Enhanced dissipation and Transition threshold for the 3-D Poiseuille flow in a channel

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The Reynolds number:

$$\text{Re} := \frac{\rho U L}{\nu},$$

\(\rho\) : density; \(U\) : velocity; \(L\) : diameter of pipe; \(\nu\) : viscosity.
Phenomenon and results of the Reynolds experiment

- Re small: laminar flow $\leftrightarrow$ (a);
- Re $\uparrow$: transition between laminar flow and turbulence $\leftrightarrow$ (b);
- Re large: turbulence $\leftrightarrow$ (c).
The classical model in \((x, y) \in \mathbb{R}^2\) and \((x, y, z) \in \mathbb{R}^3\):

\[
\begin{align*}
\partial_t v - \nu \Delta v + (v \cdot \nabla)v + \nabla q &= 0, \\
\nabla \cdot v &= 0, \\
v|_{t=0} &= v_0,
\end{align*}
\]

where

- \(\nu \sim \text{Re}^{-1} > 0\): viscosity;
- \(v \in \mathbb{R}^n (n = 2, 3)\): velocity fields;
- \(q\): pressure;
- \(\text{Re large} \leftrightarrow \text{random motion of the molecules} \leftrightarrow \text{turbulence}\).
The Euler equations

The Euler equations in \((x, y) \in \mathbb{R}^2\) and \((x, y, z) \in \mathbb{R}^3(\nu = 0\text{ in (1.1))}):

\[
\left\{
\begin{array}{l}
\partial_t v + (v \cdot \nabla)v + \nabla q = 0, \\
\nabla \cdot v = 0, \\
v|_{t=0} = v_0,
\end{array}
\right.
\]

(1.2)

where

- \(v \in \mathbb{R}^n (n = 2, 3)\): velocity fields;
- \(q\): pressure.
The shear flows:

\[ U_s = U(y) e_1 \]

for some \( U(y) \), are both the solution to N-S and Euler.

Some important shear flows in physics:

- The plane Couette flow: \( U(y) = y \);
- The plane Poiseuille flow: \( U(y) = 1 - y^2 \);
- The pipe Poiseuille flow (Hagen-Poiseuille flow): \( (1 - r^2, 0, 0) \);
- The Kolmogorov flow: \( U(y) = \sin ay \) or \( \cos ay \) for some constant \( a \in \mathbb{R} \).
Formulation of stability problem around the shear flows

Let $u = v - U_s$, then the perturbed equations around $U_s$ read as

$$
\begin{cases}
\partial_t u - \nu \Delta u + U_s \cdot \nabla u + u \cdot \nabla U_s + \nabla P = -u \cdot \nabla u, \\
\nabla \cdot u = 0, \\
| u |_{t=0} = u_0.
\end{cases}
$$

(1.3)

The linearized problem can be rewritten as

$$
\begin{cases}
\partial_t u + \mathcal{L}(t)u = 0, \\
\nabla \cdot u = 0, \\
| u |_{t=0} = u_0,
\end{cases}
$$

(1.4)

where

$$
\mathcal{L}(t)u := -\nu \Delta u + U_s \cdot \nabla u + u \cdot \nabla U_s + \nabla P.
$$
The classical analysis

Some classical concepts to study the stability problem:

- **Spectral stability and instability:** 
  \[ \sigma(L) \cap \{ c \in C : \text{Re } c > 0\} = \emptyset; \]
  \[ \sigma(L) \cap \{ c \in C : \text{Re } c > 0\} \neq \emptyset; \]
  \[ \sigma(L) \cap \{ c \in C : \text{Re } c = 0\} \neq \emptyset; \]

- **Lyapunov (nonlinear) stability:** the equilibrium \( u_E \) is called stable if for the perturbation equation: For any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \( \| u_{in} \|_X < \delta \), then \( \| u(t) \|_Y < \varepsilon \) for all \( t > 0 \);

- All the above analysis are considered for fixed viscosity (or Reynolds number)!

