Global weak solutions to a three-dimensional compressible non-Newtonian fluid with vacuum

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outline

- Models
- Motivations
- Main results
- Key points of Proofs
The equations of a compressible viscous barotropic fluid in 
\((x, t) \in \Omega \times \mathbb{R}^+\) have the following form

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) &= \text{div}(P) + \rho f.
\end{align*}
\]

(1)

\(\rho\)-density; \(u = (u_1, u_2, u_3)\)-velocity; \(P\)-the stress tensor; \(f\)-the vector of external mass forces; the operators \(\text{div}\) and \(\nabla\) act with respect to the space variables \(x\).

The initial data is given by

\[
(\rho, \rho u)|_{t=0} = (\rho_0, m_0)(x), \ x \in \Omega,
\]

and the no-slip boundary condition on the velocity

\[
u |_{\partial \Omega} = 0.
\]
Models

The system (1) must be closed by some constitutive equation for the stresses $\mathbb{P}$. Taking the Stokes axioms for the (only) criterion of its ”physical validity,” we restrict ourselves to constitutive relations of the following form

$$\mathbb{P} = \sum_{k=0}^{2} \alpha_k(\rho, \text{div} u, \|\mathbb{D}u\|^2)\mathbb{D}^k u. \quad (2)$$

$\mathbb{D}u$-the deformation velocity tensor with components

$$D_{ij}u = \frac{1}{2}(\partial_j u_i + \partial_i u_j);$$

$$\|\mathbb{D}u\|^2 \equiv \mathbb{D}u : \mathbb{D}u = \sum_{i,j=1}^{3} (D_{i,j}u)^2.$$ 

One particular case of equation (3)

$$\mathbb{P} = -p(\rho) + \lambda(\|\text{div} u\|^2)\text{div} u \mathbb{I} + 2\mu(\|\mathbb{D}u\|^2)\mathbb{D}u \quad (3)$$

which is a natural generalization of the constitutive relation in the classical fluid model.
The incompressible non-Newtonian fluids

\[
\begin{aligned}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) - \text{div}(\Gamma) + \nabla P &= \rho f, \\
\text{div} u &= 0
\end{aligned}
\]

where \( \Gamma \) denotes the viscous stress tensor and

\[
\Gamma_{ij} = (\mu_0 + \mu_1 |\mathbb{D}u|)^{r-2} \mathbb{D}_{i,j} u
\]

with \( \mu_0 > 0, \mu_1 > 0 \) are constants.

This form of \( \Gamma \) is proposed by O.A. Ladyzhenskaya 1970.
The incompressible non-Newtonian fluids

— Existence of weak solutions
Ladyzhenskaya, Lions, Nečas, Zhikov, Kaniel, Frehse, Málek, Steinhauer, Boling Guo......

— The global attractor
Boling Guo, Guoguang Lin, Yadong Shang, Caidi Zhao, Yongsheng Li,......
The compressible non-Newtonian fluids

\[
\begin{aligned}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) - \text{div}(\Gamma) + \nabla P &= \rho f.
\end{aligned}
\]

where \(\Gamma\) denotes the viscous stress tensor and

\[
\Gamma_{ij} = (\mu_0 + \mu_1 |\mathbb{D}u|)^{r-2} \mathbb{D}_{i,j} u
\]

with \(\mu_0 \geq 0, \mu_1 > 0\) are constants.
Motivations

These models are called:

- **Newtonian**, for $\mu_0 > 0, \mu_1 = 0$;
- **Rabinowitsch**, for $\mu_0, \mu_1 > 0, r = 4$;
- **Eills**, for $\mu_0, \mu_1 > 0, r > 2$;
- **Ostwald-de Waele**, for $\mu_0 = 0, \mu_1 > 0, r > 4$;
- **Bingham**, for $\mu_0, \mu_1 > 0, r = 1$;

For $\mu_0 = 0$, if $r < 2$, it is a pseudo-plastic fluid; if $r > 2$, it is a dilant fluid;

In the view of physics:

- $1 < r < 2$: shear thinning fluid
- $r > 2$: shear thickening fluid.
Motivations

**One-dimension**

- Local existence of strong/classical solution
  - Hongjun Yuan and his team, Qin Yumin, Guo Zhenhua, Fang Li, Wang Yuxin......

