{G} = (G; e, \ast) \):
Infinitary 1st-order formulas

In 1st-order languages, $\forall$ and $\exists$ range over the elements of the structure.
In infinitary languages, conjunctions and disjunctions can be infinitary.

Example: \[ torsion(x) \equiv \bigvee_{n \in \mathbb{N}} (x \ast x \ast x \ast \cdots \ast x = e), \]

In a group $G = (G; e, \ast)$:
Infinitary 1st-order formulas

In 1st-order languages, $\forall$ and $\exists$ range over the elements of the structure. In infinitary languages, conjunctions and disjunctions can be infinitary.

Example:

$$\text{torsion}(x) \equiv \bigvee_{n \in \mathbb{N}} (x \ast x \ast x \ast \cdots \ast x = e),$$

In a group $G = (G; e, \ast)$:

$$\text{divisible}(x) \equiv$$
Infinitary 1st-order formulas

In 1st-order languages, $\forall$ and $\exists$ range over the elements of the structure. In infinitary languages, conjunctions and disjunctions can be infinitary.

Example: $\text{torsion}(x) \equiv \bigvee_{n \in \mathbb{N}} (x \ast x \ast x \ast \cdots \ast x = e)$, $n \text{ times}$.

In a group $G = (G; e, \ast)$: $\text{divisible}(x) \equiv \bigwedge_{n \in \mathbb{N}} \exists y (y \ast y \ast y \ast \cdots \ast y = x)$.
Infinitary 1st-order formulas

In 1st-order languages, $\forall$ and $\exists$ range over the elements of the structure.
In infinitary languages, conjunctions and disjunctions can be infinitary.

Example: $\text{torsion}(x) \equiv \bigvee_{n \in \mathbb{N}} (x \ast x \ast x \ast \cdots \ast x = e)$,

In a group $G = (G; e, \ast)$: $\text{divisible}(x) \equiv \bigwedge_{n \in \mathbb{N}} \exists y (y \ast y \ast y \ast \cdots \ast y = x)$,

Theorem: [Scott 65] For every countable structure $\mathcal{A}$, there is an infinitary sentence $\psi_{\mathcal{A}}$ such that, for countable structures $\mathcal{C}$, $\mathcal{C} \models \psi_{\mathcal{A}} \iff \mathcal{C} \cong \mathcal{A}$. 
Infinitary 1st-order formulas

In 1st-order languages, $\forall$ and $\exists$ range over the elements of the structure.

In infinitary languages, conjunctions and disjunctions can be infinitary.

Example:

$$torsion(x) \equiv \bigvee_{n \in \mathbb{N}} (x \ast x \ast x \ast \cdots \ast x = e),$$

In a group $G = (G; e, \ast)$:

$$divisible(x) \equiv \bigwedge_{n \in \mathbb{N}} \exists y (y \ast y \ast y \ast \cdots \ast y = x),$$

Theorem: [Scott 65] For every countable structure $\mathcal{A}$, there is an infinitary sentence $\psi_{\mathcal{A}}$ such that, for countable structures $\mathcal{C}$, $\mathcal{C} \models \psi_{\mathcal{A}} \iff \mathcal{C} \cong \mathcal{A}$.

Theorem: [Scott 65] For every automorphism invariant set $B \subset \mathcal{A}^k$, there is an infinitary formula $\varphi(\bar{x})$ such that $B = \{ \bar{b} \in \mathcal{A}^k : \mathcal{A} \models \varphi(\bar{b}) \}$. 
Depths of infinitary formulas

We count alternations of $\exists$ and $\lor$ versus $\forall$ and $\land$. 
Depths of infinitary formulas

We count alternations of $\exists$ and $\lor$ versus $\forall$ and $\land$.

A $\Sigma^\text{in}_n$ formula is one of the form:

$$\lor_{i_0 \in \mathbb{N}} \exists \bar{y}_0 \land_{i_1 \in \mathbb{N}} \forall \bar{y}_1 \lor_{i_2 \in \mathbb{N}} \exists \bar{y}_2 \land_{i_3 \in \mathbb{N}} \forall \bar{y}_3 \cdots \left( \psi_{i_0, i_1, \ldots, i_n}(\bar{x}, \bar{y}_0, \bar{y}_1, \ldots, \bar{y}_n) \right)$$

$n$ alternations

finitary, quantifier free
Depths of infinitary formulas

We count alternations of $\exists$ and $\lor$ versus $\forall$ and $\land$.

A $\Pi^\infty_n$ formula is one of the form:

$$\bigwedge_{i_0 \in \mathbb{N}} \forall \bar{y}_0 \lor \bigvee_{i_1 \in \mathbb{N}} \exists \bar{y}_1 \land \bigwedge_{i_2 \in \mathbb{N}} \forall \bar{y}_2 \lor \bigvee_{i_3 \in \mathbb{N}} \exists \bar{y}_3 \cdots \left( \psi_{i_0, i_1, \ldots, i_n}(\bar{x}, \bar{y}_0, \bar{y}_1, \ldots, \bar{y}_n) \right)$$

$n$ alternations

finitary, quantifier free
Depths of infinitary formulas

We count alternations of $\exists$ and $\lor$ versus $\forall$ and $\land$.

A $\Pi^\infty_n$ formula is one of the form:

$$\land_{i_0 \in \mathbb{N}} \forall \bar{y}_0 \lor_{i_1 \in \mathbb{N}} \exists \bar{y}_1 \land_{i_2 \in \mathbb{N}} \forall \bar{y}_2 \lor_{i_3 \in \mathbb{N}} \exists \bar{y}_3 \cdots \left( \psi_{i_0, i_1, \ldots, i_n}(\bar{x}, \bar{y}_0, \bar{y}_1, \ldots, \bar{y}_n) \right)$$

$n$ alternations

finitary, quantifier free

A $\Sigma^\infty_\alpha$ formula is one of the form:

$$\lor_{i \in \mathbb{N}} \exists \bar{y} \left( \psi_i(\bar{x}, \bar{y}) \right)$$

$\Pi^\infty_\beta$ for $\beta < \alpha$
Depths of infinitary formulas

We count alternations of $\exists$ and $\lor$ versus $\forall$ and $\land$.

A $\Pi^\infty_n$ formula is one of the form:

$$\bigwedge_{i_0 \in \mathbb{N}} \forall \bar{y}_0 \lor \bigvee_{i_1 \in \mathbb{N}} \exists \bar{y}_1 \land \bigwedge_{i_2 \in \mathbb{N}} \forall \bar{y}_2 \lor \bigvee_{i_3 \in \mathbb{N}} \exists \bar{y}_3 \cdots \left( \psi_{i_0, i_1, \ldots, i_n}(\bar{x}, \bar{y}_0, \bar{y}_1, \ldots, \bar{y}_n) \right)$$

$n$ alternations

finitary, quantifier free

A $\Pi^\infty_\beta$ formula is one of the form:

$$\bigwedge_{i \in \mathbb{N}} \forall \bar{y} \left( \varphi_i(\bar{x}, \bar{y}) \right)$$

$\Sigma^\infty_\gamma$ for $\gamma < \beta$
Computable infinitary formulas

Definition: An infinitary formula is computable if it has a computable tree representation. Equivalently, if the infinitary conjunctions and disjunctions are of computable lists of formulas.

Example: \( \text{torsion}(x) \equiv \bigvee_{n \in \mathbb{N}} (x \ast x \ast x \ast \cdots \ast x = e) \),

In a group \( G = (G; e, \ast) \):
\( \text{divisible}(x) \equiv \bigwedge_{n \in \mathbb{N}} \exists y (y \ast \cdots \ast y = x) \),

We use \( Lc^\omega \) to denote the set of computably infinitary formulas.
**Definition:** An infinitary formula is *computable* if
Definition: An infinitary formula is *computable* if it has a computable tree representation.
Computable infinitary formulas

**Definition:** An infinitary formula is *computable* if it has a computable tree representation.

Equivalently, if

\[ \text{torsion} \left(x\right) \equiv \bigvee_{n \in \mathbb{N}} x \cdot x \cdot x \cdot \ldots \cdot x = e, \]

In a group \( G = (G; e, \ast) \):

\[ \text{divisible} \left(x\right) \equiv \bigwedge_{n \in \mathbb{N}} \exists y \left(y \cdot y \cdot y \cdot \ldots \cdot y = x\right), \]

We use \( L_{c, \omega} \) to denote the set of computably infinitary formulas.
Computable infinitary formulas

**Definition:** An infinitary formula is *computable* if it has a computable tree representation. Equivalently, if the infinitary conjunctions and disjunctions are of computable lists of formulas.
Computable infinitary formulas

**Definition:** An infinitary formula is *computable* if it has a computable tree representation. Equivalently, if the infinitary conjunctions and disjunctions are of computable lists of formulas.

**Example:**

In a group $G = (G; e, \cdot)$:

- *Divisible* ($x$) $\equiv \bigwedge_{n \in \mathbb{N}} \exists y (y \cdot y \cdot \ldots \cdot y = x)$.

We use $Lc_\omega$ to denote the set of computably infinitary formulas.
Computable infinitary formulas

**Definition:** An infinitary formula is *computable* if it has a computable tree representation. Equivalently, if the infinitary conjunctions and disjunctions are of computable lists of formulas.

**Example:**

\[ \text{torsion}(x) \equiv \bigvee_{n \in \mathbb{N}} \left( x \ast x \ast x \ast \cdots \ast x = e \right), \]

In a group \( G = (G; e, \ast) \):

\[ \text{divisible}(x) \equiv \bigwedge_{n \in \mathbb{N}} \exists y \left( y \ast y \ast y \ast \cdots \ast y = x \right), \]
Computable infinitary formulas

**Definition:** An infinitary formula is *computable* if it has a computable tree representation. Equivalently, if the infinitary conjunctions and disjunctions are of computable lists of formulas.

**Example:**

\[ \text{torsion}(x) \equiv \bigvee_{n \in \mathbb{N}} (x \ast x \ast x \ast \cdots \ast x = e), \]

In a group \( G = (G; e, \ast) \):

\[ \text{divisible}(x) \equiv \bigwedge_{n \in \mathbb{N}} \exists y (y \ast y \ast y \ast \cdots \ast y = x), \]

We use \( L_{c,\omega} \) to denote the set of computably infinitary formulas.
more examples

**Example:** There is a $\Pi_{2^{\alpha+1}}^c$ formula $\psi_\alpha$ such that, on a partial ordering $\mathcal{P}$,

$$\mathcal{P} \models \psi_\alpha(a) \iff \text{rk}_\mathcal{P}(a) \leq \alpha.$$
Example: There is a $\Pi^c_{2\alpha+1}$ formula $\psi_\alpha$ such that, on a partial ordering $\mathcal{P}$,

$$\mathcal{P} \models \psi_\alpha(a) \iff \text{rk}_\mathcal{P}(a) \leq \alpha.$$ 

The formula is built by transfinite recursion:

$$\psi_\alpha(x) \equiv \forall y < x \biguplus_{\gamma < \beta} \psi_\gamma(y).$$
more examples

Example: There is a $\Pi^c_{2\alpha+1}$ formula $\psi_\alpha$ such that, on a partial ordering $\mathcal{P}$,

$$\mathcal{P} \models \psi_\alpha(a) \iff \text{rk}_\mathcal{P}(a) \leq \alpha.$$  

The formula is built by transfinite recursion:

$$\psi_\alpha(x) \equiv \forall y < x \bigvee_{\gamma < \beta} \psi_\gamma(y).$$

Example: There is a $\Sigma^c_{2\alpha+1}$ sentence $\varphi_{\omega^\alpha}$ such that, for a linear ordering $\mathcal{L}$,

$$\mathcal{L} \models \varphi_{\omega^\alpha} \iff \mathcal{L} \leq \omega^\alpha.$$
Hyperarithmetic sets

Definition: A set $A \subseteq \mathbb{N}$ is hyperarithmetic if
it is definable by an infinitary computable formula $\varphi(x)$.

$$A = \{ n \in \mathbb{N} : (\mathbb{N}; 0, 1, +, \times, \leq) \models \varphi(n) \}.$$
Hyperarithmetic sets

**Definition:** A set $A \subseteq \mathbb{N}$ is *hyperarithmetic* if it is definable by an infinitary computable formula $\varphi(x)$. 

$$A = \{ n \in \mathbb{N} : \mathbf{N; 0, 1, +, \times, \leq} \models \varphi(n) \}. $$

**Theorem:** Let $A \subseteq \mathbb{N}$. The following are equivalent:

- $A$ is definable by a $\mathcal{L}_{c,\omega}$ formula
- There is a computable list $\{\varphi_n : n \in \mathbb{N}\}$ of $\mathcal{L}_{c,\omega}$ sentences over the empty vocabulary $\{\top, \bot\}$ such that $A = \{ n \in \mathbb{N} : \models \varphi_n \}$. 
Observation Deciding if “$\mathcal{M} \models \varphi$” for $\varphi$ infinitary is $\Sigma^1_1$: 

There is a valid truth-assignment to the sub-formulas making $\varphi$ true.