- However, as Reynolds experiment shows that many laminar flows are very very sensitive to Reynolds number, which maybe stable for fixed Reynolds number, but spontaneously transit into turbulence as Reynolds number tending to infinity!!! (How to define the critical Reynolds number for every laminar flow?)
The Sommerfeld paradox

For example, consider the Couette flow, the classical spectrum analysis yields that it is stable for any fixed viscosity (or Reynolds number) (V. A. Romanov, 1973).

However, in the experiment, Couette flow would be unstable under small perturbation at high Reynolds number!

This leads to a contradiction, i.e., Sommerfeld paradox!
The Sommerfeld paradox

Figure: Experiment result about the Couette flow (Trefethen et al. Science, 1993)
Examples

To understand the relationship among these stability concepts, we consider

$$\frac{\partial}{\partial t} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

(1.5)

The matrix $\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$ is not diagonalizable and has spectrum reduced to $\{0\}$, therefore it is neutrally stable, but it is not Lyapunov stable:

$$\begin{cases} x_1(t) = x_1(0) - tx_2(0), \\ x_2(t) = x_2(0), \end{cases}$$

(1.6)

Grow with $t$!
Above example can be modified to make it spectrally stable and Lyapunov stable:

$$\partial_t \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\nu & -1 \\ 0 & -\nu \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

(1.7)

The solutions are spectrally stable and Lyapunov linearly stable. However, as $\nu \to 0$, the system degenerates, and

$$\sup_t \| (x_1, x_2)(t) \| \lesssim \nu^{-1} \| (x_1, x_2)(0) \|.$$

The system is linearly stable for any fixed $\nu > 0$, but the solutions are growing very large as $\nu \to 0$ (very sensitive to Reynolds number!)
To understand the paradox and explain our problem, we introduce the quantitative asymptotic stability (instability).

**Definition (Quantitative asymptotic stability)**

Given two norms $X$ and $Y$, the equilibrium $U_s$ is called *quantitative stable* (from $X$ to $Y$) with exponent $\gamma$ if

$$\|u_0\|_X << \nu^\gamma$$

$$\implies \|u(t)\|_Y << 1, \forall t \geq 0, \text{ and } \|u(t)\|_Y \to 0 \text{ as } t \to \infty.$$  

Qualification of $\delta$ in the definition of Lyapunov stability by $\nu^{\gamma}$ (where $\gamma$ depends on the space $X$.)
The stability transition problem

A fundamental problem (Trefethen et al. Science, 1993; Masmoudi et al. 2015):

Given a norm $\| \cdot \|_X$, determine a $\gamma = \gamma(X)$ such that

$$\| u_0 \|_X \leq \nu^\gamma \implies \text{stability,}$$

$$\| u_0 \|_X \gg \nu^\gamma \implies \text{instability.}$$

$\gamma$: the transition threshold in the applied literature.
The stability transition problem

A fundamental problem (Trefethen et al. Science, 1993; Masmoudi et al. 2015):

*Given a norm $\| \cdot \|_X$, determine a $\gamma = \gamma(X)$ such that*

$$\| u_0 \|_X \leq \nu^\gamma \implies \text{stability},$$

$$\| u_0 \|_X \gg \nu^\gamma \implies \text{instability}.$$  

$\gamma$: the transition threshold in the applied literature.