- Asymptotic stability/Large-time behavior of solution
  - Guo-Fang (2016): Zero dissipation limit to rarefaction wave with vacuum
  - Guo-Dong-Liu (2019): Large-time behavior of solution to an inflow problem on the half space
  - Guo-Su-Liu: The existence and limit behavior of the shock layer for 1D steady compressible non-Newtonian fluids
  - other results ... ...
Motivations

- **Multi-dimension**

  - Existence of weak solution

  Feireisl, Liao and Málek, Zhikov and Pastukhova, Mamontov, ......
Feireisl-Liao-Málek considered the following compressible non-Newtonian fluid in \((x, t) \in \Omega \times \mathbb{R}_+\),

\[
\begin{cases}
\partial_t \rho + \text{div}(\rho u) = 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla p(\rho) \\
= \text{div}(2\mu_0 (1 + |\mathbb{D}^d u|^2)^{\frac{r-2}{2}} \mathbb{D}^d u + \eta(\text{div} u) \text{div} u \mathbb{I})
\end{cases}
\]

where

i) \(\mathbb{D}^d u = \mathbb{D} u - \frac{1}{3} (\text{tr} \mathbb{D} u) \mathbb{I}\), \(\mu_0 > 0\) is a constant, \(r \in [\frac{11}{5}, +\infty)\);

ii) the bulk viscosity coefficient \(\eta\) is a continuous function of \(\text{div} u\), \(\eta(z) : (-\frac{1}{b}, \frac{1}{b}) \to [0, +\infty)\) such that there is a convex potential \(\Lambda : \mathbb{R} \to [0, \infty]\)

\[
\begin{cases}
\Lambda(0) = 0, \\
\Lambda'(z) = z\eta(z), \\
\Lambda(z) \to \infty \text{ if } z \to \pm \frac{1}{b}, \\
\Lambda(z) = \infty \text{ if } |z| \geq \frac{1}{b};
\end{cases}
\]

iii) the pressure \(p = p(\rho)\) and the Helmholtz free energy \(\psi = \psi(\rho)\) satisfy

\[p = \rho^2 \psi'(\rho), \quad p \in C[0, \infty) \cap C^1(0, +\infty), \quad p(0) = 0, \quad p'(\rho) > 0 \text{ for } \rho > 0.\]
Motivations

The definition of weak solution in the work Feireisl-Liao-Málek

A pair of functions \((\rho, u)\) is said to be a weak solution to the problem (4) on \((0, T)\) for any fixed \(T > 0\) if the following conditions hold:

- \(\rho \in C([0, T]; L^1(\Omega)) \cap L^\infty(\Omega \times (0, T)),\)
  \(u \in L^r(0, T; W^{1,r}_0(\Omega)),\) \(\eta(|\text{div}u|)|\text{div}u|^2 \in L^1(\Omega \times (0, T));\)
- The equation of continuity in (4) is satisfied in \(D'(\Omega \times (0, T));\)
- The following weak formulation of the momentum equation

\[\left[\frac{1}{2} \int_{\Omega} \rho|u|^2 \, dx\right]_0^\tau - \left[\int_{\Omega} \rho u \cdot \varphi \, dx\right]_0^\tau + \int_0^\tau \int_{\Omega} [\rho u \cdot \partial_t \varphi + \rho u \otimes u : \nabla \varphi] \, dx \, dt\]
\[+ \int_0^\tau \int_{\Omega} |D u|^{r-1} D u : D(u - \varphi) \, dx \, dt + \int_0^\tau \int_{\Omega} p(\rho) \text{div}(\varphi - u) \, dx \, dt\]
\[\leq \int_0^\tau \int_{\Omega} [\Lambda(\text{div} \varphi) - \Lambda(\text{div} u)] \, dx \, dt \quad \text{(control the term } \eta(|\text{div} u|)\text{div} u)\]
\[\eta(\text{div} u) : (-\frac{1}{b}, -\frac{1}{b}) \to [0, +\infty)\]

for any \(\tau \in [0, T]\) and any test function \(\varphi \in C^\infty_c(\Omega \times [0, T]).\)
Feireisl-Liao-Málek (2015) showed the large-data existence result of weak solutions to the initial-boundary problem to the system (4) with nonlinear constitutive equations that guarantee that the divergence of the velocity field remains bounded, provided the initial density is without vacuum.
Zhikov-Pastukhova (2009) considered a compressible fluid in $(x, t) \in \Omega \times \mathbb{R}_+$, described by