Corollary: Hyperarithmetic sets are $\Delta^1_1$.

Given a computable list $\{M_e : e \in \mathbb{N}\}$ and a $\mathcal{L}_{\omega \omega}$-sentence $\varphi$, $\{n : M_n \models \varphi\}$ is hyperarithmetic.

Corollary: $\mathcal{O}(\leq \alpha)$ is hyperarithmetic.

Theorem: [Kleene] Let $A \subseteq \omega$. The following are equivalent:

1. $A$ is hyperarithmetic
2. $A$ is $\Delta^1_1$
3. $A \leq_m O(\leq \alpha)$ for some $\alpha < \omega_{CK}$
Observation Deciding if \( \mathcal{M} \models \varphi \) for \( \varphi \) infinitary is \( \Sigma^1_1 \):

There is a valid truth-assignment to the sub-formulas making \( \varphi \) true.
**Hyp and $\Delta^1_1$**

**Observation** Deciding if “$\mathcal{M} \models \varphi$” for $\varphi$ infinitary is $\Sigma^1_1$:
There is a valid truth-assignment to the sub-formulas making $\varphi$ true.

**Corollary:** Hyperarithmetic sets are $\Delta^1_1$.
Hyp and $\Delta^1_1$

**Observation** Deciding if “$\mathcal{M} \models \varphi$” for $\varphi$ infinitary is $\Sigma^1_1$: There is a valid truth-assignment to the sub-formulas making $\varphi$ true.

**Corollary:** Hyperarithmetic sets are $\Delta^1_1$.

Given a computable list $\{\mathcal{M}_e : e \in \mathbb{N}\}$ and a $\mathcal{L}_{c,\omega}$-sentence $\varphi$, $\{n : \mathcal{M}_n \models \varphi\}$ is hyperarithmetic.
Hyp and $\Delta^1_1$

**Observation** Deciding if “$\mathcal{M} \models \varphi$” for $\varphi$ infinitary is $\Sigma^1_1$:
There is a valid truth-assignment to the sub-formulas making $\varphi$ true.

**Corollary:** Hyperarithmetic sets are $\Delta^1_1$.

Given a computable list $\{\mathcal{M}_e : e \in \mathbb{N}\}$ and a $\mathcal{L}_{c,\omega}$-sentence $\varphi$, $\{n : \mathcal{M}_n \models \varphi\}$ is hyperarithmetic.

**Corollary:** $\mathcal{O}(\leq \alpha)$ is hyperarithmetic.
Hyp and $\Delta^1_1$

Observation: Deciding if $\mathcal{M} \models \varphi$ for $\varphi$ infinitary is $\Sigma^1_1$:
There is a valid truth-assignment to the sub-formulas making $\varphi$ true.

Corollary: Hyperarithmetic sets are $\Delta^1_1$.

Given a computable list $\{\mathcal{M}_e : e \in \mathbb{N}\}$ and a $\mathcal{L}_{c,\omega}$-sentence $\varphi$, $\{n : \mathcal{M}_n \models \varphi\}$ is hyperarithmetic.

Corollary: $\mathcal{O}_{(\leq \alpha)}$ is hyperarithmetic.

Theorem: [Kleene] Let $A \subseteq \omega$. The following are equivalent:

- $A$ is hyperarithmetic
- $A$ is $\Delta^1_1$
- $A \leq_m \mathcal{O}_{(\leq \alpha)}$ for some $\alpha < \omega^c_{1CK}$
Let $L$ be a well-ordering with domain $\subseteq \mathbb{N}$.

**Definition:** A jump hierarchy on $L$ is a set $H \subseteq L \times \mathbb{N}$ such that $H[\ell] = (H[<\ell])'$, where $X[\ell] = \{x : (\ell, x) \in X\}$ and $X[<\ell] = \{(k, x) : k < L\ell \land (k, x) \in X\}$.

**Obs:** For every well-ordering $L$, there is a unique jump hierarchy on it.
Transfinite iterations of the Turing jump

Let $\mathcal{L}$ be a well-ordering with domain $\subseteq \mathbb{N}$.

**Definition:** A *jump hierarchy* on $\mathcal{L}$ is a set $H \subseteq \mathcal{L} \times \mathbb{N}$ such that

$$H^{[\ell]} = (H^{[<\ell]})',$$

where $X^{[\ell]} = \{x : (\ell, x) \in X\}$ and $X^{[<\ell]} = \{(k, x) : k <_{\mathcal{L}} \ell \text{ and } (k, x) \in X\}$. 
Transfinite iterations of the Turing jump

Let $\mathcal{L}$ be a well-ordering with domain $\subseteq \mathbb{N}$.

**Definition:** A *jump hierarchy* on $\mathcal{L}$ is a set $H \subseteq \mathcal{L} \times \mathbb{N}$ such that

$$H^{[\ell]} = (H^{[<\ell]})',$$

where $X^{[\ell]} = \{x : (\ell, x) \in X\}$ and $X^{[<\ell]} = \{(k, x) : k \prec \mathcal{L} \ell \& (k, x) \in X\}$.

**Obs:** For every well-ordering $\mathcal{L}$ there is a unique jump hierarchy on it.
Theorem: Suppose $\alpha$ and $\beta$ are different presentations of the same ordinal. Let $H_{\alpha}$ and $H_{\beta}$ be the jump hierarchies on them. Then $H_{\alpha} \equiv_T H_{\beta}$.

Pf: Show that there is an isomorphism $\alpha \rightarrow \beta$ computable in both $H_{\alpha}$ and $H_{\beta}$.
Different presentations

**Theorem:** Suppose $\alpha$ and $\beta$ are different presentations of the same ordinal. Let $H_\alpha$ and $H_\beta$ be the jump hierarchies on them. Then $H_\alpha \equiv_T H_\beta$.

**Pf:** Show that there is an isomorphism $\alpha \to \beta$ computable in both $H_\alpha$ and $H_\beta$.

We now can define the Turing degree $0^{(\alpha)}$ for computable $\alpha < \omega_1^{CK}$.
Theorem: Suppose $\alpha$ and $\beta$ are different presentations of the same ordinal. Let $H_\alpha$ and $H_\beta$ be the jump hierarchies on them. Then $H_\alpha \equiv_T H_\beta$.

Pf: Show that there is an isomorphism $\alpha \rightarrow \beta$ computable in both $H_\alpha$ and $H_\beta$.

We now can define the Turing degree $0^{(\alpha)}$ for computable $\alpha < \omega_1^{CK}$.

Theorem: For $n \in \mathbb{N}$: $O_{(<\omega^n)} \equiv_T 0^{(2n)}$. 
Different presentations

**Theorem:** Suppose $\alpha$ and $\beta$ are different presentations of the same ordinal. Let $H_\alpha$ and $H_\beta$ be the jump hierarchies on them. Then $H_\alpha \equiv_T H_\beta$.

**Pf:** Show that there is an isomorphism $\alpha \rightarrow \beta$ computable in both $H_\alpha$ and $H_\beta$.

We now can define the Turing degree $0^{(\alpha)}$ for computable $\alpha < \omega_1^{CK}$.

**Theorem:** For $n \in \mathbb{N}$: $O_{(<\omega^n)} \equiv_T 0^{(2n)}$.

For $\alpha \in \omega_1^{CK} \setminus \mathbb{N}$: $O_{(<\omega^\alpha)} \equiv_T 0^{(2\alpha+1)}$. 
Part III

1. $\Pi^1_1$-ness and ordinals
2. Hyperarithmeticity
3. When hyperarithmetic is recursive
4. Overspill
5. A structure equivalent to its own jump
Every hyperarithmetic well-ordering is computable

Consider $E = \{ e : \leq^A \}

Then there is a bound $\alpha < \omega_{\text{CK}}$ for $E$. Then $A \leq^A \alpha$.

Theorem: If an infinitary formula has a hyperarithmetic representation, it is equivalent to a computable infinitary formula.
Every hyperarithmetic well-ordering is computable

**Theorem:** [Spector 55] If $\leq_A \subseteq \omega^2$ is a hyperarithmetic well-ordering of $\omega$, then $A = (\omega; \leq_A)$ is isomorphic to a computable well-ordering.

**Proof:**
Every hyperarithmetic well-ordering is computable

**Theorem:** [Spector 55] If $\leq_A \subseteq \omega^2$ is a hyperarithmetic well-ordering of $\omega$, then $\mathcal{A} = (\omega; \leq_A)$ is isomorphic to a computable well-ordering.

**Proof:** Consider $E = \{e : L_e \preceq A\}$. 
Every hyperarithmetic well-ordering is computable

**Theorem:** [Spector 55] If $\leq_A \subseteq \omega^2$ is a hyperarithmetic well-ordering of $\omega$, then $A = (\omega; \leq_A)$ is isomorphic to a computable well-ordering.

**Proof:** Consider $E = \{e : L_e \preceq A\}$. $E$ is $\Sigma^1_1$
Every hyperarithmetic well-ordering is computable

**Theorem:** [Spector 55] If $\leq_A \subseteq \omega^2$ is a hyperarithmetic well-ordering of $\omega$, then $A = (\omega; \leq_A)$ is isomorphic to a computable well-ordering.

**Proof:** Consider $E = \{e : \mathcal{L}_e \preceq A\}$. $E$ is $\Sigma^1_1$ and $E \subseteq \mathcal{O}$. 
Every hyperarithmetic well-ordering is computable

**Theorem:** [Spector 55] If $\leq_A \subseteq \omega^2$ is a hyperarithmetic well-ordering of $\omega$, then $\mathcal{A} = (\omega; \leq_A)$ is isomorphic to a computable well-ordering.

**Proof:** Consider $E = \{e : L_e \preceq A\}$. $E$ is $\Sigma^1_1$ and $E \subseteq \mathcal{O}$. Then there is a bound $\alpha < \omega_1^{CK}$ for $E$. 
Every hyperarithmetic well-ordering is computable

Theorem: [Spector 55] If $\leq_A \subseteq \omega^2$ is a hyperarithmetic well-ordering of $\omega$, then $\mathcal{A} = (\omega; \leq_A)$ is isomorphic to a computable well-ordering.

Proof: Consider $E = \{ e : L_e \preceq A \}$. $E$ is $\Sigma^1_1$ and $E \subseteq \mathcal{O}$. Then there is a bound $\alpha < \omega^1_{CK}$ for $E$. Then $\mathcal{A} \preceq \alpha$. 
Every hyperarithmetic well-ordering is computable

**Theorem:** [Spector 55] If $\leq_{\mathcal{A}} \subseteq \omega^2$ is a hyperarithmetic well-ordering of $\omega$, then $\mathcal{A} = (\omega; \leq_{\mathcal{A}})$ is isomorphic to a computable well-ordering.

**Proof:** Consider $E = \{e : L_e \preceq \mathcal{A}\}$. $E$ is $\Sigma^1_1$ and $E \subseteq \mathcal{O}$. Then there is a bound $\alpha < \omega^{ck}_1$ for $E$. Then $\mathcal{A} \preceq \alpha$.