Our goal:

Answer part of the above problem for the 3-D Poiseuille flow.
The Navier-Stokes equations around the 3-D Poiseuille flow \((1-y^2, 0, 0)\) in a channel \((x, y, z) \in \mathbb{T} \times I \times \mathbb{T}(I = (-1, 1))\):

\[
\begin{align*}
\partial_t u - \nu \Delta u + (u \cdot \nabla)u + (1 - y^2) \partial_x u + \begin{pmatrix}
-2yu_2 \\
0 \\
0
\end{pmatrix} + \nabla P &= 0, \\
\nabla \cdot u &= 0, \\
\partial_x u \big|_{t=0} &= u_0(x, y, z).
\end{align*}
\]  

(1.8)

The Navier-slip boundary conditions:

\[ u \cdot n = 0, \quad \omega \times n = 0 \quad \text{on} \quad \{y = \pm 1\}, \]

\[ \omega = \nabla \times u: \text{vorticity}; \quad n: \text{unit outer normal on the boundaries.} \]  

(1.9)
The important formulation in our problem:

\[
\begin{align*}
(\partial_t - \nu \Delta + (1 - y^2)\partial_x)\Delta u_2 + 2\partial_x u_2 &= -\Delta (u \cdot \nabla u_2) - \partial_y (\Delta p), \\
(\partial_t - \nu \Delta + (1 - y^2)\partial_x)\omega_2 - 2y\partial_z u_2 &= \nabla \cdot (\omega u_2 - u\omega_2), \\
(\Delta u_2, \omega_2)(t; x, \pm 1, z) &= (0, 0), \\
(\Delta u_2, \omega_2)|_{t=0} &= (\Delta u_2(0), \omega_2(0)),
\end{align*}
\]

(1.10)

\[\omega_2 = \partial_z u_1 - \partial_x u_3; p = -\Delta^{-1} \left( \sum_{i,j=1}^{3} \partial_i u_j \partial_j u_i \right).\]
The linearized problem

- The linearized Navier-Stokes equations:

\[
\begin{align*}
(\partial_t + \mathcal{L}_1) \Delta u_2 &= 0, \\
(\partial_t + \mathcal{L}_2) \omega_2 &= 2y \partial_z u_2, \\
(\Delta u_2, \omega_2)(t; x, \pm 1, z) &= (0, 0), \\
(\Delta u_2, \omega_2)|_{t=0} &= (\Delta u_2(0), \omega_2(0)).
\end{align*}
\] (1.11)

- The linearized operators:

\[
\mathcal{L}_1 = -\nu \Delta + (1 - y^2) \partial_x + 2 \partial_x \Delta^{-1} \quad \text{(with nonlocal term)}
\] (1.12)

and

\[
\mathcal{L}_2 = -\nu \Delta + (1 - y^2) \partial_x,
\] (1.13)
Some known results (Couette flow)

When $\Omega = \mathbb{T} \times \mathbb{R} \times \mathbb{T}$,

- if the perturbation is in Gevrey class, then $\gamma = 1$ (Bedrossian et al., arXiv, 2015);
- if the perturbation is in Sobolev space $H^\sigma (\sigma > 9/2)$, then $\gamma \leq \frac{3}{2}$ (Bedrossian et al., Ann. Math., 2017);
- if the perturbation is in Sobolev space $H^2$, then $\gamma \leq 1$ (Wei, Zhang, arXiv, 2018).
Some known results (Couette flow)

When $\Omega = \mathbb{T} \times \mathbb{R}$,

- if the perturbation is in Gevrey class, then $\gamma = 0$ (Bedrossian et al., ARMA, 2016);
- if the perturbation is in Sobolev space $H^s (s > 1)$, then $\gamma \leq \frac{1}{2}$ (Bedrossian et al., J. Nonlinear Sci., 2018);
- if the perturbation is in Sobolev space $H^\sigma (\sigma \geq 40)$, then $\gamma \leq \frac{1}{3}$ (Masmoudi, Zhao, arXiv, 2019)

When $\Omega = \mathbb{T} \times I (I = (-1, 1))$,

- if the perturbation is in Sobolev space $H^2$, then $\gamma \leq \frac{1}{2}$ (Chen, Li, Wei, Zhang, arXiv, 2018)
Some known results (other flows)