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla \rho^\gamma \\
&= \text{div}(|\mathbb{D}u|^{r-2}\mathbb{D}u + \nu |\text{div}u|^{r-2}I)
\end{align*}
\]

where $\nu \geq 0$ is a constant, $\gamma > 1$, $r > 1$. 
The definition of weak solution in the work Zhikov and Pastukhova

A pair of functions \((\rho, u)\) is said to be a weak solution to the problem (5) on \((0, T)\) for any fixed \(T > 0\) if the following conditions hold:

(1) \(\rho \in C([0, T]; L^1(\Omega)) \cap L^\infty(0, T; L^\gamma(\Omega))\),

\(u \in L^r(0, T; W^{1,r}_0(\Omega))\), \(\rho u \in L^\infty(0, T; L^1(\Omega))\);

(2) The equations (5) is satisfied in \(\mathcal{D}'(\Omega \times (0, T))\);

(3) \(\lim_{t \to 0} \rho = \rho_0\) in \(L^1(\Omega)\),

\(\lim_{t \to 0} \int_{\Omega} \rho u \cdot \varphi dx = \int_{\Omega} m_0 \cdot \varphi dx \quad \forall \varphi \in C^\infty_0(\Omega)\).
Zhikov and Pastukhova (2009) proved that the initial-boundary problem to the system (6) admits a weak solution such that

$$\rho u^2 \in L^{\frac{r}{r-1}}(\Omega \times (0,T)), \quad \rho^\gamma \in L^{\frac{r}{r-1}}(\Omega \times (0,T))$$

provided that $\rho_0 \in L^\gamma(\Omega), \quad \frac{m_0}{\rho_0} \in L^1(\Omega), \quad \gamma > \frac{3}{2}, \quad r \geq 3.$
Motivations

Mamontov (1999) considered a compressible fluid in $(x, t) \in \Omega \times \mathbb{R}_+$, described by

\[
\begin{aligned}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) &= \text{div}(2\mu(|Du|^2)Du + \lambda(|\text{div}u|^2)\text{div}uI)
\end{aligned}
\]  

where $\mu(s) = \exp(s^{\epsilon_0})$ ($\epsilon_0 > 0$ a constant), $\lambda(s) = \exp(\sqrt{s})$.

Mamontov showed the existence of global solutions of multidimensional equations of motion of a compressible non-Newtonian fluid in Bürgers approximation (in the absence of pressure), on the basis of the techniques of Orlicz spaces.
Motivations

Global weak solution to the multi-dimensional compressible Navier-Stokes equations for general initial data with finite energy:

Question: For general initial data with finite energy, what about the existence of global weak solution to the multi-dimensional compressible non-Newtonian fluid containing vacuum for the general case $r > 1$?
Consider the initial-boundary value problem for the isentropic compressible non-Newtonian fluid with vacuum

\[
\begin{align*}
\partial_t \rho + \text{div}_x (\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla p &= \text{div}(|\mathbb{D}u|^{r-2}\mathbb{D}u + \eta(|\text{div} u|)\text{div} u I),
\end{align*}
\]

with the initial data

\[
(\rho, \rho u)|_{t=0} = (\rho_0, m_0)(x), \quad x \in \Omega
\]

and the no-slip boundary condition on the velocity

\[
u|_{\partial \Omega} = 0.
\]

Notice: This model can describe the motion of electrons in an electric field.
Definition A pair of functions \((\rho, u)\) is said to be a finite energy weak solutions to the problem \((7)-(9)\) on \((0, T)\) for any fixed \(T > 0\) if the following conditions hold:

- \(\rho \geq 0, \quad \rho \in C([0, T]; L^1(\Omega)) \cap L^\infty(0, T; L^\gamma(\Omega)),\)
  
  \(u \in L^r(0, T; W_0^{1,r}(\Omega)), \quad \eta(|\text{div} u|)|\text{div} u|^2 \in L^1(\Omega \times (0, T));\)

- The equations (7) hold in \(\mathcal{D}'(\Omega \times (0, T))\) and
  
  \[\int_0^T \int_\Omega [\rho \partial_t \varphi + \rho u \cdot \nabla \varphi] dx dt = [\int_\Omega \rho \varphi dx] \bigg|_0^\tau\]

  for any \(\tau \in [0, T]\) and any test function \(\varphi \in C^\infty(\Omega \times [0, T])\) with \(\varphi(x, 0) = \varphi(x, T) = 0\) for \(x \in \Omega;\)

- The functions \(\rho\) and \(\rho u\) satisfy the initial conditions in the weak sense.
Remark

- Any weak solution in Definition satisfy the equations (7) hold in $\mathcal{D}'(\Omega \times (0, T))$, which is different from the definition of weak solution in the work Feireisl-Liao-Málek.

- Any weak solution in Definition satisfies the following weak formulation of the momentum equation

$$
\left[ \frac{1}{2} \int_{\Omega} \rho |u|^2 \, dx \right]^\tau_0 - \left[ \int_{\Omega} \rho u \cdot \varphi \, dx \right]^\tau_0 + \int_0^\tau \int_{\Omega} \left[ \rho u \cdot \partial_t \varphi + \rho u \otimes u : \nabla \varphi \right] \, dx \, dt \\
+ \int_0^\tau \int_{\Omega} |\mathbb{D}u|^{r-1} \mathbb{D}(u - \varphi) \, dx \, dt + \int_0^\tau \int_{\Omega} p(\rho) \text{div}(\varphi - u) \, dx \, dt \\
\leq \int_0^\tau \int_{\Omega} \left[ \Lambda(\text{div}\varphi) - \Lambda(\text{div}u) \right] \, dx \, dt
$$

for any $\tau \in [0, T]$ and any test function $\varphi \in C_c^\infty(\Omega \times [0, T])$, where $\Lambda'(z) = \eta(z) z$ and $\Lambda''(z) \geq 0$. 
Theorem 1 Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2+\nu}$ for some $\nu > 0$ and $\eta(z) = |z|^{q-1}$. Suppose that the following conditions hold:

(i) the pressure $p(\rho)$ is given by $p(\rho) = \rho^\gamma$ with the adiabatic exponent $\gamma > \frac{3}{2}$;

(ii) the initial data satisfy

$$
\left\{
\begin{array}{l}
\rho_0 \in L^\gamma(\Omega), \rho_0 \geq 0 \text{ on } \Omega, \\
\frac{|m_0|^2}{\rho_0} \in L^1(\Omega);
\end{array}
\right.
$$

(iii) the positive constants $r$ and $q$ satisfy the case that

$\frac{12}{5} \leq r < 3$ and $q > \max\{\gamma, 9\}$, or the case that $r \geq 3$ and $q > 1$.

Then, the initial-boundary value problem (7)-(9) admits a finite energy weak solution $(\rho, u)$ on $\Omega \times (0, T)$ for any given $T > 0$. 
Theorem 2 Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2+\nu}$ for some $\nu > 0$ and $\eta(z) = |z|^{q-1}$. Suppose that the following conditions hold:

(i) the pressure $p(\rho)$ is given by

\[
\begin{align*}
&p'(s) \geq a_1 s^{\gamma-1} \quad \text{for all } s > 0, \quad p(s) \leq a_2 s^{\gamma} \quad \text{for all } s \geq 0, \\
&p \in C[0, \infty) \cap C^1(0, \infty), \quad p(0) = 0
\end{align*}
\]

with the adiabatic exponent $\gamma > \frac{3}{2}$; \hfill (10)

(ii) the initial data satisfy

\[
\begin{align*}
\rho_0 &\in L^\gamma(\Omega), \rho_0 \geq \underline{\rho} > 0 \text{ on } \Omega, \\
\frac{|m_0|^2}{\rho_0} &\in L^1(\Omega);
\end{align*}
\]

(iii) the positive constants $p$ and $q$ satisfy the case that $\frac{11}{5} \leq r < 3$ and $q > \max\{\gamma, 9\}$, or the case that $r \geq 3$ and $q > 1$.