**Theorem:** If an infinitary formula has a hyperarithmetic representation it is equivalent to a computable infinitary formula.
A generalization to Linear orderings

Theorem ([M. 05])

Every hyperarithmetic linear ordering is bi-embeddable with a computable one.
A generalization to Linear orderings

**Theorem ([M. 05])**

*Every hyperarithmetic linear ordering is bi-embeddable with a computable one.*

**Obs:** The theorem generalizes Spector’s theorem:

Finally, we build a computable map from invariants to linear orderings.
A generalization to Linear orderings

**Theorem ([M. 05])**

Every hyperarithmetic linear ordering is bi-embeddable with a computable one.

**Obs:** The theorem generalizes Spector’s theorem: If an ordinal is bi-embeddable with a linear ordering, it is isomorphic.
Theorem ([M. 05])

Every hyperarithmetic linear ordering is bi-embeddable with a computable one.

**Obs:** The theorem generalizes Spector’s theorem:
If an ordinal is bi-embeddable with a linear ordering, it is isomorphic.

The proof uses Laver’s theorem on the well-quasi-orderness of linear orderings to analyze their structure under embeddability.
A generalization to Linear orderings

**Theorem ([M. 05])**

*Every hyperarithmetic linear ordering is bi-embeddable with a computable one.*

**Obs:** The theorem generalizes Spector’s theorem:
If an ordinal is bi-embeddable with a linear ordering, it is isomorphic.

The proof uses Laver’s theorem on the well-quasi-orderness of linear orderings to analyze their structure under embeddability.
We produce bi-embeddability invariants for linear orderings given by finite trees with ordinal labels.
Theorem ([M. 05])

Every hyperarithmetic linear ordering is bi-embeddable with a computable one.

**Obs:** The theorem generalizes Spector’s theorem:
If an ordinal is bi-embeddable with a linear ordering, it is isomorphic.

**The proof** uses Laver’s theorem on the well-quasi-orderness of linear orderings to analyze their structure under embeddability.
We produce bi-embeddability invariants for linear orderings given by finite trees with ordinal labels. Finally, we build a computable map from invariants to linear orderings.
Another similar behavior

**Theorem** ([Greenberg-M. 05])

*Every hyperarithmetic abelian torsion group is bi-embeddable with a computable one.*
Another similar behavior

**Theorem ([Greenberg-M. 05])**

Every hyperarithmetic abelian torsion group is bi-embeddable with a computable one.

**The proof** uses Ulm invariants, and bi-embeddability invariants defined by [Barwise, Eklof 71].
Another similar behavior

Theorem ([Greenberg-M. 05])

Every hyperarithmetic abelian torsion group is bi-embeddable with a computable one.

The proof uses Ulm invariants, and bi-embeddability invariants defined by [Barwise, Eklof 71]. The bi-embeddability invariants are finite sequences of ordinals, and we also have a computable operator from invariants to groups.
Another similar behavior

**Theorem ([Greenberg-M. 05])**

*Every hyperarithmetic abelian torsion group is bi-embeddable with a computable one.*

**The proof** uses Ulm invariants, and bi-embeddability invariants defined by [Barwise, Eklof 71]. The bi-embeddability invariants are finite sequences of ordinals, and we also have a computable operator from invariants to groups. Hyperarithmetic groups have Ulm rank $\leq \omega_1^{CK}$. 
Another similar behavior

Theorem ([Greenberg-M. 05])

Every hyperarithmetic abelian torsion group is bi-embeddable with a computable one.

The proof uses Ulm invariants, and bi-embeddability invariants defined by [Barwise, Eklof 71]. The bi-embeddability invariants are finite sequences of ordinals, and we also have a computable operator from invariants to groups.

Hyperarithmetic groups have Ulm rank $\leq \omega_1^{CK}$. If the Ulm rank is $< \omega_1^{CK}$ use the computable operator.
Another similar behavior

**Theorem ([Greenberg-M. 05])**

*Every hyperarithmetic abelian torsion group is bi-embeddable with a computable one.*

**The proof** uses Ulm invariants, and bi-embeddability invariants defined by [Barwise, Eklof 71]. The bi-embeddability invariants are finite sequences of ordinals, and we also have a computable operator from invariants to groups. Hyperarithmetic groups have Ulm rank $\leq \omega_1^{CK}$. If the Ulm rank is $< \omega_1^{CK}$ use the computable operator. If the Ulm rank is $\omega_1^{CK}$, we need to show their divisible part must be isomorphic to $\mathbb{Q}^\infty$, and hence they are bi-embeddable with $\mathbb{Q}^\infty$. 
Counterexample to Vaught’s conjecture

**Vaught’s conjecture:**
Every $L_{\omega_1,\omega}$ sentence has either countably or $2^{\aleph_0}$ many countable models.
Counterexample to Vaught’s conjecture

**Vaught’s conjecture:**
Every $L_{\omega_1,\omega}$ sentence has either countably or $2^{\aleph_0}$ many countable models.

**Def:** An $L_{\omega_1,\omega}$ sentence is a *counterexample to Vaught’s conjecture* if
Counterexample to Vaught’s conjecture

**Vaught’s conjecture:**
Every $L_{\omega_1,\omega}$ sentence has either countably or $2^{\aleph_0}$ many countable models.

**Def:** An $L_{\omega_1,\omega}$ sentence is a *counterexample to Vaught’s conjecture* if it has uncountably but not perfectly many countable models.
Counterexample to Vaught’s conjecture

**Vaught’s conjecture:**
Every $L_{\omega_1,\omega}$ sentence has either countably or $2^\aleph_0$ many countable models.

**Def:** An $L_{\omega_1,\omega}$ sentence is a *counterexample to Vaught’s conjecture* if it has uncountably but not perfectly many countable models.

**Theorem ([M. 12])**
Let $T$ be an $L_{\omega_1,\omega}$ sentence with uncountably many models. TFAE
- $T$ is a counterexample to Vaught’s conjecture.
- Relative to every oracle on a cone, every hyperarithmetic model of $T$ is isomorphic to a computable one,
Counterexample to Vaught’s conjecture

**Vaught’s conjecture:**
Every $L_{\omega_1,\omega}$ sentence has either countably or $2^{\aleph_0}$ many countable models.

**Def:** An $L_{\omega_1,\omega}$ sentence is a *counterexample to Vaught’s conjecture* if it has uncountably but not perfectly many countable models.

**Theorem ([M. 12])**
Let $T$ be an $L_{\omega_1,\omega}$ sentence with uncountably many models. TFAE

- $T$ is a counterexample to Vaught’s conjecture.
- Relative to every oracle on a cone, every hyperarithmetic model of $T$ is isomorphic to a computable one.

By “relative to every oracle on a cone”
we mean “$(\exists Y \in 2^\omega)(\forall X \geq_T Y)$ the following holds relativized to $Y$.”
Hyperarithmetic-is-recursive

Definition

An equivalence class $E$ on $2^\omega$ satisfies \textit{hyperarithmetic-is-recursive} if every hyperarithmetic real is $E$-equivalent to a computable one.
Hyperarithmetic-is-recursive

Definition
An equivalence class $E$ on $2^\omega$ satisfies *hyperarithmetic-is-recursive* if every hyperarithmetic real is $E$-equivalent to a computable one.

Examples:
- isomorphism on well-orderings;
- bi-embeddability on linear orderings;
- bi-embeddability on torsion abelian groups;
- isomorphism on models of a counterexample to Vaught’s conjecture when relativized;

Obs:
All the equivalence relations above are $\Sigma_1^1$. If we let the reals not in the class be equivalent, they are $\Sigma_1^1$-equivalence relations on $2^\omega$. 

Def:
$E$ satisfies hyperarithmetic-is-recursive trivially if every real is $E$-equivalent to a computable one.
Hyperarithmetic-is-recursive

Definition
An equivalence class $E$ on $2^\omega$ satisfies \textit{hyperarithmetic-is-recursive} if every hyperarithmetic real is $E$-equivalent to a computable one.

Examples:
\begin{itemize}
  \item isomorphism on well-orderings;
  \item bi-embeddability on linear orderings;
  \item bi-embeddability on torsion abelian groups;
  \item isomorphism on models of a counterexample to Vaught’s conjecture when relativized;
  \item $X \equiv Y \iff \omega_1^X = \omega_1^Y$.
\end{itemize}
Hyperarithmetic-is-recursive

Definition

An equivalence class $E$ on $2^\omega$ satisfies *hyperarithmetic-is-recursive* if every hyperarithmetic real is $E$-equivalent to a computable one.

Examples:

- isomorphism on well-orderings;
- bi-embeddability on linear orderings;
- bi-embeddability on torsion abelian groups;
- isomorphism on models of a counterexample to Vaught’s conjecture when relativized;

$X \equiv Y \iff \omega_1^X = \omega_1^Y$.

Obs: All the equivalence relations above are $\Sigma^1_1$. 
Hyperarithmetic-is-recursive

**Definition**
An equivalence class $E$ on $2^\omega$ satisfies *hyperarithmetic-is-recursive* if every hyperarithmetic real is $E$-equivalent to a computable one.

**Examples:**
- isomorphism on well-orderings;
- bi-embeddability on linear orderings;
- bi-embeddability on torsion abelian groups;
- isomorphism on models of a counterexample to Vaught’s conjecture when relativized;

- $X \equiv Y \iff \omega_1^X = \omega_1^Y$.

**Obs:** All the equivalence relations above are $\Sigma^1_1$.

If we let the reals not in the class be equivalent, they are $\Sigma^1_1$-equivalence relations on $2^\omega$. 
Hyperarithmetic-is-recursive

Definition
An equivalence class $E$ on $2^\omega$ satisfies \emph{hyperarithmetic-is-recursive} if every hyperarithmetic real is $E$-equivalent to a computable one.

Examples:
- isomorphism on well-orderings;
- bi-embeddability on linear orderings;
- bi-embeddability on torsion abelian groups;
- isomorphism on models of a counterexample to Vaught’s conjecture when relativized;
- $X \equiv Y \iff \omega_1^X = \omega_1^Y$.

Obs: All the equivalence relations above are $\Sigma^1_1$.
If we let the reals not in the class be equivalent, they are $\Sigma^1_1$-equivalence relations on $2^\omega$.

Def: $E$ satisfies hyperarithmetic-is-recursive \emph{trivially} if every real is $E$-equivalent to a computable one.
The question

**Question:** What makes an equivalence relation satisfy hyperarithmetic-is-recursive?
Question: What makes an equivalence relation satisfy hyperarithmetic-is-recursive?

Obs: There are odd examples:
**Question:** What makes an equivalence relation satisfy hyperarithmetic-is-recursive?

**Obs:** There are odd examples:

If $E$ is $\Sigma^1_1$ and we define $X F Y \iff (X E Y) \lor (\omega_1^X = \omega_1^Y = \omega_1^{CK})$, then the transitive closure of $F$ is $\Sigma^1_1$ and satisfies hyperarithmetic-is-recursive.
The question

**Question:** What makes an equivalence relation satisfy hyperarithmetic-is-recursive?

**Obs:** There are odd examples:
If $E$ is $\Sigma^1_1$ and we define $X$ $F$ $Y \iff (X \ E \ Y) \lor (\omega^X_1 = \omega^Y_1 = \omega^{CK}_1)$, then the transitive closure of $F$ is $\Sigma^1_1$ and satisfies hyperarithmetic-is-recursive.

**Question:** What makes an equivalence relation satisfy hyperarithmetic-is-recursive on a cone?
Martin’s measure

**Def:** A *cone* is a set of the form \( \{ X \in 2^\mathbb{N} : X \geq_T Y \} \) for some \( Y \in 2^\mathbb{N} \).
Martin’s measure

**Def:** A *cone* is a set of the form \( \{ X \in 2^\mathbb{N} : X \geq_T Y \} \) for some \( Y \in 2^\mathbb{N} \).

**Thm:** [Martin] \((0^\# \text{ exists})\)

Every \( \Sigma^1_1 \) degree-invariant \( A \subseteq 2^\mathbb{N} \) either contains or is disjoint from a cone.

**Obs:** Since in computability theory most proofs relativize:

For “natural” \( E \), \( E \) satisfies hyperarithmetic-is-recursive if, \((\exists Y)(\forall X \geq_T Y)\), every \( X \)-hyperarithmetic real is \( E \)-equivalent to an \( X \)-computable one.