Kolomogorov flow:
- if the perturbation is in Sobolev space $H^2$ and $\Omega = \mathbb{T}^2$, then $\gamma \leq \frac{2}{3} +$ (Wei, Zhao, Zhang, arXiv, 2017);
- if the perturbation is in Sobolev space $H^2$ and $\Omega = \mathbb{T}^3$, then $\gamma \leq \frac{7}{4}$ (Li, Wei, Zhang, CPAM, 2019)

Poiseuille flow:
- if the perturbation is in Sobolev space and $\Omega = \mathbb{T} \times \mathbb{R}$, then $\gamma \leq \frac{3}{4} +$ (Zelati et al., arXiv, 2019)
The main linear effects:

- 3-D lift-up effect;
- Long-wave effect: $(x, z) \in \mathbb{R} \times \mathbb{T}$ (the main reason resulting in instability at high Reynolds number);
- Boundary layer effect: $\nu \to 0$;
- Inviscid damping;
- Enhanced dissipation.
To express the effect, as an example, we consider the linearized Navier-Stokes around Couette flow \((y, 0, 0)\):

\[
\partial_t u - \nu \Delta u + y \partial_x u + (u_2, 0, 0) + \nabla P = 0.
\]

The zero mode \(\overline{u}(t; y, z) = \int_T u \, dx\) enjoys

\[
\partial_t \overline{u} - \nu \Delta \overline{u} + (\overline{u}_2, 0, 0) = 0.
\]

It is easy to see that

\[
\overline{u}_1 \sim t e^{-\nu t} \sim t \quad \text{for} \quad t \lesssim \nu^{-1}.
\]

This is so called the transient growth.
The linearized 2-D Euler equation around Couette flow $(y, 0)$:

$$\partial_t \omega + y \partial_x \omega = 0 \implies \omega(t; x, y) = \omega_0(x - ty, y).$$

The result of Orr (1907): if $\int_T \omega_0 \, dx = 0$, then

$$\|u(t)\|_{L^2} \sim t^{-1} \to 0 \text{ as } t \to \infty.$$

This is so called the inviscid damping.
Enhanced dissipation (Couette flow)

The linearized 2-D Navier-Stokes equation around Couette flow in the vorticity:

$$\partial_t \omega - \nu \Delta \omega + y \partial_x \omega = 0, \quad \omega|_{t=0} = \omega_0.$$ 

If $$\int_T \omega_0 \, dx = 0$$, then

$$\|\omega(t)\|_{L^2} \lesssim e^{-\nu \frac{1}{3} t} \|\omega_0\|_{L^2}.$$ 

This decay rate $$\nu \frac{1}{3}$$ is much faster than the heat diffusion rate $$\nu$$, which is so called the enhanced dissipation.
Our main results (Linear enhanced dissipation)

Theorem (D.-Lin-Zhang, 2019)

For any $0 < \nu \leq 1$, and $k = (k_1, k_3) \in \mathbb{Z}^2$ with $k_1 \neq 0$, the solution of (1.11) satisfies the following estimates

$$\|\hat{\Phi}(t; k_1, \cdot, k_3)\|_{L^2} \leq Ce^{-at}\|\hat{\Phi}(0; k_1, \cdot, k_3)\|_{L^2},$$

and

$$\|\hat{\omega}_2(t; k_1, \cdot, k_3)\|_{L^2} \leq Ce^{-at}\|\hat{\omega}_2(0; k_1, \cdot, k_3)\|_{L^2}$$

$$\quad + \frac{Ce^{-at}(1 + at)}{|k_1|}\|\hat{\Phi}(0; k_1, \cdot, k_3)\|_{L^2},$$

where $a = c|k_1\nu|^{\frac{1}{2}} + \nu|k|^2$ with $c \in (0, 1)$, $\Phi := \Delta u_2$; $\hat{\phi}(k_1, y, k_3)$: Fourier transform $\phi(x, y, z)$ with respect to the directions $(x, z)$. 