Then, the initial-boundary value problem (7)-(9) admits a finite energy weak solution $(\rho, u)$ on $\Omega \times (0, T)$ for any given $T > 0$. 

Remark

(i) The solution constructed in Theorem 1 -2 admits that

\[ \rho^\gamma \in L^{\frac{q+1}{q}}(\Omega \times (0,T)) \quad \text{(when } \frac{12}{5} \leq r < 3 \text{ and } q > \max\{\gamma, 9\} \text{),} \]

or \[ \rho^\gamma \in L^{\frac{r+1}{r}}(\Omega \times (0,T)) \quad \text{(when } r \geq 3 \text{ and } q > 1 \text{),} \]

\[ \nabla u \in L^{q+1}(\Omega \times (0,T)) \text{ and } \nabla u \in L^r(\Omega \times (0,T)). \]

(ii) The solution constructed in Theorem 1 and Theorem 2 will satisfy the continuity equation in the sense of re-normalized solutions.

(iii) Our results also hold for the bulk viscosity coefficient

\[ \eta(|\nabla u|) \sim |\nabla u|^{q-1}, \text{ where the symbol } \sim \text{ refers that there exist positive constants } C_1 \text{ and } C_2 \text{ such that} \]

\[ C_1|\nabla u|^{q-1} \leq \eta(|\nabla u|) \leq C_2|\nabla u|^{q-1} \]

and \( \eta(z) + z\eta'(z) > 0 \) holds for any \( z > 0. \)
Main difficulties and the countermeasure in proof of Theorems

(1) The initial density containing vacuum and strong degeneracy of the term $\text{div}(|\mathbb{D}u|^{r-2}\mathbb{D}u)$ in momentum equations

Inspired by Jiang-Zhang-2001 and Chapter 7 in Feireisl-2004, we introduce an approximate problem

\[
\begin{aligned}
\partial_t \rho + \text{div}(\rho u) &= \epsilon \Delta \rho, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla (p + \delta \rho^\beta) + \epsilon \nabla u \cdot \nabla \rho &= \text{div}((\delta + |\mathbb{D}u|^2)^{\frac{r-2}{2}} \mathbb{D}u + \eta(|\text{div}u|)\text{div}u\mathbb{I})
\end{aligned}
\]

with the initial-boundary conditions

\[
\begin{aligned}
\nabla \rho \cdot n|_{\partial \Omega} &= 0, \quad \rho|_{t=0} = \rho_{0,\delta}, \\
\rho u|_{t=0} &= m_{0,\delta},
\end{aligned}
\]

(12)
(2) The strong nonlinearity for the pressure term $p(\rho)$ and the term $\text{div}(|\mathcal{D}u|^{r-2} \mathcal{D}u)$.

The difficulty comes from the nonlinear term

$$(\delta + |\mathcal{D}u|^2)^{\frac{r-2}{2}} \mathcal{D}u + \eta(|\text{div}u|)\text{div}u I$$

in the approximate problem.

Recall that $\eta(z) = |z|^{q-1}$, $\Lambda'(z) = \eta(z)z$ and $\Lambda''(z) \geq 0$, one can deduce that

$$(\eta(\text{div}u)\text{div}u) = \eta(\text{div}u)\text{div}u.$$ 

So we need focus on the first term of the above nonlinear term.
Step 1. Limit in the Galerkin approximation

The approximate problem (11)-(12) with fixed positive parameters $\epsilon$ and $\delta$ can be solved by means of a modified Faedo-Galerkin method.

Step 2. The vanishing limits of the artificial viscosity $\epsilon \to 0$.