\(\quad\)
Martin’s measure

Def: A cone is a set of the form \( \{ X \in 2^\mathbb{N} : X \geq_T Y \} \) for some \( Y \in 2^\mathbb{N} \).

Thm:[Martin] (\( 0^\# \) exists) Every \( \Sigma^1_1 \) degree-invariant \( A \subseteq 2^\mathbb{N} \) either contains or is disjoint from a cone.

Def: A degree-invariant \( A \subseteq 2^\mathbb{N} \) has Martin measure 1 if it contains a cone, and Martin measure 0 if it doesn’t.
Martin’s measure

Def: A cone is a set of the form \( \{ X \in 2^\mathbb{N} : X \geq_T Y \} \) for some \( Y \in 2^\mathbb{N} \).

Thm: [Martin] (0\(^\#\) exists)
Every \( \Sigma^1_1 \) degree-invariant \( A \subseteq 2^\mathbb{N} \) either contains or is disjoint from a cone.

Def: A degree-invariant \( A \subseteq 2^\mathbb{N} \) has Martin measure 1 if it contains a cone, and Martin measure 0 if it doesn’t.

Def: \( E \) satisfies hyperarithmetical-is-recursive on a cone if,
\[
(\exists Y)(\forall X \geq_T Y),
\]
every \( X \)-hyperarithmetical real is \( E \)-equivalent to an \( X \)-computable one.
Martin’s measure

Def: A *cone* is a set of the form $\{ X \in 2^\mathbb{N} : X \geq_T Y \}$ for some $Y \in 2^\mathbb{N}$.

Thm: [Martin] (0$^#$ exists)
Every $\Sigma^1_1$ degree-invariant $A \subseteq 2^\mathbb{N}$ either contains or is disjoint from a cone.

Def: A degree-invariant $A \subseteq 2^\mathbb{N}$ *has Martin measure 1* if it contains a cone, and *Martin measure 0* if it doesn’t.

Def: $E$ satisfies *hyperarithmetic-is-recursive on a cone* if,

$$(\exists Y)(\forall X \geq_T Y),$$

every $X$-hyperarithmetic real is $E$-equivalent to an $X$-computable one.

Obs: Since in computability theory most proofs relativize:
For “natural” $E$,

$E$ satisfies *hyperarithmetic-is-recursive* $\iff$ it does on a cone.
A sufficient condition: a first attempt

Def: For $K \subseteq 2^\omega$, $(K, \varpi, r)$ is a ranked equivalence relation if $\varpi$ is an equivalence relation on $K$, and $r: K/\varpi \to 2^\omega$.

Def: $(K, \varpi, r)$ is scattered if $r$ is $1$-scattered for each $\varpi_i$ contains countably many equivalence classes.

Def: $(K, \varpi, r)$ is projective if $K$ and $\varpi$ are projective and $r$ has a projective presentation.

Theorem (M. ZFC+PD) Let $(K, \varpi, r)$ be scattered projective ranked equivalence relation such that $8\alpha\in 2^\omega$, $r(\alpha) < \omega_1$.

For every $X$ on a cone, (i.e. $9\exists Y : X \subseteq T_Y$, ) every equivalence class with an $X$-hyperarithmetic member has an $X$-computable member.

Lemma: [Martin] (ZFC+PD) If $f: 2^\omega \to 2^\omega$ is projective and $f(X) < \omega_1$, then $f$ is constant on a cone.

Antonio Montalbán (U.C. Berkeley) When hyperarithmetic is recursive Sept. 2012 15 / 28

Antonio Montalbán (U.C. Berkeley) Higher Recursion and computable structures May 2019 32 / 50
A sufficient condition: a first attempt

A sufficient condition for hyp-is-rec.

**Def:** For $\mathbb{K} \subseteq 2^\omega$, $(\mathbb{K}, \equiv, r)$ is a **ranked equivalence relation** if

$\equiv$ is an equivalence relation on $\mathbb{K}$, and $r: \mathbb{K}/\equiv \to \omega_1$.

**Def:** $(\mathbb{K}, \equiv, r)$ is **scattered** if

$r^{-1}(\alpha)$ contains countably many equivalence classes for each $\alpha \in \omega_1$.

**Def:** $(\mathbb{K}, \equiv, r)$ is **projective** if

$\mathbb{K}$ and $\equiv$ are projective and $r$ has a projective presentation $2^\omega \to 2^\omega$.

**Theorem ([M.] (ZFC+PD))**

Let $(\mathbb{K}, \equiv, r)$ be scattered projective ranked equivalence relation

such that $\forall Z \in \mathbb{K}$, $r(Z) < \omega_1^Z$.

For every $X$ on a cone, (i.e. $\exists Y \forall X \geq_T Y$,) every equivalence class

with an $X$-hyperarithmetic member has an $X$-computable member.

**Lemma:** [Martin] (ZFC+PD) If $f: 2^\omega \to \omega_1$ is projective and $f(X) < \omega_1^X$,

then $f$ is constant on a cone.
The main theorem

Theorem ([M. 13] (ZFC + (0♯ exists) + ¬CH))

Let $E$ be a $\Sigma^1_1$-equivalence relation on $2^\omega$. TFAE
The main theorem

**Theorem ([M. 13] (ZFC + (0♯ exists) + ¬CH))**

Let $E$ be a $\Sigma^1_1$-equivalence relation on $2^\omega$. TFAE

1. $E$ satisfies **hyperarithmetic-is-recursive** on a cone non-trivially.
The main theorem

**Theorem ([M. 13] (ZFC + (0♯ exists) + ¬CH))**

Let $E$ be a $\Sigma^1_1$-equivalence relation on $2^\omega$. TFAE

1. $E$ satisfies *hyperarithmetic-is-recursive* on a cone non-trivially.
2. $E$ has $\aleph_1$ many equivalence classes.
The main theorem

Theorem ([M. 13] (ZFC + (0♯ exists) + ¬CH)

Let $E$ be a $\Sigma^1_1$-equivalence relation on $2^\omega$. TFAE

1. $E$ satisfies hyperarithmetic-is-recursive on a cone non-trivially.
2. $E$ has $\aleph_1$ many equivalence classes.

This theorem applies to all the examples mentioned before.

Examples:
- isomorphism on well-orderings;
- bi-embeddability on linear orderings;
- bi-embeddability on torsion abelian groups;
- isomorphism on models of a counterexample to Vaught’s conjecture;
- $X \equiv Y \iff \omega_1^X = \omega_1^Y$. 
The $\neg$CH assumption.
The $\neg$CH assumption.

**Theorem:** [Burgess 78] Let $E$ be $\Sigma^1_1$-equivalence relation on $2^\omega$.
Either $E$ has perfectly many classes, or it has at most $\aleph_1$ many classes.

Recall: $E$ has *perfectly many classes* if there is a perfect tree all whose paths are $E$-inequivalent.
The $\neg$CH assumption.

**Theorem:** [Burgess 78] Let $E$ be $\Sigma^1_1$-equivalence relation on $2^\omega$.
Either $E$ has perfectly many classes, or it has at most $\aleph_1$ many classes.

Recall: $E$ has *perfectly many classes* if there is a perfect tree all whose paths are $E$-inequivalent.

**Theorem ([M. 13] (ZFC + (0$^\#$ exists)))

Let $E$ be a $\Sigma^1_1$-equivalence relation on $2^\omega$. TFAE
The $\neg$CH assumption.

**Theorem:** [Burgess 78] Let $E$ be $\Sigma^1_1$-equivalence relation on $2^\omega$. Either $E$ has perfectly many classes, or it has at most $\aleph_1$ many classes.

Recall: $E$ has *perfectly many classes* if there is a perfect tree all whose paths are $E$-inequivalent.

**Theorem ([M. 13] (ZFC + (0$\# \text{ exists})])**

Let $E$ be a $\Sigma^1_1$-equivalence relation on $2^\omega$. TFAE

1. $E$ satisfies *hyperarithmetic-is-recursive on a cone*. 
The \( \neg \text{CH} \) assumption.

**Theorem:** [Burgess 78] Let \( E \) be \( \Sigma^1_1 \)-equivalence relation on \( 2^\omega \).

Either \( E \) has perfectly many classes, or it has at most \( \aleph_1 \) many classes.

Recall: \( E \) has *perfectly many classes* if there is a perfect tree all whose paths are \( E \)-inequivalent.

**Theorem ([M. 13] (ZFC + (0\(^\#\) exists))**

Let \( E \) be a \( \Sigma^1_1 \)-equivalence relation on \( 2^\omega \). TFAE

1. \( E \) satisfies *hyperarithmetic-is-recursive* on a cone.
2. \( E \) does not have perfectly many equivalence classes.
The sharp assumption

**Def:** \( S \subseteq 2^\omega \) is **cofinal** \((\text{in the Turing degrees})\) if \( \forall Y \exists X \geq_T Y \ (X \in S) \).
The sharp assumption

Def: $S \subseteq 2^\omega$ is **cofinal (in the Turing degrees)** if $\forall Y \exists X \geq_T Y \ (X \in S)$.

Thm: [Martin]$(0^\# \text{ exists})$. If $S$ is degree invariant and cofinal, it contains a cone.
The sharp assumption

**Def:** \( S \subseteq 2^\omega \) is *cofinal (in the Turing degrees)* if \( \forall Y \exists X \geq_T Y \ (X \in S) \).

**Thm:** [Martin](0^# exists). If \( S \) is degree invariant and cofinal, it contains a cone.

**Theorem ([M. 13] (ZF))**

Let \( \mathcal{E} \) be a \( \Sigma^1_1 \)-equivalence relation on \( 2^\omega \). TFAE
The sharp assumption

**Def:** \( S \subseteq 2^\omega \) is **cofinal** (in the Turing degrees) if \( \forall Y \exists X \geq_T Y \ (X \in S) \).

**Thm:** [Martin](0^{#} exists). If \( S \) is degree invariant and cofinal, it contains a cone.

**Theorem ([M. 13] (ZF))**

Let \( E \) be a \( \Sigma_1^1 \)-equivalence relation on \( 2^\omega \). TFAE

1. \( E \) satisfies hyperarithmetic-is-recursive relative to a cofinal set of oracles.
The sharp assumption

**Def:** $S \subseteq 2^\omega$ is **cofinal (in the Turing degrees)** if $\forall Y \, \exists X \geq_T Y \ (X \in S)$.

**Thm:** [Martin]($0^\#$ exists). If $S$ is degree invariant and cofinal, it contains a cone.

**Theorem ([M. 13] (ZF))**

Let $\mathcal{E}$ be a $\Sigma^1_1$-equivalence relation on $2^\omega$. TFAE

1. $E$ satisfies hyperarithmetic-is-recursive relative to a cofinal set of oracles.
2. $E$ does not have perfectly many equivalence classes.
The sharp assumption

**Def:** $S \subseteq 2^\omega$ is **cofinal (in the Turing degrees)** if $\forall Y \exists X \geq_T Y$ ($X \in S$).

**Thm:** [Martin](0# exists). If $S$ is degree invariant and cofinal, it contains a cone.

**Theorem ([M. 13] (ZF))**

Let $\mathcal{E}$ be a $\Sigma^1_1$-equivalence relation on $2^\omega$. TFAE

1. $E$ satisfies hyperarithmetic-is-recursive relative to a cofinal set of oracles.
2. $E$ does not have perfectly many equivalence classes.
The sharp assumption is necessary for “on a cone” version
The sharp assumption is necessary for “on a cone” version

Theorem ([M. 13])

The following are equivalent over ZF.

1. Every $\Sigma^1_1$-equivalence relation without perfectly many classes satisfies hyperarithmetic-is-recursive on a cone.

2. $0^\#$ exists.
The sharp assumption is necessary for “on a cone” version

**Theorem ([M. 13])**

The following are equivalent over ZF.

1. Every $\Sigma^1_1$-equivalence relation without perfectly many classes satisfies hyperarithmetic-is-recursive on a cone.

2. $0^#$ exists.

The key result in this proof is:

**Thm:**[Sami 99] Let $S = \{ Y \in 2^\omega : \exists Z (\forall W \leq_{hyp} Z (W \leq_T Y) \& \omega_1^Z = \omega_1^Y) \}$. If $S$ contains a cone, then $0^#$ exists.
The sharp assumption is necessary for “on a cone” version

Theorem ([M. 13])

The following are equivalent over ZF.