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Our main results (Stability threshold of Poiseuille flow $(1 - y^2, 0, 0)$)

Theorem (D.-Lin-Zhang, 2019)

Suppose that there exists $c_0 \in (0, 1)$ such that $\|u_0\|_{H^2} + \|\partial_z u_0\|_{H^2} \leq c_0 \nu^{\frac{7}{4}}$, then the solution $u$ of (1.10) is global in time with

$$
\|\partial_z u_2\|_{H^2} + \|u\|_{H^2} + e^{c' \sqrt{\nu} t} \|(\Delta u_2)\|_{L^2} \\
+ e^{c' \sqrt{\nu} t} \|\partial_z (\Delta u_2)\|_{L^2} + e^{c' \sqrt{\nu} t} \|\partial_x \omega_2\|_{L^2} \\
+ \|P_0 u_3\|_{H^1} + \|\partial_z P_0 u_3\|_{H^1} \leq C(\|u_0\|_{H^2} + \|\partial_z u_0\|_{H^2}),
$$

and

$$
\|u\|_{H^2} + \|\partial_z u\|_{H^2} \leq C \nu^{-1}(\|u_0\|_{H^2} + \|\partial_z u_0\|_{H^2}),
$$

where

$$
P_0 f = \int_T f(x, y, z) dx, \quad f_\neq = f - P_0 f.
$$
Some remarks

- The decay rate $\sqrt{\nu}$ in enhanced dissipation corresponds to the structure of Poiseuille flow (Couette flow: $\nu^{1/3}$), which is faster than that of hear diffusion;
- The transition threshold $\gamma \leq \frac{7}{4}$ also holds for the 3-D Kolmogorov flow and it may not be optimal;
- Compared with the results obtained by Zelati et al. for the 2-D Poiseuille flow in $\mathbb{T} \times \mathbb{R}$, there is no logarithmic loss in our results.
Main idea of our proof

- **Resolvent estimates (sharp):** including the resolvent estimates for Orr-Sommerfeld equations with and without the nonlocal term;
- **Pseudospectra bounds and semigroup bound (sharp):** obtained via Gearhart-Prüss type lemma (Wei, 2019, SCM) \( \Rightarrow \) enhanced dissipation
- **Continuity argument (for nonlinear stability):**
  Suppose that there exists small \( \epsilon_1 \in (0, 1) \) (independent of \( \nu, k, \lambda \)), such that for any \( 0 < \nu < 1, T > 0 \), if the solution \( \partial^k_z u \in C([0, T]; H^2) \cap L^2([0, T; H^3]) (k = 0, 1) \) of (1.10) satisfies

  \[
  E_0(T) \leq \epsilon_1 \nu, \quad E_1(T) \leq \epsilon_1 \nu^{\frac{3}{4}} \quad \text{(needed in estimates)},
  \]

  \[
  \implies E_0(T) \leq C_1(\|u_0\|_{H^2} + \|\partial_z u_0\|_{H^2}),
  \]

  \[
  E_1(T) \leq C_1 \nu^{-1}(\|u_0\|_{H^2} + \|\partial_z u_0\|_{H^2}),
  \]
Main idea of our proof

where $C_1$ is independent of $\nu, k, \lambda$ and

$$E_0(T) = \sup_{t \in [0, T]} \left( \| \partial_z u_2 \|_{H^2} + \| u \|_{H^2} + e^{c' \sqrt{\nu} t} \| (\Delta u_2) \neq \| L^2 \right.$$  
$$+ e^{c' \sqrt{\nu} t} \| \partial_z (\Delta u_2) \neq \| L^2 + e^{c' \sqrt{\nu} t} \| \partial_x \omega_2 \|_{L^2}$$
$$+ \| P_0 u_3 \|_{H^1} + \| \partial_z P_0 u_3 \|_{H^1} \right),$$

$$E_1(T) = \sup_{t \in [0, T]} \left( \| u \|_{H^2} + \| \partial_z u \|_{H^2} \right).$$