The common difficulty in the Step 1 and the Step 2:

The nonlinear term

$$(\delta + |\mathbb{D}u|^2)^{\frac{r-2}{2}} \mathbb{D}u$$

in the approximate problem.
The key tool to deal with the nonlinear term above
Let $A_\epsilon(x, \xi)$ and $A(x, \xi)$ be Carathéodory vector functions. $A_\epsilon(x, \xi)$, $A(x, \xi) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, where $\Omega$ is a bounded domains in $\mathbb{R}^d$. The Carathéodory property means continuity with respect to $\xi \in \mathbb{R}^d$ for a.e. $x \in \Omega$ and measurability with respect to $x$ for any $\xi$. These vector functions are assumed to satisfy the minimal monotonicity and convergence conditions

$$(A_\epsilon(x, \xi) - A_\epsilon(x, \eta)) \cdot (\xi - \eta) \geq 0,\quad A_\epsilon(x, 0) \equiv 0$$

$$|A_\epsilon(x, \xi)| \leq c_0(|\xi|) < \infty,\quad \lim_{\epsilon \to 0} A_\epsilon(x, \xi) = (x, \xi)$$

for a.e. $x \in \Omega$ and any $\xi \in \mathbb{R}^d$. 
Lemma

Suppose that \( v_\epsilon \rightharpoonup v, \ A_\epsilon(x, v_\epsilon) \rightharpoonup z \) in \( L^1(\Omega) \). Let \( K \subset \Omega \) be a measurable set such that \( z \cdot v \in L^1(K) \). Then

\[
\liminf_{\epsilon \to 0} \int_K A_\epsilon(x, v_\epsilon) \cdot v_\epsilon \, dx \geq \int_K z \cdot v \, dx
\]

and, in the case of equality,

\[
z|_K = A|_K, \quad A = A(x, v).
\]
We employ the following weak formulation of the momentum equation in the approximate problem

\[
\left[ \frac{1}{2} \int_{\Omega} \rho |u|^2 \, dx \right]_0^T - \left[ \int_{\Omega} \rho u \cdot \varphi \, dx \right]_0^T \\
+ \int_0^T \int_{\Omega} (\rho u \cdot \partial_t \varphi + \rho u \otimes u : \nabla \varphi) \, dx \, dt \\
+ \int_0^T \int_{\Omega} (p(\rho) + \delta \rho^\beta) (\text{div} \varphi - \text{div} u) \, dx \, dt \\
- \int_0^T \int_{\Omega} \epsilon \nabla \rho \cdot \nabla u \cdot \varphi \, dx \, dt \\
+ \int_0^T \int_{\Omega} \left( (\delta + |D u|^2) \frac{r-2}{2} |D u|^2 - (\delta + |D u|^2) \frac{r-2}{2} D u : D \varphi \right) \, dx \, dt \\
\leq \int_0^T \int_{\Omega} [\Lambda(\text{div} \varphi) - \Lambda(\text{div} u)] \, dx \, dt \text{ for a.e.} \tau \in [0, T]
\]
The family of regularized Kernels

\[ \eta_h(t) := \frac{1}{h} \mathbb{I}_{[-h,0]}(t) \quad \text{and} \quad \eta_{-h}(t) := \frac{1}{h} \mathbb{I}_{[0,h]}(t) \quad (h > 0), \]

together with the cut-off functions

\[ \xi_\sigma \in C^\infty_c(0,\tau), \quad 0 \leq \xi \leq 1, \quad \xi_\sigma(t) = 1 \]

whenever \( t \in [\sigma, \tau - \sigma], \quad \sigma > 0. \) Noticing that

\[ \eta_h * u = \frac{1}{h} \int_t^{t+h} u ds \in W^{1,r}(0,T;W_0^{1,r}(\Omega)), \]

we can take the quantities

\[ \varphi_{h,\sigma} = \xi_\sigma \eta_{-h} * \eta_h * (\xi_\sigma u) \quad (\sigma, h > 0) \]

as test functions in (13).
By careful calculation, we can arrive at

$$\int_0^\tau \int_\Omega \left( (\delta + |\mathbb{D}u|^2)^{\frac{r-2}{2}} |\mathbb{D}u|^2 - (\delta + |\mathbb{D}u|^2)^{\frac{r-2}{2}} \mathbb{D}u : \mathbb{D}u \right) dx\,dt \leq 0,$$