1. Every $\Sigma^1_1$-equivalence relation without perfectly many classes satisfies hyperarithmetic-is-recursive on a cone.

2. $0^#$ exists.

The key result in this proof is:

Thm:[Sami 99] Let $S = \{ Y \in 2^\omega : \exists Z (\forall W \leq_{hyp} Z (W \leq_T Y) \& \omega^Z_1 = \omega^Y_1) \}$. If $S$ contains a cone, then $0^#$ exists.

The proof of our result uses the following equivalence: $X \equiv Y$ iff

- $X$ and $Y$ are code structures $L^\alpha(A)$ and $L^\beta(B)$ with $\alpha = \beta$ and $\omega^A_1 = \omega^B_1$,
- or neither $X$ nor $Y$ are presentations of the form $L^\alpha(A)$ for $\alpha \in \omega_1$, $A \in 2^\omega$. 
The sharp assumption is necessary for “on a cone” version

Theorem ([M. 13])

The following are equivalent over ZF.

1. Every $\Sigma^1_1$-equivalence relation without perfectly many classes satisfies hyperarithmetic-is-recursive on a cone.

2. $0^#$ exists.

The key result in this proof is:

Thm:[Sami 99] Let $S = \{ Y \in 2^\omega : \exists Z \ (\forall W \leq_{hyp} Z \ (W \leq_T Y) \ & \ \omega^Z_1 = \omega^Y_1 \}$. If $S$ contains a cone, then $0^#$ exists.

The proof of our result uses the following equivalence: $X \equiv Y$ iff

- $X$ and $Y$ are code structures $L_\alpha(A)$ and $L_\beta(B)$ with $\alpha = \beta$ and $\omega^A_1 = \omega^B_1$,
- or neither $X$ nor $Y$ are presentations of the form $L_\alpha(A)$ for $\alpha \in \omega_1$, $A \in 2^\omega$.

It then uses Barwise compactness to put us in the hypothesis of Sami’s theorem:
Part IV

1. $\Pi^1_1$-ness and ordinals
2. Hyperarithmeticy
3. When hyperarithmetic is recursive
4. Overspill
5. A structure equivalent to its own jump
Theorem

There is a computable ill-founded linear ordering with a jump hierarchy.

Proof:
Theorem

There is a computable ill-founded liner ordering with a jump hierarchy.

Proof: Let \( J = \{ e \in \mathbb{N} : \exists H \text{ jump hierarchy on } L_e \} \) where \( L_e \) is eth comp. LO.
Ill-founded hierarchies

Theorem

There is a computable ill-founded linear ordering with a jump hierarchy.

Proof: Let \( J = \{ e \in \mathbb{N} : \exists H \text{ jump hierarchy on } \mathcal{L}_e \} \) where \( \mathcal{L}_e \) is eth comp. LO. \( J \) is \( \Sigma^1_1 \)
Ill-founded hierarchies

Theorem

There is a computable ill-founded linear ordering with a jump hierarchy.

Proof: Let $J = \{ e \in \mathbb{N} : \exists H \text{ jump hierarchy on } \mathcal{L}_e \}$ where $\mathcal{L}_e$ is the $e$th computable LO. $J$ is $\Sigma^1_1$ and $J \subseteq O$.

---

Antonio Montalbán (U.C. Berkeley)  Higher Recursion and computable structures  May 2019  38 / 50
Ill-founded hierarchies

**Theorem**

*There is a computable ill-founded linear ordering with a jump hierarchy.*

**Proof:** Let \( J = \{ e \in \mathbb{N} : \exists H \text{ jump hierarchy on } L_e \} \) where \( L_e \) is \( e \)-th comp. LO. \( J \) is \( \Sigma^1_1 \) and \( J \subseteq O \). But \( O \) not \( \Sigma^1_1 \).
Theorem

There is a computable ill-founded linear ordering with a jump hierarchy.

Proof: Let $J = \{ e \in \mathbb{N} : \exists H$ jump hierarchy on $L_e \}$ where $L_e$ is eth comp. LO. $J$ is $\Sigma^1_1$ and $J \subseteq O$. But $O$ not $\Sigma^1_1$, so $J \subsetneq O$. Any $L_e$ for $e \in J \setminus O$ is as wanted.
Ill-founded hierarchies

Theorem

There is a computable ill-founded linear ordering with a jump hierarchy.

Proof: Let $J = \{ e \in \mathbb{N} : \exists H \text{ jump hierarchy on } \mathcal{L}_e \}$ where $\mathcal{L}_e$ is the $e$th computable LO. $J$ is $\Sigma^1_1$ and $J \subseteq \mathcal{O}$. But $\mathcal{O}$ not $\Sigma^1_1$, so $J \subsetneq \mathcal{O}$. Any $\mathcal{L}_e$ for $e \in J \setminus \mathcal{O}$ is as wanted.

Theorem: [Spector 59][Gandy 60] For $A \subseteq \mathbb{N}$, TFAE:

- $A$ is definable by formula of the form: $\forall X \ (\ldots \text{arithmetic} \ldots)$
- $A$ is definable by formula of the form: $\exists \text{hyp } X \ (\ldots \text{arithmetic} \ldots)$

Proof:
Ill-founded hierarchies

Theorem

There is a computable ill-founded linear ordering with a jump hierarchy.

Proof: Let $J = \{ e \in \mathbb{N} : \exists H \text{ jump hierarchy on } L_e \}$ where $L_e$ is eth comp. LO. $J$ is $\Sigma^1_1$ and $J \subseteq O$. But $O$ not $\Sigma^1_1$, so $J \subset O$. Any $L_e$ for $e \in J \setminus O$ is as wanted.

Theorem: [Spector 59][Gandy 60] For $A \subseteq \mathbb{N}$, TFAE:

- $A$ is definable by formula of the form: $\forall X \ (\ldots \text{arithmetic} \ldots)$
- $A$ is definable by formula of the form: $\exists \ \text{hyp } X \ (\ldots \text{arithmetic} \ldots)$

Proof:

($\Rightarrow$) Use that $L_e$ is well-founded $\iff$ there is a hyp jump hierarchy on it.
Ill-founded hierarchies

**Theorem**

There is a computable ill-founded linear ordering with a jump hierarchy.

**Proof:** Let $J = \{ e \in \mathbb{N} : \exists H \text{ jump hierarchy on } L_e \}$ where $L_e$ is the $e$th computable LO.

$J$ is $\Sigma^1_1$ and $J \subseteq \mathcal{O}$. But $\mathcal{O}$ is not $\Sigma^1_1$, so $J \not\subseteq \mathcal{O}$. Any $L_e$ for $e \in J \setminus \mathcal{O}$ is as wanted.

**Theorem:** [Spector 59][Gandy 60] For $A \subseteq \mathbb{N}$, TFAE:

- $A$ is definable by formula of the form: $\forall X$ (...arithmetic...
- $A$ is definable by formula of the form: $\exists \text{ hyp } X$ (...arithmetic..)

**Proof:**

$(\Rightarrow)$ Use that $L_e$ is well-founded $\iff$ there is a hyp jump hierarchy on it.

$(\Leftarrow)$ Use that the set of indices for hyp reals is $\Pi^1_1$. 
Theorem: There is an ill-founded computable linear ordering with no hyperarithmetic descending sequences.

Proof:
Harrison’s linear ordering

**Theorem:** There is an ill-founded computable linear ordering with no hyperarithmetic descending sequences.

**Proof:** Let \( J = \{ e \in \mathbb{N} : \mathcal{L}_e \text{ has no hyp. des. seq.} \} \)
Harrison’s linear ordering

**Theorem:** There is an ill-founded computable linear ordering with no hyperarithmetic descending sequences.

**Proof:** Let $J = \{ e \in \mathbb{N} : \mathcal{L}_e \text{ has no hyp. des. seq.} \}$

$J$ is $\Sigma^1_1$
Harrison’s linear ordering

**Theorem:** There is an ill-founded computable linear ordering with no hyperarithmetic descending sequences.

**Proof:** Let \( J = \{ e \in \mathbb{N} : L_e \text{ has no hyp. des. seq.} \} \)
\( J \) is \( \Sigma^1_1 \) and \( J \subseteq \mathcal{O} \).
Harrison’s linear ordering

**Theorem:** There is an ill-founded computable linear ordering with no hyperarithmetic descending sequences.

**Proof:** Let $J = \{ e \in \mathbb{N} : \mathcal{L}_e \text{ has no hyp. des. seq.} \}$

$J$ is $\Sigma^1_1$ and $J \subset \mathcal{O}$. Therefore $J \not\subset \mathcal{O}$. 
Harrison’s linear ordering

**Theorem:** There is an ill-founded computable linear ordering with no hyperarithmetic descending sequences.

**Proof:** Let $J = \{ e \in \mathbb{N} : \mathcal{L}_e \text{ has no hyp. des. seq.} \}$

$J$ is $\Sigma_1^1$ and $J \subseteq \mathcal{O}$. Therefore $J \not\subseteq \mathcal{O}$.

**Theorem:** Every such linear ordering is isomorphic to

$$\omega_1^{CK} + \omega_1^{CK} \cdot \mathbb{Q} + \beta \text{ for some } \beta < \omega_1^{CK}.$$
Harrison’s linear ordering

**Theorem:** There is an ill-founded computable linear ordering with no hyperarithmetic descending sequences.

**Proof:** Let $J = \{ e \in \mathbb{N} : L_e \text{ has no hyp. des. seq.}\}$

$J$ is $\Sigma^1_1$ and $J \subseteq \mathcal{O}$. Therefore $J \not\subseteq \mathcal{O}$.

**Theorem:** Every such linear ordering is isomorphic to

$$\omega_1^{CK} + \omega_1^{CK} \cdot \mathbb{Q} + \beta$$

for some $\beta < \omega_1^{CK}$.

**Definition:** $\omega_1^{CK} + \omega_1^{CK} \cdot \mathbb{Q}$ is called the *Harrison linear ordering*. 
Harrison’s linear ordering

**Theorem:** There is an ill-founded computable linear ordering with no hyperarithmetic descending sequences.

**Proof:** Let $J = \{ e \in \mathbb{N} : \mathcal{L}_e \text{ has no hyp. des. seq.} \}$

$J$ is $\Sigma^1_1$ and $J \subseteq \mathcal{O}$. Therefore $J \subsetneq \mathcal{O}$.

**Theorem:** Every such linear ordering is isomorphic to

$$\omega_1^{CK} + \omega_1^{CK} \cdot \mathbb{Q} + \beta$$

for some $\beta < \omega_1^{CK}$.

**Definition:** $\omega_1^{CK} + \omega_1^{CK} \cdot \mathbb{Q}$ is called the **Harrison linear ordering**.

It has a computable presentation, but the $\omega_1^{CK}$ cut is not even hyp.
Barwise compactness

\( \mathcal{L}_{c,\omega} \) does not satisfy compactness.
Barwise compactness

$\mathcal{L}_{c,\omega}$ does not satisfy compactness. Consider the vocabulary $\{0, 1, 2, 3\ldots\}$. 
Barwise compactness

$L_{c,\omega}$ does not satisfy compactness. Consider the vocabulary $\{0, 1, 2, 3\ldots\}$. The list $\{\forall n \in \mathbb{N} \ c = n; \ c \neq 0, c \neq 1, c \neq 2, c \neq 3, \ldots\}$ is finitely satisfiable but not satisfiable.
Barwise compactness

\( \mathcal{L}_{c,\omega} \) does not satisfy compactness. Consider the vocabulary \( \{0, 1, 2, 3\ldots\} \).
The list \( \{\bigwedge_{n \in \mathbb{N}} c = n; \ c \neq 0, \ c \neq 1, \ c \neq 2, \ c \neq 3, \ldots\} \)
is finitely satisfiable but not satisfiable.

Let \( \mathcal{H} \) be the Harrison linear ordering and \( \omega_{1}^{CK} \) be its well-founded part.