Continuity argument: Let $(\| u_0 \|_{H^2} + \| \partial_z u_0 \|_{H^2}) \leq c_0 \nu^{7/4}$, $E_0(T) \leq \varepsilon_1 \nu$, $E_1(T) \leq \varepsilon_1 \nu^{3/4}$. Then choosing $c_0 = \frac{\varepsilon_1}{2C_1}$ yields $E_0(T) \leq \frac{\varepsilon_1}{2} \nu$ and $E_1(T) \leq \frac{\varepsilon_1}{2} \nu^{3/4}$ from above estimates!!! (This is the continuity argument!)
The Orr-Sommerfeld equations with Navier-slip boundary conditions

- The Orr-Sommerfeld equations without the nonlocal term:

\[
\begin{align*}
-\nu (\partial_y^2 - |k|^2)w + ik_1 (1 - y^2 - \lambda)w &= F_1, \\
w(\pm 1) &= 0.
\end{align*}
\]  
(1.14)

- The Orr-Sommerfeld equations with the nonlocal term:

\[
\begin{align*}
-\nu (\partial_y^2 - |k|^2)w + ik_1 [(1 - y^2 - \lambda)w + 2\varphi] &= F, \\
w(\pm 1) &= 0,
\end{align*}
\]  
(1.15)

where \((\partial_y^2 - |k|^2)\varphi = w\) with \(\varphi(\pm 1) = 0\), the nonlocal term: \(\varphi = (\partial_y^2 - |k|^2)^{-1}w\).
Sketch of the proof

Step 1: Resolvent estimates.

**Lemma (Without the nonlocal term)**

Suppose that \( w \in H^2(I) \) is the solution of (1.14) with \( \lambda \in \mathbb{R} \) and \( F_1 \in L^2(I) \). Then there holds that

\[
\nu \frac{1}{2} \left| k_1 \right| \frac{1}{2} \| w \|_{L^2} \leq C \| F_1 \|_{L^2}. \tag{1.16}
\]

**Lemma (With the nonlocal term)**

Suppose that \( w \in H^2(I) \) is the solution of (1.15) with \( \lambda \in \mathbb{R} \) and \( F \in L^2(I) \). Then there holds that

\[
\nu \frac{1}{2} \left| k_1 \right| \frac{1}{2} \| w \|_{L^2} \leq C \| F \|_{L^2}. \tag{1.17}
\]
The proofs for the above two Lemmas are very lengthy (19 pages!) since the Poiseuille flow is a non-monotone flow. Especially for the case with nonlocal term, the proof is much more complicated. An example without nonlocal term:

For $\delta \sim \nu^{\frac{1}{4}}|k_1|^{-\frac{1}{4}}$ and $\lambda > 1$, key estimate:

$$\int_{-1}^{1} (\lambda - 1 + y^2)|w|^2 dy \geq \int_{[-1,1]\setminus(-\delta,\delta)} (\lambda - 1 + y^2)|w|^2 dy$$

$$\geq \delta^2 \int_{[-1,1]\setminus(-\delta,\delta)} |w|^2 dy,$$

Other case can be obtained by similar arguments.
Sketch of the proof: Semigroup bound

Step 2: Pseudospectra bounds, semigroup bound and enhanced dissipation.