based on the fact that

$$\lim_{\sigma \to 0} \lim_{h \to 0} \int_0^\tau \int_\Omega (\Lambda(\text{div}\varphi_{h,\sigma}) - \Lambda(\text{div}u)) \,dx\,dt = 0,$$

$$\left[ \int_\Omega \rho u \cdot \varphi_{h,\sigma} \,dx \right]_0^\tau = 0 \text{ (for all } \sigma, h > 0),$$
So together with Fatou’s lemma,

\[
\lim_{\sigma \to 0} \lim_{h \to 0} \int_0^T \int_\Omega \left( (\delta + |D u|^2)^{\frac{r-2}{2}} |D u|^2 - (\delta + |D u|^2)^{\frac{r-2}{2}} D u : D \varphi_{h,\sigma} \right) \, dx \, dt \geq 0.
\]

it ensures that

\[
(\delta + |D u|^2)^{\frac{r-2}{2}} D u = (\delta + |D u|^2)^{\frac{r-2}{2}} D u
\]

and

\[
(\delta + |D u|^2)^{\frac{r-2}{2}} |D u|^2 = (\delta + |D u|^2)^{\frac{r-2}{2}} |D u|^2.
\]
Step 3. The artificial pressure coefficient \( \delta \to 0 \).

3.1 The density estimates.

**Lemma**

There exists a positive constant \( C \), independence of \( \delta \), such that

\[
\int_0^T \int_\Omega \left( \rho_\delta^{\frac{q+1}{q}} \gamma + \delta \rho_\delta^{\beta+\frac{\gamma}{q}} \right) dx dt \leq C,
\]

holds for the case that \( \frac{11}{5} \leq r < 3 \) and \( q > \max\{\gamma, 9\} \), and

\[
\int_0^T \int_\Omega \left( \rho_\delta^{\frac{r}{r-1}} \gamma + \delta \rho_\delta^{\beta+\frac{\gamma}{r-1}} \right) dx dt \leq C,
\]

holds for the case that \( r \geq 3 \) and \( q > 1 \).
3.2 The amplitude of oscillations. 
For the cut-off operators introduced in Feireisl-2004 and Jiang-Zhang-2001, we consider a family of functions

\[ T_k(z) = kT\left(\frac{z}{k}\right) \text{ for } z \in \mathbb{R}, \ k = 1, 2, \cdots \]  

(14) 

where \( T \in C^\infty(\mathbb{R}) \) is chosen so that

\[ T(z) = z \text{ for } z \leq 1, \ T(z) = 2 \text{ for } z \geq 3, \ T \text{ concave.} \]
Lemma

There exists a positive constant $C$, independence of $k$, such that

$$\lim_{\delta \to 0} \sup \| T_k(\rho_\delta) - T_k(\rho) \|_{L^{q+\frac{1}{\gamma}}(\Omega \times (0,T))} \leq C$$

holds for the case that $\frac{12}{5} \leq r < 3$ and $q > \max\{\gamma, 9\}$,

$$\lim_{\delta \to 0} \sup \| T_k(\rho_\delta) - T_k(\rho) \|_{L^{\gamma+1}(\Omega \times (0,T))} \leq C$$

holds for the case that $r \geq 3$ and $q > 1$.

Remark. The limit functions $\rho$ and $u$ satisfy the continuity equation $(4)_1$ in the sense of renormalized solutions.
3.3 The momentum equation.

The first key point is to prove $\bar{\rho}^\gamma = \rho^\gamma$. The following integrability properties of the limit functions $\rho$ and $u$, play an important role, which are stated as

$$
\rho^\gamma \in L^{\frac{q+1}{q}}(\Omega \times (0, T)) \quad (\frac{12}{5} \leq r < 3 \text{ and } q > \max\{\gamma, 9\}),
$$
or $$
\rho^\gamma \in L^{\frac{r}{r-1}}(\Omega \times (0, T)) \quad (r \geq 3 \text{ and } q > 1),
$$
\div u \in L^{q+1}(\Omega \times (0, T)) \text{ and } \nabla u \in L^r(\Omega \times (0, T)),
$$
$$
\rho u^2 \in L^{\frac{r}{r-1}}(\Omega \times (0, T)).
$$
The second key equality $|\mathbf{D}u|^r = |\mathbf{D}u|^r$ is obtained by applying the technique in the following inequality