**Theorem:** [Barwise] Let \( \{\varphi_{f(e)} : e \in \omega_{1}^{CK}\} \) be a computable list of
\( \mathcal{L}_{c,\omega} \)-sentences such that, for every \( \alpha < \omega_{1}^{CK} \), \( \{\varphi_{f(e)} : e < \alpha\} \) is satisfiable.
Barwise compactness

\( \mathcal{L}_{c,\omega} \) does not satisfy compactness. Consider the vocabulary \{0, 1, 2, 3...\}. The list \( \{ \exists n \in \mathbb{N} \ c = n; \ c \neq 0, \ c \neq 1, \ c \neq 2, \ c \neq 3, \ldots \} \) is finitely satisfiable but not satisfiable.

Let \( \mathcal{H} \) be the Harrison linear ordering and \( \omega_{1}^{CK} \) be its well-founded part.

**Theorem:** [Barwise] Let \( \{ \varphi_{f(e)} : e \in \omega_{1}^{CK} \} \) be a computable list of \( \mathcal{L}_{c,\omega} \)-sentences such that, for every \( \alpha < \omega_{1}^{CK} \), \( \{ \varphi_{f(e)} : e < \alpha \} \) is satisfiable. Then \( \{ \varphi_{f(e)} : e \in \omega_{1}^{CK} \} \) is satisfiable.
**Barwise compactness**

\( \mathcal{L}_{c,\omega} \) does not satisfy compactness. Consider the vocabulary \( \{0, 1, 2, 3\ldots\} \). The list \( \{ \bigwedge_{n \in \mathbb{N}} c = n; \ c \neq 0, \ c \neq 1, \ c \neq 2, \ c \neq 3, \ldots \} \) is finitely satisfiable but not satisfiable.

Let \( \mathcal{H} \) be the Harrison linear ordering and \( \omega_{1}^{CK} \) be its well-founded part.

**Theorem:** [Barwise] Let \( \{ \varphi_{f(e)} : e \in \omega_{1}^{CK} \} \) be a computable list of \( \mathcal{L}_{c,\omega} \)-sentences such that, for every \( \alpha < \omega_{1}^{CK} \), \( \{ \varphi_{f(e)} : e < \alpha \} \) is satisfiable. Then \( \{ \varphi_{f(e)} : e \in \omega_{1}^{CK} \} \) is satisfiable.

**Proof:** Let \( J = \{ e \in \mathcal{H} : \exists A (A \models \bigwedge_{e < \alpha} \varphi_{f(e)}) \} \).
Barwise compactness

\( \mathcal{L}_{c,\omega} \) does not satisfy compactness. Consider the vocabulary \( \{0, 1, 2, 3 \ldots \} \).

The list \( \{ \forall n \in \mathbb{N} \ c = n; \ c \neq 0, \ c \neq 1, \ c \neq 2, \ c \neq 3, \ldots \} \) is finitely satisfiable but not satisfiable.

Let \( \mathcal{H} \) be the Harrison linear ordering and \( \omega_1^{CK} \) be its well-founded part.

**Theorem:** [Barwise] Let \( \{ \varphi_f(e) : e \in \omega_1^{CK} \} \) be a computable list of \( \mathcal{L}_{c,\omega} \)-sentences such that, for every \( \alpha < \omega_1^{CK} \), \( \{ \varphi_f(e) : e < \alpha \} \) is satisfiable. Then \( \{ \varphi_f(e) : e \in \omega_1^{CK} \} \) is satisfiable.

**Proof:** Let \( J = \{ e \in \mathcal{H} : \exists A \ (A \models \bigwedge_{e < \alpha} \varphi_f(e)) \} \). \( J \) is \( \Sigma^1_1 \).
**Barwise compactness**

\(\mathcal{L}_{c,\omega}\) does not satisfy compactness. Consider the vocabulary \(\{0, 1, 2, 3\ldots\}\).

The list \(\bigwedge_{n \in \mathbb{N}} c = n; \quad c \neq 0, \ c \neq 1, \ c \neq 2, \ c \neq 3, \ldots\) is finitely satisfiable but not satisfiable.

Let \(\mathcal{H}\) be the Harrison linear ordering and \(\omega^\text{CK}_1\) be its well-founded part.

**Theorem:** [Barwise] Let \(\{\varphi_f(e) : e \in \omega^\text{CK}_1\}\) be a computable list of \(\mathcal{L}_{c,\omega}\)-sentences such that, for every \(\alpha < \omega^\text{CK}_1\), \(\{\varphi_f(e) : e < \alpha\}\) is satisfiable. Then \(\{\varphi_f(e) : e \in \omega^\text{CK}_1\}\) is satisfiable.

**Proof:** Let \(J = \{e \in \mathcal{H} : \exists A (A \models \bigwedge_{e < \alpha} \varphi_f(e))\}\). \(J\) is \(\Sigma^1_1\) and \(\mathcal{H} \upharpoonright \omega^\text{CK}_1 \subseteq J\).
Barwise compactness

\( \mathcal{L}_{c,\omega} \) does not satisfy compactness. Consider the vocabulary \( \{0, 1, 2, 3, \ldots\} \).

The list \( \{ \bigwedge_{n \in \mathbb{N}} c = n; \ c \neq 0, \ c \neq 1, \ c \neq 2, \ c \neq 3, \ldots \} \)

is finitely satisfiable but not satisfiable.

Let \( \mathcal{H} \) be the Harrison linear ordering and \( \omega_1^{CK} \) be its well-founded part.

**Theorem:** [Barwise] Let \( \{ \varphi_f(e) : e \in \omega_1^{CK} \} \) be a computable list of \( \mathcal{L}_{c,\omega} \)-sentences such that, for every \( \alpha < \omega_1^{CK} \), \( \{ \varphi_f(e) : e < \alpha \} \) is satisfiable. Then \( \{ \varphi_f(e) : e \in \omega_1^{CK} \} \) is satisfiable.

**Proof:** Let \( J = \{ e \in \mathcal{H} : \exists A ( A \models \bigwedge_{e < \alpha} \varphi_f(e)) \} \). \( J \) is \( \Sigma_1 \) and \( \mathcal{H} \upharpoonright \omega_1^{CK} \subseteq J \).

Furthermore, we can get \( A \) to be low for \( \omega_1 \). I.e. \( \omega_1^A = \omega_1^{CK} \).
Barwise compactness

\( L_{c,\omega} \) does not satisfy compactness. Consider the vocabulary \( \{0, 1, 2, 3\ldots\} \). The list \( \{\forall n \in \mathbb{N} \; c = n; \; c \neq 0, \; c \neq 1, \; c \neq 2, \; c \neq 3, \ldots\} \) is finitely satisfiable but not satisfiable.

Let \( \mathcal{H} \) be the Harrison linear ordering and \( \omega^{CK}_{1} \) be its well-founded part.

**Theorem:** [Barwise] Let \( \{\varphi_{f(e)} : e \in \omega^{CK}_{1}\} \) be a computable list of \( L_{c,\omega} \)-sentences such that, for every \( \alpha < \omega^{CK}_{1} \), \( \{\varphi_{f(e)} : e < \alpha\} \) is satisfiable. Then \( \{\varphi_{f(e)} : e \in \omega^{CK}_{1}\} \) is satisfiable.

**Proof:** Let \( J = \{e \in \mathcal{H} : \exists A \; (A \models \bigwedge_{e < \alpha} \varphi_{f(e)})\} \). \( J \) is \( \Sigma_{1} \) and \( \mathcal{H} \rceil \omega^{CK}_{1} \subseteq J \).

Furthermore, we can get \( A \) to be low for \( \omega_{1} \). I.e. \( \omega_{1}^{A} = \omega^{CK}_{1} \).

**Corollary:** [Kreisel] Let \( S \) be a \( \Pi_{1}^{1} \) set of \( L_{c,\omega} \)-formulas. If every hyperarithmetic subset of \( S \) is satisfiable, then so is \( S \).
A different formulation for overspill arguments

**Theorem:** There is an $\omega$-model $\mathcal{M}$ of ZFC whose well-ordered part is $\omega_1^{CK}$.
A different formulation for overspill arguments

**Theorem:** There is an $\omega$-model $\mathcal{M}$ of ZFC whose well-ordered part is $\omega_1^{CK}$.

That is, $\omega^\mathcal{M} \cong \omega$, and $ON^\mathcal{M} \cong \omega_1^{CK} + \omega_1^{CK} \cdot \mathbb{Q}$.
A different formulation for overspill arguments

**Theorem:** There is an $\omega$-model $\mathcal{M}$ of ZFC whose well-ordered part is $\omega_1^{CK}$.

That is, $\omega^\mathcal{M} \cong \omega$, and $\text{ON}^\mathcal{M} \cong \omega_1^{CK} + \omega_1^{CK} \cdot \mathbb{Q} \cong (\omega_1^{CK})^\mathcal{M}$. 
A different formulation for overspill arguments

**Theorem:** There is an $\omega$-model $\mathcal{M}$ of ZFC whose well-ordered part is $\omega_1^{CK}$.

That is, $\omega^\mathcal{M} \cong \omega$, and $ON^\mathcal{M} \cong \omega_1^{CK} + \omega_1^{CK} \cdot \mathbb{Q} \cong (\omega_1^{CK})^\mathcal{M}$.

**Proof:** The set of countable models of ZFC & $\forall x \in \omega \ (\forall n \in \mathbb{N} \ x = n)$ is $\Sigma_1^1$. 
A different formulation for overspill arguments

**Theorem:** There is an $\omega$-model $\mathcal{M}$ of ZFC whose well-ordered part is $\omega_1^{CK}$.

That is, $\omega^\mathcal{M} \cong \omega$, and $\text{ON}^\mathcal{M} \cong \omega_1^{CK} + \omega_1^{CK} \cdot \mathbb{Q} \cong (\omega_1^{CK})^\mathcal{M}$.

**Proof:** The set of countable models of ZFC & $\forall x \in \omega \ (\forall n \in \mathbb{N} \ x = n)$ is $\Sigma^1_1$.
So there is such a model with $\omega_1^\mathcal{M} = \omega_1^{CK}$.
Part V

1. $\Pi_1^1$-ness and ordinals
2. Hyperarithmeticity
3. When hyperarithmetical is recursive
4. Overspill
5. A structure equivalent to its own jump
The jump of a structure

Given a structure $A$, we define $A'$ by adding relations $R_{i,j}$ for $i,j \in \omega$, $(a_1,...,a_j) \in R_{i,j} \iff A|_{\Sigma^{e,j}(a_1,...,a_j)} = \varphi_{\Sigma_{e,j}}$.

Definition: We call $A'$ the jump of $A$.

Lemma: (1st Jump inversion theorem) $(\forall A)(\exists B) B' \equiv A \oplus 0'$

Lemma: (2nd Jump inversion theorem) $DgSp(A') = \{ X' : X \in DgSp(A) \}$

Examples:
If $L$ a Linear ordering, then $L' \equiv (L, succ, 0')$.
If $B$ a Boolean algebra, then $B' \equiv (B, atom, 0')$. 
The jump of a structure

Given a structure $\mathcal{A}$, we define $\mathcal{A}'$ by adding relations $R_{i,j}$ for $i, j \in \omega$,

$$(a_1, \ldots, a_j) \in R_{i,j} \iff \mathcal{A} \models \varphi_{e,j}(a_1, \ldots, a_j),$$

where $\varphi_{e,j}$ be the $e$th $\Sigma_1$ formula on the variables $x_1, \ldots, x_j$. 

Definition: We call $\mathcal{A}'$ the jump of $\mathcal{A}$.

Lemma: (1st Jump inversion theorem) $(\forall \mathcal{A}) (\exists \mathcal{B}) \mathcal{B}' \equiv \mathcal{A} \oplus 0'$

Lemma: (2nd Jump inversion theorem) $DgSp(\mathcal{A}') = \{ X' : X \in DgSp(\mathcal{A}) \}$

Examples:
- If $L$ a Linear ordering, then $L' \equiv (L, \text{succ}, 0')$.
- If $B$ a Boolean algebra, then $B' \equiv (B, \text{atom}, 0')$. 