Recall

\[ \tilde{\mathcal{L}}_1 = -\nu (\partial_y^2 - |k|^2) + ik_1[(1 - y^2) + 2(\partial_y^2 - |k|^2)^{-1}] \]

and

\[ \tilde{\mathcal{L}}_2 = -\nu (\partial_y^2 - |k|^2) + ik_1(1 - y^2). \]

Introduce

\[ \Psi(A) := \inf\{\|(A - i\lambda)\| : f \in D(A), \lambda \in \mathbb{R}, \|f\| = 1\}. \]

Lemma (Wei, SCM, 2019)

Let \( A \) be an \( m \)-accretive operator on a Hilbert spaces \( X \), then \( \|e^{-tA}\| \leq e^{-t\Psi(A) + \frac{\pi}{2}} \) for any \( t \geq 0 \).
Sketch of the proof: Enhanced dissipation

Lemma (Semigroup bound)

There exist constants $C, c > 0$, independent of $\nu$ such that

$$\left\| e^{-t\mathcal{L}_i} g \right\|_{L^2} \leq C e^{-c\nu \frac{1}{2} t - \nu t} \left\| g \right\|_{L^2}, \ i = 1, 2. \quad (1.18)$$

$$\implies \left\| \hat{\Phi}(t; k_1, \cdot, k_3) \right\|_{L^2} \leq C e^{-c\nu \frac{1}{2} t - \nu t} \left\| \hat{\Phi}(0; k_1, \cdot, k_3) \right\|_{L^2}. $$

This is the enhanced dissipation for $(\Delta u_2) \neq 0$. (follows from the semigroup bound directly)
Sketch of the proof: Enhanced dissipation

To study the enhanced dissipation for $g$, we consider

$$\begin{aligned}
(\partial_t + \hat{L}_1)\phi &= 0, \\
(\partial_t + \hat{L}_2)g &= 2ik_3y(\partial_y^2 - |k|^2)^{-1}\phi, \\
(\phi, g)|_{t=0} &= (\phi_0, g_0).
\end{aligned} \tag{1.19}$$

Idea (deal with the nonlocal term in (1.19)$_2$):

$$g_1 = g + \frac{k_3}{k_1} y\phi, \implies (\partial_t + \hat{L}_2)g_1 = -\nu \frac{k_3}{k_1} y\partial_y \phi. \implies OK!$$

The enhanced dissipation can be obtained by semigroup bound.
Sketch of the proof: Nonlinear problem

Step 3: Nonlinear stability (*Space-time estimates and closing the energy estimates*).

We consider the nonlinear problem.

\[
\begin{align*}
(\partial_t + L_1) \Delta u_2 &= \nabla \cdot f, \\
(\partial_t + L_2) \omega_2 - 2y\partial_z u_2 &= \nabla \cdot G, \\
\Delta u_2(t; x, \pm 1, z) &= \omega_2(t; x, \pm 1, z) = 0, \\
(\Delta u_2, \omega_2)|_{t=0} &= (\Delta u_2(0), \omega_2(0)),
\end{align*}
\]

where \( f = -\nabla (u \cdot \nabla u_2) - (0, \Delta p, 0) \) and \( G = (\omega_1, \omega_2, \omega_3)u_2 - (u_1, u_2, u_3)\omega_2. \)
We establish the following estimates.

**Lemma**

\[
\| \partial_z^k (\Delta u_2) \neq 0 \|_{L^2}^2 \lesssim \| \partial_z^k (\Delta u_2) \neq 0 \|_{L^2}^2 + \nu^{-1} \| e^{c' \sqrt{\nu t}} \partial_z^k f \neq 0 \|_{L^2}^2,
\]  

(1.21)

and

\[
\| \partial_x \omega_2 \|_{L^2}^2 \lesssim \| \partial_x \omega_2(0) \|_{L^2}^2 + \| (\Delta u_2) \neq 0 \|_{L^2}^2 + \| \partial_z (\Delta u_2) \neq 0 \|_{L^2}^2 \\
+ \nu^{-1} \left( \| e^{c' \sqrt{\nu t}} \partial_x G \|_{L^2}^2 + \| e^{c' \sqrt{\nu t}} f \neq 0 \|_{L^2}^2 + \| e^{c' \sqrt{\nu t}} \partial_z f \neq 0 \|_{L^2}^2 \right),
\]  