$$
\left[ \frac{1}{2} \int_{\Omega} \rho |u|^2 \, dx \right]_0^\tau - \left[ \int_{\Omega} \rho u \cdot \varphi \, dx \right]_0^\tau \\
+ \int_0^\tau \int_{\Omega} (\rho u \cdot \partial_t \varphi + \rho u \otimes u : \nabla \varphi) \, dx \, dt \\
+ \int_0^\tau \int_{\Omega} \rho^\gamma (\text{div} \varphi - \text{div} u) \, dx \, dt \\
+ \int_0^\tau \int_{\Omega} \left( |\mathbf{D}u|^r - |\mathbf{D}u|^{r-2} \mathbf{D}u : \mathbf{D} \varphi \right) \, dx \, dt \\
\leq \int_0^\tau \int_{\Omega} (\Lambda(\text{div} \varphi) - \Lambda(\text{div} u)) \, dx \, dt \text{ for a.e. } \tau \in [0, T]
$$

(15)
In Theorem 2, the pressure $p(\rho)$ is given by

$$
\begin{align*}
&\left\{\begin{array}{l}
p'(s) \geq a_1 s^{\gamma - 1} \text{ for all } s > 0, \\
p(s) \leq a_2 s^{\gamma} \text{ for all } s \geq 0,
\end{array}\right. \\
p &\in C[0, \infty) \cap C^1(0, \infty), \\
p(0) &= 0
\end{align*}
$$

with the adiabatic exponent $\gamma > \frac{3}{2}$; and the initial data satisfy

$$
\begin{align*}
\rho_0 &\in L^{\gamma}(\Omega), \rho_0 \geq \underline{\rho} > 0 \text{ on } \Omega, \\
\frac{|m_0|^2}{\rho_0} &\in L^1(\Omega).
\end{align*}
$$
The approximate problem is still adopted as

$$\begin{cases} 
\partial_t \rho + \text{div}(\rho u) = \epsilon \Delta \rho, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla (p + \delta \rho^\beta) + \epsilon \nabla u \cdot \nabla \rho = \text{div}( (\delta + |\mathbb{D}u|^2)^{\frac{r-2}{2}} \mathbb{D}u + \eta(|\text{div}u|)\text{div}u \mathbb{I}) 
\end{cases}$$

with the initial-boundary conditions

$$\begin{cases} 
\nabla \rho \cdot n |_{\partial \Omega} = 0, \quad \rho|_{t=0} = \rho_{0,\delta}, \\
u|_{\partial \Omega} = 0, \quad \rho u|_{t=0} = m_{0,\delta},
\end{cases}$$
The difference between proof of Theorem 2 and Theorem 1 is the pressure term. The limit in the Galerkin approximation and the vanishing limits of the artificial viscosity $\epsilon \to 0$ can be settled by the similar way in proof of Theorem 1.
In the artificial pressure coefficient $\delta \to 0$, one can arrive at

$$\int_0^\tau \int_\Omega (p(\rho)\text{div}u - p(\rho)\text{div}u)dxdt \leq 0 \text{ for a.a } \tau \in [0, T].$$

To deal with $\rho P(\rho) - \rho P(\rho)$, the convexity of the function $zP(z)$ with $P(z) = \int_1^z \frac{p(s)}{s^2}ds$ and the initial density without vacuum play an important role.

Since the initial density is without vacuum, the convexity of the function $zP(z)$ implies that there exists a certain $\alpha > 0$ such that

$$\int_\Omega [\rho P(\rho) - \rho P(\rho)]dx \geq \alpha \limsup_{\delta \to 0} \int_\Omega |\rho_\delta - \rho|^2 dx.$$ 

The constant $\alpha > 0$ in above inequality depends on the positive lower bound of the initial density. This is different way to deal with the initial density being without vacuum.
Thank you for your attention!