Antonio Montalbán (U.C. Berkeley)  Higher Recursion and computable structures  May 2019  43 / 50
The jump of a structure

Given a structure $\mathcal{A}$, we define $\mathcal{A}'$ by adding relations $R_{i,j}$ for $i, j \in \omega$,

$$(a_1, ..., a_j) \in R_{i,j} \iff \mathcal{A} \models \varphi_{e,j}(a_1, ..., a_j),$$

where $\varphi_{e,j}$ be the $e$th $\Sigma_1^c$ formula on the variables $x_1, ..., x_j$.

**Definition:** We call $\mathcal{A}'$ the *jump of $\mathcal{A}$*. 
The jump of a structure

Given a structure $\mathcal{A}$, we define $\mathcal{A}'$ by adding relations $R_{i,j}$ for $i, j \in \omega$,

$$(a_1, ..., a_j) \in R_{i,j} \iff \mathcal{A} \models \varphi^\Sigma_{e,j}(a_1, ..., a_j),$$

where $\varphi^\Sigma_{e,j}$ be the $e$th $\Sigma_1^c$ formula on the variables $x_1, ..., x_j$.

**Definition:** We call $\mathcal{A}'$ the *jump of $\mathcal{A}$.*

**Lemma:** (1st Jump inversion theorem) $(\forall \mathcal{A})(\exists \mathcal{B}) \mathcal{B}' \equiv \mathcal{A} \oplus 0'$
The jump of a structure

Given a structure $\mathcal{A}$, we define $\mathcal{A}'$ by adding relations $R_{i,j}$ for $i, j \in \omega$,

$$(a_1, \ldots, a_j) \in R_{i,j} \iff \mathcal{A} \models \varphi_{e,j}(a_1, \ldots, a_j),$$

where $\varphi_{e,j}$ be the $e$th $\Sigma^c_1$ formula on the variables $x_1, \ldots, x_j$.

**Definition:** We call $\mathcal{A}'$ the *jump of $\mathcal{A}$*.

**Lemma:** (1st Jump inversion theorem) $(\forall \mathcal{A})(\exists \mathcal{B}) \quad \mathcal{B}' \equiv \mathcal{A} \oplus 0'$

**Lemma:** (2nd Jump inversion theorem) $DgSp(\mathcal{A}') = \{X' : X \in DgSp(\mathcal{A})\}$. 
The jump of a structure

Given a structure $\mathcal{A}$, we define $\mathcal{A}'$ by adding relations $R_{i,j}$ for $i, j \in \omega$,

$$(a_1, ..., a_j) \in R_{i,j} \iff \mathcal{A} \models \varphi_{e,j}^\Sigma(a_1, ..., a_j),$$

where $\varphi_{e,j}^\Sigma$ be the $e$th $\Sigma_1^c$ formula on the variables $x_1, ..., x_j$.

**Definition:** We call $\mathcal{A}'$ the *jump of $\mathcal{A}$*.

**Lemma:** (1st Jump inversion theorem) $(\forall \mathcal{A})(\exists \mathcal{B}) \mathcal{B}' \equiv \mathcal{A} \oplus 0'$

**Lemma:** (2nd Jump inversion theorem) $DgSp(\mathcal{A}') = \{X' : X \in DgSp(\mathcal{A})\}$.

**Examples:**
- If $\mathcal{L}$ a Linear ordering, then $\mathcal{L}' \equiv (\mathcal{L}, \text{succ}, 0')$.
- If $\mathcal{B}$ a Boolean algebra, then $\mathcal{B}' \equiv (\mathcal{B}, \text{atom}, 0')$. 
Does the jump jump?

Question: Is there a structure equivalent to its own jump?

Answer: It depends of what you mean by “equivalent.”

Definition: A is Muchnik reducible to B if every $X \in 2^\omega$ that computes a copy of B also computes a copy of A.

Definition: A is Medvedev reducible to B if there is a computable operator that given copy of B outputs a copy of A.

Theorem: No structure is Medvedev equivalent to its own jump.

Theorem: There is a structure Muchnik equivalent to its own jump.
Does the jump jump?

**Question:** Is there a structure equivalent to its own jump?
Does the jump jump?

**Question:** Is there a structure equivalent to its own jump?

**Answer:**

It depends on what you mean by “equivalent.”

**Definition:**

A structure \( A \) is Muchnik reducible to a structure \( B \) if every computable operator that computes a copy of \( B \) also computes a copy of \( A \).

**Definition:**

A structure \( A \) is Medvedev reducible to a structure \( B \) if there is a computable operator that given a copy of \( B \) outputs a copy of \( A \).

**Theorem:**

No structure is Medvedev equivalent to its own jump.

**Theorem:**

There is a structure Muchnik equivalent to its own jump.
Does the jump jump?

**Question:** Is there a structure equivalent to its own jump?

**Answer:** It depends
Does the jump jump?

**Question:** Is there a structure equivalent to its own jump?

**Answer:** It depends of what you mean by “equivalent.”
Does the jump jump?

**Question:** Is there a structure equivalent to its own jump?

**Answer:** It depends of what you mean by “equivalent.”

**Definition:** $\mathcal{A}$ is Muchnik reducible to $\mathcal{B}$ if

\[
every \ X \in 2^\omega \ that \ computes \ a \ copy \ of \ \mathcal{B} \ also \ computes \ a \ copy \ of \ \mathcal{A}.
\]
Does the jump jump?

Question: Is there a structure equivalent to its own jump?

Answer: It depends of what you mean by “equivalent.”

Definition: \( A \) is Muchnik reducible to \( B \) if every \( X \in \mathbb{2}^\omega \) that computes a copy of \( B \) also computes a copy of \( A \).

Definition: \( A \) is Medvedev reducible to \( B \) if there is a computable operator that given copy of \( B \) outputs a copy of \( A \).
Does the jump jump?

**Question:** Is there a structure equivalent to its own jump?

**Answer:** It depends of what you mean by “equivalent.”

**Definition:** $\mathcal{A}$ is **Muchnik** reducible to $\mathcal{B}$ if every $X \in 2^\omega$ that computes a copy of $\mathcal{B}$ also computes a copy of $\mathcal{A}$.

**Definition:** $\mathcal{A}$ is **Medvedev** reducible to $\mathcal{B}$ if there is a computable operator that given copy of $\mathcal{B}$ outputs a copy of $\mathcal{A}$.

**Theorem:** No structure is Medvedev equivalent to its own jump.
Does the jump jump?

**Question:** Is there a structure equivalent to its own jump?

**Answer:** It depends of what you mean by “equivalent.”

**Definition:** \( \mathcal{A} \) is **Muchnik** reducible to \( \mathcal{B} \) if every \( X \in 2^\omega \) that computes a copy of \( \mathcal{B} \) also computes a copy of \( \mathcal{A} \).

**Definition:** \( \mathcal{A} \) is **Medvedev** reducible to \( \mathcal{B} \) if there is a computable operator that given copy of \( \mathcal{B} \) outputs a copy of \( \mathcal{A} \).

**Theorem:** No structure is Medvedev equivalent to its own jump.

**Theorem:** There is a structure Muchnik equivalent to its own jump.
There is a structure Muchnik equivalent to its own jump.
There is a structure Muchnik equivalent to its own jump

**Proof 1:** [Montalbán 2011]
Uses the existence of $0^\#$ and builds a model of $ZFC + V = L$. 
There is a structure Muchnik equivalent to its own jump

**Proof 1:** [Montalbán 2011]
Uses the existence of $0^\#$ and builds a model of $ZFC + V = L$.

**Proof 2:** [Puzarenko 2011]
Uses $\omega_{1}^{CK}$ iterates of power set and builds a model of $KP + V = L$. 
There is a structure Muchnik equivalent to its own jump

**Proof 1:** [Montalbán 2011]
Uses the existence of $0^\#$ and builds a model of $\text{ZFC} + V = L$.

**Proof 2:** [Puzarenko 2011]
Uses $\omega_1^{CK}$ iterates of power set and builds a model of $\text{KP} + V = L$.

**Proof 3:** [Montalbán, Schweber, Turetski 2018]
Uses $\omega_1^{CK}$ iterates of power set and builds a jump hierarchy.
There is a structure Muchnik equivalent to its own jump

Proof 1: [Montalbán 2011]
Uses the existence of $0^\#$ and builds a model of $ZFC + V = L$.

Proof 2: [Puzarenko 2011]
Uses $\omega_1^{CK}$ iterates of power set and builds a model of $KP + V = L$.

Proof 3: [Montalbán, Schweber, Turetski 2018]
Uses $\omega_1^{CK}$ iterates of power set and builds a jump hierarchy.

Theorem ([Montalbán 2011])

*Infinitely many iterates of the power set are needed to prove this theorem.*
Jump-hierarchy structures

Def: A jump-hierarchy structure is a structure \( \mathcal{J} = (A; \leq, R_{i,j} : i, j \in \omega) \), where \( A = (A; \leq) \) is a linear ordering and

\[
\mathcal{A} \models R_{i,j}(a, b_1, ..., b_j) \iff b_1, ..., b_j < a \quad \& \quad \mathcal{L} \upharpoonright a \models \varphi_{i,j}(b_1, ..., b_j).
\]
Jump-hierarchy structures

Def: A jump-hierarchy structure is a structure $\mathcal{J} = (\mathcal{A}; \leq, R_{i,j} : i, j \in \omega)$, where $\mathcal{A} = (\mathcal{A}; \leq)$ is a linear ordering and

$$\mathcal{A} \models R_{i,j}(a, b_1, \ldots, b_j) \iff b_1, \ldots, b_j < a \quad \& \quad \mathcal{L} \upharpoonright a \models \varphi_{i,j}^\Sigma(b_1, \ldots, b_j).$$

Obs: If $a + 1$ is the successor of $a$ in $\mathcal{A}$,

$$\mathcal{J} \upharpoonright a + 1 \equiv_{\text{Muchnik}} (\mathcal{J} \upharpoonright a)'.$$
Jump-hierarchy structures

**Def:** A **jump-hierarchy structure** is a structure $\mathcal{J} = (A; \leq, R_{i,j} : i, j \in \omega)$, where $\mathcal{A} = (A; \leq)$ is a linear ordering and

$$\mathcal{A} \models R_{i,j}(a, b_1, \ldots, b_j) \iff b_1, \ldots, b_j < a \quad \& \quad \mathcal{L} \upharpoonright a \models \varphi_{i,j}^{\Sigma}(b_1, \ldots, b_j).$$

**Obs:** If $a + 1$ is the successor of $a$ in $\mathcal{A}$,

$$\mathcal{J} \upharpoonright a + 1 \equiv_{\text{Muchnik}} (\mathcal{J} \upharpoonright a)'.$$

**Obs:** If $\mathcal{J} \upharpoonright a \cong \mathcal{J} \upharpoonright b$ for some $a < b \in L$,

$$\mathcal{J} \upharpoonright a \equiv_{\text{Muchnik}} (\mathcal{J} \upharpoonright a)'.$
Jump-hierarchy structures

**Def:** A jump-hierarchy structure is a structure \( \mathcal{J} = (A; \leq, R_{i,j} : i, j \in \omega) \), where \( A = (A; \leq) \) is a linear ordering and

\[
A \models R_{i,j}(a, b_1, \ldots, b_j) \iff b_1, \ldots, b_j < a \quad \& \quad L \upharpoonright a \models \varphi^\Sigma_{i,j}(b_1, \ldots, b_j).
\]

**Obs:** If \( a + 1 \) is the successor of \( a \) in \( A \), \( \mathcal{J} \upharpoonright a + 1 \equiv_{\text{Muchnik}} (\mathcal{J} \upharpoonright a)' \).

**Obs:** If \( \mathcal{J} \upharpoonright a \cong \mathcal{J} \upharpoonright b \) for some \( a < b \in L \), \( \mathcal{J} \upharpoonright a \equiv_{\text{Muchnik}} (\mathcal{J} \upharpoonright a)' \).

**Obs:** If \( A \) is well-ordered, there is a jump-hierarchy structure over \( A \).
The Hanf number of computably infinitary logic

**Lemma:** [Morley][Barwise] If a $\mathcal{L}_{c,\omega}$-sentence $\varphi$ has a model of size $\beth_{\omega_1^{CK}}$, it has a countable model with a non-trivial isomorphism.