(1.22)

where \( k = 0, 1 \).
Sketch of the proof: Nonlinear problem

The energy functional:

$$\|u\|_{X^{c'}}^2 = \|e^{c'\sqrt{\nu}t}u\|_{L^\infty L^2}^2 + \sqrt{\nu}\|e^{c'\sqrt{\nu}t}u\|_{L^2 L^2}^2 + \nu\|e^{c'\sqrt{\nu}t}\nabla u\|_{L^2 L^2}^2.$$ 

In the norm $\|\cdot\|_{X^{c'}}$, the first two parts correspond to the enhanced dissipation whose decay rate $e^{-c\sqrt{\nu}t}$ is resulted from the sharp resolvent estimates, and the last part is due to the combined effect of heat diffusion and enhanced dissipation.
Sketch of the proof: Nonlinear problem

For the nonlinear problem, the enhanced dissipation with rate $\sqrt{\nu}$ plays an important role in nonlinear problem. For example:

\[
\| e^{c'} \sqrt{\nu t} \partial_x G \|_{L^2 L^2}^2 + \| e^{c'} \sqrt{\nu t} f \# \|_{L^2 L^2}^2 + \| e^{c'} \sqrt{\nu t} \partial_z f \# \|_{L^2 L^2}^2 \\
\lesssim E_0^2 \nu^{-1} \| \partial_x \omega_2 \|_{X_{c'}}^2 + E_0^2 \nu^{-1} \| \Delta u_2 \|_{Y_0}^2 \\
+ E_1^2 \nu^{-\frac{1}{2}} (\| (\Delta u_2) \# \|_{X_{c'}}^2 + \| \partial_z (\Delta u_2) \# \|_{X_{c'}}^2 + \| \partial_x \omega_2 \|_{X_{c'}}^2).
\]

and

\[
(\Delta u_2) \# \|_{X_{c'}}^2 + \| \partial_z (\Delta u_2) \# \|_{X_{c'}}^2 + \| \partial_x \omega_2 \|_{X_{c'}}^2 \\
\lesssim (\| u(0) \|_{H^2} + \| \partial_z u(0) \|_{H^2})^2 + \nu^{-1} \left( E_0^2 \nu^{-1} \| \partial_x \omega_2 \|_{X_{c'}}^2 + E_0^2 \nu^{-1} \| \Delta u_2 \|_{Y_0}^2 \\
+ E_1^2 \nu^{-\frac{1}{2}} (\| (\Delta u_2) \# \|_{X_{c'}}^2 + \| \partial_z (\Delta u_2) \# \|_{X_{c'}}^2 + \| \partial_x \omega_2 \|_{X_{c'}}^2)\right).
\]
To close the energy estimates, the following estimates are needed:

$$
\sum_{k=0}^{1} \left( \| \partial_z^k P_0 \Delta u_2 \|^2_{Y_0} + \| \partial_z^k \nabla P_0 u_3 \|^2_{Y_0} + \nu^{-1} \| \partial_z^k \Delta p \|^2_{L^2 L^2} 
+ \| \partial_z^k (\Delta u_2) \|^2_{X_{c'}} \right) + \| \partial_x \omega_2 \|^2_{X_{c'}} \leq C(\| u(0) \|_{H^2} + \| \partial_z u(0) \|_{H^2})^2,
$$

$$
\sum_{k=0,1} (\| \partial_z^k \Delta P_0 u_3 \|^2_{Y_0} + \| \partial_z^k \Delta u \|^2_{X_{c'}}) \lesssim \nu^{-3/2} (\| u_0 \|_{H^2} + \| \partial_z u_0 \|_{H^2})^2,
$$

$$
E_0 \leq C(\| u_0 \|_{H^2} + \| \partial_z u_0 \|_{H^2}), \quad E_1 \leq C \nu^{-1} (\| u_0 \|_{H^2} + \| \partial_z u_0 \|_{H^2}).
$$

Finally, the standard continuity argument yields the conclusion.
Thanks for your attention!