**Proof:**

1. Consider structures of the form $(M, L, E, e, f)$ where $M\models \varphi$.
2. $L$ is a linear ordering with a first element 0.
3. $E \subseteq L \times M < \omega \times M < \omega$.
4. For each $\alpha \in L$, $E(\alpha, \cdot, \cdot)$ is an equivalence relation on $M < \omega$.
5. $E(0, \bar{a}, \bar{b})$ if $\bar{a}$ and $\bar{b}$ satisfy the same atomic formulas.
6. $E(\alpha, \bar{a}, \bar{b})$ if $\forall \beta < \alpha \forall d \in M \exists c \in M E(\beta, \bar{a}c, \bar{b}d)$.
7. $e \neq f \in M$ and, for all $\alpha \in L$, $E(\alpha, e, f)$.
8. and, for all $\alpha < \omega_{1^{CK}}$ there is an $a \in L$ such that $L \upharpoonright a \sim = \alpha$.
The Hanf number of computably infinitary logic

**Lemma:** [Morley][Barwise] If a $\mathcal{L}_{c,\omega}$-sentence $\varphi$ has a model of size $\beth_{\omega_1}^{ck}$, it has a countable model with a non-trivial isomorphism.

**Proof:** Consider structures of the form $(M, \mathcal{L}, E, e, f)$ where
The Hanf number of computably infinitary logic

**Lemma:** [Morley][Barwise] If a $L_{c,\omega}$-sentence $\varphi$ has a model of size $\beth_{\omega_1}^{CK}$, it has a countable model with a non-trivial isomorphism.

**Proof:** Consider structures of the form $(M, \mathcal{L}, E, e, f)$ where

1. $M \models \varphi$
The Hanf number of computably infinitary logic

**Lemma:** [Morley][Barwise] If a $\mathcal{L}_{c,\omega}$-sentence $\varphi$ has a model of size $\beth_{\omega_1}^{ck}$, it has a countable model with a non-trivial isomorphism.

**Proof:** Consider structures of the form $(\mathcal{M}, \mathcal{L}, E, e, f)$ where

1. $\mathcal{M} \models \varphi$
2. $\mathcal{L}$ is a linear ordering with a first element 0.
The Hanf number of computably infinitary logic

**Lemma:** [Morley][Barwise] If a $\mathcal{L}_{c,\omega}$-sentence $\varphi$ has a model of size $\beth_{\omega_1}^{CK}$, it has a countable model with a non-trivial isomorphism.

**Proof:** Consider structures of the form $(\mathcal{M}, \mathcal{L}, E, e, f)$ where

1. $\mathcal{M} \models \varphi$
2. $\mathcal{L}$ is a linear ordering with a first element 0.
3. $E \subseteq \mathcal{L} \times M^{\omega} \times M^{\omega}$. 
The Hanf number of computably infinitary logic

Lemma: [Morley][Barwise] If a $\mathcal{L}_{c,\omega}$-sentence $\varphi$ has a model of size $\beth_{\omega_1}^{CK}$, it has a countable model with a non-trivial isomorphism.

Proof: Consider structures of the form $(\mathcal{M}, \mathcal{L}, E, e, f)$ where

1. $\mathcal{M} \models \varphi$
2. $\mathcal{L}$ is a linear ordering with a first element 0.
3. $E \subseteq \mathcal{L} \times \mathcal{M}^{<\omega} \times \mathcal{M}^{<\omega}$.
4. For each $\alpha \in \mathcal{L}$, $E(\alpha, \cdot, \cdot)$ is an equivalence relation on $\mathcal{M}^{<\omega}$.
5. $E(0, \bar{a}, \bar{b})$ if $\bar{a}$ and $\bar{b}$ satisfy the same atomic formulas.
Lemma: [Morley][Barwise] If a $L_{c,\omega}$-sentence $\varphi$ has a model of size $\beth_{\omega_1^{CK}}$, it has a countable model with a non-trivial isomorphism.

Proof: Consider structures of the form $(M, L, E, e, f)$ where

1. $M \models \varphi$
2. $L$ is a linear ordering with a first element 0.
3. $E \subseteq L \times M^{<\omega} \times M^{<\omega}$.
4. For each $\alpha \in L$, $E(\alpha, \cdot, \cdot)$ is an equivalence relation on $M^{<\omega}$.
5. $E(0, \bar{a}, \bar{b})$ if $\bar{a}$ and $\bar{b}$ satisfy the same atomic formulas.
6. $E(\alpha, \bar{a}, \bar{b})$ if $\forall \beta < \alpha \forall d \in M \exists c \in M \ E(\beta, \bar{ac}, \bar{bd})$.
   and $\forall c \in M \exists c \in M \ E(\beta, \bar{ac}, \bar{bd})$. 
The Hanf number of computably infinitary logic

Lemma: [Morley][Barwise] If a $\mathcal{L}_{c,\omega}$-sentence $\varphi$ has a model of size $\beth_{\omega_1}^{CK}$, it has a countable model with a non-trivial isomorphism.

Proof: Consider structures of the form $(M, \mathcal{L}, E, e, f)$ where

1. $M \models \varphi$
2. $\mathcal{L}$ is a linear ordering with a first element 0.
3. $E \subseteq \mathcal{L} \times M^{<\omega} \times M^{<\omega}$.
4. For each $\alpha \in \mathcal{L}$, $E(\alpha, \cdot, \cdot)$ is an equivalence relation on $M^{<\omega}$.
5. $E(0, \bar{a}, \bar{b})$ if $\bar{a}$ and $\bar{b}$ satisfy the same atomic formulas.
6. $E(\alpha, \bar{a}, \bar{b})$ if $\forall \beta < \alpha \forall d \in M \exists c \in M E(\beta, \bar{a}c, \bar{b}d)$. and $\forall c \in M \exists c \in M E(\beta, \bar{a}c, \bar{b}d)$.
7. $e \neq f \in M$ and, for all $\alpha \in L$, $E(\alpha, e, f)$. 
The Hanf number of computably infinitary logic

**Lemma:** [Morley][Barwise] If a $\mathcal{L}_{c,\omega}$-sentence $\varphi$ has a model of size $\beth_{\omega_1^{CK}}$, it has a countable model with a non-trivial isomorphism.

**Proof:** Consider structures of the form $(\mathcal{M}, \mathcal{L}, E, e, f)$ where

1. $\mathcal{M} \models \varphi$
2. $\mathcal{L}$ is a linear ordering with a first element 0.
3. $E \subseteq \mathcal{L} \times \mathcal{M}^{<\omega} \times \mathcal{M}^{<\omega}$.
4. For each $\alpha \in \mathcal{L}$, $E(\alpha, \cdot, \cdot)$ is an equivalence relation on $\mathcal{M}^{<\omega}$.
5. $E(0, \bar{a}, \bar{b})$ if $\bar{a}$ and $\bar{b}$ satisfy the same atomic formulas.
6. $E(\alpha, \bar{a}, \bar{b})$ if $\forall \beta < \alpha \forall d \in \mathcal{M} \exists c \in \mathcal{M} \ E(\beta, \bar{a}c, \bar{b}d)$. and $\forall c \in \mathcal{M} \exists c \in \mathcal{M} \ E(\beta, \bar{a}c, \bar{b}d)$.
7. $e \neq f \in \mathcal{M}$ and, for all $\alpha \in \mathcal{L}$, $E(\alpha, e, f)$.

and, for all $\alpha < \omega_1^{CK}$

- there is an $a \in \mathcal{L}$ such that $\mathcal{L} \upharpoonright a \cong \alpha$. 

Antonio Montalbán (U.C. Berkeley)  
Higher Recursion and computable structures  
May 2019  47 / 50
Claim 1: If $\mathcal{L}$ is ill-founded, then $e$ and $f$ are automorphic.
Finishing the proof

**Claim 1:** If $\mathcal{L}$ is ill-founded, then $e$ and $f$ are automorphic.

**Proof:** If $C \subseteq L$ has no least element,

$$\{(\bar{a}, \bar{b}) : (\exists \alpha \in C) \ E(\alpha, \bar{a}, \bar{b})\}$$

has back-and-forth property.
Finishing the proof

Claim 1: If \( L \) is ill-founded, then \( e \) and \( f \) are automorphic.

Proof: If \( C \subseteq L \) has no least element,
\[
\{(\bar{a}, \bar{b}) : (\exists \alpha \in C) \ E(\alpha, \bar{a}, \bar{b})\}
\]
has back-and-forth property.

Claim 2: If \( L \) is well-ordered, there is such structure satisfying 1-6.
Finishing the proof

**Claim 1:** If $\mathcal{L}$ is ill-founded, then $e$ and $f$ are automorphic.

**Proof:** If $C \subseteq L$ has no least element,
\[
\{(\bar{a}, \bar{b}) : (\exists \alpha \in C) \ E(\alpha, \bar{a}, \bar{b})\}
\]
has back-and-forth property.

**Claim 2:** If $\mathcal{L}$ is well-ordered, there is such structure satisfying 1-6.

**Claim 3:** If $\alpha$ well-ordered, $E(\alpha, \cdot, \cdot)$ has at most $\beth_\alpha$ equivalence classes.
Finishing the proof

**Claim 1:** If $\mathcal{L}$ is ill-founded, then $e$ and $f$ are automorphic.

**Proof:** If $C \subseteq L$ has no least element,

$$\{(\bar{a}, \bar{b}) : (\exists \alpha \in C) \ E(\alpha, \bar{a}, \bar{b})\}$$

has back-and-forth property.

**Claim 2:** If $\mathcal{L}$ is well-ordered, there is such structure satisfying 1-6.

**Claim 3:** If $\alpha$ well-ordered, $E(\alpha, \cdot, \cdot)$ has at most $\beth_\alpha$ equivalence classes.

**Claim 4:** If $\mathcal{L} < \omega_1^{CK}$ there is such structure satisfying 1-7.
Finishing the proof

**Claim 1:** If $\mathcal{L}$ is ill-founded, then $e$ and $f$ are automorphic.

**Proof:** If $C \subseteq L$ has no least element, 
\[ \{ (\bar{a}, \bar{b}) : (\exists \alpha \in C) \ E(\alpha, \bar{a}, \bar{b}) \} \] has back-and-forth property.

**Claim 2:** If $\mathcal{L}$ is well-ordered, there is such structure satisfying 1-6.

**Claim 3:** If $\alpha$ well-ordered, $E(\alpha, \cdot, \cdot)$ has at most $\beth_\alpha$ equivalence classes.

**Claim 4:** If $\mathcal{L} < \omega_1^{CK}$ there is such structure satisfying 1-7.

**Claim 5:** There is a model $\mathcal{B}$ of 1-8 with $\omega_1^\mathcal{B} = \omega_1^{CK}$ and $\mathcal{L} \cong \mathcal{H}$.

**Proof:** Use Barwise compactness.
The necessity of infinitely many iterations of the power set

**Theorem:** [Montalbán 11]

\[
ZFC - (\text{Power set axiom}) + \left( \mathcal{P}(\mathcal{P}(\cdots \mathcal{P}(\omega)\cdots)) \right) \text{ exists} \quad \text{does not prove}
\]

the existence of a structure Muchnik equivalent to its own jump.
The necessity of infinitely many iterations of the power set

**Theorem:** [Montalbán 11]

\[ \text{ZFC - (Power set axiom) + } \left( \mathcal{P}(\mathcal{P}(\cdots \mathcal{P}(\omega) \cdots )) \right) \text{ exists} \]

\[ \text{does not prove} \]

the existence of a structure Muchnik equivalent to its own jump.

**Proof** of case \( n = 1 \):
The necessity of infinitely many iterations of the power set

**Theorem:** [Montalbán 11]

$$\text{ZFC - (Power set axiom) + } \left( \mathcal{P} \left( \mathcal{P} \left( \cdots \mathcal{P}(\omega) \cdots \right) \right) \right) \text{ exists}$$

does not prove

the existence of a structure Muchnik equivalent to its own jump.

**Proof** of case $n = 1$: Show that if $\mathcal{A} \equiv_{\text{Muchnik}} \mathcal{A}'$, then

$$\{ X \subseteq \omega : X \text{ is c.e. in every copy of } \mathcal{A} \}$$

forms an $\omega$-model of 2nd-order arithmetic.